Semistar Dedekind domains. (English. English summary)


Let $D$ be an integral domain with quotient field $K$ and $\mathcal{F}(D)$ the set of nonzero $D$-submodules of $K$. A semistar operation $\star$ is a function $\mathcal{F}(D) \to \mathcal{F}(D)$ such that for all $x \in K \setminus \{0\}$ and for all $E, F \in \mathcal{F}(D)$

1. $(xE)^\star = xE^\star$,
2. $E \subseteq E^\star$ and $E \subseteq F$ implies that $E^\star \subseteq F^\star$,
3. $(E^\star)^\star = E^\star$.

Semistar operations sound similar to star operations which are defined on the set $F(D)$ of nonzero fractional ideals of $D$, but they are not quite the same. (See [R. Gilmer, Multiplicative ideal theory, Dekker, New York, 1972; MR0427289 (55 #323) (Sections 32, 34)] for a quick review of the star operations.) For star operations it has been a popular area of study to take a particular star operation of finite type, say the $t$-operation, and a classical notion, say a Dedekind domain, and look for what a $t$-Dedekind domain (for each $A \in F(D)$ $A_t$ is $t$-invertible) looks like, and that a $t$-Dedekind domain is a Krull domain. Similarly a $t$-Noetherian (integral $t$-ideals satisfy ACC) is a Mori domain and a $t$-Prüfer ($D_M$ is a valuation domain for each maximal $t$-ideal $M$) is the now well-known PVMD. Indeed, the above procedure can be repeated with any $\star$ of finite character [see, e.g., E. G. Houston, Jr., S. B. Malik and J. L. Mott, Canad. Math. Bull. 27 (1984), no. 1, 48–52; MR0725250 (85d:13026)] or with $\star_f$ for any star operation $\star$. Here, $A^{\star_f} = \bigcup \{F^\star: 0 \neq F$ is a finitely generated $D$-submodule of $A\}$. It is well known that a Mori PVMD is Krull ($t$-Dedekind) and $P^{\star_f}$MD’s are known to be PVMD’s.

The authors of the paper under review have taken the same route for a semistar operation $\star$, albeit with some modifications to ensure a smooth theory. ($\star$ denotes a semistar operation henceforth.) For instance they introduce the notion of a quasi-$\star$-ideal as an ideal $I$ such that $I = I^{\star} \cap D$, and define prime and maximal quasi-semistar ideals in the obvious fashion. Then an integral domain with ACC on quasi-$\star$-ideals is a $\star$-Noetherian domain, which they prove to be a Mori domain in some cases. They call $D$ a $P^{\star_f}$MD if $D_M$ is a valuation domain for every quasi-$\star_f$-maximal ideal $M$. They study $\star$-almost Dedekind domains ($D_M$ is a discrete rank one valuation ring for every quasi-$\star_f$-maximal ideal $M$). They call $D$ a $\star$-Dedekind
domain \((\ast\text{-DD})\) if \(D\) is \(\ast\)-Mori and \(P\ast\text{-MD}\) and show that the \(\ast\text{-DD}'s\) fail to be integrally closed in general. An interested reader may find the following papers useful reading: [D. D. Anderson and M. Zafrullah, Comm. Algebra 21 (1993), no. 4, 1189–1201; MR1209927 (94c:13023); J. L. Mott and M. Zafrullah, Arch. Math. (Basel) 56 (1991), no. 6, 559–568; MR1106498 (92d:13012)].

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