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Nagata transforms and localizing systems. (English. English summary)

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Let R be an integral domain with quotient field K. A set $\mathcal{F}$ of ideals of R is said to be a localizing system if (1) $I\in\mathcal{F}$ and $J\supseteq I$ implies $J\in\mathcal{F}$ and (2) $I\in\mathcal{F}$ and J an ideal of R such that $(J{:}_R\,a)\in\mathcal{F}$ for each $a\in I$ implies $J\in\mathcal{F}$. It is easy to show that a localizing system is a special type of generalized multiplicative system and that therefore one can form the ring $R_{\mathcal{F}}=\{x\in K\mid xI\subseteq R \text{ for some } I\in\mathcal{F}\}$. Now recall that the Nagata transform of an ideal I of R is the ring $T_R(I)=\bigcup_{n=1}^{\infty}(R{:}\,I^n)$. It is well known that if I is a finitely generated ideal, then $T_R(I)=\bigcap R_P$, where the intersection is taken over those prime ideals of R which do not contain I [J. Brewer, Math. Z. **107** (1968), 301–306; MR0236158 (38 #4456)]; it is also easy to see that this intersection of localizations equals $R_{\mathcal{F}}$ for an appropriate localizing system $\mathcal{F}$. The purpose of the paper under review is to characterize when the Nagata transform of an ideal is of the form $R_{\mathcal{F}}$. Denoting by $\mathcal{N}(I)$ the smallest localizing system of R which contains I, the authors' main result is the equivalence of the following statements: (1) $T(I)=R_{\mathcal{F}}$ for some localizing system $\mathcal{F}$; (2) $T(I)=R_{\mathcal{N}(I)}$; and (3) $T_R(I)=T_{T_R(I)}(IT_R(I))$. The proofs require transfinite induction. The paper concludes with an example of an R and I for which $T_R(I)$ is not of the form $R_{\mathcal{F}}$. *Evan G. Houston* (1-NC3)

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