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The Prime Spaces as Spectral Spaces (*).

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1. – Introduction.

The prime spectrum of a general commutative unitary ring, endowed with the Zariski topology, is a well established tool in algebraic geometry. In the case of integral domains, an older topological tool introduced by O. ZARISKI, namely the abstract Riemann surface (whose underlying set is the collection of all valuation domains of some field containing the given integral domain, cf. [Z-S, p. 113]), is also available, but seems to have fallen into disuse.

The important role of valuation domains and the increasing needs of commutative ring theory have motivated several researchers to consider possible extensions of the concept of «valuation», looking at its multiform aspects (cf. for instance [S], [M], [J], [Co], [Gr 1], [Gr 2], [B], [Fr], [H] and [H-V]). In particular P. SAMUEL [S] and, successively, I. CONNELL [Co] have introduced similar extensions of the notation of «place» in a non-integral context, which have led to the construction of a new topological space (denoted Gam by Connell; the definition is given later), extending at the same time the abstract Riemann surface and the prime spectrum.

The purpose of this paper is twofold: (a) to pursue the topological study of the space Gam, initiated by Connell, establishing a very close connection with Hochster's theory of spectral spaces (cf. [Ho]); (b) to apply the theory of Gam spaces for a better understanding of Gilmer's $D+m$ constructions (cf. [Gi 1, Appendix 2]).

Before stating the main theorems of this paper (cf. the following Theorems 2.1, 2.2 and 2.5), we need to recall several known results from the already quoted papers by P. SAMUEL, I. CONNELL and M. HOCHSTER.

The following three properties were introduced by P. SAMUEL ([S]; see also [B, ch. 6, § 1. Ex. 6-8, pp. 169-170]) in order to extend the notion of valuation to the possibly non-integral domain case.

Let A be a subring of a ring (possibly not a domain) B.

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(P_1) There exist $\mathfrak{p} \in \mathrm{Spec}(A)$ such that for every pair $(C, \mathfrak{q})$ with $A \subsetneq C \subset B$ and $\mathfrak{q} \in \mathrm{Spec}(C)$ then $\mathfrak{q} \cap A \neq \mathfrak{p}$.

(P_2) The complement $S := B \setminus A$ is a multiplicatively closed subset of B.

(P_3) For every dominated polynomial $f(T_1, \ldots, T_r) \in A[T_1, \ldots, T_r]$, $r \geq 1$ (cf. [S, p. 123]), and for every family of r elements $\{s_1, \ldots, s_r\} \subset S := B \setminus A$, then $f(s_1, \ldots, s_r) \neq 0$.

We say that A is a (P_i)-*ring* in B if the property (P_i) is satisfied for the pair $A \subset B$, $(1 \leq i \leq 3)$.

Samuel has shown that:

$$(P_1) \Rightarrow (P_3) \Rightarrow (P_2)$$

and that the converse of these implications does not hold in general. Moreover, if $B = K$ is a field, then for the ring A the following equivalences hold:

$$(P_1)\text{-ring} \Leftrightarrow (P_2)\text{-ring} \Leftrightarrow (P_3)\text{-ring} \Leftrightarrow \text{valuation domain}.$$

It is also known (see [M] and [Gr 1] or also [Hu, Th. 5.1 and Th. 5.5]) that:

$$A \text{ is } (P_1) \text{ in } B \Leftrightarrow A \text{ is a valuation ring in } B;$$

$$A \text{ is } (P_3) \text{ in } B \Leftrightarrow A \text{ is a paravaluation ring in } B;$$

where A is called a *paravaluation* (resp., *valuation*) ring in B if there exists a map (resp., a surjective map) $v: B \to G \cup \{\infty\}$ from B to a totally ordered abelian group G extended with ∞, satisfying the classical properties:

(V_1) $v(xy) = v(x) + v(y)$, $\forall x, y \in B$;

(V_2) $v(x + y) \geq \min(v(x), v(y))$, $\forall x, y \in B$;

(V_3) $v(1) = 0$ and $v(0) = \infty$,

such that $A = \{b \in B \mid v(b) \geq 0\}$.

We collect (see [S] or [B]) some properties that we will use later. Let A be a (P_2)-ring in B and let $S := B \setminus A$. Suppose that $S \neq \emptyset$, then:

(1) $\mathfrak{f} := \{a \in A \mid as \in A, \ \forall s \in S\}$ is the conductor of $A \subset B$ and it is a prime ideal of A and B.

(2) A is a paravaluation ring in B if and only if $A/\mathfrak{f}$ is a paravaluation ring in $B/\mathfrak{f}$.

(3) Suppose now that B is an integral domain. Then, A is a paravaluation ring in B if and only if $A = V \cap B$, where V is a valuation domain of the quotient field of B.

Another possible extension of the notion of valuation was given by I. G. CONNELL [Co] by means of a generalized concept of place for commutative rings.

Let A be a fixed ring and let B be an A-algebra (for the sake of simplicity, we suppose $A \subset B$). A *specialization* ϕ *of an* A-*algebra* B (to another A-algebra C) or simply an *A-specialization* in B is an A-algebras homomorphism defined on a sub-A-algebra $\overline{B}$ of B, to another A-algebra C such that:

$$x, y \in B, \quad xy \in \overline{B}, \quad x \in B \setminus \overline{B} \Rightarrow y \in \mathrm{Ker}(\phi).$$

We will denote an A-specialization by $\phi: B \dashrightarrow C$ or by $\phi: \overline{B} \to C$, where $\overline{B} \subset B$ is the domain of definition of ϕ. When C is a field, then ϕ is called in *A-preplace* of B. The kernel of a A-preplace of B is called an *A-prime* of B. The set of all A-primes of B is denoted by $\mathrm{Gam}_A(B)$ and is called the *prime space of the pair* $A \subset B$.

An interesting and useful characterization of A-primes of B is the following:

A subset $\mathfrak{P}$ *of* B *is a A-prime of* B *if and only if* (see [Co, Prop. 1]):

(Pr_1) $\mathfrak{P}$ *is an A-module*, $1 \notin \mathfrak{P}$;

(Pr_2) $\mathfrak{P}\mathfrak{P} \subset \mathfrak{P}$;

(Pr_3) $(B \setminus \mathfrak{P})(B \setminus \mathfrak{P}) \subset (B \setminus \mathfrak{P})$.

We notice that the specializations and the preplaces «define» the same class of rings. More precisely, it is not difficult to deduce from Connell's results the following:

PROPOSITION 1.1. – *Let* $\overline{B}$ *be a sub-A-algebra of* B. *The following statements are equivalent*:

(i) $\overline{B}$ *is a domain of definition of an A-specialization in* B;

(ii) *For every* $x, y \in B$, $xy \in \overline{B}$, $x \in B \setminus \overline{B} \Rightarrow y \in \overline{B}$;

(iii) $\overline{B}$ *is a* (P_2)*-ring in* B;

(iv) $\overline{B}$ *is a domain of definition of an A-preplace of* B.

Any sub-A-algebra $\overline{B}$ *of* B, *verifying* (*one of*) *the previous statements is integrally closed in* B. ■

Finally we recall that an *A-place* of B, ϕ, is an A-preplace of B $\phi: \overline{B} \to C$ where C is a field, satisfying the following universal property: if $\phi': \overline{B}' \to C'$ is another A-preplace of B with $\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi')$, then there exists a unique A-algebras homomorphism $\lambda: C \to C'$ such that $\phi'(b) = \lambda \circ \phi(b)$, whenever the right-hand side is defined.

Two A-places of B given by the homomorphisms $\phi: \overline{B} \to C$, and $\phi': \overline{B}' \to C'$ are said to be *equivalent* (in symbols $\phi \sim \phi'$) if $\overline{B} = \overline{B}'$ and there exists an A-algebras isomorphism $\lambda: C \to C'$ such that the following diagram:

$$\begin{array}{ccc} \overline{B} & \xrightarrow{\phi} & C \\ \Vert & & \downarrow \lambda \\ \overline{B}' & \xrightarrow{\phi'} & C' \end{array}$$

commutes.

It can be shown that the natural map:

$$\mathrm{Places}_A(B) \to \mathrm{Gam}_A(B), \qquad \phi \mapsto \mathrm{Ker}(\phi),$$

defines a bijection, modulo the previous equivalence relation $\sim$, that is:

$$\mathrm{Places}_A(B)/\sim \; \tilde{\to} \; \mathrm{Gam}_A(B).$$

It is possible to introduce a topology in $\mathrm{Gam}_A(B)$ in a natural way. For every finite subset E of B, we consider:

$$\Delta(E) := \{\mathfrak{P} \in \mathrm{Gam}_A(B) \mid \mathfrak{P} \cap E = \emptyset\}.$$

If $E = \{b_i : 1 \leq i \leq n\}$, then:

$$\Delta(E) = \bigcap_{i=1}^{n} \Delta(b_i).$$

It is easy to see that $\mathcal{T} := \{\Delta(E) \mid E \subset B, \#E < \infty\}$ gives rise to a basis for the open sets, because $\Delta(E) \cap \Delta(E') = \Delta(E \cup E')$. The topology defined by $\mathcal{T}$ is called the *Zariski topology of* $\mathrm{Gam}_A(B)$.

For every subset E (possibly not finite) of B, we can consider:

$$\mathcal{W}(E) := \{\mathfrak{P} \in \mathrm{Gam}_A(B) \mid \mathfrak{P} \supset E\} = \bigcap_{f \in E} \mathcal{W}(f) = \mathrm{Gam}_A(B) \setminus \bigcup_{f \in E} \Delta(f).$$

Thus $\mathcal{W}(E)$ is a closed set in $\mathrm{Gam}_A(B)$.

Notice immediately the following difference between the behaviour of the space $\mathrm{Gam}_A(B)$ and the prime spectrum of a ring:

REMARK 1.2. – (*a*) For every subset $Y \subset \mathrm{Gam}_A(B)$, we define $\mathcal{J}(Y) := \bigcap_{\mathfrak{P} \in Y} \mathfrak{P}$. It is easy to see that $\mathcal{J}(Y)$ is a sub-A-module of B and is multiplicatively closed. It can be shown that:

$$Y = \mathcal{W}(E) \Rightarrow Y = \mathcal{W}(\mathcal{J}(Y)).$$

It is not true in general, that every closed set of $\mathrm{Gam}_A(B)$ is of the form $\mathcal{W}(E)$, for some $E \subset B$.

(*b*) If $f, f' \in B$, then in general we have that:

$$\Delta(f) \cap \Delta(f') = \Delta(f, f') \subset \Delta(ff').$$

It is not very hard to prove that:

PROPOSITION 1.3 (I. G. Connell, 1968). – (*a*) *The following maps*

$$\mathrm{Spec}(B) \hookrightarrow \mathrm{Gam}_A(B) \twoheadrightarrow \mathrm{Spec}(B),$$

$$\mathfrak{q} \mapsto \mathfrak{q}$$

$$\mathfrak{P} \mapsto (\mathfrak{P} :_B B),$$

where $(\mathfrak{P}:_B B) := \{s \in B \mid sB \subset \mathfrak{P}\}$ *is the unique, largest, prime ideal of* B *contained in* $\mathfrak{P}$, *are continuous and their composition is the identity map.*

(*b*) $\mathrm{Spec}(B)$ *is a dense subspace of* $\mathrm{Gam}_A(B)$. ■

Finally we recall that a topological space X is called *a spectral space* if it is homeomorphic to the prime spectrum of some ring, endowed with the Zariski topology. In a famous paper M. HOCHSTER (cf. [Ho]) gives several characterizations of spectral spaces. Among them we recall the following:

THEOREM 1.4 (M. Hochster, 1969). – *A topological space* X *is spectral if and only if the following properties hold*:

(S_1) X *is* $\boldsymbol{T}_0$;

(S_2) X *is quasi-compact;*

(S_3) X *is sober* (cf. [G-D]);

(S_4) X *has a basis of quasi-compact open sets closed under finite intersections.* ■

2. – The prime spaces as spectral spaces.

Several topological results given by Connell show that a general prime space $\mathrm{Gam}_A(B)$ is «close to being» a spectral space. Moreover, for some special pairs $A \subset B$, the prime space $\mathrm{Gam}_A(B)$ is already known to be spectral space.

(*a*) If B is integral over A (in particular, if $A = B$) then the canonical map $\mathrm{Gam}_A(B) \to \mathrm{Spec}(B)$ (cf. Proposition 1.3) is the identity map, because in this case, for every A-prime $\mathfrak{P}$ of B, $(\mathfrak{P}:_B \mathfrak{P})$ (the largest sub-A-algebra of B in which $\mathfrak{P}$ is a prime ideal) coincides with B (cf. Proposition 1.1 or [Co, Cor. 2, p. 82]).

(*b*) If A is a Noetherian 0-dimensional ring (in particular, if $A = K$ is a field) and if $B = A[x]$ is the polynomial ring in one variable over A, then the canonical map $\mathrm{Gam}_A(B) \to \mathrm{Spec}(B)$ (cf. Proposition 1.3) is the identity map. The reason in this case is similar, but less evident. As a matter of fact, $\mathrm{Gam}_{A_{\mathrm{red}}}(B_{\mathrm{red}})$ is canonically homeomorphic to $\mathrm{Gam}_A(B)$, because the prime radical of B coincides with $\cap\{\mathfrak{P} \mid \mathfrak{P} \in \mathrm{Gam}_A(B)\}$. Furthermore, in the present situation A_{red} is a finite product of fields, thus $\mathrm{Gam}_{A_{\mathrm{red}}}(B_{\mathrm{red}})$ is canonically homeomorphic to a finite disjoint union of prime spaces of the type $\mathrm{Gam}_K(K[x])$, where K is a field. Finally, the canonical map $\mathrm{Gam}_K(K[x]) \to \mathrm{Spec}(K[x])$ is the identity map, since also in this case $(\mathfrak{P}:_{K[x]}\mathfrak{P}) = K[x]$ for every K-prime $\mathfrak{P}$ of $K[x]$, because no integrally closed sub-K-algebra of $K[x]$ is properly contained between K and $K[x]$ (cf. Proposition 1.1 and [Co, Prop. 17]). By the way, we remark that the previous property does not hold for polynomial rings in several variables. For instance, $\mathrm{Gam}_K(K[x, y]) \not\to \mathrm{Spec}(K[x, y])$, because $xK[x]$ is a K-prime, but not a prime ideal, in $K[x, y]$.

(*c*) Let $B = K$ be a field and hence A is an integral domain. Let $X(A, K) := \{V \mid A \subset V \subset K,\ V \text{ a valuation domain of } K\}$ denote the abstract Riemann surface of the pair $A \subset K$ (cf. [Z-S, p. 113]). Then, it is known that the canonical map $X(A, K) \to \mathrm{Gam}_A(K)$, $V \mapsto \mathfrak{m}_V$ (where $\mathfrak{m}_V$ is the maximal ideal of V) is an homeomorphism (cf. [Co, Prop. 10]). Furthermore, when K is the field of quotients of A, it is known that $X(A, K)$ is a spectral space canonically homeomorphic to $\mathrm{Spec}(A'^b)$ where A'^b is the Kronecker function ring, with respect to the b-operation (or completion) (cf. [Gi2, Sect. 32]) of the integral closure A' of A into K (that is:

$$A'^b = \{f/g \mid f,\ g \in A'[T],\ g \neq 0 \text{ and } c(f)^b \subset c(g)^b\}$$

where, for every polynomial $h \in A'[T]$, $c(h)$ is the ideal of A' generated by the coefficients of h); (cf. [D-F-F] and [D-F, Th. 2]).

Even though there is no hope that the canonical map of Proposition 1.3 $\mathrm{Gam}_A(B) \to \mathrm{Spec}(B)$ is the identity map in general (cf. (*b*) and also Remark 1.2), nevertheless it is natural to investigate whether $\mathrm{Gam}_A(B)$ is a spectral space and «how far» $\mathrm{Gam}_A(B)$ is from $\mathrm{Spec}(B)$.

Our first result, which generalizes [D-F-F, Th. 2.5 and Th. 4.1] and completes some partial result by Connell (cf. [Co, Propp. 6, 14, 20]), gives a positive answer to the first question.

THEOREM 2.1. – *For every A-algebra B,* $A \subset B$,

(*a*) $\mathrm{Gam}_A(B)$ *is a spectral space;*

(*b*) *the canonical map* $\rho_A: \mathrm{Gam}_A(B) \to \mathrm{Spec}(A)$, $\mathfrak{P} \mapsto \mathfrak{P} \cap A$, *is a surjective closed spectral map.*

PROOF. – (*a*) Trivially $\mathrm{Gam}_A(B)$ is $\boldsymbol{T}_0$, because if $\mathfrak{P} \neq \mathfrak{P}'$ then for instance $\mathfrak{P} \not\subset \mathfrak{P}'$ therefore for every $b \in \mathfrak{P} \setminus \mathfrak{P}'$, $\mathfrak{P}' \in \Delta(b)$ and $\mathfrak{P} \notin \Delta(b)$. Moreover, Connell has shown that if F is an irreducible closed subspace of $\mathrm{Gam}_A(B)$ then $\mathcal{J}(F) \in \mathrm{Gam}_A(B)$ and $F = W(\mathcal{J}(F))$ [Co, Prop. 6]; thus $\mathcal{J}(F)$ is the unique generic point of F, hence $\mathrm{Gam}_A(B)$ is a sober space. An argument by Connell proves also that $\mathrm{Gam}_A(B)$ is a quasi-compact space [Co, Prop. 14]. Therefore $\mathrm{Gam}_A(B)$ verifies the properties (S_1), (S_2) and (S_3) of Theorem 1.4. To conclude the proof of (*a*) we need to verify (S_4).

We know that the basis for the open sets $\{\Delta(E) \mid E \subset B,\ \#E < \infty\}$ is closed under finite intersections, so the only remaining fact to show is that $\Delta(b_1, \ldots, b_n)$ is quasi-compact, for every $\{b_1, \ldots, b_n\} \subset B$, $n \geq 1$. We use mathematical induction. When $n = 1$, $\Delta(E) = \Delta(b)$ with $b = b_1 \in B$. We claim that the map:

$$\lambda: \mathrm{Gam}_{A[1/b]}\left(B\left[\frac{1}{b}\right]\right) \to \mathrm{Gam}_A(B), \qquad \mathfrak{Q} \mapsto f^{-1}(\mathfrak{Q}),$$

where $f: B \hookrightarrow B[1/b]$, $b' \mapsto b'/1$ is the canonical A-homomorphism, is continuous and $\mathrm{Im}(\lambda) = \Delta(b)$. The previous claim implies that $\Delta(b)$ is quasi-compact, as it is the continuous image of a quasi-compact space. We notice that λ is continuous because it is obtained as the restriction to a subspace of the «dual» map of the canonical homomorphism f, that is:

$$\lambda\colon \mathrm{Gam}_{A[1/b]}\left(B\left[\frac{1}{b}\right]\right) \subset \mathrm{Gam}_A\left(B\left[\frac{1}{b}\right]\right) \xrightarrow{\mathrm{Gam}_A(f)} \mathrm{Gam}_A(B).$$

To show that $\mathrm{Im}(\lambda) = \Delta(b)$, we begin by noticing that $\mathrm{Im}(\lambda) \subset \Delta(b)$; otherwise there would exist a $A[1/b]$-prime $\mathfrak{Q}$ in $B[1/b]$ such that $b \in f^{-1}(\mathfrak{Q})$ and hence $1 = 1/b \cdot b \in \mathfrak{Q}$, which is a contradiction.

To prove that $\mathrm{Im}(\lambda) \supset \Delta(b)$ is enough to show that:

i) $\mathfrak{P} \in \Delta(b) \Rightarrow \mathfrak{P}[1/b] \in \mathrm{Gam}_{A[1/b]}(B[1/b])$;

ii) $\mathfrak{P} \in \Delta(b) \Rightarrow f^{-1}(\mathfrak{P}[1/b]) = \mathfrak{P}$.

i) It is evident that $\mathfrak{P}[1/b]$ is a sub-$A[1/b]$-module of $B[1/b]$ not containing the identity element, which is multiplicatively closed. Moreover if:

$$\frac{c}{b^k} \quad \text{and} \quad \frac{d}{b^m} \in B\left[\frac{1}{b}\right] \setminus \mathfrak{P}\left[\frac{1}{b}\right] \quad \text{with} \ \frac{c\cdot d}{b^{k+m}} \in \mathfrak{P}\left[\frac{1}{b}\right],$$

then the necessarily c and d would be in $B \setminus \mathfrak{P}$ with $c \cdot d \in \mathfrak{P}$, which is impossible. Thus $\mathfrak{P}[1/b] \in \mathrm{Gam}_{A[1/b]}(B[1/b])$.

ii) It is easy to see that $\mathfrak{P} \subset f^{-1}(\mathfrak{P}[1/b])$. On the other hand, let $\beta \in f^{-1}(\mathfrak{P}[1/b])$ then we can write:

$$\beta = p_0 + \frac{p_1}{b} + \ldots + \frac{p_r}{b^r}, \qquad \text{with } p_i \in \mathfrak{P},\ i \geqslant 0$$

and we can suppose that r is minimal with such a property. If $r = 0$ then $\beta = p_0 \in \mathfrak{P}$. If $r \geqslant 1$ then, after multiplying both members of the previous identity by b^r, we obtain:

$$\beta \cdot b^r = p_0 b^r + p_1 b^{r-1} + \ldots + p_r$$

and then

$$(\beta \cdot b^{r-1} - p_0 b^{r-1} - \ldots - p_{r-1})\, b = p_r \in \mathfrak{P}.$$

We know that $b \notin \mathfrak{P}$, thus

$$\beta \cdot b^{r-1} - p_0 b^{r-1} - \ldots - p_{r-1} =: p \in \mathfrak{P},$$

since $B \setminus \mathfrak{P}$ is multiplicatively closed. Therefore:

$$\beta = p_0 + \frac{p_1}{b} + \ldots + \frac{p_{r-1} + p}{b^{r-1}},$$

gives another expression of β as an element of $f^{-1}(\mathfrak{P}[1/b])$, contradicting the minimality of r. We conclude that $r = 0$, hence $\beta \in \mathfrak{P}$.

Now, we can suppose that $\Delta(b_1, \ldots, b_{n-1})$ is quasi-compact as continuous image under the natural map:

$$\lambda' : \mathrm{Gam}_{A[1/b_1, \ldots, 1/b_{n-1}]}\left(B\left[\frac{1}{b_1}, \ldots, \frac{1}{b_{n-1}}\right]\right) \to \mathrm{Gam}_A(B).$$

We consider the following diagram of continuous natural maps:

$$\begin{array}{ccc} & \mathrm{Gam}_{A[1/b_1, \ldots, 1/b_{n-1}]}\left(B\left[\frac{1}{b_1}, \ldots, \frac{1}{b_{n-1}}\right]\right) & \\ \nearrow & & \searrow^{\lambda'} \\ \mathrm{Gam}_{A[1/b_1, \ldots, 1/b_n]}\left(B\left[\frac{1}{b_1}, \ldots, \frac{1}{b_{n-1}}\right] \otimes_A B\left[\frac{1}{b_n}\right]\right) & \xrightarrow{\quad\lambda\quad} & \mathrm{Gam}_A(B) \\ \searrow & & \nearrow_{\lambda''} \\ & \mathrm{Gam}_{A[1/b_n]}\left(B\left[\frac{1}{b_n}\right]\right) & \end{array}$$

It is straightforward to see that the image of the continuous map λ is

$$\mathrm{Im}(\lambda) = \mathrm{Im}(\lambda') \cap \mathrm{Im}(\lambda'') = \Delta(b_1, \ldots, b_{n-1}) \cap \Delta(b_n) = \Delta(b_1, \ldots, b_n).$$

We notice that if, in the previous proof, we had that $\{b_1, \ldots, b_n\} \subset A$, then the conclusion would be obtained more easily, extending the well known properties of localization. As a matter of fact, if S is a multiplicative subset of A, then restriction to a subspace of the «dual» map of the canonical A-homomorphism $h: B \to S^{-1}B$ is a continuous map:

$$\lambda_S : \mathrm{Gam}_{S^{-1}A}(S^{-1}B) \subset \mathrm{Gam}_A(S^{-1}B) \xrightarrow{\mathrm{Gam}_A(h)} \mathrm{Gam}_A(B), \qquad \mathfrak{Q} \mapsto h^{-1}(\mathfrak{Q}).$$

Moreover, in this case:

$$\mathrm{Im}(\lambda_S) = \{\mathfrak{P} \in \mathrm{Gam}_A(B) : \mathfrak{P} \cap S = \emptyset\},$$

because it can be shown that for every $\mathfrak{P} \in \mathrm{Gam}_A(B)$:

$$\mathfrak{P} \cap S = \emptyset \Leftrightarrow h^{-1}(S^{-1}\mathfrak{P}) = \mathfrak{P}.$$

(The proof is a straightforward modification, to the case of A-primes, of an argument concerning the behaviour of prime ideals under localization.)

(*b*) We begin by noticing that $\rho_A : \mathrm{Gam}_A(B) \to \mathrm{Spec}(A)$ is a continuous and spectral map. As a matter of a fact, ρ_A is the «dual» of the structure homomorphism $A \subset B$, and moreover $\mathrm{Gam}_A(A) = \mathrm{Spec}(A)$. More explicitly, by using the remark at the end of the proof of the previous point (*a*), for $S = \{a^n : n \geqslant 0\}$ and $a \in A$, we obtain that:

$$\rho_A^{-1}(D(a)) = \Delta(a),$$

(where $D(a) := \{\mathfrak{p} \in \mathrm{Spec}(A) | \ a \notin \mathfrak{p}\}$ and $\Delta(a) := \{\mathfrak{P} \in \mathrm{Gam}_A(B) | \ a \notin \mathfrak{P}\}$). Therefore, the inverse image, under ρ_A, of a quasi-compact open subspace of Spec(A) is a quasi-compact open subspace of $\mathrm{Gam}_A(B)$ and this is exactly the definition of a spectral map between spectral spaces [Ho].

To show that ρ_A is a closed and surjective map, we will prove that:

(i) for every closed subspace F of $\mathrm{Gam}_A(B)$, $\rho_A(F)$ is a patch subspace (cf. [Ho, Sect. 2]) of Spec(A);

(ii) for every pair $\mathfrak{p} \subset \mathfrak{q}$ of prime ideals of Spec(A) there exists at least an A-prime $\mathfrak{P}$ of B such that $\rho_A(\mathfrak{P}) = \mathfrak{p}$; moreover, for every fixed A-prime $\mathfrak{P}$ of B such that $\rho_A(\mathfrak{P}) = \mathfrak{p}$, there exists an A-prime $\mathfrak{Q}$ of B such that $\rho_A(\mathfrak{Q}) = \mathfrak{q}$ and $\mathfrak{P} \subset \mathfrak{Q}$.

The first statement of (ii) shows the surjectivity of ρ_A. By the corollary of the Theorem 1 in [Ho], the second statement in (ii), together with (i), implies that $\rho_A(F)$ is closed in Spec(A).

(i) We prove that $\mathrm{Spec}(A) \setminus \rho_A(F)$ is an open set of Spec(A) endowed with the patch topology. Let $y \in \mathrm{Spec}(A) \setminus \rho_A(F)$. Then for every $x \in \rho_A(F)$ there exists a subspace G_x of Spec(A) such that $y \in G_x$ $x \notin G_x$ and $G_x \in \{D(f), V(f) | \ f \in A\}$ is a basic patch set of Spec(A). Set $G := \cap\{G_x \mid x \in \rho_A(F)\}$. Clearly $y \in G$, $G \cap \rho_A(F) = \emptyset$ and G is a closed space in the patch topology of Spec(A). We consider:

$$\rho_A^{-1}(G) = \bigcap \{\rho_A^{-1}(G_x) | \ x \in \rho_A(F)\}.$$

The subspace $\rho_A^{-1}(G)$ is a closed set in the patch topology of $\mathrm{Gam}_A(B)$, having empty intersection with the closed set F. By the finite intersection property in the compact space $\mathrm{Gam}_A(B)$, endowed with the patch topology, we deduce that there exists a finite set $\{x_1, \ldots, x_n\} \subset \rho_A(F)$ such that $G_{x_1} \cap \ldots \cap G_{x_n} \cap \rho_A(F) = \emptyset$. Hence, $G_{x_1} \cap \ldots \cap G_{x_n}$ is an open set in the patch topology of Spec(A) containing y, which has an empty intersection with $\rho_A(F)$.

(ii) Let $\mathfrak{p} \in \mathrm{Spec}(A)$. By Zorn's Lemma applied to the set $\mathcal{S} := \{\mathfrak{M} | \ \mathfrak{M}$ is a sub-A-module of B, $\mathfrak{M}$ is multiplicatively closed, $\mathfrak{M} \cap (A \setminus \mathfrak{p}) = \emptyset$, $\mathfrak{M}(A \setminus \mathfrak{p}) \subset \mathfrak{M}\}$, we can find a maximal element $\mathfrak{P}$ in $\mathcal{S}$. A straightforward verification shows that $\mathfrak{P}$ is an A-prime of B and $\mathfrak{P} \cap A = \mathfrak{p}$. Hence ρ_A is a surjective map. To check the second part of the statement, it is enough to show that:

$$\rho_A(\mathcal{W}(\mathfrak{P})) = V(\mathfrak{P} \cap A).$$

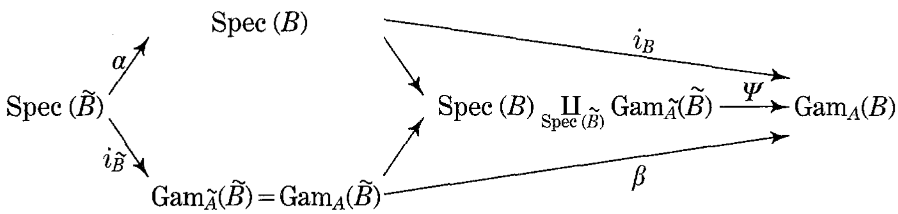
It is clear that $\rho_A(\mathcal{W}(\mathfrak{P})) \subset V(\mathfrak{P} \cap A)$. Let $\mathfrak{q} \in V(\mathfrak{P} \cap A)$. The set $E := \mathfrak{P} + \mathfrak{q}$ is a multiplicatively closed sub-A-module of B. Moreover $(A \setminus \mathfrak{q})E \subset E$ and $(A \setminus \mathfrak{q}) \cap E = \emptyset$. By applying Zorn's Lemma to the non empty set $\mathcal{T} := \{\mathfrak{M} | \ \mathfrak{M}$ is a sub-A-module of B, $\mathfrak{M}$ is multiplicatively closed, $\mathfrak{M} \cap (A \setminus \mathfrak{q}) = \emptyset$, $\mathfrak{M}(A \setminus \mathfrak{q}) \subset \mathfrak{M}$, $\mathfrak{M} \supset E\}$, we can find a maximal element $\mathfrak{Q}$ in $\mathcal{T}$, which turns out to be an A-prime of B such that $\rho_A(\mathfrak{Q}) = \mathfrak{q}$ and $\mathfrak{Q} \supset \mathfrak{P}$. ■

The following theorem gives an answer to the question: «how far is $\mathrm{Gam}_A(B)$ from $\mathrm{Spec}(B)$?» The idea is to obtain $\mathrm{Gam}_A(B)$ as an amalgamated sum of topological spaces in which $\mathrm{Spec}(B)$ is one of the summands.

THEOREM 2.2. – *Let B be an A-algebra, $A \subset B$ and let $\mathfrak{f}$ be the conductor of $A \subset B$. Then there exists the following natural homeomorphism of topological spaces*:

$$\mathrm{Gam}_A(B) \cong \mathrm{Spec}(B) \coprod_{\mathrm{Spec}(B/\mathfrak{f})} \mathrm{Gam}_{A/\mathfrak{f}}(B/\mathfrak{f}).$$

PROOF. – Let $\pi_A : A \to \tilde{A} := A/\mathfrak{f}$ and $\pi_B : B \to \tilde{B} := B/\mathfrak{f}$ be the canonical projections. We consider the following commutative diagram of topological spaces:



where $i_{\tilde{B}}$ and i_B are the inclusion maps (cf. Proposition 1.3), $\alpha := \mathrm{Spec}(\pi_B)$, $\beta := \mathrm{Gam}_A(\pi_B)$ and Ψ is the (unique) continuous map arising from the universal property of the amalgamated sum of topological spaces (cf. [D, Ch. VI, Sect. 6]). Therefore:

$$\Psi(x) = \begin{cases} i_B(\mathfrak{p}), & \text{if } x = \mathfrak{p} \in \mathrm{Spec}(B) \setminus \alpha(\mathrm{Spec}(\tilde{B}));\\ \beta(\tilde{\mathfrak{P}}), & \text{if } x = \tilde{\mathfrak{P}} \in \mathrm{Gam}_A(\tilde{B}) \setminus i_{\tilde{B}}(\mathrm{Spec}(\tilde{B}));\\ i_B \circ \alpha(\tilde{\mathfrak{p}}) = \beta \circ i_{\tilde{B}}(\tilde{\mathfrak{p}}), & \text{if } x \text{ is the image of an element } \tilde{\mathfrak{p}} \in \mathrm{Spec}(\tilde{B}). \end{cases}$$

We claim that:

$$\mathfrak{P} \in \mathrm{Gam}_A(B), \quad \mathfrak{P} \not\supseteq \mathfrak{f} \Rightarrow \mathfrak{P} \in i_B(\mathrm{Spec}(B)).$$

By Proposition 1.3, it is enough to show that

$$(\mathfrak{P} :_B B) = \mathfrak{P}.$$

Obviously, $(\mathfrak{P} :_B B) \subset \mathfrak{P}$. Conversely, suppose that there exists $p \in \mathfrak{P}$, $p \notin (\mathfrak{P} :_B B)$. Hence, for some $\beta \in B$, $p\beta \notin \mathfrak{P}$. Let $t \in \mathfrak{f} \setminus \mathfrak{P}$, then $(p\beta)t \notin \mathfrak{P}$. From the other side $(p\beta)t = p(\beta t) \in \mathfrak{P}A \subset \mathfrak{P}$: Contradiction.

Now, we can define the following map:

$$\tilde{\Psi} : \mathrm{Gam}_A(B) \to \mathrm{Spec}(B) \coprod_{\mathrm{Spec}(\tilde{B})} \mathrm{Gam}_{\tilde{A}}(\tilde{B}), \qquad \mathfrak{P} \mapsto \begin{cases} \tilde{\mathfrak{P}} := \mathfrak{P}/\mathfrak{f}, & \text{if } \mathfrak{P} \supset \mathfrak{f};\\ \mathfrak{p}, & \text{if } \mathfrak{P} \not\supseteq \mathfrak{f} \text{ and } i_B(\mathfrak{p}) = \mathfrak{P}. \end{cases}$$

It is easy to show that $\widetilde{\Psi}\circ\Psi$ and $\Psi\circ\widetilde{\Psi}$ are the identity maps, thus Ψ is a bijection.

To show that Ψ is a homeomorphism, we prove that Ψ is a closed map. In order to prove that Ψ is a closed map, it is enough to show that if $F \subset \operatorname{Spec}(B) \coprod_{\operatorname{Spec}(\widetilde{B})} \operatorname{Gam}_{\widetilde{A}}(\widetilde{B})$ is such that:

$$F' := F \cap \operatorname{Spec}(B) = V(b), \qquad \text{with } b \in B;$$
$$F'' := F \cap \operatorname{Gam}_{\widetilde{A}}(\widetilde{B}) = W(\widetilde{b}_1, \ldots, \widetilde{b}_n) \quad \text{with } \widetilde{b}_1, \ldots, \widetilde{b}_n \in \widetilde{B},\ n \geq 1,$$

then $\Psi(F)$ is a closed subspace of $\operatorname{Gam}_A(B)$. We claim that:

$$\Psi(F) = W(b) \cup \left(W(b_1, \ldots, b_n) \cap \left(\bigcap_{f \in \mathfrak{f}} W(f) \right) \right),$$

where b_i is an element of B chosen in the class $\widetilde{b}_i$, that is $\widetilde{b}_i = b_i + \mathfrak{f}$, for $i = 1, \ldots, n$.

As a matter of fact

$$\Psi(F) \subset W(b) \cup \left(W(b_1, \ldots, b_n) \cap \left(\bigcap_{f \in \mathfrak{f}} W(F) \right) \right),$$

because if $x \in F$ then:

$$\Psi(x) = \begin{cases} i_B(\mathfrak{p}), & \text{if } x = \widetilde{\mathfrak{p}} \in F' = V(b), \\ \beta(\widetilde{\mathfrak{P}}), & \text{if } x = \widetilde{\mathfrak{P}} \in F'' \in W(\widetilde{b}_1, \ldots, \widetilde{b}_n). \end{cases}$$

On the other hand, if $\mathfrak{P} \in W(b) \cup \left(W(b_1, \ldots, b_n) \cap \left(\bigcap_{f \in \mathfrak{f}} W(f) \right) \right)$, then $\mathfrak{P} = \Psi(x)$ where

$$x := \begin{cases} \mathfrak{P}/\mathfrak{f} \in W(\widetilde{b}_1, \ldots, \widetilde{b}_n) = F'', & \text{if } \mathfrak{P} \notin i_B(\operatorname{Spec}(B)), \\ \mathfrak{p} \in V(b) = F', & \text{if } \mathfrak{P} = i_B(\mathfrak{p}) \in i_B(\operatorname{Spec}(B)), \end{cases}$$

thus $x \in F$. ■

The previous theorem reduces the study of a general space $\operatorname{Gam}_A(B)$ to the space $\operatorname{Gam}_{\widetilde{A}}(\widetilde{B})$, where $\widetilde{B}$ is an $\widetilde{A}$-algebra with zero conductor. In some relevant cases, that we now intend to investigate, this space turns out to be a «classical» abstract Riemann surface.

COROLLARY 2.3. – *Let $\mathfrak{m}$ be a maximal ideal of a ring B, let $\phi: B \to \boldsymbol{k}(B) := B/\mathfrak{m}$ be the natural projection onto the residue field of B in $\mathfrak{m}$ and let D be a subring of $\boldsymbol{k}(B)$.*

If A is the subring of B obtained as a pullback in the following way:

$$\begin{array}{ccc} A := \phi^{-1}(D) & \twoheadrightarrow & D \\ \downarrow & & \downarrow \\ B & \twoheadrightarrow & \boldsymbol{k}(B) \end{array}$$

then there exists a natural homeomorphism of topological spaces

$$\mathrm{Gam}_A(B) \cong \mathrm{Spec}(B) \coprod_{\mathrm{Spec}(\boldsymbol{k}(B))} \mathrm{Gam}_D(\boldsymbol{k}(B)) .$$

PROOF. – It is enough to notice that, in the present situation, the maximal ideal $\mathfrak{m}$ of B is the conductor of $A \subset B$. The conclusion follows from Theorem 2.2. ∎

COROLLARY 2.4. – *Let B, $\boldsymbol{k}(B)$, ϕ, D and A as the statement of Corollary 2.3. Furthermore, suppose that B is an integral domain, the integral closure B' of B is a Prüfer domain and that $\boldsymbol{k}(B)$ is the quotient field of D. Then there exists a natural homeomorphism of* $\mathrm{Gam}_A(B)$ *with:*

$$\mathrm{Spec}(B(T) \times_{\boldsymbol{k}(B)(T)} D'^b) ,$$

where T is an indeterminate over B and $\boldsymbol{k}(B)$, $B(T)$ is the Nagata ring of B (i.e. $B(T) := B[T]_S$, where S is the set of polynomials with unit content) and D'^b is the Kronecker function ring of the integral closure of D, with respect to the b-operation (cf. also the previous point (c)).

PROOF. – In [A-D-F] it is shown that, given an integral domain B, the continuous map $\mathrm{Spec}(B(T)) \to \mathrm{Spec}(B)$, $\mathfrak{p} \mapsto \mathfrak{p} \cap B$, is a homeomorphism if and only if B' is Prüfer. Moreover, by the previous point (*c*), we know that the natural maps establish the following homeomorphisms:

$$\mathrm{Gam}_D(\boldsymbol{k}(B)) \cong X(D, \boldsymbol{k}(B)) \cong \mathrm{Spec}(D'^b) .$$

We consider the following pullback diagram of natural ring homomorphisms:

$$\begin{array}{ccc} B(T) \times_{\boldsymbol{k}(B)(T)} D'^b \cong \phi(T)^{-1}(D'^b) & \twoheadrightarrow & D'^b \\ \downarrow & & \downarrow \\ B(T) & \overset{\phi(T)}{\twoheadrightarrow} & \boldsymbol{k}(B)(T) . \end{array}$$

We are in the hypotheses of [F, Th. 1.4], thus we can deduce the following natural homeomorphism of topological spaces:

$$\mathrm{Spec}\,(B(T) \times_{k(B)(T)} D'^{b}) \cong \mathrm{Spec}\,(B(T)) \coprod_{\mathrm{Spec}\,(k(B)(T))} \mathrm{Spec}\,(D'^{b})\,.$$

The conclusion now follows by the previous remarks and Corollary 2.3. ■

We notice that the hypotheses of Corollary 2.3 are satisfied by the rings intervening in the classical $D + \mathfrak{m}$ construction (cf. [Gi 1, Appendix 2]). As a matter of fact, let V be a valuation domain and suppose that $V = \boldsymbol{k} + \mathfrak{m}$, where $\mathfrak{m}$ is its maximal ideal and $\boldsymbol{k}$ a field (isomorphic to the residue field $\boldsymbol{k}(V)$ of V). Let D be a subring of $\boldsymbol{k}$, then the domain $D + \mathfrak{m}$ coincides with the following pullback:

$$\begin{array}{ccc} D + \mathfrak{m} \cong \phi^{-1}(D) & \longrightarrow & D \\ \downarrow & & \downarrow \\ V = \boldsymbol{k} + \mathfrak{m} & \xrightarrow{\phi} & \boldsymbol{k}(V) \cong \boldsymbol{k}\,. \end{array}$$

If we apply Corollary 2.3 and [F, Th. 1.4] to the case in which $B := \mathbb{Q}[\![T]\!]$ is the discrete valuation domain of power series in one variable T, with coefficients in the field of rationals number $\mathbb{Q}$ and $D := \mathbb{Z}$, then we obtain the following natural homeomorphism:

$$\mathrm{Gam}_A\,(B) \cong \mathrm{Spec}\,(\mathbb{Q}[\![T]\!]) \coprod_{\mathrm{Spec}\,(\mathbb{Q})} \mathrm{Spec}\,(\mathbb{Z}) \cong \mathrm{Spec}\,(\mathbb{Q}[\![T]\!] \times_{\mathbb{Q}} \mathbb{Z}) \cong \mathrm{Spec}\,(A)\,.$$

The previous situation is a very particular case of the class of a pair of domains $A \subset B$ for which the spectral map $\rho_A : \mathrm{Gam}_A\,(B) \to \mathrm{Spec}\,(A)$ is the identity map.

THEOREM 2.5. – *Let B, $\boldsymbol{k}(B)$, ϕ, D and A as in the statement of Corollary 2.3. Furthermore suppose that B is a Prüfer domain and that $\boldsymbol{k}(B)$ is the quotient field of D. Then the following properties are equivalent:*

(i) $\mathrm{Gam}_A\,(B) = \mathrm{Spec}\,(A)$ (*i.e.* ρ_A *is the identity map*);

(ii) $\mathrm{Gam}_D\,(\boldsymbol{k}(B)) = \mathrm{Spec}\,(D)$ (*i.e.* ρ_D *is the identity map*);

(iii) *D is an i-domain (that is, for every, overring R of D, the canonical map* $\mathrm{Spec}\,(R) \to \mathrm{Spec}\,(D)$ *is injective*; *cf.* [Pa]);

(iv) *A is an i-domain.*

PROOF. – We consider the following commutative diagram of topological spaces and continuous maps:

$$\begin{array}{ccc} \mathrm{Gam}_A\,(B) \cong \mathrm{Spec}\,(B) \displaystyle\coprod_{\mathrm{Spec}\,(\boldsymbol{k}(B))} \mathrm{Gam}_D\,(\boldsymbol{k}(B)) \\ \downarrow \rho_A \qquad\qquad \downarrow \tilde{\rho}_D \\ \mathrm{Spec}\,(A) \cong \mathrm{Spec}\,(B) \displaystyle\coprod_{\mathrm{Spec}\,(\boldsymbol{k}(B))} \mathrm{Spec}\,(D)\,, \end{array}$$

where the homeomorphism in the first row is given by Corollary 2.3, the homeomorphism in second row is obtained from [F, Th. 1.4] and the map $\tilde{\rho}_D$ is the natural extension of ρ_D with the identity map of Spec(B). Therefore, it is clear that (i) $\Leftrightarrow$ (ii). The equivalence (ii) $\Leftrightarrow$ (iii) is proven in [D-F-F, Prop. 2.2]. The show that (iii) $\Leftrightarrow$ (iv) we use the fact that a domain R is an i-domain if and only if its integral closure R' is Prüfer and the map Spec$(R') \to$ Spec(R) is bijective [Pa, Prop. 2.14]. In the present situation, the integral closure A' of A is isomorphic to $B \times_{k(B)} D'$, where D' is the integral closure of D, thus A' is Prüfer if and only if D' is Prüfer (cf. also [F, Prop. 2.2(10) and Th. 2.4]). Moreover it is clear that Spec$(A') \to$ Spec(A) is a bijection if and only if Spec$(D') \to$ Spec(D) is a bijection (cf. [F, Th. 1.4]). ■

From the previous theorem and, in particular from the proof of (iii) $\Leftrightarrow$ (iv), we deduce immediately the following:

COROLLARY 2.6. – *Let B, $\boldsymbol{k}(B)$, ϕ, D and A be as in the statement of Corollary* 2.3. *Furthermore, suppose that B is a Prüfer domain and D is integrally closed in its field of quotients $\boldsymbol{k}(B)$. Then the following properties are equivalent:*

(i) $\text{Gam}_A(B) = \text{Spec}(A)$ (*i.e.* ρ_A *is the identity map*);

(ii) $\text{Gam}_D(\boldsymbol{k}(B)) = \text{Spec}(D)$ (*i.e.* ρ_D *is the identity map*);

(iii) *A is a Prüfer domain;*

(iv) *D is a Prüfer domain.* ■

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