DIVISORIAL PRIME IDEALS IN PRÜFER DOMAINS

BY

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ABSTRACT. Given a Prüfer domain $R$ and a prime ideal $P$ in $R$, we study some conditions which force $P$ to be a divisorial ideal of $R$. This paper extends some recent work of Huckaba and Papick.

1. Introduction. Let $R$ be an arbitrary Prüfer domain and $P \in \text{Spec}(R)$. In this paper we study some conditions which force $P$ to be divisorial, i.e., $P = P_v$. This work expands upon a recent paper of Huckaba and Papick [5]. In particular we generalize [5, Proposition 3.10] and [5, Proposition 3.11]. Unexplained terminology and unreferenced facts about Prüfer domains may be found in [3].

2. Some sufficient conditions for $P$ to be divisorial. Let $R$ be an arbitrary Prüfer domain with quotient field $K$, and $P$ a nonzero prime ideal of $R$. It is known that if $P$ is maximal, then $P$ is divisorial if and only if $P$ is invertible [5, Corollary 3.4]. Hence, we shall concentrate on nonzero, non-maximal prime ideals of $R$.

Let $P$ be a nonzero, non-maximal prime ideal of $R$. We know that $P^1$ is a subring of $K$ [5, Theorem 3.8] and in particular $P^{-1} = (P :_K P)$ [5, Proposition 2.3], as well as $P^{-1} = R_P \cap (\bigcap \alpha R_{M_\alpha})$, where $\{M_\alpha\}$ is the set of maximal ideals of $R$ not containing $P$ [5, Theorem 3.2]. Hence we have the following inclusion of rings:

$$R \subseteq P^{-1} \subseteq S \subseteq K \cap \left(\bigcap \alpha R_{M_\alpha}\right).$$

We shall prove that if $P^{-1} \not\subseteq S$, then $P$ is divisorial. However, first let us consider a somewhat novel result which is at the opposite extreme of our Prüfer setting.

PROPOSITION 2.0. Let $R$ be an arbitrary integral domain with quotient field $K$ and $(0) \neq P \in \text{Spec}(R)$. If $P^{-1}$ is not a subring of $K$, then $P$ is divisorial.

Proof. Since $P^{-1}$ is not a subring of $K$, then $(P :_K P) \not\subseteq P^{-1}$. Let $J = (R : P^{-1})$. Recall that $J = P_v$ [5, Lemma 2.1]. To complete the proof we will show that
J = P. It suffices to prove that \( J \subseteq P \). Let \( r \in J \). Since \( rP^{-1}P \subseteq P \) and \( PP^{-1} \not\subseteq P \), it follows that \( r \in P \). Hence, \( J = P \).

We are now prepared to state our main result.

**Theorem 2.1.** Let \( R \) be a Prufer domain with quotient field \( K \), and \( P \) a nonzero, non-maximal prime ideal of \( R \). If \( P^{-1} \not\subseteq S = K \cap (\bigcap M_\alpha) \), where \( \{M_\alpha\} \) is the set of maximal ideals of \( R \) not containing \( P \), then \( P \) is divisorial.

Before we establish Theorem 2.1, a lemma is needed.

**Lemma 2.2.** Same notation as the theorem. Then \( P^{-1} \not\subseteq S \) if and only if there exists a finitely generated ideal \( I \) of \( R \) such that \( I \subseteq P \) and \( I \not\subseteq M_\alpha \) for each \( \alpha \).

**Proof.** Recall that \( P^{-1} = R_p \cap S \), and use [4, Corollary 2].

**Proof of Theorem 2.1.** Since \( R \) is a Prufer domain, it suffices to show that \( P \) is an intersection of finitely generated ideals of \( R \). Let \( I \) be a finitely generated ideal of \( R \) such that \( I \subseteq P \) and \( I \not\subseteq M_\alpha \) for each \( \alpha \). For \( a \in R \setminus P \), we claim that \( P \subseteq (I, a) \). It is enough to check this assertion locally. For \( M \in \{M_\alpha\} \), we obviously have \( R_M = (I, a) R_M = PR_M \). If \( M \not\in \{M_\alpha\} \), then \( PR_M \subseteq aR_M = (I, a) R_M \) in the valuation ring \( R_M \). Finally, we wish to show that \( P = \bigcap \{(I, r) : r \in R \setminus P\} \). Since \( P \) is non-maximal, it will suffice to show for \( M \) maximal with \( P \subseteq M \), and \( r \in M \setminus P \) that \( r \notin (I, r^2) \). This follows since \( r \notin (r^2) R_M = (I, r^2) R_M \).

**Corollary 2.3.** Same notation as the theorem. If \( P \not\subseteq \bigcup M_\alpha \), then \( P \) is divisorial.

**Proof.** Let \( a \in P \setminus \bigcup M_\alpha \) and set \( I = (a) \). The desired conclusion follows from Lemma 2.2 and Theorem 2.1.

**Corollary 2.4.** Same notation as the theorem. If \( P \) is the radical of an invertible ideal \( I \), then \( P \) is divisorial.

**Proof.** Apply Lemma 2.2 and Theorem 2.1.

**Corollary 2.5** [5, Proposition 3.10]. Same notation as the theorem. If \( P \) is contained in all but a finite number of maximal ideals, then \( P \) is divisorial.

**Proof.** Use Corollary 2.3 and Theorem 2.1 to obtain the result.

Before stating our final corollary, we need some terminology. A domain \( R \) has property \((\#)\) if \( \bigcap_{M \in V_1} R_M \neq \bigcap_{M \in V_2} R_M \) for any two distinct subsets \( V_1 \) and \( V_2 \) of \( \text{Max}(R) \); \( \text{Max}(R) \) being the set of maximal ideals of \( R \).

**Corollary 2.6.** Let \( R \) be a Prufer domain having each overring satisfy property \((\#)\). If \( P \) is a nonzero, non-maximal prime ideal of \( R \), then \( P \) is divisorial.

**Proof.** This follows immediately from [4, Theorem 3], Lemma 2.2, and Theorem 2.1.
Corollary 2.7. Same notation as the theorem. If $P = PR_p$, then $P$ is divisorial.

Proof. The fact that $P = PR_p$, implies that $P$ is comparable with all ideals of $R$, and in particular, $P$ is contained in each maximal ideal of $R$. Hence $P$ is divisorial by Corollary 2.5.

Remark 2.8. There exists a nonzero, non-maximal prime ideal $P$ of the ring of entire functions $R$ ($R$ is a Bézout domain) such that $P$ is not divisorial. In fact, $P^{-1} = R$ [5, Example 3.12].

3. The ideal transform of $P$. In this final section we study an interesting special case arising from the previous section. More specifically, let $R$ be a Prüfer domain and $P$ a nonzero, non-maximal prime ideal of $R$. Recall the ideal transform of $P$, $T(P) = \bigcup_{n=1}^{\infty} (R :_{P} P^n)$, and note that $T(P) = P_0 \cap (\bigcap_{\lambda} R_{M_\lambda})$, where $P_0 = \bigcap_{n=1}^{\infty} P^n$ and $\{M_\lambda\}$ is the set of maximal ideals of $R$ not containing $P$ [3, Exercise 11, p. 331]. Hence, since $P^{-1} = R_0 \cap (\bigcap_{\lambda} R_{M_\lambda})$ [5, Theorem 3.2], we have the following tower of rings:

$$R \subseteq P^{-1} \subseteq T(P) \subseteq S.$$ 

Note that if $P^{-1} \neq T(P)$, it is immediate from Theorem 2.1 that $P$ is divisorial. It is our intent to study when $P^{-1} \neq T(P)$, and as one consequence of our efforts we will give a different proof of the fact that $P$ is divisorial in this setting.

Lemma 3.0. Let $R$ be a Prüfer domain and $P$ a nonzero, non-maximal prime ideal of $R$. Then, $P$ is a prime ideal of $P^{-1}$. (Recall that $P$ is an ideal of $P^{-1}$, since $P^{-1} = (P :_{P} P)$ [5, Proposition 2.3].)

Proof. Since $P \in \text{Spec}(R)$, we know that $PR(x) \in \text{Spec}(R(x))$, where $R(x) = R[x]_U$, $U = \{f \in R[x] : c(f) = R\}$ [1, Theorem 4]. Also, $R(x)$ is a Bézout domain, as $R$ is a Prüfer domain [1, Theorem 4 and p. 558]. Hence the overring $P^{-1}(x)$ is a quotient ring of $R(x)$. Notice that $P(P^{-1}(x)) \neq P^{-1}(x)$ [3, Proposition 33.1(4)]. Hence, $PR(x)(P^{-1}(x)) = P(P^{-1}(x))$ is a prime ideal of $P^{-1}(x)$. Whence, there exists a $Q \in \text{Spec}(P^{-1})$ such that $P(P^{-1}(x)) = Q(P^{-1}(x))$ [1, Theorem 4]. Therefore $P = Q$ [3, Proposition 33.1(4)], and so $P$ is a prime ideal of $P^{-1}$.

We are now ready to analyze when $P^{-1} \notin T(P)$.

Theorem 3.1. Let $R$ be a Prüfer domain and $P$ a nonzero, non-maximal prime ideal of $R$. If $P^{-1} \notin T(P)$, then

(a) $P^{-1} \notin T(P)$ is a minimal extension, i.e., there are no rings properly between $P^{-1}$ and $T(P)$.

(b) $P$ is an invertible maximal ideal of $P^{-1}$.

(c) $P$ is a divisorial ideal of $R$. 

(d) \( T(P) = \bigcap Q = S' \) where \( \{Q_\alpha\} \) is the set of prime ideals of \( R \) not containing \( P \).

(e) \( P^{-n} \) is never a ring for \( n > 1 \).

**Proof.**  (a). Let us suppose \( A \) is a ring satisfying \( P^{-1} \subseteq A \subseteq T(P) \). Since \( T(P) \) and \( A \) are intersections of localizations of \( R \) at certain prime ideals of \( R \) (\( R \) is a Prüfer domain), there exists a prime ideal \( Q \) in \( R \) such that \( A \subseteq R_Q \) and \( T(P) \subseteq R_Q \). We claim \( P \subseteq Q \), for if \( P \not\subseteq Q \) there exists \( Q \in \text{Spec}(T(P)) \) such that \( T(P)_Q = R_Q \) [6, Exercise 16(c), p. 149]. This contradiction establishes our claim. Hence \( A \subseteq R_Q \subseteq R_P \), and so \( A \subseteq R_P \cap (\bigcap \alpha R_{M_\alpha}) = P^{-1} \) [5, Theorem 3.2]. Therefore \( A = P^{-1} \), and the proof is complete.

(b) Assume \( P \) is not a maximal ideal of \( P^{-1} \). (Recall by Lemma 3.0 that \( P \) is a prime ideal of \( P^{-1} \).) Since \( P^{-1} \subseteq T(P) \) is a minimal extension, we know that \( P^{-1} = (P_R :_{P^{-1}}) = P^{-2} \). However \( P^{-1} = P^{-2} \). Thus, since \( P^{-n} = (R : P^n) = ((R : P^{n-1}) : P) \), we can conclude by induction that \( P^{-n} = P^{-1} \) for each positive integer \( n \). Therefore \( P^{-1} = T(P) \), the desired contradiction.

(c) As \( P \) is a non-maximal prime ideal of \( R \), we see by (b) that \( P^{-1} \neq R \), and thus \( P \neq R \). Therefore, \( P = R_P \), as \( P \) is an ideal of \( P^{-1} \) [5, Lemma 2.1].

(d) Since \( T(P) \subseteq \bigcap \alpha R_{Q_\alpha} = S' \) [6, Exercise 16(d), p. 149], it suffices to show \( S' \subseteq T(P) \). Assume otherwise. As in part (a), there exists a prime ideal \( Q \in \text{Spec}(R) \) such that \( T(P) \subseteq R_Q \) and \( S' \not\subseteq R_Q \). Hence \( P \subseteq Q \), and so \( T(P) \subseteq R_Q \subseteq R_P \). Whence, \( T(P) \subseteq R_P \cap (\bigcap \alpha R_{M_\alpha}) = P^{-1} \), a contradiction. Therefore, \( T(P) = S' \).

(e) Suppose \( P^{-n} \) is a ring for some \( n > 1 \). Then \( P^{-n} = R_P \cap (\bigcap \alpha R_{M_\alpha}) = P^{-1} \) [5, Theorem 3.2], and by induction \( P^{-1} = T(P) \). This contradiction completes the proof.

**Remark 3.2.**  (a) Let \( R \) be an arbitrary integral domain with quotient field \( K \), and \( P \in \text{Spec}(R) \). Note that if \( P^{-1} \) is a subring of \( K \), then \( P^{-1} = (P : K) \) [5, Proposition 2.3]. Hence \( P \) is an ideal of \( P^{-1} \), but \( P \) need not be a prime ideal of \( P^{-1} \) [5, Example 2.5]. However, if \( R \) is a Prüfer domain, then Lemma 3.0 shows that \( P \in \text{Spec}(P^{-1}) \).

(b) The converse of Theorem 3.1 (b) is valid, i.e.; under the assumptions of Theorem 3.1, if \( P \) is an invertible maximal ideal of \( P^{-1} \), then \( P^{-1} \subseteq T(P) \). To see this notice that \( P^{-1} \subseteq (P^{-1} :_K P) = P^{-2} \subseteq T(P) \).

(c) The converse of Theorem 3.1(c) is not generally true. Let \( R \) be a valuation domain, and \( P \) a nonzero, non-maximal prime ideal of \( R \) such that \( P = P^2 \). Then \( P^{-1} = T(P) \), yet \( P = P_v \) (Corollary 2.5).
(d) The converse of Theorem 3.1(d) is not generally true. Let $R$ be a valuation domain and $P$ a nonzero, non-maximal prime ideal of $R$ such that $P$ is unbranched, i.e., $P = \bigcup_{Q \in \text{Spec}(R)} Q$. Observe that $P^{-1} = R_p$ [5, Corollary 3.6] and $S' = \bigcap_{Q \in \text{Spec}(R)} R_Q = R_p$. Therefore,

$$P^{-1} = R_p \subseteq T(P) \subseteq S' = R_p,$$

and so $T(P) = S'$, yet $P^{-1} = T(P)$.

(e) The converse of Theorem 3.1(e) is obviously true.

REFERENCES