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Semistar Dedekind domains. (English. English summary)

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Let D be an integral domain with quotient field K and $\overline{F}(D)$ the set of nonzero D-submodules of K. A semistar operation $*$ is a function $*\colon E \to E^*$ on $\overline{F}(D)$ such that for all $x \in K \setminus \{0\}$ and for all $E, F \in \overline{F}(D)$

(1) $(xE)^* = xE^*$,
(2) $E \subseteq E^*$ and $E \subseteq F$ implies that $E^* \subseteq F^*$,
(3) $(E^*)^* = E^*$.

Semistar operations sound similar to star operations which are defined on the set $F(D)$ of nonzero fractional ideals of D, but they are not quite the same. (See [R. Gilmer, *Multiplicative ideal theory*, Dekker, New York, 1972; MR0427289 (55 #323) (Sections 32, 34)] for a quick review of the star operations.) For star operations it has been a popular area of study to take a particular star operation of finite type, say the t-operation, and a classical notion, say a Dedekind domain, and look for what a t-Dedekind domain (for each $A \in F(D)$ A_t is t-invertible) looks like, and that a t-Dedekind domain is a Krull domain. Similarly a t-Noetherian (integral t-ideals satisfy ACC) is a Mori domain and a t-Prufer (D_M is a valuation domain for each maximal t-ideal M) is the now well-known PVMD. Indeed, the above procedure can be repeated with any $*$ of finite character [see, e.g., E. G. Houston, Jr., S. B. Malik and J. L. Mott, Canad. Math. Bull. **27** (1984), no. 1, 48–52; MR0725250 (85d:13026)] or with $*_f$ for any star operation $*$. Here, $A^{*_f} = \bigcup\{F^*\colon 0 \neq F$ is a finitely generated D-submodule of $A\}$. It is well known that a Mori PVMD is Krull (t-Dedekind) and P$*$MD's are known to be PVMD's.

The authors of the paper under review have taken the same route for a semistar operation $*$, albeit with some modifications to ensure a smooth theory. ($*$ denotes a semistar operation henceforth.) For instance they introduce the notion of a quasi-$*$-ideal as an ideal I such that $I = I^* \cap D$, and define prime and maximal quasi-semistar ideals in the obvious fashion. Then an integral domain with ACC on quasi-$*$-ideals is a $*$-Noetherian domain, which they prove to be a Mori domain in some cases. They call D a P$*$MD if D_M is a valuation domain for every quasi-$*_f$-maximal ideal M. They study $*$-almost Dedekind domains (D_M is a discrete rank one valuation ring for every quasi-$*_f$-maximal ideal M). They call D a $*$-Dedekind domain ($*$-DD) if D is $*$-Mori and P $*$ MD and show that the $*$-DD's fail to be integrally closed in general. An interested reader may find the following papers useful reading: [D. D. Anderson and M. Zafrullah, Comm. Algebra **21** (1993), no. 4, 1189–1201; MR1209927 (94c:13023); J. L. Mott and M. Zafrullah, Arch. Math. (Basel) **56** (1991), no. 6, 559–568; MR1106498 (92d:13012)].

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[References]

1. D.D. Anderson, D.F. Anderson, Some remarks on star operations and the class group, J. Pure Appl. Algebra 51 (1988) 27–33. MR0941887 (89f:13024)
2. D.D. Anderson, S.J. Cook, Two star-operations and their induced lattices, Comm. Algebra 28 (2000) 2461–2475. MR1757473 (2001c:13033)
3. D.F. Anderson, E.G. Houston, M. Zafrullah, Pseudo-integrality, Canad. Math. Bull. 34 (1991) 15–22. MR1108923 (92f:13025)
4. K.E. Aubert, Theory of x-ideals, Acta Math. 107 (1962) 1–52. MR0148773 (26 #6279)
5. V. Barucci, Mori domains, in: S. Chapman, S. Glaz (Eds.), Non Noetherian Commutative Ring Theory, Kluwer Academic Publishers, Dordrecht, 2000, pp. 57–73 (Chapter 3). MR1858157 (2002h:13028)
6. V. Barucci, S. Gabelli, How far is a Mori domain from being a Krull domain? J. Pure Appl. Algebra 45 (1987) 101–112. MR0889586 (88j:13025)
7. P.-J. Cahen, E.G. Houston, T.G. Lucas, Discrete valuation overrings of Noetherian domains, Proc. Am. Math. Soc. 124 (1996) 1719–1721. MR1317033 (96h:13057)
8. C. Chevalley, La notion d'anneau de décomposition, Nagoya Math. J. 7 (1954) 21–33. MR0067866 (16,788g)
9. S. El Baghdadi, M. Fontana, Semistar linkedness and flatness, Prüfer semistar multiplication domains, Comm. Algebra 32 (2004) 1101–1126. MR2063800
10. W. Fanggui, On w-dimension of domains, Comm. Algebra 27 (1999) 2267–2276. MR1683866 (2000f:13046)
11. W. Fanggui, On w-dimension of domains, II, Comm. Algebra 29 (2001) 2419–2428. MR1845120 (2002k:13003)
12. W. Fanggui, On UMT-domains and w-integral dependence, preprint.
13. W. Fanggui, R.L. McCasland, On w-modules over strong Mori domains, Comm. Algebra 25 (4) (1997) 1285–1306. MR1437672

(98g:13025)
14. W. Fanggui, R.L. McCasland, On strong Mori domains, J. Pure Appl. Algebra 135 (2) (1999) 155–165. MR1667555 (99m:13044)
15. M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. 123 (4) (1980) 331–355. MR0581935 (81j:13001)
16. M. Fontana, J.A. Huckaba, Localizing systems and semistar operations, in: S. Chapman, S. Glaz (Eds.), Non Noetherian Commutative Ring Theory, Kluwer Academic Publishers, Dordrecht, 2000, pp. 169–197 (Chapter 8). MR1858162 (2002k:13001)
17. M. Fontana, P. Jara, E. Santos, Prüfer $\star$-multiplication domains and semistar operations, J. Algebra Appl. 2 (2003) 21–50. MR1964763 (2004b:13021)
18. M. Fontana, K.A. Loper, Kronecker function rings: a general approach, in: Ideal Theoretic Methods in Commutative Algebra (Columbia, MO, 1999), Lecture Notes in Pure and Appl. Math., Vol. 220, Dekker, New York, 2001, pp. 189–205. MR1836601 (2002h:13029)
19. M. Fontana, K.A. Loper, A Krull-type theorem for the semistar integral closure of an integral domain. AJSE Theme Issue, Commutative Algebra 26C (2001) 89–95. MR1843459 (2002e:13019)
20. M. Fontana, K.A. Loper, Nagata rings, Kronecker function rings and related semistar operations, Comm. Algebra 31 (2003) 4775–4805. MR1998028 (2004e:13034)
21. M. Fontana, G. Picozza, Semistar invertibility on integral domains, preprint.
22. S. Gabelli, E.G. Houston, Coherentlike conditions in pullbacks, Michigan Math. J. 44 (1997) 99–123. MR1439671 (98d:13019)
23. J.M. García, P. Jara, E. Santos, Prüfer $\ast$-multiplication domains and torsion theories, Comm. Algebra 27 (1999) 1275–1295. MR1669156 (2000a:13028)
24. R. Gilmer, Multiplicative Ideal Theory, M. Dekker, New York, 1972. MR0427289 (55 #323)
25. S. Glaz, W. Vasconcelos, Flat ideals, II, Manuscripta Math. 22 (1977) 325–341. MR0472797 (57 #12487)
26. M. Griffin, Some results on v-multiplication rings, Canad. J. Math. 19 (1967) 710–721. MR0215830 (35 #6665)
27. F. Halter-Koch, Generalized integral closures, in: D.D. Anderson (Ed.), Factorization in Integral Domains, M. Dekker Lect. Notes Math. Pure Appl. 187 (1997) 349–358. MR1460786 (98e:13014)
28. F. Halter-Koch, Ideal systems, in: Monographs and Textbooks in Pure and Applied Mathematics, Vol. 211, Marcel Dekker, New

York, 1998. MR1828371 (2001m:13005)
29. F. Halter-Koch, Construction of ideal systems with nice Noetherian properties, in: S. Chapman, S. Glaz (Eds.), Non Noetherian Commutative Ring Theory, Kluwer Academic, Dordrecht, 2000, pp. 271–285 (Chapter 12). MR1858166 (2002h:13002)
30. F. Halter-Koch, Localizing systems, module systems, and semistar operations, J. Algebra 238 (2001) 723–761. MR1823782 (2002a:13012)
31. F. Halter-Koch, Kronecker function rings and generalized integral closures, Comm. Algebra 31 (2003) 45–59. MR1969212 (2004e:13004)
32. F. Halter-Koch, Characterization of Prüfer multiplication monoids and domain by means of spectral module theory, Monatsh. Math. 139 (2003) 19–31. MR1981115 (2004f:20102)
33. J.R. Hedstrom, E.G. Houston, Some remarks on star-operations, J. Pure Appl. Algebra 18 (1980) 37–44. MR0578564 (81m:13008)
34. E.G. Houston, Prime t-ideals in $R[X]$, in: P.-J. Cahen, D.G. Costa, M. Fontana, S-E. Kabbaj (Eds.), Commutative Ring Theory, Dekker Lecture Notes 153, 1994, 163–170. MR1261887 (94k:13028)
35. E.G. Houston, S.B. Malik, J.L. Mott, Characterization of $\star$-multiplication domains, Canad. Math. Bull. 27 (1984) 48–52. MR0725250 (85d:13026)
36. E.G. Houston, M. Zafrullah, Integral domains in which each t-ideal is divisorial, Michigan Math. J. 35 (1988) 291–300. MR0959276 (89i:13027)
37. P. Jaffard, Les Systèmes d'Idéaux, Dunod, Paris, 1960. MR0114810 (22 #5628)
38. B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J. Algebra 123 (1989) 151–170. MR1000481 (90e:13017)
39. B.G. Kang, On the converse of a well-known fact about Krull domains, J. Algebra 124 (1989) 284–289. MR1011595 (90g:13033)
40. D.J. Kwak, Y.S. Park, On t-flat overrings, Chinese J. Math. 23 (1995) 17–24. MR1336275 (96f:13030)
41. R. Matsuda, I. Sato, Note on star-operations and semistar operations, Bull. Fac. Sci. Ibaraki Univ. 28 (1996) 155–161. MR1408283 (97f:13003)
42. R. Matsuda, T. Sugatani, Semistar operations on integral domains II, J. Toyama Univ. 18 (1995) 155–161. MR1369703 (97b:13002)
43. J.L. Mott, M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. 35 (1981) 1–26. MR0627923 (83d:13026)
44. A. Okabe, R. Matsuda, Star operations and generalized integral

closures, Bull. Fac. Sci. Ibaraki Univ. Ser. A 24 (1992) 7–13. MR1177280 (93h:13006)

45. A. Okabe, R. Matsuda, Semistar operations on integral domains, Math. J. Toyama Univ. 17 (1994) 1–21. MR1311837 (95k:13027)
46. M. Zafrullah, Ascending chain conditions and star operations, Comm. Algebra 17 (6) (1989) 1523–1533. MR0997154 (90b:13023)