Journal of Pure and Applied Algebra 39 (1986) 95-103
North-Holland


# SOME PROPERTIES OF DIVISORIAL PRIME IDEALS IN PRÜFER DOMAINS

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Communicated by C.A. Weibel
Received 10 July 1984

## 1. Introduction

Let $P$ be a prime ideal of a Prüfer domain $R$. In [3] we studied when $P$ is a divisorial ideal; i.e., when $(P^{-1})^{-1}=P$. If $P$ is a maximal ideal, then it is known that $P$ is divisorial if and only if $P$ is finitely generated, [10, Corollary 3.4]. When $P$ is a non-maximal prime ideal, we gave several sufficient conditions for $P$ to be divisorial, [3]. However, the characterization of non-maximal divisorial prime ideals was left open.

In Section 2 of this paper, we estabish in Proposition 9 the desired characterization of divisorial prime ideals. Also, in Proposition 7, we give equivalent conditions for a non-idempotent prime ideal with the property that $P^{-1}=T(P)$ to be divisorial, and in Theorem 8, we characterize those prime ideals for which each power is divisorial.

In Section 3, applications are given for a special class of Prüfer domains. In particular, for Prüfer domains for which each overring satisfies (#), (see [6]), it is proved that the product of divisorial prime ideals is divisorial. Finally, examples are given to show that each prime ideal of a Prüfer domain may be divisorial, yet not all ideals of the ring are divisorial. That is, there is no Cohen type theorem for divisorial ideals in Prüfer domains.

The following notation will be fixed throughout this paper. Let $R$ be a Prüfer domain with quotient field $K$, and let Spec($R$) denote the set of prime ideals of $R$. If $P\in \mathrm{Spec}(R)$, let $\{M_\alpha\}$ denote the set of maximal ideals of $R$ that do not contain $P$. Define $S=(\bigcap R_{M_\alpha})\cap K$. The ideal transform of $P$ is $T(P)=\bigcup_{n=1}^{\infty}(R_K:P^n)$. When no ambiguity may arise, write $(R:I)$ instead of $(R_K:I)$. For the prime ideal $P$,

* This work was supported in part by a NATO Senior Fellowship.
** This work was supported in part from a University of Missouri Summer Research Fellowship.

define $P_0=\bigcap_{n=1}^{\infty}P^n$. It is well known that $P_0$ is a prime ideal in $R$ and if $P\neq P^2$, then $P_0$ is the unique prime ideal of $R$ that is maximal with respect to being properly contained in $P$. Finally, we follow the common practice of writing $P_v$ for $(P^{-1})^{-1}$ and $P^{-n}$ for $(P^n)^{-1}$.

The following proposition collects the known results that are needed for this paper.

**Proposition 0.** *Let $P$ be a non-maximal prime ideal in $R$.*

(A) [10, Theorems 3.2 and 3.8, and Prop. 3.9]. *The fractional ideal $P^{-1}$ is a ring, in fact, $P^{-1}=R_P\cap(\bigcap R_{M_\alpha})=(P:P)$.*

(B) [5, Ex. 11, p. 331]. *The following ring inclusions hold: $R\subseteq P^{-1}\subseteq T(P)\subseteq S$.*

(C) [3, Theorem 2.1]. *If $P^{-1}\neq S$, then $P$ is a divisorial ideal.*

(D) [6, Corollary 2]. *A necessary and sufficient condition for $P^{-1}\neq S$ is the existence of a finitely generated ideal $I$ such that $I\subseteq P$ and $I\nsubseteq M_\alpha$, for each $\alpha$.*

(E) [3, Theorem 3.1]. *If $P^{-1}\neq T(P)$, then:*

(i) *$T(P)$ is a minimal extension of $P^{-1}$; i.e., there are no rings properly between $P^{-1}$ and $T(P)$.*

(ii) *$P$ is an invertible prime ideal of $P^{-1}$.*

(iii) *$P$ is a divisorial prime ideal of $R$.*

(iv) *$T(P)=\bigcap R_{Q_\alpha}$, where $\{Q_\alpha\}$ is the set of prime ideals of $R$ not containing $P$.*

(v) *$P^{-n}\equiv(R:P^n)$ is not a ring for $n>1$.*

## 2. Powers of prime ideals

We begin with two elementary lemmas.

**Lemma 1.** *Let $M$ be a maximal ideal of $R$. Then $M^n$ is a divisorial ideal for each $n\geq 1$ if and only if $M$ is finitely generated.*

**Proof.** ($\Rightarrow$) Use Corollary 3.4 of [10].

($\Leftarrow$) If $M$ is finitely generated, so is $M^n$. Thus each $M^n$ is an invertible ideal of $R$. □

Note that for each prime ideal $P$ of $R$,

$$(R:T(P))=\left[R:\bigcup_{n=1}^{\infty}(R:P^n)\right]=\bigcap_{n=1}^{\infty}[R:(R:P^n)]=\bigcap_{n=1}^{\infty}(P^n)_v.$$

Thus we have proved the following lemma:

**Lemma 2.** *If $P\in\operatorname{Spec}(R)$, then $(R:T(P))=\bigcap_{n=1}^{\infty}(P^n)_v$.*

**Proposition 3.** *Let $P\in\operatorname{Spec}(R)$. Then $P^n$ is divisorial for each $n\geq 1$ if and only if $(R:T(P))=P_0$.*

**Proof.** ($\Rightarrow$) By Lemma 2, $P_0=\bigcap P^n=\bigcap (P^n)_v=(R:T(P))$.

($\Leftarrow$) There are three cases to consider.

*Case* 1. Assume that $P=M$ is a maximal ideal of $R$. Note that $R\subsetneqq M^{-1}$. For suppose that $R=M^{-1}$. Then by induction on $n$, we have

$$M^{-n}=(R:M^n)=[(R:M):M^{n-1}]=(R:M^{n-1})=M^{-(n-1)}=R.$$

Thus, $T(M)=\bigcup M^{-n}=R$. But this contradicts the hypothesis that $(R:T(M))=M_0\subsetneqq R$. It follows from Corollary 3.4 of [10] that $M$ is finitely generated. Lemma 1 completes the proof of this case.

*Case* 2. Assume that $P$ is a non-maximal prime ideal of $R$ such that $P^{-1}=T(P)$. Then $P_0=(R:T(P))=(R:P^{-1})=P_v$. Thus, $P=P^n=P_v$ for each $n\geq 1$.

*Case* 3. Assume that $P$ is a non-maximal prime ideal of $R$ such that $P^{-1}\subsetneqq T(P)$. From Proposition 0, $P^{-1}$ is an overring of $R$, $P$ is divisorial in $R$, and $P$ is an invertible prime ideal in $P^{-1}$. Using these facts, we see that

$$\begin{aligned}(P^n)_v&=R:(R:P^n)=R:((R:P):P^{n-1})\\&=R:(P^{-1}:P^{n-1})=R:(P^{-1}:P^{n-1})P^{-1}\\&=(R:P^{-1}):(P^{-1}:P^{n-1})=(P:P^{-n}).\end{aligned}$$

Let $z\in(P^n)_v$. Then $zP^{-n}\subseteq P$, and so $zP^{-n}P^{n-1}\subseteq PP^{n-1}=P^n$. Hence $zP^{-1}\subseteq P^n$, since $P^{n-1}$ is invertible in $P^{-1}$. Therefore, $z\in P^n$ and thus $(P^n)_v=P^n$. □

**Corollary 4.** *Each power of a prime ideal $P$ is a divisorial ideal of $R$ if and only if $P^2$ is a divisorial ideal of $R$.*

**Proof.** Assume that $P^2$ is a divisorial ideal of $R$. If $P=M$ is a maximal ideal of $R$ and if $M$ is not finitely generated, then $M^{-1}=M^{-2}=R$, so $M^2$ is not divisorial, a contradiction. Now apply Lemma 1. Let $P$ be a non-maximal prime ideal of $R$. If $P^{-1}=T(P)$, then Case 2 of Proposition 3 shows that $P^2=P$. Hence $P^n=P$ is divisorial for each $n$. If $P^{-1}\subsetneqq T(P)$, Case 3 of Proposition 3 gives the desired conclusion. □

If a prime ideal $P$ is divisorial, is $P^n$ divisorial for each $n\geq 1$? The example presented below gives a negative answer to this question.

**Example 5.** We construct a Prüfer domain $R$ with a divisorial prime ideal $P$ such that $P\neq P^2$, yet $P^{-1}=T(P)$. It follows that $(P^2)_v=P$ (see Proposition 7); and hence $P^2$ is not divisorial.

Let $S$ be the ring of entire functions. It is well known that $S$ is a Bezout domain. We use the notation and results from M. Henriksen [8,9]. Let $M$ be a maximal free ideal of $S$ and let $K=S/M$. The field $K$ is a proper extension of the complex numbers $\mathbb{C}$. Hence, if $t\in K$ is transcendental over $\mathbb{C}$ and if $V_0$ is a nontrivial valuation domain on $\mathbb{C}(t)$, then $V_0$ can be extended to a nontrivial valuation domain $V$ on $K$.

Consider the pull-back

$$(*)\qquad \begin{array}{ccc} R=\phi^{-1}(V) & \longrightarrow & V \\ \big\downarrow \cap & & \big\downarrow \cap \\ S & \xrightarrow[\phi]{} & K=S/M \end{array}$$

where $\phi$ is the canonical homomorphism and the vertical arrows are inclusion maps. The ideal $M$ is the conductor of $S$ relative to $R$, so $R$ and $S$ have the same quotient field. It follows from properties of pullbacks that $M$ is a non-maximal prime ideal in the domain $R$, [2].

If $P\in \operatorname{Spec}(S)$, then $R_{P\cap R}=S_P$. (If $P\not\supseteq M$, then the equality holds by [2, p. 335] or [11, p. 46]. If $P\supseteq M$, that is $P=M$, let $s\in S$. From the pullback (*) we know that $V=R/M$ and that the quotient field of $V$ is $K=S/M$. Thus $s+M=(r+M)/(r'+M)$, where $r,r'\in R$ and $r'\notin M$. Then $sr'-r=m\in M$ which implies that $s=(r+m)/r'\in R_M$. Therefore, $S_M=R_M$.) We have proved that $S$ is a flat $R$-module. We claim that $R$ is a Prüfer domain. Let $I$ be a finitely generated ideal of $R$. We need only show that $I$ is $R$-projective. But $I$ is $R$-projective if and only if $I\otimes_R S$ is $S$-projective and $I/MI$ is $R/M$-projective, [12, Theorem 1.1]. Since $S$ is a flat $R$-module, $I\otimes_R S\cong IS$, and since $S$ is a Prüfer domain, $IS$ is a projective $S$-module. For the second condition, note that $(IM)S=I(MS)=IM$. Thus

$$I/IM\subseteq IS/IMS\cong IS\otimes_S S/M,$$

which is a vector space over $S/M$. It follows that $I/IM$ is a finitely generated torsion-free $R/M$-module and by [1, p. 133], $I/IM$ is a projective $R/M$-module. This proves the claim.

Let $Q=\bigcap M^n$. It is known that $Q$ is a prime ideal of $S$, and hence of $R$, that is properly contained in $M$. Treat both $Q$ and $M$ as $R$-ideals. Thus $Q^{-1}$ means $(R:Q)$ instead of $(S:Q)$. By Proposition 0, $Q^{-1}=R_Q\cap(\bigcap_{M_\alpha\not\supseteq Q}R_{M_\alpha})$. Let $N_\alpha$ be the unique maximal ideal in $S$ such that $N_\alpha\cap R=M_\alpha$. We have

$$Q^{-1}=R_Q\cap\left(\bigcap_{M_\alpha\not\supseteq Q}R_{M_\alpha}\right)=S_Q\cap(\bigcap S_{N_\alpha})=S,$$

since $S$ is the ring of entire functions, the set of fixed maximal ideals of $S$ is contained in $\{N_\alpha\}$, and $R_{P\cap R}=S_P$ for each $P\in\operatorname{Spec}(S)$. Thus, $Q_v=R:(R:Q)=(R:S)=M$. Therefore, $R\subsetneqq Q^{-1}$ and $Q_v\neq Q$.

It is easy to see that $M^{-1}=S$, so $M_v=M$; i.e., $M$ is a divisorial ideal in $R$. By [5, p. 133],

$$T(M)=R_Q\cap\left(\bigcap_{M_\alpha\not\supseteq Q}R_{M_\alpha}\right)=S=M^{-1}.$$

Thus $M$ is the prime ideal of $R$ with the desired properties.

**Remark 6.** A question that arose from our earlier study of prime ideals [3], was whether $P$ being divisorial is equivalent to $P^{-1}$ being different from $R$. The ideal $Q$ in Example 5 shows that this is not the case.

The next result characterizes those divisorial prime ideals of $R$ that arise as in Example 5.

**Proposition 7.** *Let $P$ be a non-idempotent prime ideal of $R$. Then the following conditions are equivalent:*

(1) $P=(P^n)_v$, *for each* $n \geq 1$.

(2) $P=P_v$ *and* $P^2 \neq (P^2)_v$.

(3) $R \subsetneq P^{-1}=T(P)$ *and* $(R:T(P))=P$.

*Furthermore, if $P$ satisfies any, and hence all, of these equivalent conditions, then $P$ is not a maximal ideal of $R$.*

**Proof.** Suppose that $M$ is a maximal ideal of $R$ satisfying either (1), (2), or (3). Then $M$ is finitely generated. Therefore $M^2$ is divisorial, a contradiction.

(1) ⇒ (2). Clear.

(2) ⇒ (3). Since $P$ is a divisorial ideal, $R \subsetneq P^{-1}$. Case 3 of Proposition 3 implies that $P^{-1}=T(P)$. Thus, $P=P_v=(R:P^{-1})=(R:T(P))$.

(3) ⇒ (1). Since $P^{-1}=T(P)$, it follows that $P^{-n}=P^{-1}$. Thus,

$$(P^n)_v=P_v=(R:P^{-1})=(R:T(P))=P. \qquad \square$$

We are ready for the main theorem.

**Theorem 8.** *Let $P$ be a prime ideal of a Prüfer domain $R$. The following conditions are equivalent:*

(1) $P^n$ *is divisorial, for each* $n \geq 1$.

(2) $(R:T(P))=P_0$.

(3) $P^2=(P^2)_v$.

(4) *Either* (a) $P^{-1} \subsetneq T(P)$, *or* (b) $P$ *is a divisorial idempotent ideal.*

**Proof.** We have already proved that (1), (2), and (3) are equivalent.

(1) ⇒ (4). Assume that $P^{-1}=T(P)$. Clearly $R \subsetneq P^{-1}=T(P)$ and $(R:T(P))=P$. Thus $P$ is an idempotent ideal of $R$, or else Proposition 7 is contradicted.

(4) ⇒ (3). First assume that $P$ is a non-maximal prime ideal. If $P^{-1} \subsetneq T(P)$ use Case 3 of Proposition 3. If $P=P^2$ and $P$ is divisorial, obviously $P^2=(P^2)_v$. If $P$ is a maximal ideal of $R$, then (b) cannot hold. Thus $P^{-1} \subsetneq T(P)$ implies $P$ is finitely generated. $\square$

We end this section by characterizing those prime ideals of a Prüfer domain that are divisorial.

**Proposition 9.** *Let $P$ be a prime ideal of a Prüfer domain $R$. Then $P$ is divisorial if and only if either $P^{-1} \neq T(P)$, or $(R : T(P)) = P$.*

**Proof.** ($\Rightarrow$) Assume that $P$ is a divisorial prime ideal such that $P^{-1} = T(P)$. Then $P = P_v = (R : P^{-1}) = (R : T(P))$.

($\Leftarrow$) If $P^{-1} \neq T(P)$, use Case 3 of Proposition 3. If $P^{-1} = T(P)$, then $P = (R : T(P)) = (R : P^{-1}) = P_v$.

## 3. Applications

Let Max($R$) be the set of maximal ideals of $R$. Define $R$ to be a (#)-*domain* if $\Delta_1$ and $\Delta_2$ are distinct subsets of Max($R$), then $\bigcap_{M \in \Delta_1} R_M \neq \bigcap_{M \in \Delta_2} R_M$. Say that $R$ is a (# #)-*domain* in case each overring of $R$ is a (#)-domain. These rings were studied by Gilmer and Heinzer in [6]. They proved that $R$ is a (# #)-domain if and only if for each prime ideal $P$ of $R$, there exists a finitely generated ideal $A \subseteq P$ such that each maximal ideal of $R$ containing $A$ contains $P$. It is shown in [10, Proposition 3.11] that if $R$ is a (# #)-domain, then each non-maximal prime ideal of $R$ is divisorial. We now give several other interesting properties of (# #)-domains.

**Lemma 10.** *Let $P \subsetneq Q$ be prime ideals of a Prüfer domain $R$. Then $Q$ blows up in the ring $P^{-1}$ (i.e., $QP^{-1} = P^{-1}$) if and only if there exists a finitely generated ideal $I$ of $R$ such that $P \subsetneq I \subseteq Q$.*

**Proof.** ($\Leftarrow$) If $P^{-1} \neq T(P)$, then $P$ is an invertible prime ideal of $P^{-1}$ (Proposition 0), and hence $P$ is maximal in $P^{-1}$. Thus, $QP^{-1} = P^{-1}$. Assume that $P^{-1} = T(P)$ and that $P \subsetneq I \subseteq Q$ for a finitely generated ideal $I$ of $R$. Extending to $P^{-1}$, $P = PP^{-1} \subseteq IP^{-1} \subseteq QP^{-1}$, which implies that

$$(P^{-1} : P) \supseteq (P^{-1} : IP^{-1}) \supseteq (P^{-1} : QP^{-1}) \supseteq P^{-1}.$$

But $(P^{-1} : P) = P^{-2}$ and $P^{-2} = P^{-1}$ (since $P^{-1} = T(P)$). Hence $(P^{-1} : IP^{-1}) = P^{-1}$. Since $I$ is an invertible ideal in the Prüfer domain $P^{-1}$, this can only happen if $IP^{-1} = P^{-1}$. Therefore, $P^{-1} = QP^{-1}$.

($\Rightarrow$) Assume that $QP^{-1} = P^{-1}$. Write $1 = \sum c_i u_i$, where $c_i \in Q$ and $u_i \in P^{-1}$. Let $I = (c_1, \ldots, c_n)$. Note that $P \subsetneq I \subseteq Q$, since if $p \in P$, then $p = \sum c_i(pu_i) \in I$. □

Say that $R$ has the *separation property* if for each pair of prime ideals $P \subsetneq Q$ there exists a finitely generated ideal $I$ such that $P \subsetneq I \subseteq Q$.

**Proposition 11.** *If $R$ is a (# #)-domain, then $R$ has the separation property.*

**Proof.** It suffices to show that for each non-maximal prime ideal $P$ of $R$, $P$ is maximal in $P^{-1}$. Suppose not. Say that $P$ is a non-maximal prime ideal in the Prüfer domain $P^{-1}$. Clearly the (# #) condition is inherited by overrings. Hence, $P$ is divisorial as an ideal of $P^{-1}$. It follows that $P^{-2}=(P^{-1}:P)\supsetneq P^{-1}$, thus $P^{-1}\neq T(P)$. Therefore, $P$ is an invertible prime ideal in $P^{-1}$ (Proposition 0). This contradiction completes the proof. □

From Example 5, we know that in an arbitrary Prüfer domain that the product of divisorial prime ideals may not be divisorial.

**Proposition 12.** *Let $R$ be a (# #)-domain. Then the product of finitely many divisorial prime ideals is a divisorial ideal.*

**Proof.** We first prove that if $P$ is a divisorial prime ideal in a (# #)-domain $R$, then $P^2$ is divisorial. We can assume that $P$ is not a maximal ideal. If $P^{-1}\neq T(P)$, then Theorem 8 implies that $P^2$ is divisorial. If $P^2=P$, there is nothing to prove. We are left with the case where $P\neq P^2$, $P^{-1}=T(P)$. We prove that this cannot happen in a (# #)-domain. Suppose this situation does occur. Then

$$R_p\cap(\bigcap R_{M_\alpha})=P^{-1}=T(P)=R_{P_0}\cap(R_{M_\alpha}).$$

The (# #)-property implies there exists a finitely generated ideal $A\subseteq P$ such that $A\not\subseteq M_\alpha$, for each $\alpha$. By [6, Corollary 1], $R_P\cap(\bigcap R_{M_\alpha})\neq R_{P_0}\cap(\bigcap R_{M_\alpha})$, a contradiction; hence the claim is established.

Consider the product $P_1^{t_1}\cdot P_2^{t_2}\cdots P_n^{t_n}$, where the $P_i$ are distinct divisorial prime ideals. Without loss of generality we may assume that the $P_i$ are incomparable, and hence pairwise comaximal. For suppose that $P_i\subsetneq P_j$. Then $P_iP_j$ can be replaced by $P_i$, [5, Theorem 23.3]. The claim and Corollary 4 imply that $P_i^{t_i}$ is divisorial for $i=1,\ldots,n$. Finally $P_1^{t_1}\cdot P_2^{t_2}\cdots P_n^{t_n}=\bigcap_{i=1}^n P_i^{t_i}$; and the intersection of divisorial ideals is divisorial. □

The converses of Propositions 11 and 12 are readily seen to be false. Let $D$ be an almost Dedekind domain that is not Dedekind. Then $D$ is not a (#)-domain, [4, Theorem 3]. Since $D$ is a one-dimensional ring, if $P\subsetneq Q$ are prime ideals in $D$, we have $P=(0)$ and $Q$ is maximal. Hence, $D$ is a separated domain. As for Proposition 12, the nonzero divisorial prime ideals of $D$ (same $D$ as above) are the finitely generated maximal ideals of $D$. Clearly, the product of finitely many of these ideals is still finitely generated and therefore is divisorial.

In [7], Heinzer characterized those Prüfer domains for which every ideal is divisorial. The question remains as to whether there is a 'Cohen type theorem' for divisorial ideals - that is, if each prime ideal of $R$ is divisorial is every ideal of $R$ divisorial? The answer is negative. In particular, $R$ may be a (# #)-domain in which each prime ideal is divisorial, yet there exist ideals of $R$ which are not divisorial. Let $V=\mathbb{Q}[[X]]=\mathbb{Q}+M$, where $\mathbb{Q}$ is the field of rational numbers and $M=XV$. Let $R=\mathbb{Z}+M$, where $\mathbb{Z}$ is the ring of integers. Then $R$ is a 2-dimensional Prüfer domain for which each maximal ideal is principal. In addition $R$ is a (# #)-domain, [6]. The prime ideal $M$ of $D$ is divisorial and is contained in each maximal ideal of $D$; hence, in infinitely many maximal ideals of $D$. Therefore, some ideal of $D$ is not divisorial, [7].

We conclude this paper with two results:

First, a technical sufficient condition for $P=P_v$ (Proposition 13); and second, a different characterization for $P=P_v$ in Prüfer domains of dimension $\leq 2$.

Consider the following conditions on a prime ideal $P$.

($\Delta$)
(i) There exists an ideal $I$ such that $P \subsetneqq I \subsetneqq R$.
(ii) If $\{N_\beta\}$ is the set of maximal ideals of $R$ that contain $P$ but not $I$ and if $\{P_\kappa\}$ is the set of minimal prime ideals of $I$, then $N_\beta, P_\kappa \subseteq \bigcup M_\alpha$ for each $\beta$ and $\kappa$.
(iii) Neither $N_\beta$ nor $P_\kappa$ is finitely generated.

**Proposition 13.** *If $P$ is a non-maximal prime ideal of $R$ such that $P \neq P_v$, then $P$ satisfies ($\Delta$).*

**Proof.** If $P \neq P_v$, then for every ideal $I$ between $P$ and $P_v$, $I_v = P_v$; and hence $I^{-1} = P^{-1}$. In particular, $I^{-1}$ is a ring. Thus,

$$I^{-1} = (\bigcap R_{P_\kappa}) \cap (\bigcap R_{N_\beta}) \cap (\bigcap R_{M_\alpha}).$$

By Proposition 0, $P \neq P_v$ implies that $P^{-1} = \bigcap R_{M_\alpha}$. It follows that

$$(*) \qquad R_{P_\kappa} \cap (\bigcap R_{M_\alpha}) = \bigcap R_{M_\alpha} = R_{N_\beta} \cap (\bigcap R_{M_\alpha})$$

for each $\kappa$ and $\beta$. From this we deduce that $P_\kappa, N_\beta \subseteq \bigcup M_\alpha$, for each $\kappa$ and $\beta$. (To see this, suppose that $f \in P_\kappa \setminus \bigcup M_\alpha$. Then $1/f \in \bigcap R_{M_\alpha} \setminus R_{P_\kappa}$ which contradicts (*). Therefore, $P_\kappa \subseteq \bigcup M_\alpha$.) Finally, part D of Proposition 0 implies that neither $P_\kappa$ nor $N_\beta$ is finitely generated.

**Proposition 14.** *Let $R$ be a Prüfer domain of dimension $\leq 2$ and let $P \in \operatorname{Spec}(R)$. Then $P = P_v$ if and only if $P^{-1} \neq R$.*

**Proof.** ($\Rightarrow$) Clear.

($\Leftarrow$) We need only prove this result for a non-maximal prime ideal $P$ of $R$. We know that $P^{-1} = R_p \cap (\bigcap R_{M_\alpha})$. Let $\{M_\beta\}$ be the maximal ideals of $R$ containing $P$. Separate $\{M_\beta\}$ into two disjoint sets $\{M_\varepsilon\}, \{M_\delta\}$, where $P_v \subseteq M_\varepsilon$ and $P_v \nsubseteq M_\delta$. If $P \neq P_v$, then each $M_\varepsilon$ is a minimal prime of $P_v$. By [10, Theorem 3.2],

$$P_v^{-1} = (\bigcap R_{M_\alpha}) \cap (\bigcap R_{M_\delta}) \cap (\bigcap R_{M_\varepsilon}) = R \subsetneqq P^{-1},$$

Therefore, $P_v = P$.

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