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The Radical Trace Property and Primary Ideals

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INTRODUCTION

Let R be an integral domain and let M be an R-module. Then the trace of M is the ideal generated by the set $\{fm | f \in \mathrm{Hom}_R(M, R) \text{ and } m \in M\}$. For an ideal I of R, the trace is simply the product of I and I^{-1}. We call an ideal J a *trace ideal* if J is the trace of some R-module. An elementary result which will be used freely throughout this paper is that if J is a trace ideal, then $JJ^{-1} = J$; i.e., J^{-1} equals the ring $(J : J)$ [6, Proposition 7.2]. Thus the trace ideals of R are precisely those ideals J for which J^{-1} equals $(J : J)$. (Such ideals are also referred to as being "strong"; see, for example, [3].) If R is a valuation domain and M is an R-module, then the trace of M is either R or a prime ideal of R [14, Proposition 2.1]. Extracting the conclusion of this result, Fontana *et al*. give the following definition: A domain R is said to satisfy the *trace property* (or to be a TP domain) if for each R-module M, the trace of M is equal to either R or a prime ideal of R [14, p. 169] (see also [1, Theorem 2.8]). Theorem 3.5 of [14] gives a characterization of Noetherian TP domains. Namely, for a Noetherian domain R, R is a TP domain if and only if R is one-dimensional, has at most one non-invertible maximal ideal M, and, if such a maximal ideal exists, then M^{-1} equals the integral closure of R (or, equivalently, $M^{-1} = (M : M)$ is a Dedekind domain). In Section 2 of [17], Gabelli shows that by replacing "integral closure" with "complete integral closure," the same list of conditions characterizes the class of Mori domains which satisfy the trace property. Recall that a Mori domain is an integral domain which satisfies the ascending chain condition on divisorial ideals.

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In [22], Heinzer and Papick relaxed the requirement that each proper trace ideal be a prime ideal to require only that each be a radical ideal, thus creating the radical trace property and the class of RTP domains. For Noetherian domains, they prove that if R is a Noetherian domain, then it satisfies the radical trace property if and only if R_P is a TP domain for each prime P [22, Proposition 2.1]. Gabelli extended this result to Mori domains [17, Theorem 2.14].

For Prüfer domains, there are results concerning the trace property in [14] and the radical trace property in [22]. The so-called $(\#\#)$ property for Prüfer domains is involved in the positive results of both papers. Recall that a Prüfer domain R is said to satisfy $(\#)$ if for any two distinct sets of maximal ideals $\mathscr{M}$ and $\mathscr{N}$, $\bigcap_{M \in \mathscr{M}} R_M \neq \bigcap_{N \in \mathscr{N}} R_N$ [19]. If, in addition, each overring of R satisfies $(\#)$, then R is said to satisfy $(\#\#)$. There are two positive results concerning Prüfer domains in [14]. The first is that if R is a Prüfer domain which satisfies $(\#\#)$, then R is a TP domain if and only if the non-invertible prime ideals are linearly ordered [14, Theorem 4.2]. The second is that if R is a finite-dimensional Prüfer domain, then R is a TP domain if and only if R satisfies the $(\#\#)$ property and the non-invertible prime ideals are linearly ordered [14, Theorem 4.6]. For the radical trace property, Theorem 2.7 of [22] states that the following are equivalent for a Prüfer domain R which satisfies acc on primes:

(1) R has the radical trace property.

(2) R has Noetherian spectrum.

(3) R satisfies $(\#\#)$.

In [20] an example is given of an almost Dedekind domain R with exactly one non-invertible maximal ideal M. Since R_M is a discrete rank one valuation domain, $M^2 \neq M$. But since $M^{-1} = (M : M) = R$, the same is true for $(M^2)^{-1}$. Thus M^2 is a trace ideal of R which is neither prime nor radical. Whence R is neither a TP nor an RTP domain [14, Example 4.3; 22, p. 115].

In Theorem 23, we show that if R is a Prüfer domain, then the following are equivalent:

(1) R satisfies the radical trace property.

(2) For each primary ideal Q, either Q is invertible or QQ^{-1} is a prime ideal.

(3) For each primary ideal Q, if Q^{-1} is a ring, then Q is prime.

(4) Each branched prime is the radical of a finitely generated ideal.

Using this result and Theorem 3 of [19], we prove that every Prüfer domain with $(\#\#)$ is an RTP domain.

From the second and third equivalent statements in Theorem 23, we extract the following definitions. We say that a domain R satisfies the *trace property for primary ideals* (and refer to R as a TPP domain) if for each primary ideal Q, either Q is invertible or QQ^{-1} is prime. We will show that this is equivalent to the property that for each primary ideal Q, either $QQ^{-1} = \sqrt{Q}$ or Q is invertible and $\sqrt{Q}$ is maximal (Corollary 8). In Theorem 4, we show that every RTP domain is a TPP domain. Moreover, Theorems 3, 5, 7, 9, and 10 together with the statement and proof of Lemma 32 lead us to conjecture that the two properties are equivalent. We will refer to R as a *PRIP domain* (or say that R has PRIP) if for each primary ideal Q, Q^{-1} a ring implies Q is prime. In Example 30 we give an example of a Noetherian TP domain which is not a PRIP domain.

For two non-empty subsets B and C of a field K, $(B:C) = \{x \in K | xC \subseteq B\}$. Of course, for an ideal I of domain D, the commonly used notation for $(D:I)$ is "I^{-1}." To avoid confusion we shall reserve this notation exclusively for the situation where the domain in question is R, a localization of R, or a homomorphic image of R. For any other ring T we shall always use $(T:I)$. For subsets, we use "$\subset$" to denote proper subset and "$\subseteq$" to indicate subset with possible equality. (For the most part, $\subset$ is used whenever equality is not possible, but when the question of equality is not relevant, $\subseteq$ may appear even though equality is not possible.)

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The first two lemmas provide useful tools for our work. The first gives a slight generalization of Lemma 3.7 in [16]. Results similar to the second are used in the proofs of Proposition 2.10 in [22] and Lemma 1.1 in [17].

LEMMA 0 (cf. [16, Lemma 3.7]). *Let R be an integral domain and let Q be a primary ideal with radical P. If J is an ideal of R which contains Q and is not contained in P, then $J^{-1} \subseteq (Q:Q)$.*

Proof. Let $r \in J \setminus P$. Then $(R:J)Qr \subseteq P$. Hence $(R:J)Q \subseteq Q$ since Q is P-primary and $r \notin P$. ∎

LEMMA 1. *Let R be an integral domain and let Q be a primary ideal of R with $\sqrt{Q} = P$. Let $a \in QQ^{-1} \setminus P$ and let $I = aR + Q$. Then $I^{-1} = (QQ^{-1})^{-1} = (QQ^{-1} : QQ^{-1}) = (Q:Q)$.*

Proof. Since I contains Q and is contained in QQ^{-1}, we have $(Q:Q) \subseteq (QQ^{-1} : QQ^{-1}) = (QQ^{-1})^{-1} \subseteq I^{-1}$. Equality follows from Lemma 0. ∎

Our first use of the above lemmas is to show that in an RTP domain, non-maximal primes are divisorial.

THEOREM 2. *Let R be an RTP domain. Then each non-maximal prime ideal is divisorial and for each such prime P, $P^{-1} = (P : P)$.*

Proof. Let P be a non-maximal prime ideal. First, we must have $(P : P) \neq R$, for otherwise the combination of the radical trace property and Lemma 1 implies that every ideal containing P is a radical ideal. Hence, among other things, P cannot be invertible.

Assume $PP^{-1} \neq P$ and let $I = a^2R + P$ where $a \in PP^{-1} \setminus P$. By Lemma 1, we have $I^{-1} = (PP^{-1})^{-1} = (PP^{-1} : PP^{-1}) = (P : P)$. By RTP, II^{-1} is a radical ideal and thus $a = p + ua^2$ for some $p \in P$ and some $u \in I^{-1} = (PP^{-1})^{-1}$. Whence $p = a(1 - ua)$. As $a \in PP^{-1} \setminus P$ and $u \in (PP^{-1} : PP^{-1}$, we have $1 - ua \in P \subset I \subseteq PP^{-1}$. It follows that $PP^{-1} = R$ and we have the contradictory statement that $(P : P) = R$. Hence, $PP^{-1} = P$ and $P^{-1} = (P : P)$.

Since $P^{-1} \neq R$, P_v is a proper ideal of R and $P_v^{-1} = (P_v : P_v) = (P : P) = P^{-1}$. As above, by setting $I = a^2R + P$ for some $a \in P_v \setminus P$ we arrive at a contradiction. Hence $P = P_v$. ∎

THEOREM 3. *Let R be an integral domain. Then the following are equivalent.*

(1) *R is an RTP domain.*

(2) *For each multiplicative set S, R_S is an RTP domain.*

(3) *For each prime ideal P, R/P is an RTP domain.*

Proof. Assume R is an RTP domain and let S be a multiplicative set. Let I be an ideal of R, $J' = IR_S IR_S^{-1}$, and $J = J' \cap R$. Then $J' = JR_S$ so it suffices to show that J is a radical ideal of R. Let $u \in J^{-1}$. Then $u \in (R_S : J')$. Hence $uJ' \subseteq J'$. Since $uJ \subseteq R$ and $J' = JR_S$, $uJ \subseteq J$. As R is an RTP domain, J must be a radical ideal.

Assume R is an RTP domain and let P be a non-maximal prime ideal of R. Let I be an ideal of R properly containing P and let $J' = (I/P)(R/P : I/P)$. Then $J' = J/P$ for some ideal J of R (properly containing P). As above, it suffices to show that J is a radical ideal. Since J^{-1} is contained in $(P : P)$, each element of J^{-1} is well defined in R/P and $J^{-1}/P \subseteq (R/P : J')$. Hence for each $u \in J^{-1}$, $uJ/P = uJ' \subseteq J'$. Thus $uJ \subseteq J$. Since R is an RTP domain, J is a radical ideal. ∎

For each ideal I of an RTP domain R, it is always the case that II^{-1} contains the radical of I. Thus for a P-primary ideal Q, $P \subseteq QQ^{-1}$.

Our next result establishes the first link between the radical trace property and the trace property for primary ideals. Later we will show that the two properties are equivalent for both Prüfer domains and Mori domains.

THEOREM 4. *Let R be an RTP domain and let Q be a P-primary ideal. Then either $QQ^{-1} = P$ or $QQ^{-1} = R$ and P is maximal*; *i.e.*, *R is a TPP domain.*

Proof. To begin, assume Q is invertible. Then $(Q:Q) = R$. Hence for each ideal J containing Q and not contained in P, $J^{-1} = R$. Thus J must be a radical ideal of R since R is an RTP domain. This implies that P is maximal.

Now assume Q is not invertible. Let M be a prime containing Q. By Theorem 3, R_M is an RTP domain. Hence, $QR_M QR_M^{-1}$ is a radical ideal of R_M. By the above, if QR_M is invertible, then $M = P$. It follows that if M is minimal over QQ^{-1}, then there can be no primes properly between P and M. Thus we assume M is minimal over QQ^{-1} and not equal to P. Then $QR_M QR_M^{-1} = MR_M$ since R_M is an RTP domain. Furthermore, by Lemma 1 we have $IR_M^{-1} = MR_M^{-1} = (MR_M : MR_M) = (QR_M : QR_M)$ for each ideal I containing Q and contained in M but not contained in P. In particular, this would be true for the ideal $I = a^2R + P$ where $a \in M \setminus P$. As in the proof of Theorem 2, the radical trace property guarantees the existence of an element $u \in IR_M^{-1}$ and an element $p \in PR_M$ such that $a = ua^2 + p$. Hence $p = a(1 - ua)$. As $IR_M^{-1} = (MR_M : MR_M)$, $1 - ua$ is a unit of R_M which implies $a \in RP_M$, a contradiction. Hence it must be that $QQ^{-1} = P$. ∎

Before we present our next results, note that if Q is a primary ideal in a TPP domain, then QQ^{-1} always contains $\sqrt{Q}$.

Our next result concerns domains with the trace property for primary ideals. It provides more circumstantial evidence for the conjectured equivalence of the radical trace property and the trace property for primary ideals. For one-dimensional domains, it shows that the radical trace property and the trace property for primary ideals are equivalent.

THEOREM 5. *Let R be a TPP domain and let I be an ideal such that $I^{-1} = (I:I)$. Then for each prime P minimal over I, $IR_P = PR_P$.*

Proof. Let P be minimal over I and let $Q = IR_P \cap R$. Then $Q^{-1} \subseteq I^{-1}$ and, since R is a TPP domain, $P \subseteq QQ^{-1}$. Thus $IR_P \subseteq PR_P \subseteq Q^{-1}QR_P \subseteq I^{-1}IR_P = IR_P$. ∎

COROLLARY 6. *Let R be a one-dimensional integral domain. Then R is an RTP domain if and only if R is a TPP domain.*

As we have seen above in Theorem 3, the radical trace property is stable under localizations and the formation of quotient rings. Our next three results deal with establishing the same stability for the trace property for primary ideals. We first show that if M is a prime ideal of a TPP domain R, then R_M is a TPP domain.

THEOREM 7. *Let R be a TPP domain and let M be a prime ideal of R. Then R_M is a TPP domain.*

Proof. Let Q be a primary ideal contained in M with $\sqrt{Q} = P$. If $P = M$, then there is nothing to prove since $MR_M \subseteq QQ^{-1}R_M \subseteq QR_M QR_M^{-1}$. Thus we may assume $P \neq M$. To complete the proof we shall prove $QR_M QR_M^{-1} = PR_M$.

If QR_M is invertible, then $(QR_M : QR_M) = R_M$. Hence, by Lemma 1, $JR_M^{-1} = R_M$ for each ideal J between Q and M that is not contained in P. Thus $J^{-1}R_M = R_M$ as well. It follows that $JJ^{-1}R_M = JR_M$ for each such ideal J. But this last statement implies that every primary ideal between P and M is prime. Thus it must be that QR_M is not invertible. Moreover, if N is a prime ideal with $P \subset N \subset M$, then $QR_M QR_M^{-1}$ must be contained in NR_M for otherwise, QR_N will be an invertible ideal of R_N. Thus since our goal is to show $QR_M QR_M^{-1} = PR_M$, we may assume $\mathrm{ht}(M/P) = 1$.

Since $\mathrm{ht}(M/P) = 1$ and QR_M is not invertible, if $QR_M QR_M^{-1} \neq PR_M$, then for some M-primary ideal J, $QR_M QR_M^{-1} = JR_M$. In fact, we must have $J = M$, since $JR_M JR_M^{-1} = JR_M$ and for each M-primary ideal I, $M \subseteq II^{-1}$. Let $a \in M \setminus P$ and let I be the M-primary ideal obtained by contracting the ideal $a^2R_M + QR_M$ into R. Then $I \neq M$ but by Lemma 1, $I^{-1}R_M \subseteq IR_M^{-1} = MR_M^{-1} = (MR_M : MR_M) = (QR_M : QR_M)$. Since $M \subseteq II^{-1}$, $a = ua^2 + q$ for some $u \in MR_M^{-1}$ and some $q \in QR_M$. Hence $q = a(1 - ua)$. But, as in the proof of Theorem 4, $a \in M \setminus P$ and $1 - ua$ is a unit of R_M since $ua \in MR_M$. Thus we must have $QR_M QR_M^{-1} = PR_M$. ∎

The following corollary is derived from the proof of Theorem 7.

COROLLARY 8. *Let R be an integral domain. Then R is a TPP domain if and only if for each primary ideal Q, either $QQ^{-1} = \sqrt{Q}$ or Q is invertible and $\sqrt{Q}$ is maximal.*

THEOREM 9. *Let R be an integral domain. Then the following are equivalent.*

(1) *R is a TPP domain.*

(2) *For each multiplicative set S, R_S is a TPP domain.*

(3) *For each prime ideal P, R/P is a TPP domain.*

Proof. Assume R is a TPP domain.

We first show that R_S is a TPP domain for each multiplicative set S.

Let S be a multiplicative set. That R_S is a TPP domain follows from Theorem 7 and Corollary 8, since the primary ideals of R_S are all of the form QR_S for some primary ideal Q of R.

Now let $P \subset N$ be a pair of prime ideals and let $\overline{R} = R/P$. Let $\overline{Q}$ be an $\overline{N}$-primary ideal of $\overline{R}$. Then Q is an N-primary ideal of R containing P

and $Q^{-1} \subseteq P^{-1} = (P:P)$. Thus we have $\overline{N} \subset \overline{Q}\,\overline{Q}^{-1}$. There is nothing to prove if N is maximal. Hence we may assume N is not maximal, in which case $QQ^{-1} = N$.

Assume $\overline{Q}\,\overline{Q}^{-1} \neq \overline{N}$. Let M be a prime ideal which properly contains N and let $a \in M \setminus N$ be such that $\bar{a} \in \overline{Q}\,\overline{Q}^{-1}$. As our goal is to show $\overline{Q}\,\overline{Q}^{-1} = \overline{N}$, we may assume R is quasilocal with maximal ideal M since R_M has TPP. Moreover, we may assume M is minimal over the ideal $I = a^2R + Q$. Thus I is an M-primary ideal of R. Since $\overline{Q}$ is a primary ideal of $\overline{R}$, $\bar{I}^{-1} \subseteq (\overline{Q}:\overline{Q})$. If $\overline{Q}$ is invertible, then we have $(\overline{Q}:\overline{Q}) = \overline{R}$. Hence $\bar{I}^{-1} = \overline{R}$. Since $P \subset Q \subset I$, $I^{-1} \subset (P:P)$. Therefore $I^{-1} = R$ and $I = M$. Since R is quasilocal, this leads to the contradictory statement that $a = a^2t + q$ for some $t \in R$ and some $q \in Q$. Thus $\overline{Q}$ is not invertible. Moreover, we must have $\mathrm{ht}(M/N) = 1$. Since M is the maximal ideal of R, $\overline{Q}\,\overline{Q}^{-1}$ is $\overline{M}$-primary. Thus there is an M-primary ideal J containing Q where $\bar{J} = \overline{Q}\,\overline{Q}^{-1}$. Thus $\bar{J}^{-1} = (\bar{J}:\bar{J})$. It follows that $J^{-1} = (J:J)$. Hence $J = M$ and $\overline{Q}\,\overline{Q}^{-1} = \overline{M}$. By Lemma 1, $\bar{I}^{-1} = (\overline{Q}:\overline{Q}) = (\overline{M}:\overline{M})$. Since I is M-primary and $\bar{I}^{-1}$ is a ring, $\bar{I}\bar{I}^{-1} = \overline{M}$. Following the same line of reasoning as in the proof of Theorem 7, we obtain the contradictory statement that $\bar{a} = \bar{q}/(1 - \overline{au}) \in \overline{Q}$ (where as before $\bar{u}$ is an element of $(\overline{M}:\overline{M})$). ∎

THEOREM 10. *Let R be an integral domain with TPP. Then each non-maximal prime ideal of R is divisorial and for each such prime P, $P^{-1} = (P:P)$.*

Proof. Let P be a non-maximal prime ideal of R. Since R has TPP, $PP^{-1} = P$. Thus $P^{-1} = (P:P)$. By Proposition 2.2 of [25], $P^{-1} = (P_v : P_v)$. Assume P is not divisorial. Since P is not maximal, there is an element $a \in P_v \setminus P$ such that the ideal $I = a^2R + P$ is a proper ideal of R. Since $P^{-1} = (P_v : P_v)$ and $P \subset I \subseteq P_v$, $I^{-1} = P^{-1} = (P_v : P_v)$. Moreover, by Proposition 2.2 of [25], $I^{-1} = (II^{-1})^{-1} = (II^{-1} : II^{-1})$. By Theorem 5, $II^{-1}R_N = NR_N$ for each prime N minimal over II^{-1}. It follows that $a = ba^2/t + p/t$ for some $b \in I^{-1}$ and some $t \in R \setminus N$. Since $P_v^{-1} = I^{-1} = (II^{-1})^{-1} = (II^{-1} : II^{-1})$, $ba \in R$ and $b^2a^2 \in II^{-1}$. Thus $ba \in N$ and $t - ba \in R \setminus N$. Whence, $a = p/(t - ba) \in PR_N$ which is impossible. Therefore P is divisorial. ∎

Our last result of this section shows that for Mori domains the radical trace property and the trace property for primary ideals are equivalent. Before proving this result, we recall some facts about Mori domains.

THEOREM 11. *Let R be a Mori domain. Then*

(a) *For each ideal I of R, there is a finitely generated ideal $J \subseteq I$ such that $I_v = J_v$. Moreover, if $I_v \neq R$, then I is contained in a maximal divisorial ideal* [27, *Corollaire* 1].

(b) *Each maximal divisorial ideal is prime* [5, *Proposition* 2.1].

(c) *For each multiplicative set* S, R_S *is a Mori domain* [28, *Sect*. 3, *Corollaire* 1].

(d) *If* P *is an invertible prime ideal of* R, *then* $\mathrm{ht}(P) = 1$ [5, *Theorem* 2.5].

(e) *If* I *is an ideal of* R *such that* I^{-1} *is a ring*, *then* I^{-1} *is a Mori domain* [29, *page* 11; 3, *Corollary* 11; 25, *Proposition* 2.2].

The proof we provide for our next theorem is almost identical to the proof Gabelli gives to establish Theorem 2.5 *in* [17].

THEOREM 12. *Let* R *be a Mori domain. Then* R *is an RTP domain if and only if* R *has TPP*.

Proof. Assume R has TPP. By Corollary 9, it suffices to show that R is one-dimensional. Let M be a maximal ideal of R. Then R_M is a quasilocal Mori domain with TPP. Thus we may assume R is quasilocal. If M is invertible, then it has height one [5, Theorem 2.5]. Thus we may assume M is not invertible. Thus $M^{-1} = (M : M)$. Let $J \subseteq M$ be a finitely generated ideal with $J_v = M_v$. By Theorem 10, each non-maximal prime ideal of R is divisorial. Hence J is M-primary. Let N be a maximal ideal of M^{-1}. Since R is quasilocal, $M \subseteq N$ [8, Lemma 5] or [4, p. 104]. Hence NJ is an M-primary ideal of R and an ideal of M^{-1}. If N is not invertible as an ideal of M^{-1}, then $(M^{-1} : N) = (N : N) \subseteq (NJ : NJ)$. But since $M^{-1} = J^{-1}$, we have $(R : NJ) = (J^{-1} : N) = (M^{-1} : N) = (NJ : NJ)$. Thus $NJ = M$ since R has TPP. But this implies $J \subseteq NJ$ which is impossible since J is finitely generated. Hence each maximal ideal of M^{-1} is invertible. Since M^{-1} is a Mori domain, invertible primes have height one and therefore M^{-1} is a Dedekind domain [5, Theorem 2.5]. Thus $M^{-1} = (M : M)$ is the complete integral closure of R. Whence for each non-zero ideal I of R, $(I : I) \subseteq (M : M)$. That M has height one now follows from Theorem 10. ∎

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The first two results of this section also appear in [24].

LEMMA 13 (cf. [24, Proposition 2.1]). *Let* I *be an ideal of an integral domain* R *and let* P *be a minimal prime of* I. *If* I^{-1} *is a ring*, *then* P^{-1} *is a ring*.

Proof. Assume I^{-1} is a ring and let $t \in I^{-1}$ and $r \in I$. Then $t^2r^2 = (t^2r)r \in I$ since I^{-1} is a ring. It follows that $II^{-1} \subseteq \sqrt{I}$. Let $s \in P^{-1}$ and let $p \in P$. Since P is minimal over I, there is an integer $n \geq 1$ and an element $r \in R \setminus P$ such that $rp^n \in I$. Thus since I^{-1} is a ring, $s^{2n}rp^n \in$

II^{-1} so that $r(sp)^{2n} \in P$. But since $r \notin P$ and $sp \in R$, we get $sp \in P$. Therefore $P^{-1} = (P : P)$ is a ring. ∎

LEMMA 14 (cf. [24, Theorem 3.4]). *Let P and Q be a pair of prime ideals of an integral domain R. If both P^{-1} and Q^{-1} are rings, then $(P \cap Q)^{-1}$ is a ring.*

Proof. Assume P^{-1} and Q^{-1} are rings. Then $P^{-1} = (P : P)$ and $Q^{-1} = (Q : Q)$ since both are prime ideals [25, Proposition 2.3]. Let $I = P \cap Q$ and let $t \in I^{-1}$. Then for each $p \in P$ and each $q \in Q$, $tpq \in R$. Thus $tp \in Q^{-1}$ and $tq \in P^{-1}$. Hence, $tpr, tqr \in I$ for each $r \in I$. Thus $t^2pr, t^2qr \in R$ and we have $t^2r \in P^{-1} \cap Q^{-1}$. It follows that $t^2r^2 \in I$. Whence $tr \in I$ since I is a radical ideal and $tr \in R$. ∎

Theorem 3.3 of [15] states that if I is an ideal of a semi-normal domain R, then $(\sqrt{I} : \sqrt{I})$ is the largest subring of I^{-1}. An obvious consequence of this theorem is the following useful lemma. (The "semi-normal" version of Lemma 15 is Corollary 3.4 of [15].)

LEMMA 15. *Let R be an integrally closed domain and let I be an ideal of R. Then I^{-1} is a ring if and only if $I^{-1} = \sqrt{I}^{\,-1} = (\sqrt{I} : \sqrt{I})$.*

Recall from above that a domain R is said to be a PRIP domain if for each primary ideal Q, Q^{-1} a ring implies Q is prime. Our next five results concern PRIP domains. All but one deal specifically with Prüfer domains which have PRIP. For an ideal I of a Prüfer domain R, Huckaba and Papick prove that I^{-1} is a ring if and only if $I^{-1} = (\bigcap R_{P_\alpha}) \cap (\bigcap R_{M_\beta})$ where $\{P_\alpha\}$ is the set of minimal primes of I and $\{M_\beta\}$ is the set of maximal ideals which do not contain I [25, Theorem 3.2]. We shall make frequent use of this result. In particular, we use it repeatedly in the proof of our next theorem. We shall also use the property that for an overring S of a Prüfer domain R, the prime and primary ideals of S are all extended from R [18, Theorem 26.1].

THEOREM 16. *Let R be a Prüfer domain. If R has PRIP, then every overring has PRIP.*

Proof. Let S be an overring of R and let J be a primary ideal of S. Since every prime ideal of S is extended from R, $\sqrt{J} = PS$ for some prime P of R. Moreover, $J = IS$ where $I = J \cap R$. Thus I is P-primary. If $(S : J)$ is a ring, then $(S : J) = (S : PS)$ since S is integrally closed (or by [25, Theorem 3.2]). Thus by [25, Theorem 3.2], $(S : J) \subseteq S_{PS} = R_P$. Since $J = IS$, $I^{-1} = (R : I) \subseteq (S : J) \subseteq R_P$. Hence I^{-1} is a ring (again by [25, Theorem 3.2]). By PRIP $I = P$ and therefore $J = PS$ is a prime ideal. ∎

LEMMA 17. *If R is a domain with PRIP and P is a non-maximal prime ideal, then $P^{-1} \neq R$.*

Proof. Let M be a maximal ideal which contains P. Since P is not maximal there is a primary ideal Q between P and M which is not prime. If $P^{-1} = R$, then $Q^{-1} = R$ and QQ^{-1} is not prime. ∎

THEOREM 18. *Let R be a Prüfer domain with PRIP. Then every non-maximal prime is divisorial and for each prime ideal P, P is a maximal ideal of $(P : P)$.*

Proof. Let P be a prime ideal of R. Since R is Prüfer, P is a prime ideal of $(P : P)$. If P is a maximal ideal, $(P : P) = R$. Thus we may assume P is not maximal. In this case $P^{-1} = (P : P)$. Combining Theorem 16 and Lemma 17, we get that if P is not maximal in $(P : P)$, then $P^{-1} \neq (P^{-1} : P) = (R : P^2)$. But by [12, Theorem 3.1], this implies P is invertible in P^{-1} and hence maximal in P^{-1}. Therefore P is maximal in $(P : P)$ and it follows that $P = P_v$. ∎

THEOREM 19. *Let R be a Prüfer domain with PRIP and let I be an ideal for which I^{-1} is a ring. Then every prime minimal over I extends to a maximal ideal of I^{-1}.*

Proof. Since R is integrally closed, I^{-1} a ring implies $I^{-1} = \sqrt{I}^{\,-1} = (\sqrt{I} : \sqrt{I})$. Hence we may assume $I = \sqrt{I}$. Let P be a prime minimal over I. Since I^{-1} is a ring, $P^{-1} = (P : P)$ by Lemma 13. Thus $(P : P) = P^{-1} \subseteq I^{-1} = (I : I)$. Since P is a maximal ideal of $(P : P)$, if it survives in $(I : I)$ it will extend to a maximal ideal of $(I : I)$. But by Theorem 3.2 of [25], I^{-1} is contained in R_p. ∎

For a prime ideal P of a domain R, let $\mathscr{N}(P)$ be the set of maximal ideals of R which do not contain P and let $T = \bigcap_{N \in \mathscr{N}(P)} R_N$. It is always the case that $Q^{-1} \subseteq T$ for each P-primary ideal Q. Thus in the event that $P^{-1} = T$, then $Q^{-1} = T$ for each P-primary ideal Q.

LEMMA 20. *Let R be a Prüfer domain and let P be a branched prime ideal. If R is either a TPP domain or a PRIP domain, then $P^{-1} \neq T$.*

Proof. Assume $P^{-1} = T$. Since R is a Prüfer domain the primary ideals of P^{-1} are extended from R. In particular, $QP^{-1} = Q$ for each P-primary ideal Q since $PP^{-1} = P$. But if $P^{-1} = T$, then $Q^{-1} = T = (Q : Q)$. Thus if R is either a TPP domain or a PRIP domain, then $P^{-1} \neq T$. ∎

The following lemma and its proof are extracted from the proof of Theorem 2.5 of [22].

LEMMA 21. *Let I be an ideal of a Prüfer domain R and let P be a prime minimal over I. If $IR_P \neq PR_P$ and P is the radical of a finitely generated ideal C, then I^{-1} is not a ring.*

Proof. Assume $P = \sqrt{C}$ where C is finitely generated and that $IR_P \neq PR_P$. Then there is an element $r \in P$ such that $IR_P \subseteq rR_P$. Then for each maximal ideal M containing P, $IR_M \subset rR_M$; and for each maximal ideal $N \in \mathscr{N}$, $IR_N \subseteq CR_N = R_N$. It follows that $I \subset J = rR + C \subseteq P$. Since R is Prüfer, I^{-1} a ring implies $I^{-1} \subseteq R_P$ which in this case implies $1 \in JJ^{-1} \subseteq PI^{-1} \subseteq PR_P$. Thus I^{-1} is not a ring (see also [22, Lemma 2.4]). ∎

Our next lemma is related to Lemma 10 of [13].

LEMMA 22. *Let $P \subset M$ be prime ideals of a Prüfer domain R. If P is the radical of a finitely generated ideal, then $MP^{-1} = P^{-1}$.*

Proof. Assume $P = \sqrt{C}$ where C is finitely generated. Let $J = rR + C$ where $r \in M \setminus P$. Since $PR_{M_\alpha} \cap R = P$ for each maximal ideal M_α containing P, $PR_{M_\alpha} \subset rR_{M_\alpha} = JR_{M_\alpha}$. On the other hand, for each $N \in \mathscr{N}$, $R_N = CR_N = PR_N = JR_N$. Hence $P \subset J \subseteq M$. Therefore $JP^{-1} = MP^{-1} = P^{-1}$ since J is invertible and $J^{-1} \subset P^{-1}$. ∎

We are now ready to characterize the class of Prüfer domains with the radical trace property.

THEOREM 23. *Let R be a Prüfer domain. Then the following are equivalent*

(1) *R is an RTP domain.*

(2) *R is a TPP domain.*

(3) *R is a PRIP domain.*

(4) *Each branched prime is the radical of a finitely generated ideal.*

Proof. That (1) implies (2) is true in general (Theorem 4).

There are a number of ways to prove that (2) implies (3). Since R is Prüfer, the combination of Lemma 15 and Lemma 4.4 of [15] implies that if Q^{-1} is a ring for a P-primary ideal Q, then $Q^{-1} = (Q:Q) = P^{-1} = (P:P)$. Thus if R is a TPP domain and Q is a primary ideal, Q^{-1} a ring implies Q is prime.

We next prove the equivalence of (1) and (4).

First assume R is an RTP domain. Let P be a branched prime and let Q be a proper P-primary ideal. Since P is branched and R is Prüfer, P is minimal over a finitely generated ideal A [18, Theorem 23.3]. If P^{-1} is not a ring, then P is invertible [25, Theorem 3.8]. Thus we may assume P^{-1} is a ring, in which case, $P^{-1} = R_P \cap T$ by Theorem 3.2 of [25]. By Lemma 20, $P^{-1} \neq T$. Hence R_P does not contain $T = \bigcap_{N \in \mathscr{N}} R_N$. Thus there is a finitely generated ideal B which is contained in P and not contained in any $N \in \mathcal{N}$ [19, Corollary 2]. It follows that $P = \sqrt{A + B}$.

Now assume each branched prime is the radical of a finitely generated ideal and let I be an ideal such that $I^{-1} = (I : I)$. By Lemma 21, $IR_P = PR_P$ for each prime P minimal over I. If I is not a radical ideal, then there is an element $r \in \sqrt{I} \setminus I$. Since $IR_P = PR_P$ for each prime minimal over I, no prime minimal over I contains the ideal $J = (I :_R r)$.

Let M be a prime ideal that contains J and let P be a prime contained in M and minimal over I. By Lemma 22, $MP^{-1} = P^{-1}$ and therefore $MI^{-1} = I^{-1}$. Hence $JI^{-1} = I^{-1}$. Let $j_1, j_2, \ldots, j_n \in J$ and $u_1, u_2, \ldots, u_n \in I^{-1}$ be such that $j_1u_1 + j_2u_2 + \cdots + j_nu_n = 1$. But then $r = rj_1u_1 + rj_2u_2 + \cdots + rj_nu_n \in I$ since $rJ \subseteq I$ and $I^{-1} = (I : I)$. Hence I is a radical ideal and R is an RTP domain.

Finally we show that (3) implies (1).

Assume R is a PRIP domain and let I be an ideal for which $I^{-1} = (I : I)$. As in the proof of (4) implies (1), it suffices to show $IR_P = PR_P$ for each prime P minimal over I since P is a maximal ideal of $(P : P)$ by Theorem 19.

If $IR_P \neq PR_P$ for some P minimal over I, then for each maximal ideal M containing P, $IR_M \neq PR_M$. Since R_m is a valuation domain for each M, P is branched and therefore is minimal over some finitely generated ideal [18, Theorems 17.3 and 23.3]. By Lemma 20, $P^{-1} \neq T$. As in the proof of (1) implies (4), we get that P is the radical of a finitely generated ideal. Since I^{-1} is a ring we get a contradiction by Lemma 21. Hence $IR_P = PR_P$ for each prime P minimal over I. ∎

COROLLARY 24. *Let R be a Prüfer domain. If R is an RTP domain, then every overring in an RTP domain.*

In [19], Gilmer and Heinzer prove that for a Prüfer domain R, R has (♯) if and only if for each maximal ideal M there is a finitely generated ideal A such that M is the only maximal ideal containing A [19, Theorem 1]. In the same paper they also prove that R has (♯♯) if and only if for each prime ideal P there exists a finitely generated ideal $A \subseteq P$ such that each maximal ideal containing A contains P [19, Theorem 3]. Our next two results follow from combining these two theorems with our Theorem 23.

COROLLARY 25. *Let R be a Prüfer domain which has (♯♯). Then R is an RTP domain.*

Proof. Let P be a branched prime. Then P is minimal over a finitely generated ideal A [18, Theorem 23.3]. Moreover, since R has (♯♯), we may assume that each maximal ideal containing A contains P [19, Theorem 3]. It follows that $\sqrt{A} = P$ since Spec(R) is treed. Thus R is an RTP domain. ∎

As a partial converse to Corollary 25, we have the following.

COROLLARY 26. *Let R be an RTP Prüfer domain. If every maximal ideal of R is branched, then R has ($\sharp$). If every prime ideal of R is branched, then R has ($\sharp\sharp$).*

Proof. Since R is an RTP Prüfer domain, each branched prime is the radical of a finitely generated ideal. The result now follows from Theorems 1 and 3 of [19]. ∎

Only the first implication in Corollary 26 is new, the other is the same as (1) implies (3) in (the previously mentioned) Theorem 2.7 of [22] since in a Prüfer domain every prime being branched is equivalent to the domain having acc on prime ideals.

A Prüfer domain R is said to have the separation property if for each pair of comparable primes $P \subset M$, there is a finitely generated ideal I such that $P \subset I \subseteq M$ [13, p. 100]. It is known that every Prüfer domain with ($\sharp\sharp$) has the separation property [13, Proposition 11]. Our next theorem shows more generally that every Prüfer RTP domain has the separation property.

THEOREM 27. *Let R be a Prüfer RTP domain and let $P \subset M$ be a pair of prime ideals. Then there is a finitely generated ideal I such that $P \subset I \subseteq M$.*

Proof. Since $P \subset M$ there is a branched prime Q properly containing P and contained in M. Thus, as in the proof of Corollary 25, there is a finitely generated ideal I such that $Q = \sqrt{I}$. Since P is properly contained in Q, we have $PR_N \subseteq IR_N$ for each maximal ideal N. Hence $P \subset I \subseteq M$. ∎

THEOREM 28. *Let R be a Prüfer domain. Then the following are equivalent.*

(1) *R is a TP domain.*

(2) *R is an RTP domain and the non-invertible prime ideals are linearly ordered.*

(3) *R is a TPP domain and the non-invertible prime ideals are linearly ordered.*

(4) *R is a PRIP domain and the non-invertible prime ideals are linearly ordered.*

(5) *Each branched prime is the radical of a finitely generated ideal and the non-invertible prime ideals are linearly ordered.*

Proof. Since R is a Prüfer domain, if P is a non-invertible prime, then P^{-1} is a ring. Hence, if R is a TP domain, then the non-invertible prime ideals must be linearly ordered by Lemma 14. On the other hand, if the non-invertible primes are linearly ordered and I is a radical ideal with I^{-1} a ring, then I must be prime by Lemma 13. ∎

COROLLARY 29. *Let R be a Prüfer domain. If R is a TP domain, then every overring is a TP domain.*

Our first example shows that there are Noetherian RTP domains which do not have PRIP.

EXAMPLE 30. Let $R = K[[X^3, X^4, X^5]]$. Then R is a Noetherian TP domain but does not have PRIP.

Proof. Let $M = (X^3, X^4, X^5)$. Then $M^{-1} = K[[X]]$. Thus R is a TP domain by Theorem 3.5 of [25]. But the ideal $I = (X^3, X^4)$ is a proper M-primary ideal with $I^{-1} = K[[X]]$. Hence R does not have PRIP. ∎

We next give two ways to construct RTP domains. The first involves a pullback construction beginning with a valuation domain. The second involves a semi-quasi local Prüfer domain and a subfield.

Before presenting our first construction, we need to set a little notation. Let M be an ideal of a domain T and let f be the canonical homomorphism from T onto T/M. Let $D \subset T/M$ be a subring of T/M and let R be the pullback of the following diagram:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & T/M \end{array}$$

In Theorem 31, we take T to be a valuation domain with maximal ideal M. In Theorem 34, T will be a semi-quasilocal Prüfer domain with Jacobson radical M.

THEOREM 31. *Let (V, M) be a valuation domain with $K = V/M$. Let $R = f^{-1}(D)$ be the pullback of the subring D of K where f is the canonical homomorphism from V onto K. Then R is an RTP domain if and only if D is an RTP domain. The same equivalence holds for TPP.*

Proof. Note that since V is a valuation domain, every ideal of R compares with M [10, Proposition 2.1]. Moreover, V is the largest ring which has M as an ideal and thus, $V = (M : M) = (R : M)$.

If R is an RTP domain, then so is D by Theorem 3. Assume D is an RTP domain and let I be an ideal of R such that $I^{-1} = (I : I)$.

If I is contained in M, then I is an ideal of V since $V = M^{-1} \subseteq I^{-1}$. If $I_v \neq M$, then $(V : I)$ is a ring by Theorem 7 of [23]. Whence I is prime by [25, Proposition 3.5]. If $I_v = M$ and $I \neq M$, then there is an element $m \in M \setminus I$. But since V is a valuation domain, we have $m^{-1}I \subseteq M$. Thus $m^{-1} \in I^{-1}$ which is impossible since we assumed that $I^{-1} = M^{-1}$.

If I properly contains M, then it is easy to show that $I = f^{-1}(J)$ for some ideal J of D. It follows from [11, Proposition 1.8] or [23, Proposition 6] that $I^{-1} = f^{-1}((D:J))$ and $(I:I) = f^{-1}((J:J))$.

Now since $I^{-1} = (I:I)$, we have $(D:J) = (J:J)$. As D is an RTP domain, J and hence I are radical ideals. Therefore R is an RTP domain if and only if D is an RTP domain.

For TPP, let Q be a P-primary ideal of R.

If $P \subset M$, then $QQ^{-1} = P$ since $R_P = V_P$ implies Q is also an ideal of V (and every valuation domain satisfies the trace property). On the other hand if $M \subset P$, then there is a prime ideal P' of D and a P'-primary ideal Q' such that $P = f^{-1}(P')$ and $Q = f^{-1}(Q')$. As above $Q^{-1} = f^{-1}((D:Q'))$. Hence $QQ^{-1} = f^{-1}(Q')f^{-1}((D:Q')) = f^{-1}(Q'(D:Q'))$. It follows that $QQ^{-1} = P$ if and only if $Q'(D:Q') = P'$, and that $QQ^{-1} = R$ and P is maximal if and only if $Q'(D:Q') = D$ and P' is maximal.

The remaining case is when $P = M$.

Let $F = qf(D)$. Since $R_M \subseteq V$, $R_M = f^{-1}(F)$. Thus M is the maximal ideal of $f^{-1}(F)$. Hence Q is the primary ideal of both R and R_M. If Q is not an invertible ideal of R_M, then $Q(QR_M)^{-1} = M$ and it follows that $QQ^{-1} \neq R$. If Q is invertible, then Q is principal as an ideal of R_M; i.e., $Q = mf^{-1}(F)$ for some $m \in M$. So $Q^{-1} = (1/m)f^{-1}((D:F))$. If $D = F$, then $(D:F) = F$, $R = R_M$, and Q is invertible with M maximal. If $D \neq F$, then $(D:F) = (0)$ so $Q^{-1} = (1/m)M$. In this case $QQ^{-1} = f^{-1}(F)M = M$.

Combining all three cases we have that if D has TPP, then so does R. The converse holds by Theorem 9. ∎

While Theorem 31 provides a way to make RTP domains which are neither Mori nor Prüfer, it also shows that the classical $D + M$ construction of [7] will not be of use in trying to decide whether or not every TPP domain is also an RTP domain.

For an ideal I and a prime ideal P not containing I, there is a unique prime ideal of $T = (I:I)$ which contracts to P; namely, the ideal $P' = (P:_T I)$ $[= (P:I)$ when $I^{-1} = (I:I)]$ (see, for example, [10, Theorem 1.4(c); 4, pp. 104–105]). The following lemma provides information about the primes of $(I:I)$ which contain I when I is a trace ideal of a TPP domain.

LEMMA 32. *Let R be a TPP domain and let I be an ideal of R such that $I^{-1} = (I:I)$. If $P' \subset N'$ are a pair of primes of I^{-1} which contain I, then $P' \cap R = N' \cap R$.*

Proof. Let $T = I^{-1}$ and let $P' \subset N'$ be primes of T containing I. Let $P = P' \cap R$ and $N = N' \cap R$. Assume $P \neq N$ and let $r \in N \setminus P$. Without loss of generality we may assume that N' is minimal over $rT + P'$. Thus the ideal $Q' = (r^2T + P')T_{N'} \cap T$ is a N'-primary ideal of T which does not contain r. Hence neither does the N-primary ideal $Q = Q' \cap R$. But since $I \subset Q$ and R is a TPP domain, $Q^{-1} \subseteq I^{-1}$ and therefore $QT = QQ^{-1}T = NT$. As $r \in N \setminus Q'$ we have a contradiction. ▮

LEMMA 33. *Let R be a TPP domain and let $\mathcal{J}$ be the set of those ideals I of R such that $I^{-1} = (I : I)$. If for each $I \in \mathcal{J}$, the pair R and $(I : I)$ satisfy INC, then R is an RTP domain; i.e., every ideal in $\mathcal{J}$ is a radical ideal of R.*

Proof. Let I be an ideal of R such that $I^{-1} = (I : I)$ and $I \neq \sqrt{I}$. Let $t \in \sqrt{I} \setminus I$, $T = I^{-1}$, and $J' = (I :_T)$. We begin by showing if M' is a prime of T minimal over I, then M' does not contain J'. Let $M = M' \cap R$, $Q' = IT_{M'} \cap T$, and $Q = Q' \cap R$. Then Q' is M'-primary and hence Q is M-primary. Since $I \subset Q$, $Q^{-1} \subseteq I^{-1} = T$. As $M \subseteq QQ^{-1}$, we have that $MT = QT \subseteq Q'$. It follows that $t \in IT_{M'}$ and, hence, J' is not contained in M'.

Let $J = J' \cap R$. By Theorem 5, no minimal prime of I in R contains J. But since $IT = I$, $J' \neq T$. Let N' be a minimal prime of J' in T and let $N = N' \cap R$. Then by the above N' is not minimal over I. But by Lemma 32, if $M' \subset N'$ with M' a prime minimal over I, then $N = M' \cap R$ contradicting the assumption that R and $(I : I)$ satisfy INC. Hence, I must be a radical ideal of R. ▮

So far all of the examples of RTP domains have had treed spectrum. In our next theorem we show that this is not always the case. The theorem also shows that for a non-maximal prime P of an RTP domain there may be prime ideals of P^{-1} which contain P but do not contract to P.

THEOREM 34. *Let T be a semi-quasilocal Prüfer domain which contains a field K. Let $M_1, M_2, \ldots, M_n$ be the maximal ideals of T and let $R = K + M$ where $M = M_1 \cap M_2 \cap \cdots \cap M_n$. Then R is quasilocal and both T and R are RTP domains.*

Proof. If $n = 1$, we are in the same situation as Theorem 31. Thus we may assume $n > 1$.

Since T has only finitely many maximal ideals, R is quasilocal [10, Sect. 3], and $T = (R : M) = (M : M)$ has (♯♯) [19, Corollary 3]. Thus T is an RTP domain.

To prove that R is an RTP domain, we will first show that R is a TPP domain.

Let P be a non-maximal prime ideal of R. Since R is quasilocal and $T = M^{-1}$, there is a unique prime P' of T such that $P = P' \cap R$. Furthermore, $R_P = T_{P'}$ so $P' = PR_P \cap T$. Thus for each P-primary ideal Q, there is a unique P'-primary ideal Q' such that $Q = Q' \cap R$. Moreover since T has only finitely many maximal ideals, only finitely many of the maximal ideals of T do not contain P'; say, $M_1, M_2, \ldots, M_k$. Thus $P = P'N = P' \cap N$ and $Q = Q'N = Q' \cap N$ where $N = M_1 \cap M_2 \cap \cdots \cap M_k = M_1 M_2 \cdots M_k$.

Since T is an RTP domain, $Q'(T : Q') = P'$. So $Q(T : Q') = Q'N(T : Q') = P'N = P$. Thus $N(T : Q') \subseteq (R : Q)$ and $P \subseteq Q(R : Q)$. For the reverse containment, note that $(R : Q) \subseteq (T : Q) = (T : Q' \cap N) = (T : Q') + (T : N)$ and $(T : N)Q \cap R = (T : N)Q'N \cap R \subseteq Q' \cap R = Q$.

Let J be an M-primary ideal and let $J' = JT$. Then either $J'(T : J') = T$ or $J'(T : J') = M'$ is a radical ideal of T. In the first case, $M = MT = MJ'(T : J') = MJ(T : J')$ and, hence, $M(T : J') \subseteq (R : J)$. In the second case, let $M_1, M_2, \ldots, M_k$ be the maximal ideals of T which do not contain $J'(T : J')$. If no such ideals exist, $M' = M$, $(T : J') = (R : J)$ and $J(R : J) = M$. For $k \geq 1$, let $N = M_1 M_2 \cdots M_k$. Then $M = NM' = NJ'(T : J') = NJ(T : J)$. Thus $N(T : J) \subseteq (R : J)$ and again $M \subseteq J(R : J)$. Therefore R is a TPP domain.

Let I be an ideal of R for which $I^{-1} = (I : I)$. Since $T = (R : M)$ is a Prüfer domain and R and T satisfy INC, R and $(I : I)$ satisfy INC. Thus by Lemma 33, R is an RTP domain. ∎

Papick gives an example of a domain R where the integral closure of R is a Prüfer domain but the spectrum of R is not treed [26, Example 2.28]. The domain in Papick's example fits the hypotheses of the above theorem and thus is an RTP domain. Unlike Prüfer RTP domains, this domain also has a pair of comparable primes where the larger survives in the inverse of the smaller. To illustrate this fact we present the ring R as our next example.

EXAMPLE 35. Let K be a field, let X and Y be two indeterminates, and let $G = \mathbf{Z} \oplus \mathbf{Z}$ (ordered lexicographically). Let $T = V_1 \cap V_2$ where V_1 and V_2 are the valuation domains arising from the respective valuations v_1 and v_2 from $K(X, Y)$ to G defined by $v_1(X) = (1, 0)$, $v_2(Y) = (0, 1)$ and $v_2(X) = (0, 1)$, $v_2(Y) = (1, 0)$. Finally, let R be the subring $K + M$. Then R is a quasilocal RTP domain with two height one primes P_x and P_y and M survives in both P_x^{-1} and P_y^{-1}.

Proof. That R is a quasilocal RTP domain follows from Theorem 34. From the proof of Theorem 34 (see also [10, Sect. 3]), we see that R has three non-zero prime ideals, the unique maximal ideal $M = M_1 \cap M_2$, and two incomparable height one primes $P_x = P_1 \cap M_2$ and $P_y = M_1 \cap P_2$ (where P_i is the height one prime of V_i). Thus R is not a TP domain. As in the proof of Theorem 34, $P_x^{-1} = P_1^{-1} = T_{P_1} \cap T_{M_2}$ and $P_y^{-1} = P_2^{-1} = T_{P_2} \cap T_{M_1}$ since $P_x = P_1 \cap M_2$ and $P_y = P_2 \cap M_1$. Hence M survives in both P_x^{-1} and P_y^{-1}. ∎

In general, it is not the case that each overring of an RTP domain is also an RTP domain. For example, if V is a valuation domain of the form $k(X,Y) + M$, then it follows from Theorem 31 that the ring $S = k + M$ is a TP domain while the ring $T = k[X,Y] + M$ is not (cf. [22, p. 120]). The ring $S = k + M$ is an example of a pseudo-valuation domain (or PVD, for short). By [21], a domain is a *pseudo-valuation domain* if it has the same spectrum as some valuation overring. Proposition 2.6 of [2] characterizes PVD in terms of pullbacks. In the notation of Theorem 31, the aforementioned proposition means that the domain R is a PVD if and only if $R = f^{-1}(k)$ for some subfield k of V/M. In our next two theorems, we show how PVDs are related to whether or not every overring can be an RTP domain.

As mentioned above, if R is a Noetherian RTP domain, then for each non-invertible prime P, PR_P^{-1} is a Dedekind domain. Thus if $R = K[X^{k_1}, X^{k_2}, \ldots, X^{k_n}]$ where $1 < k_1 < k_2 < \cdots < k_n$ are positive integers, then R can be an RTP domain only if $(X^{k_1}, X^{k_2}, \ldots, X^{k_n})^{-1} = K[X]$. We use this fact to prove our next result.

THEOREM 36. *Let W be an integrally closed PVD which is not a valuation domain. Then there is an overring of W which is not an RTP domain.*

Proof. Let V be the corresponding valuation domain containing W with the same maximal ideal M. Then $V = M^{-1}$ [21, Theorem 10]. Let $L = V/M$ and $K = W/M$. Then there is an element $x \in L$ which is transcendental over K. The ring $K[x^2, x^5]$ is not an RTP domain since $(X^2, X^5)^{-1} \neq K[X]$. Thus by Theorem 31 the pullback of $K[X^2, X^5]$ gives a ring between V and W which is not an RTP domain. ∎

THEOREM 37. *If every overring of R is an RTP (TP) domain, then the integral closure of R is an RTP (TP) Prüfer domain.*

Proof. By Proposition 2.7 of [9], either R' is Prüfer or there is an integrally closed PVD overring of R which is not a valuation domain. The result now follows from the previous theorem. ∎

Our final result involves the construction of a Prüfer TP domain which does not satisfy (♯).

Let $K = F(X, Y)$ where $X = \{X_1, X_2, \ldots\}$ and $Y = \{Y_1, Y_2, \ldots\}$. For each $n \geq 1$, let $G_n = \sum_{k=1}^n \mathbf{Z}$ ordered lexicographically. For each n, let V_n be the valuation domain corresponding to the valuation v_n determined by setting $v_n(X_1^{r_1}X_2^{r_2} \cdots X_n^{r_n}) = (r_1, r_2, \ldots, r_n) \in G_n$ and $v_n(X_k) = v_n(Y_i) = (0, 0, \ldots, 0)$ for each $k > n$ and each $i \geq 1$. Similarly, let W_n be the valuation domain corresponding to the valuation w_n determined by setting $w_n(X_1^{r_1}X_2^{r_2} \cdots X_n^{r_n}Y_n^{s_n}) = (r_1, r_2, \ldots, r_n, s_n) \in G_{n+1}$ and $w_n(X_k) = w_n(Y_i) = (0, 0, \ldots, 0)$ for each $k > n$ and each $i \neq n$. Obviously, $W_n \subset V_n$ for each n.

Thus $\bigcap W_n \subseteq \bigcap V_n$. But it is easy to check that $V = \bigcap V_n$ is a valuation domain which properly contains $\bigcap W_n$. Moreover the value group associated with V is the group $G = \sum_{n=1}^{\infty} \mathbf{Z}$ ordered lexicographically.

EXAMPLE 38. Let $R = \bigcap W_n$. Then R is a Prüfer TP domain that does not satisfy (♯).

Proof. For each $n \geq 1$, let $K_n = K(X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n)$ and $R_n = \bigcap_{k=1}^{n} W_{n,k}$ where $W_{n,k} = W_k \cap K_n$. Since the $W_{n,k}$'s are incomparable (discrete) valuation domains, each R_n is a Prüfer domain [18, Theorem 11.11]. Moreover, for each n, the maximal ideals of R_n are the principal ideals $Y_1 R_n, Y_2 R_n, \ldots, Y_n R_n$. Since $R_n \subset R_{n+1}$, the union $\bigcup R_n$ is a Prüfer domain [18, Proposition 22.6]. As $R_n = (\bigcap_{k=1}^{k=n} W_k) \cap K_n$, $R = \bigcup R_n$ so R is a Prüfer domain. Furthermore, the principal ideals $Y_n R$ are maximal ideals of R.

For each pair of integers $k \leq n$, let $V_{n,k} = V_k \cap K_n$ and let $P'_{n,k}$ be the maximal ideal of $V_{n,k}$. Then $V_k = \bigcup_{n=k}^{\infty} V_{n,k}$ with maximal ideal $P'_k = \bigcup_{n=k}^{\infty} P'_{n,k}$. Let $P_k = P'_k \cap R$. Then for $k \leq n$, $P_k \subset Y_n R$. On the other hand, for $k > n$, $P_k + Y_n R = R$. Thus $P = \bigcup P_k$ is an unbranched maximal ideal of R and P is the only other maximal ideal besides the principal ones $Y_n R$. Also $R_P = \bigcup V_{n,n} = \bigcap V_n = V$. Thus R does not satisfy (♯).

The non-zero non-maximal primes of R are the P_k's and for each k, P_k is the only prime minimal over the set $\{X_1, X_2, \ldots, X_k\}$. Since the P_k's are linearly ordered and each branched prime is the radical of a finitely generated ideal, R is a TP domain. ∎

We end with a number of questions concerning RTP domains.

(1) Is the trace property for primary ideals equivalent to the radical trace property?

(2) If I is an ideal of an RTP domain, is $(I : I)$ an RTP domain?

(3) If the answer to (2) is "No," is the answer "Yes" if we assume that $I^{-1} = (I : I)$ or that I is prime?

(4) If I is a trace ideal of an RTP domain R, does the pair R and $(I : I)$ satisfy INC?

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