An amalgamated duplication of a ring along an ideal: the basic properties

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Dedicated to Luigi Salce, on his 60th birthday

Abstract

We introduce a new general construction, denoted by $R \ast E$, called the amalgamated duplication of a ring $R$ along an $R$–module $E$, that we assume to be an ideal in some overring of $R$. (Note that, when $E^2 = 0$, $R \ast E$ coincides with the Nagata’s idealization $R \ltimes E$.)

After discussing the main properties of the amalgamated duplication $R \ast E$ in relation with pullback–type constructions, we restrict our investigation to the study of $R \ast E$ when $E$ is an ideal of $R$. Special attention is devoted to the ideal-theoretic properties of $R \ast E$ and to the topological structure of its prime spectrum.

1 Introduction

If $R$ is a commutative ring with unity and $E$ is an $R$–module, the idealization $R \ltimes E$, introduced by Nagata in 1956 (cf. Nagata’s book [16], page 2), is a new ring, containing $R$ as a subring, where the module $E$ can be viewed as an ideal such that its square is $(0)$.

This construction has been extensively studied and has many applications in different contexts (cf. e.g. [17], [6], [9], [11]). Particularly important is the generalization given by Fossum, in [5], where he defined a commutative extension of a ring $R$ by an $R$–module $E$ to be an exact sequence of abelian groups:

$$0 \to E \xrightarrow{\iota} S \xrightarrow{\pi} R \to 0$$

where $S$ is a commutative ring, the map $\pi$ is a ring homomorphism and the $R$–module structure on $E$ is related to $S$ and to the maps $\iota$ and $\pi$ by the

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equation $s \cdot \iota(e) = \iota(\pi(s) \cdot e)$ (for all $s \in S$ and $e \in E$). It is easy to see that the idealization $R \ltimes E$ is a very particular commutative extension of $R$ by the $R$–module $E$ (called trivial extension of $R$ by $E$ in [5]).

In this paper, we will introduce a new general construction, called the amalgamated duplication of a ring $R$ along an $R$–module $E$ (that we assume to be an ideal in some overring of $R$ and so $E$ is an $R$-submodule of the total ring of fractions $T(R)$ of $R$) and denoted by $R \rtimes E$ (see Lemma 2.4).

When $E^2 = 0$, the new construction $R \rtimes E$ coincides with the idealization $R \ltimes E$. In general, however, $R \rtimes E$ it is not a commutative extension in the sense of Fossum. One main difference of this construction, with respect to the idealization (or with respect to any Fossum’s commutative extension) is that the ring $R \rtimes E$ can be a reduced ring (and, in fact, it is always reduced if $R$ is a domain).

Motivations and some applications of the amalgamated duplication $R \rtimes E$ are discussed more in detail in two recent papers [1], [2]. More precisely, M. D’Anna [1] has studied some properties of this construction in case $E = I$ is a proper ideal of $R$, in order to construct reduced Gorenstein rings associated to Cohen-Macaulay rings and he has applied this construction to curve singularities. M. D’Anna and M. Fontana in [2] have considered the case of the amalgamated duplication of a ring, in a not necessarily Noetherian setting, along a multiplicative-canonical ideal in the sense of Heinzer-Huckaba-Papick [10].

The present paper is devoted to a more systematic investigation of the general construction $R \rtimes E$, with a particular consideration to the ideal-theoretic properties and to the topological structure of its prime spectrum. More precisely, the paper is divided in two parts: in Section 2 we study the main properties of the amalgamated duplication $R \rtimes E$. In particular we give a presentation of this ring as a pullback (cf. Proposition 2.4) and from this fact (cf. also [4], [7]) we obtain several connections between the properties of $R$ and the properties of $R \rtimes E$ and some useful information about Spec$(R \rtimes E)$ (cf. Remark 2.13).

In Section 3 we consider the case when $E = I$ is an ideal of $R$; this situation allows us to deepen the results obtained in Section 2; in particular we give a complete description of Spec$(R \rtimes I)$ (cf. Theorems 3.5 and 3.8).

### 2 The general construction

In this section we will study the construction of the ring $R \rtimes E$ in a general setting. More precisely, $R$ will always be a commutative ring with unity, $T(R) := \{\text{regular elements}\}^{-1}R$ its total ring of fractions and $E$ an $R$-submodule of $T(R)$. Moreover, in order to construct the ring $R \rtimes E$, we are interested in those $R$-submodules of $T(R)$ such that $E \cdot E \subseteq E$.

**Lemma 2.1** Let $E$ be an $R$-submodule of $T(R)$ and let $J$ be an ideal of $R$.

(a) $E \cdot E \subseteq E$ if and only if there exists a subring $S$ of $T(R)$ containing $R$ and $E$, such that $E$ is an ideal of $S$.
(b) If $E \cdot E \subseteq E$ then:

$$R+E := \{ z = r + e \in T(R) \mid r \in R, \; e \in E \}$$

is a subring of $(E : E) := \{ z \in T(R) \mid zE \subseteq E \} \subseteq T(R)$, containing $R$ as a subring and $E$ as an ideal.

(c) Assume that $E \cdot E \subseteq E$; the canonical ring homomorphism $\varphi : R \hookrightarrow R+E \to (R+E)/E$, $r \mapsto r + E$, is surjective and $\text{Ker}(\varphi) = E \cap R$.

(d) Assume that $E \cdot E \subseteq E$; the set $J+E := \{ j + e \mid j \in J, \; e \in E \}$ is an ideal of $R+E$ containing $E$ and $(J+E) \cap R = \text{Ker}(R \hookrightarrow R+E \to (R+E)/(J+E)) = J+(E \cap R)$.

**Proof.**  (a) It is clear that the implication “if” holds. Conversely, set $S := (E : E)$. The hypothesis that $E \cdot E \subseteq E$ implies that $E$ is an ideal of $S$ and that $S$ is a subring of $T(R)$ containing $R$ as a subring.

(b) It is obvious that $R+E$ is an $R$-submodule of $(E : E)$ containing $R$ and $E$. Moreover, let $r, s \in R$ and $e, f \in E$, if $z := r + e$ and $w := s + f \in R+E$ then $zw = rs + (rf + se + ef) \in R+E$ and $zf = rf + ef \in E$.

(c) and (d) are straightforward.  

From now on we will always assume that $E \cdot E \subseteq E$.

In the $R$-module direct sum $R \oplus E$ we can introduce a multiplicative structure by setting:

$$(r, e)(s, f) := (rs, rf + se + ef), \quad \text{where } r, s \in R \text{ and } e, f \in E.$$  

We denote by $R \hat{\oplus} E$ the direct sum $R \oplus E$ endowed also with the multiplication defined above.

The following properties are easy to check:

**Lemma 2.2** With the notation introduced above, we have:

(a) $R \hat{\oplus} E$ is a ring.

(b) The map $j : R \hat{\oplus} E \to R \times (R+E)$, defined by $(r, e) \mapsto (r, r + e)$, is an injective ring homomorphism.

(c) The map $i : R \to R \hat{\oplus} E$, defined by $r \mapsto (r, 0)$, is an injective ring homomorphism.  

**Remark 2.3** (a) With the notation of Lemma 2.1 note that if $E = S$ is a subring of $T(R)$ containing as a subring $R$, then $R+S = S$. Also, if $I$ is an ideal of $R$, then $R+I = R$.

(b) In the statement of Lemma 2.1 (d), note that, in general, $J+E$ does not coincide with the extension of $J$ in $R+E$: we have $J(R+E) = \{ j + \alpha \mid j \in J, \; \alpha \in J E \} \subseteq J+E$, but the inclusion can be strict (cf. Lemma 3.4 (a), (d) and (e)).
(c) For an arbitrary $R$-module $E$, M. Nagata introduced in 1955 the idealization of $E$ in $R$, denoted here by $R \bowtie E$, which is the $R$-module $R \oplus E$ endowed with a multiplicative structure defined by:

$$(r, e)(s, f) := (rs, rf + se), \quad \text{where } r, s \in R \text{ and } e, f \in E$$

(cf. [15] and also Nagata’s book [16] page 2 and Huckaba’s book [11] Chapter VI, Section 25). The idealization $R \bowtie E$, called also the trivial extension of $R$ by $E$ [5], is a ring such that the canonical embedding $R \hookrightarrow R \bowtie E$, $r \mapsto (r, 0)$, defines a subring of $R \bowtie E$ isomorphic to $R$ and the embedding $E \hookrightarrow R \bowtie E$, $e \mapsto (0, e)$, defines an ideal $E^\times$ in $R \bowtie E$ (isomorphic as an $R$-module to $E$), which is nilpotent of index 2 (i.e. $E^\times \cdot E^\times = 0$). Therefore, even if $R$ is reduced, the idealization $R \bowtie E$ is not a reduced ring, except in the trivial case for $E = (0)$, since $R \bowtie (0) = R$. Moreover, if $p_R : R \bowtie E \to R$ is the canonical projection (defined by $(r, e) \mapsto r$), then

$$0 \to E \to R \bowtie E \xrightarrow{p_R} R \to 0$$

is an exact sequence.

Note that the idealization $R \bowtie E$ coincides with the ring $R \bowtie E$ (Lemma 2.2) if and only if $E$ is an $R$-submodule of $T(R)$ that is nilpotent of index 2 (i.e. $E \cdot E = (0)$).

**Lemma 2.4** With the notation of Lemma 2.2, note that $\delta := j \circ i : R \hookrightarrow R \times (R + E)$ is the diagonal embedding and set:

$$R^\Delta := (j \circ i)(R) = \{(r, r) \mid r \in R\} \quad \text{and} \quad R \bowtie E := j(R \bowtie E) = \{(r, r + e) \mid r \in R, \ e \in E\}.$$

We have:

(a) The canonical maps $R \cong R^\Delta \subseteq R \bowtie E \subseteq R \times T(R)$ are ring homomorphisms.

(b) $R \bowtie E$ is a subdirect product of the rings $R$ and $(R + E)$, i.e. if $\pi_i \ (i = 1, 2)$ are the projections of $R \times (R + E)$ onto $R$ and $R + E$, respectively, and if $\mathfrak{D}_i := \text{Ker}(\pi_i|_{R \bowtie E})$, then $(R \bowtie E)/\mathfrak{D}_1 \cong R$, $(R \bowtie E)/\mathfrak{D}_2 \cong R + E$ and $\mathfrak{D}_1 \cap \mathfrak{D}_2 = (0)$.

**Proof.** (a) is obvious. For (b) recall that $S$ is a subdirect product of a family of rings $\{R_i \mid i \in I\}$ if there exists a ring monomorphism $\varphi : S \to \prod_i R_i$ such that, for each $i \in I$, $\pi_i \circ \varphi : S \to R_i$ is a surjection (where $\pi_i : \prod_i R_i \to R_i$ is the canonical projection) [13] page 30. Note also that $\mathfrak{D}_1 = \{(0, e) \mid e \in E\}$ and $\mathfrak{D}_2 = \{(e, 0) \mid e \in E \cap R\}$. The conclusion is straightforward (cf. also [13] Proposition 10).

We will call the ring $R \bowtie E$, defined in Lemma 2.4 the amalgamated duplication of a ring along an $R$ module $E$; the reason for this name will be clear after studying the prime spectrum of $R \bowtie E$ and comparing it with the prime spectrum of $R$ (see Proposition 2.13). The following is an easy consequence of the previous lemma.

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Corollary 2.5 With the notation of Lemma 2.4, the following properties are equivalent:

(i) $R$ is a domain;
(ii) $R + E$ is a domain;
(iii) $\mathfrak{O}_1$ is a prime ideal of $R \times E$;
(iv) $\mathfrak{O}_2$ is a prime ideal of $R \times E$;
(v) $R \times E$ is a reduced ring and $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are prime ideals of $R \times E$.

We will see in a moment that $R$ is a domain if and only if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal prime ideals $R \times E$ (cf. Remark 2.5).

Proposition 2.6 Let $v : R \times (R + E) \rightarrow R \times ((R + E)/E)$ and $u : R \rightarrow R \times ((R + E)/E)$ be the natural ring homomorphisms defined, respectively, by $v((x, r + e)) := (x, r + E)$ and $u(r) := (r, r + E)$, for each $x, r \in R$ and $e \in E$. Then $v^{-1}(u(R)) = R \times E$. Therefore, if $v' := v|_{R \times E} : R \times E \rightarrow R$ is the canonical map defined by $(r, r + e) \mapsto r$ (cf. Lemma 2.4) and $u' : R \times E \rightarrow R \times (R + E)$ is the natural embedding, then the following diagram:

$$
\begin{array}{ccc}
R \times E & \xrightarrow{v'} & R \\
\downarrow{v'} & & \downarrow{u} \\
R \times (R + E) & \xrightarrow{v} & R \times ((R + E)/E)
\end{array}
$$

is a pullback.

Proof. Since $E$ is an ideal of $R + E$ (Lemma 2.1 (b)), $\mathfrak{O}_1 = (0) \times E$ is a common ideal of $v^{-1}(u(R))$ and $R \times (R + E)$. Moreover, by definition, if $x, r \in R$ and $e \in E$, then $(x, r + e) \in v^{-1}(u(R))$ if and only if $(x, r + E) \in u(R)$, that is $x = r \in E$. Therefore we conclude that $v^{-1}(u(R)) = R \times E$. The second part of the statement follows easily from the fact that $v^{-1}(u(R)) = R \times E$ and $(R \times E)/\mathfrak{O}_1 \cong R$, with $\mathfrak{O}_1 = \text{Ker}(v')$ (Proposition 2.4 (b)).

Corollary 2.7 The ring $R \times (R + E)$ is a finitely generated $(R \times E)$-module. In particular, $R \times E \subseteq R \times (R + E)$ is an integral extension and $\dim(R \times E) = \dim(R \times (R + E)) = \sup\{\dim(R), \dim(R + E)\}$.

Proof. Clearly $u : R \rightarrow R \times ((R + E)/E)$ is a finite ring homomorphism, since $R \times ((R + E)/E)$ is generated by $(1, 0)$ and $(0, 1)$ as $R$-module. Since $u$ is finite, also $u' : R \times (R + E) \rightarrow R \times ((R + E)/E)$ is a finite ring homomorphism [4 Corollary 1.5 (4)]. Last statement follows from [12 Theorems 44 and 48] and from the fact that $\text{Spec}(R \times (R + E))$ is homeomorphic to the disjoint union of $\text{Spec}(R)$ and $\text{Spec}(R + E)$ (cf. also Remark 2.5).
Remark 2.8 Recall that every ideal of the ring $R \times (R+E)$ is a direct product of ideals $I \times J$, with $I$ ideal of $R$ and $J$ ideal of $R+E$. In particular, every prime ideal $Q$ of $R \times (R+E)$ is either of the type $I \times (R+E)$ or $R \times J$, with $I$ prime ideal of $R$ and $J$ prime ideal of $(R+E)$. Therefore, in the situation of Lemma 2.4 if $R$ is an integral domain (and so $R+E$ also is an integral domain by Corollary 2.5), then $(0) \times (R+E)$ and $R \times (0)$ are necessarily the only minimal primes of $R \times (R+E)$. By the integrality property (Corollary 2.7 and [12, Theorem 46]), then $\mathfrak{O}_1 = ((0) \times (R+E)) \cap (R \times E) = (0) \times E$ and $\mathfrak{O}_2 = (R \times (0)) \cap (R \times E) = (R \cap E) \times (0)$ are the only minimal primes of $R \times E$.

Conversely, if $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal primes of $R \times E$, then clearly $R \times E$ is a reduced ring (Lemma 2.4 (b)) and, by Corollary 2.5, $R$ is an integral domain.

Corollary 2.9 The following statements are equivalent:

(i) $R$ and $R+E$ are Noetherian;

(ii) $R \times (R+E)$ is Noetherian;

(iii) $R \times E$ is Noetherian.

Proof. Clearly (i) and (ii) are equivalent. The statements (ii) and (iii) are equivalent by the Eakin-Nagata Theorem [14, Theorem 3.7], since $R \times (R+E)$ is a finitely generated $(R \times E)$–module (Corollary 2.7).

Remark 2.10 (a) In the situation of Proposition 2.6, the pullback degenerates in two cases:

(1) $u^\prime: R \times E \to R$ is an isomorphism if and only if $E = 0$;

(2) $\nu^\prime: R \times E \to R \times (R+E)$ is an isomorphism if and only if $E$ is an overring of $R$ (i.e., if and only if $E = R+E$).

(b) By the previous remark, we deduce easily that $R$ Noetherian does not imply in general that $R+E$ is Noetherian and, conversely, $R+E$ Noetherian does not imply that $R$ is Noetherian: take, for instance, $E$ to be an arbitrary overring of $R$. However, if we assume that $R+E$ is a finitely generated $R$-module (cf. also the following Corollary 2.11), then by the Eakin-Nagata Theorem [14, Theorem 3.7] $R$ is Noetherian if and only if $R+E$ is Noetherian.

This same situation described above (i.e. when $E$ is an arbitrary overring of $R$) shows that, in Corollary 2.7, we may have that $\dim(R \times E) = \dim(R)$ or that $\dim(R \times E) = \dim(R+E)$ (with $\dim(R) \neq \dim(R+E)$).

Corollary 2.11 Assume that $E$ is a fractional ideal of $R$ (i.e. there exists a regular element $d \in R$ such that $dE \subseteq R$); then the following statements are equivalent:

(i) $R$ is a Noetherian ring;

(ii) $R+E$ is a Noetherian $R$-module;

(iii) $R \times (R+E)$ is a Noetherian ring;
(iv) $R \otimes E$ is a Noetherian ring.

**Proof.** By Corollary 2.12 and by previous Remark 2.10 (b), it is sufficient to show that, in this case, $R$ is a Noetherian ring if and only if $R+ E$ is a Noetherian $R$-module. Clearly, if $R$ is Noetherian, then $E$ is a finitely generated $R$-module and so $R + E$ is also a finitely generated $R$-module and thus it is a Noetherian $R$-module. Conversely, assume that $R + E$ is a Noetherian $R$-module; since it is faithful, by [14, Theorem 3.5] it follows that $R$ is a Noetherian ring. □

**Corollary 2.12** In the situation described above:

(a) Let $R'$ and $(R+ E)'$ be the integral closures of $R$ and $R+ E$ in $T(R)$. Then $R \otimes E$ and $R \times (R+ E)$ have the same integral closure in $T(R) \times T(R)$, which is precisely $R' \times (R+ E)'$. Moreover, if $R+ E$ is a finitely generated $R$-module, then the integral closure of $R^\bigtriangleup$ in $T(R) \times T(R)$ (Lemma 2.7) also coincides with $R' \times (R+ E)'$.

(b) If $E \cap R$ contains a regular element, then $T(R \otimes E) = T(R \times (R+ E)) = T(R) \times T(R)$ and, moreover, $R \otimes E$ and $R \times (R+ E)$ have the same complete integral closure in $T(R) \times T(R)$.

**Proof.** (a) It is clear that $(x, y) \in T(R) \times T(R)$ is integral over $R \times (R+ E)$ if and only if $(x, y) \in R' \times (R+ E)'$. Since the extension $R \otimes E \hookrightarrow R \times (R+ E)$ (⊂ $T(R) \times T(R)$) is integral (Corollary 2.7), we have the first statement. If, in addition, we assume that $R + E$ is a finitely generated $R$-module, then the ring extension $R^\bigtriangleup \hookrightarrow R \times (R+ E)$ (Lemma 2.4) is finite (so, in particular, integral) and thus we have the second statement.

(b) Since $E$ is an $R$-submodule of $T(R)$, then clearly $T(R) = T(R+ E)$, hence it is obvious that $T(R \times (R+ E)) = T(R) \times T(R)$. If $e$ is a nonzero regular element of $E \cap R$, then $(e, e)$ is a nonzero regular element belonging to $(E \cap R) \times E$, which is a common ideal of $R \otimes E$ and $R \times (R+ E)$. From this fact it follows that $R \otimes E$ and $R \times (R+ E)$ have the same total quotient ring [8, page 326] and so $T(R \otimes E) = T(R) \times T(R)$. The last statement follows from [8, Lemma 26.5]. □

Note that, in Corollary 2.12 (b), the assumption that $E \cap R$ contains a regular element is essential, since if $E$ is the ideal (0) of an integral domain $R$ with quotient field $K$, then $R \otimes (0) \cong R$ and so $T(R \otimes (0)) \cong K$, but $T(R \times R) = K \times K$.

**Remark 2.13** Using Theorem 1.4 (c) and Corollary 1.5 (1) of [4], the previous Proposition 2.6 and Corollary 2.7 can be used to give a scheme-theoretic description of Spec($R \otimes E$) and Spec($R \times (R+ E)$). We do not give here many details, since in the following Section 3 we will prove directly and in a more elementary way the most part of the statements contained in this remark for the case $E = I$ is an ideal of $R$.

Recall that if $f : A \to B$ is a ring homomorphism, $f^a : \text{Spec}(B) \to \text{Spec}(A)$ denotes, as usual, the continuous map canonically associated to $f$, i.e. $f^a(Q) :=$
The restriction of the map \( f : X \rightarrow Y \) are naturally associated to a given ideal \( J \) of \( R \) and giving an example of the general construction.

**Proposition 2.14** In the situation of Proposition 2.14 and with the notation of Lemma 2.1, for each ideal \( J \) of \( R \) we can consider the following ideals of \( R \times E \):

\[
\mathcal{J}_1 := v^{-1}(J), \quad \mathcal{J}_2 := u^{-1}(R \times J(R+E)) \quad \text{and} \quad \mathcal{J}_0 := J^c := J(R \times E).
\]

Then we have:

(a) \( \mathcal{J}_1 = u^{-1}(J \times (R+E)) = u^{-1}(J \times (J+E)) = \{(j, j + e) \mid j \in J, \ e \in E\} \).

(b) \( \mathcal{J}_0 = \{(j, j + \alpha) \mid j \in J, \ \alpha \in JE\} \).

(c) \( \mathcal{J} := \mathcal{J}_1 \cap \mathcal{J}_2 = u^{-1}(J \times J(R+E)) \).

(d) \( \mathcal{J}_0 \subseteq \mathcal{J}_1 \cap \mathcal{J}_2 \).

**Proof.** (a) and (b) are straightforward. Statement (c) is obvious, since \( J \times J(R+E) = (J \times (R+E)) \cap (R \times J(R+E)) \). (d) follows from (c) and from the fact that \( J(R \times E) \subseteq u^{-1}(J(R \times (R+E))) = u^{-1}(J \times J(R+E)) \).

**Example 2.15** Let \( R := k[t^4, t^6, t^7, t^9] \) (where \( k \) is a field and \( t \) an indeterminate), \( S := k[t^2, t^3] \) and \( E := (t^2, t^3)S = t^2k[t] \). We have that \( R + E = S \) and hence

\[
R \times E = \{(f(t), g(t)) \mid f \in R, \ g \in S \ and \ g - f \in E\} = \{(f(t), g(t)) \mid f \in R, \ g \in S \ and \ f(0) = g(0)\}.
\]
Since $E$ is a maximal ideal of $S$, the prime ideals in $R \times S$ containing $\mathfrak{O}_1$ are either of the form $P \times S$, for some prime ideal $P$ of $R$, or $R \times E$; hence the primes not containing $\mathfrak{O}_1$ are of the form $R \times Q$, with $Q \in \text{Spec}(S)$ and $Q \neq E$.

By Remark 2.13 and Proposition 2.14, we have that if $P$ is a prime in $R$, the ideal $\mathcal{P}_1 = (v')^{-1}(P) = (u')^{-1}(P \times S) = \{(p, p + e) \mid p \in P, e \in E\}$ is a prime in $R \otimes E$, containing $\mathfrak{O}_1$, and $R \otimes E/\mathcal{P}_1 \cong R/P$. Moreover, with the notation of Proposition 2.13 in this way we describe completely $V_S(\mathfrak{O}_1)$. Notice also that, if we set $M := (t^4, t^6, t^7)R$, then the maximal ideals $M \times S$ and $R \times E$ of $R \times S$ have the same trace in $R \otimes E$, i.e. $(R \times E) \cap (R \otimes E) = \{(r, r + e) \mid r \in R \cap E, e \in E\} = (M \times S) \cap (R \otimes E)$.

On the other hand, again by Remark 2.13, we have that $Y \setminus V_Y(\mathfrak{O}_1)$ is homeomorphic to $Z \setminus V_Z(\mathfrak{O}_1)$. Hence the prime ideals of $R \otimes E$ not containing $\mathfrak{O}_1$ are of the form $(R \times Q) \cap (R \otimes E)$, for some prime ideal $Q$ of $S$, with $Q \neq E$.

3 The prime spectrum of $R \otimes I$

In this section we study the case when the $R$-module $E = I$ is an ideal of $R$ (that we will assume to be proper and different from $(0)$, to avoid the trivial cases); in this situation $R + I = R$. We start with applying to this case some of the results we obtained in the general situation.

**Proposition 3.1** Using the notation of Proposition 2.6, the following commutative diagram of canonical ring homomorphisms

\[
\begin{array}{ccc}
R \otimes I & \overset{v'}{\longrightarrow} & R \\
\downarrow{u'} & & \downarrow{u} \\
R \times R & \overset{v}{\longrightarrow} & R \times (R/I)
\end{array}
\]

is a pullback. The ideal $\mathfrak{O}_1 = (0) \times I = \text{Ker}(v') = \text{Ker}(v)$ is a common ideal of $R \otimes I$ and $R \times R$, the ideal $\mathfrak{O}_2 = \text{Ker}(R \otimes I \overset{u'}{\longrightarrow} R \times R \overset{\pi_2}{\longrightarrow} R)$ coincides with $I \times (0) = (I \times (0)) \cap (R \otimes I)$ and $(R \otimes I)/\mathfrak{O}_1 \cong R$, for $i = 1, 2$.

In particular, if $R$ is a domain then $R \otimes I$ is reduced and $\mathfrak{O}_1$ and $\mathfrak{O}_2$ are the only minimal primes of $R \otimes I$.

**Proof.** The first part is an easy consequence of Lemma 2.4(b) and Proposition 2.6 the last statement follows from Corollary 2.6.

**Remark 3.2** Note that, when $I \subseteq R$, then $R \otimes I := \{(r, r + i) \mid r \in R, i \in I\} = \{(r + i, r) \mid r \in R, i \in I\}$. It follows that we can exchange the roles of $\mathfrak{O}_1$ and $\mathfrak{O}_2$ (and that $\mathfrak{O}_2$ is also a common ideal of $R \otimes I$ and $R \times R$).

If we specialize to the present situation Corollary 2.7 Corollary 2.11 and Corollary 2.12 then we obtain:

**Corollary 3.3** Let $R'$ (respectively, $R^*$) be the integral closure (respectively, the complete integral closure) of $R$ in $T(R)$, we have:
(a) \( \dim(R \otimes I) = \dim(R) \).

(b) \( R \) is Noetherian if and only if \( R \otimes I \) is Noetherian.

(c) The integral closure of \( R^\triangleleft \) and of \( R \otimes I \) in \( T(R) \times T(R) \) coincide with \( R' \times R' \).

(d) If \( I \) contains a regular element, then \( T(R \otimes I) = T(R) \times T(R) \) and the complete integral closure of \( R \otimes I \) in \( T(R) \times T(R) \) coincide with \( R^* \times R^* \), which is the complete integral closure of \( R \times R \) in \( T(R) \times T(R) \).

The next goal is to investigate directly the relations among \( \text{Spec}(R \times R) \), \( \text{Spec}(R \otimes I) \), and \( \text{Spec}(R) \), under the canonical maps associated to natural embeddings, i.e. the diagonal embedding \( \delta: R \hookrightarrow R \otimes I \) (with \( (r \mapsto (r,r)) \)) and the inclusion \( R \otimes I \hookrightarrow R \times R \). With a slight abuse of notation, we identify \( R \) with its isomorphic image \( R^\triangleleft \) in \( R \otimes I \) (\( \subseteq R \times R \)) under the diagonal embedding (Lemma 2.4) and we denote the contraction to \( R \) of an ideal \( \mathcal{H} \) of \( R \otimes I \) (or, \( H \) of \( R \times R \)) by \( \mathcal{H} \cap R \) (or, by \( H \cap R \)).

We start with an easy lemma.

**Lemma 3.4** With the notation of Proposition 2.14, let \( J \) be an ideal of \( R \). Then:

(a) \( J_1 := u'^{-1}(J) = u'^{-1}(J \times R) = u'^{-1}(J \times (J + I)) = \{(j, j + i) \mid j \in J, i \in I\} := J \otimes I \). If \( J = I \), then \( I \otimes I := I \times I \) is a common ideal of \( R \otimes I \) and \( R \times R \).

(b) \( J_2 := u'^{-1}(R \times J) = \{(j + i, j) \mid j \in J, i \in I\} \).

(c) \( J := J_1 \cap J_2 = u'^{-1}(J \times J) = \{(j, j + i') \mid j \in J, i' \in I \cap J\} = \{(j_1, j_2) \mid j_1, j_2 \in J, j_1 - j_2 \in I\} \).

(d) \( J_0 := J(R \otimes I) = \{(j, j + i'') \mid j \in J, i'' \in JJ\} \) (cf. \[1\] Lemma 8).

(e) \( J_0 \subseteq J_1 \cap J_2 \).

(f) \( J_1 = J_2 \Leftrightarrow I \subseteq J \).

(g) \( J_0 + J = R \Rightarrow J_0 = J_1 \cap J_2 \).

(h) \( J_1 \cap R = J_2 \cap R = J_0 \cap R = J \cap R = J \).

**Proof.** (a) is a particular case of Proposition 2.14 (a). The second part is straightforward.

(b) Let \( r \in R \) and \( j \in J \); we have that \( (r, j) \in R \otimes I \) if and only if \( (r, j) = (s, s + i) \), for some \( s \in R \) and \( i \in I \). Therefore \( r = s - i \) and \( (r, j) = (j + i', j) \) for some \( i' \in I \).

(c) Let \( j_1, j_2 \in J \); we have that \( (j_1, j_2) \in R \otimes I \) if and only if \( (j_1, j_2) = (s, s + i) \), for \( s \in R \) and \( i \in I \). Therefore \( j_1 = s, j_2 = j_1 + i \) and \( j_2 - j_1 = i \in I \).

Statements (d) and (e) are particular cases of Proposition 2.14 (b) and (d)).
(f) follows easily from (a) and (b), since:
\[ J_1 = J_2 \Rightarrow J + I = J \Rightarrow I \subseteq J \Rightarrow J_1 = J_2. \]

(g) is a consequence of (c) and (d), since \( J + I = R \) implies that \( J \cap I = JI \).

(h) It is obvious that \( J_1 \cap R = J = J_2 \cap R \) and hence, by (c) and (e), we also have \( J \cap R = J_0 \cap R = J \).

With the help of the previous lemma we pass to describe the prime spectrum of \( R \otimes I \). In the following, the residue field at the prime ideal \( Q \) of a ring \( A \) (i.e. the field \( A_Q/QA_Q \)) will be denoted by \( k_A(Q) \). Part of the next theorem is contained in [1, Proposition 5].

**Theorem 3.5** (1) Let \( P \) be a prime ideal of \( R \) and consider the ideals \( \mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_0 \) and \( \mathcal{P} \) of \( R \otimes I \) as in Lemma 3.4 (with \( P = J \)). Then:

1. (a) \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are the only prime ideals of \( R \otimes I \) lying over \( P \).
2. (b) If \( P \supseteq I \), then \( \mathcal{P}_1 = \mathcal{P}_2 = \mathcal{P} = \sqrt{\mathcal{P}_0} = P \otimes I \). Moreover, \( k_R(P) \cong k_{R \otimes I}(\mathcal{P}) \).
3. (c) If \( P \not\supseteq I \) then \( \mathcal{P}_1 \neq \mathcal{P}_2 \). Moreover \( \mathcal{P} = \sqrt{\mathcal{P}_0} \) and \( k_R(P) \cong k_{R \otimes I}(\mathcal{P}_1) \cong k_{R \otimes I}(\mathcal{P}_2) \).
4. (d) If \( P \) is a maximal ideal of \( R \) then \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \) are maximal ideals of \( R \otimes I \).
5. (e) If \( R \) is a local ring with maximal ideal \( M \) then \( R \otimes I \) is a local ring with maximal ideal \( \mathcal{M} = \sqrt{\mathcal{M}_0} = M \otimes I \) (using again the notation of Lemma 3.4 for \( M = J \)).
6. (f) \( R \) is reduced if and only if \( R \otimes I \) is reduced.

(2) Let \( Q \) be a prime ideal of \( R \otimes I \) and let \( \mathfrak{O}_1 \) be as in Proposition 3.4. Two cases are possible either \( Q \not\supseteq \mathfrak{O}_1 \) or \( Q \supseteq \mathfrak{O}_1 \).

1. (a) If \( Q \not\supseteq \mathfrak{O}_1 \), then there exists a unique prime ideal \( Q \) of \( R \times R \) such that \( Q = Q \cap (R \otimes I) \) with \( Q = R \times P \), where \( P := Q \cap R \) (and \( P \not\supseteq I \)). In this case, with the notation of the previous part (1), \( \mathcal{P}_1 \neq \mathcal{P}_2 \) and
   \[ Q = \mathcal{P}_2 = \{(p + i, p) \mid p \in P, i \in I\}. \]

Furthermore, the canonical ring homomorphisms \( R \otimes I \to R \times R \xrightarrow{\pi_2} R \) induce for the localizations the following isomorphisms:
\[ (R \otimes I)_Q \cong (R \times R)_Q = (R \times R)_{R \times P} \cong R \quad \text{(thus} \ k_{R \otimes I}(Q) \cong k_R(P)). \]

1. (b) If \( Q \supseteq \mathfrak{O}_1 \), then there exists a unique prime ideal \( P \) of \( R \) such that \( Q = v^{-1}(P) \) (or, equivalently, \( P = v(Q) \)). With the notation...
of the previous part (1), if \( P \supseteq I \) then \( Q = P_1 = P_2 \). On the other hand, if \( P \not\supseteq I \) then \( Q = P_1 (\neq P_2) \). In both cases,

\[
Q = \{(p, p + i) \mid p \in P, \ i \in I\}.
\]

Furthermore, the canonical ring homomorphism \( \nu' : R \times I \to R \) induces the following isomorphism:

\[
(R \times I)/Q \cong R/P \quad (\text{thus } k_{R \times I}(Q) \cong k_R(P)).
\]

**Proof.** Note that the composition of the diagonal embedding \( \delta : R \hookrightarrow R \times I, (r \mapsto (r, r)) \), with the inclusion \( R \times I \subseteq R \times R, ((r, r + i) \mapsto (r, r + i)) \), coincides with the diagonal embedding \( R \hookrightarrow R \times R, (r \mapsto (r, r)) \), which is a finite ring homomorphism. Thus, in particular, both \( R \hookrightarrow R \times I \) and \( R \times I \subseteq R \times R \) are integral homomorphisms. Note also that if \( Q \) is a prime ideal of \( R \times R \) lying over \( P \), then necessarily \( Q \in \{P \times R, R \times P\} \) (Remark 2.8).

(1, a) Note that \( P_1 = \nu'^{-1}(P \times R) \) and \( P_2 = \nu'^{-1}(R \times P) \) (Lemma 3.4); hence \( P_1 \) and \( P_2 \) are prime ideals lying over \( P \). By integrality, if \( Q \subseteq \text{Spec}(R \times I) \) and \( Q \cap R = P \), then there exists \( \overline{Q} \subseteq \text{Spec}(R \times R) \) such that \( \overline{Q} \cap (R \times I) = Q \). Therefore \( \overline{Q} \subseteq \{P \times R, R \times P\} \) and \( Q \subseteq \{P_1, P_2\} \).

(1, b) We know already by Lemma 3.4 (f) and (c) that, if \( P \supseteq I \), then \( P_1 = P_2 = P \), hence by part (1, a) we conclude easily that \( P = \sqrt{P_0} \). Moreover we have the following sequence of canonical homomorphisms:

\[
\frac{R}{P} \subseteq \frac{R \times I}{\sqrt{P_0}} = \frac{R \times I}{P} \subseteq \frac{R \times R}{P \times R} \cong \frac{R}{P} \subseteq \frac{R \times R}{R \times P},
\]

from which we deduce the last part of the statement.

(1, c) By Lemma 3.4 (e) and (f) we know that, if \( P \not\supseteq I \), then \( P_1 \neq P_2 \) and \( P_0 \subseteq P = P_1 \cap P_2 \). By part (1, a) and by the integrality of \( R \hookrightarrow R \times I \), we conclude easily that \( P = \sqrt{P_0} \). Finally, as in part (1, b), it is easy to see that \( k_R(P) \cong k_{R \times I}(P_1) \cong k_{R \times I}(P_2) \).

(1, d) follows by the integrality of \( R \subseteq R \times I \).

(1, e) follows immediately by part (1, d) and part (1, b).

(1, f) follows by integrality of \( R \hookrightarrow R \times I \) and \( R \times I \subseteq R \times R \) and from the fact that \( R \) is reduced if and only if \( R \times R \) is reduced.

(2) If \( P = Q \cap R \), then necessarily \( Q \in \{P_1, P_2\} \) by (1, a).

(2, a) Since \( Q \not\subseteq \mathfrak{O}_1 \), then \( Q = P_2 \), because \( P_1 \supseteq \mathfrak{O}_1 \). Note that \( P_2 = (R \times P) \cap R \times I \); it is easy to see that \( Q := R \times P \) is the unique prime of \( R \times R \) contracting over \( Q \). The elementwise description of \( P_2 \) is a particular case of Lemma 3.4 (b). Last statement follows from the following canonical inclusions of localizations \( R_P \hookrightarrow (R \times I)/Q \hookrightarrow (R \times R)/Q = (R \times R)_{R \times P} \cong R_P \).

(2, b) The first and the last statements are trivial consequences of the fact that \( \nu' \) induces an isomorphism between \( R \times I/\mathfrak{O}_1 \) and \( R \). It is easy to see that the prime \( P \) is such that \( P = Q \cap R \). Therefore the second statement follows from (1, b). If \( P \not\supseteq I \) (and \( Q \supseteq \mathfrak{O}_1 \)) then \( Q = P_1 (\neq P_2) \), since \( Q \) does not contain
Lemma 3.4 (a). In the situation of Theorem 3.5, note that, if \( R \) is a radical ideal of \( R \), then by integrality of \( R \leftarrow R \otimes I \subseteq R \times R \), inside the ring \( R \times R \), the prime ideals \( P \times R \) and \( R \times P \) are the only minimal prime ideals of \( P \times P = \mathcal{P}_0(R \times R) = P(R \times R) \), and so

\[
\mathcal{P}_0(R \times R) = P \times P = (P \times R) \cap (R \times P) = \sqrt{\mathcal{P}_0(R \times R)}
\]

is a radical ideal of \( R \times R \), with

\[
(P \times P) \cap (R \otimes I) = ((P \times R) \cap (R \times P)) \cap (R \otimes I) = \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P}.
\]

Next example shows that in \( R \otimes I \), in general, \( \mathcal{P}_0 \) is not a radical ideal (i.e. it may happen that \( \mathcal{P}_0 \not\subseteq \sqrt{\mathcal{P}_0} = \mathcal{P} \)).

Example 3.7 Let \( V \) be a valuation domain with a nonzero non maximal non idempotent prime ideal \( P \). (An explicit example can be constructed as follows: let \( k \) be a field and let \( X, Y \) be two indeterminates over \( k \), then take \( V := k[X,Y] + Yk(X)[Y]_Y \) and \( P := Yk(X)[Y]_Y \). It is well known that \( V \) is discrete valuation domain of dimension 2, and \( P \) is the height 1 prime ideal of \( V \).)

In this situation, it is easy to see that the ideal \( P \times P \) is a common (radical) ideal of \( V \otimes P \) and of its overring \( V \times V \). Moreover, note that \( \mathcal{P}_0 = P(V \otimes P) = \{(p, p + x) \mid p \in P, x \in P^2\} \) (Lemma 3.4(d)) and that \( P(V \times V) = P \times P \subseteq V \otimes P \). More precisely, by Lemma 3.4(c), we have:

\[
P \times P = (P \times P) \cap (V \otimes P) = (P \times V) \cap (V \times P) \cap (V \otimes P) = \mathcal{P}_1 \cap \mathcal{P}_2 = \mathcal{P} = \{(p, p + y) \mid p \in P, y \in P \cap P = P\}.
\]

Clearly, since \( P^2 \neq P \), then \( \mathcal{P}_0 \not\subseteq \mathcal{P} \); for instance if \( z \in P \setminus P^2 \), then \( (p, p + z) \in \mathcal{P} \setminus P(V \otimes P) \).

We complete now the description of the affine scheme \( \text{Spec}(R \otimes I) \), initiated in Theorem 3.5 determining in particular the localizations of \( R \otimes I \) in each of its prime ideals. Part of the next theorem is contained in [11 Proposition 7].

Theorem 3.8 Let \( X := \text{Spec}(R) \), \( Y := \text{Spec}(R \otimes I) \) and \( Z := \text{Spec}(R \times R) \cong \text{Spec}(R) \times \text{Spec}(R) \) and let \( \alpha : Z \to Y \) and \( \gamma : Y \to X \) be the canonical surjective maps associated to the integral embeddings \( R \otimes I \hookrightarrow R \times R \) and \( R \cong R^h \hookrightarrow R \otimes I \) (proof of Theorem 3.5).

(a) The restrictions of \( \alpha \)

\[
\alpha |_{Z \setminus V_Z(\mathcal{O}_X)} : Z \setminus V_Z(\mathcal{O}_X) \longrightarrow Y \setminus V_Y(\mathcal{O}_Y)
\]
(for \( i = 1, 2 \)) are scheme isomorphisms, and clearly
\[
Z \setminus V_2(\mathfrak{O}_i) \cong X \setminus V_X(I) .
\]

In particular, for each prime ideal \( P \) of \( R \), such that \( P \not\supseteq I \), if we set \( \mathfrak{P}_1 := P \times R \) and \( \mathfrak{P}_2 := R \times P \) we have \( \mathfrak{P}_i := \mathfrak{P}_1 \cap (R \otimes I) \), for \( 1 \leq i \leq 2 \) and the following canonical ring homomorphisms are isomorphisms:
\[
R_P \longrightarrow (R \otimes I)_{\mathfrak{P}_1} \longrightarrow (R \times R)_{\mathfrak{P}_1}, \quad \text{for } 1 \leq i \leq 2.
\]

(b) The restriction of \( \gamma \)
\[
\gamma |_{V_Y(\mathfrak{O}_1) \cap V_Y(\mathfrak{O}_2)} : V_Y(\mathfrak{O}_1) \cap V_Y(\mathfrak{O}_2) \longrightarrow V_X(I)
\]
is a scheme isomorphism.

(c) If \( P \in \text{Spec}(R) \) is such that \( P \supseteq I \) and \( \mathfrak{P} \in \text{Spec}(R \otimes I) \) is the unique prime ideal such that \( \mathfrak{P} \cap R = P \), the following diagram of canonical homomorphisms:
\[
\begin{array}{ccc}
(R \otimes I)_{\mathfrak{P}} & \longrightarrow & R_P \\
\downarrow & & \downarrow u_P \\
R_P \times R_P & \overset{u_P}{\longrightarrow} & R_P \times (R_P/I_P)
\end{array}
\]
is a pullback (where \( I_P := IR_P \), \( u_P(x) := (x, x + I_P) \) and \( v_P((x,y)) := (x, y + I_P) \), for \( x, y \in R_P \)), i.e. \( (R \otimes I)_{\mathfrak{P}} \cong R_P \otimes I_P \) (Proposition 3.1).

Proof. (a) Since \( \mathfrak{O}_1 = \{0\} \times I \) (respectively, \( \mathfrak{O}_2 = I \times \{0\} \)) is a common ideal of \( R \times R \) and \( R \otimes I \), this statement follows from the general results on pullbacks [4] Theorem 1.4] and from Theorem 3.5 (and its proof). Note that \( Z \setminus V_2(\mathfrak{O}_1) \cong ((X \times X) \setminus ((X \times (V_X(I)))) = X \setminus V_X(I) = ((X \times X) \setminus (V_X(I) \times X)) \cong Z \setminus V_2(\mathfrak{O}_2) \).

(b) Note that \( V_Y(\mathfrak{O}_1) \cap V_Y(\mathfrak{O}_2) = V_Y(\mathfrak{O}_1 + \mathfrak{O}_2) \) and \( \mathfrak{O}_1 + \mathfrak{O}_2 = I \times I \). Therefore the present statement follows from the fact that the canonical surjective homomorphism \( R \otimes I \rightarrow R/I \), defined by \( (r, r + i) \mapsto r + I \) (for each \( r \in R \) and \( i \in I \)) has kernel equal to \( I \times I \).

(c) If we start from the pullback diagram considered in Proposition 3.1 and we apply the tensor product \( R_P \otimes_R \rightarrow \), then by [4] Proposition 1.9] we get the following pullback diagram:
\[
\begin{array}{ccc}
R_P \otimes_R (R \otimes I) & \overset{id \otimes v'}{\longrightarrow} & R_P \otimes_R R \\
\downarrow id \otimes u' & & \downarrow id \otimes u \\
R_P \otimes_R (R \times R) & \overset{id \otimes v}{\longrightarrow} & R_P \otimes_R (R \times (R/I))
\end{array}
\]

Note that, by the properties of the tensor product, we deduce immediately the following canonical ring isomorphisms: \( R_P \otimes_R (R \times R) \cong R_P \times R_P \), \( R_P \otimes_R R \cong
$R_P$ and that $R_P \otimes_R (R \times (R/I)) \cong R_P \times (R_P \otimes_R (R/I)) \cong R_P \times (R_P/I R_P)$. Therefore, the previous pullback diagram gives rise to the following pullback of canonical homomorphisms:

$$
\begin{array}{ccc}
R_P \otimes_R (R \times I) & \longrightarrow & R_P \\
\downarrow & & \downarrow u_P \\
R_P \times R_P & \longrightarrow & R_P \times (R_P/I_P).
\end{array}
$$

On the other hand, recall that Spec$(R_P \otimes_R (R \times I))$ can be canonically identified (under the canonical homeomorphism associated to the natural ring homomorphism $R \otimes I \rightarrow R_P \otimes_R (R \otimes I)$) with the set of all prime ideals $\mathcal{H} \in$ Spec$(R \times I)$ such that $\mathcal{H} \cap R \subseteq P$. Since we know already that, in the present situation, there exists a unique prime ideal $\mathcal{P} \in$ Spec$(R \otimes I)$ such that $\mathcal{P} \cap R = P$ (Theorem 3.5 (1, b)) and that the canonical embedding $R \hookrightarrow R \times I$ has the going-up property, we deduce that Spec$(R_P \otimes_R (R \times I))$ can be canonically identified with the set of all the prime ideals of $R \times I$ contained in $\mathcal{P}$. Therefore $R_P \otimes_R (R \times I)$ is a local ring with a unique maximal ideal corresponding to the prime ideal $\mathcal{P}$ of $R \times I$ and thus we deduce that the canonical ring homomorphism $(R \otimes I)_{\mathcal{P}} \rightarrow R_P \otimes_R (R \otimes I)$ is an isomorphism. □

**Proposition 3.9** The ring $R \otimes I$ can be obtained as a pullback of the following diagram of canonical homomorphisms:

$$
\begin{array}{ccc}
R \otimes I & \longrightarrow & R/I \\
\bar{v}' & \downarrow \bar{u} & \\
R \times R & \longrightarrow & R/I \times R/I
\end{array}
$$

where $\bar{u}$ is the diagonal embedding, $\bar{v}$ is the canonical surjection $(x, y) \mapsto (x + I, y + I)$, $\bar{v}'$ is the natural inclusion and $\bar{v}'$ is defined by $(x, x + i) \mapsto x + I$, for all $x, y \in R$ and $i \in I$.

**Proof.** By Proposition 3.1 we know that

$$
\begin{array}{ccc}
R \otimes I & \longrightarrow & R \\
\downarrow & & \downarrow u \\
R \times R & \longrightarrow & R \times R/I
\end{array}
$$

is a pullback. On the other hand, it is easy to verify that the following diagram:

$$
\begin{array}{ccc}
R & \longrightarrow & R/I \\
\downarrow & & \downarrow \bar{u} \\
R \times R/I & \longrightarrow & R/I \times R/I
\end{array}
$$

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is a pullback, where \( w \) is the canonical surjection \((x, y) \mapsto (x + I, y)\) and \( \varphi \) is the natural projection \( x \mapsto x + I \), for each \( x \in R \) and for each \( y \in R/I \). The conclusion follows by juxtaposing two pullbacks.

**Corollary 3.10** If \( R \) is a local ring, integrally closed in \( T(R) \) with maximal ideal \( M \) and residue field \( k \), then \( R \bowtie M \) is seminormal in its integral closure inside \( T(R) \times T(R) \) (which, in this situation, coincides with \( R \times R \)).

**Proof.** By the previous proposition \( R \bowtie M \) (which is a local ring) can be obtained as a pullback of the following diagram of canonical homomorphisms:

\[
\begin{array}{ccc}
R \bowtie M & \xrightarrow{\bar{\psi}'} & k' \\
\downarrow \bar{\psi}' & & \downarrow \bar{u}' \\
R \times R & \xrightarrow{\bar{\psi}} & k \times k
\end{array}
\]

The statement follows from the fact that, in this case, the integral closure of \( R \bowtie M \) in \( T(R) \times T(R) \) coincides with \( R \times R \) (Corollary 3.3 (c)). Therefore, since \( \bar{\psi}' \) is a minimal extension, then \( \bar{\psi}' \) is also minimal \([3, \text{Lemme 1.4 (ii)}]\), and thus the conclusion follows from \([3, \text{Théorème 2.2 (ii)}]\) and from \([18, (1.1)]\) (keeping in mind Theorem 3.5 (c)).

**Example 3.11** (a) Let \( R := k[[t]] \) (where \( k \) is a field and \( t \) an indeterminate) and let \( I := t^n R \). Using Proposition 3.9, if we denote by \( h^{(i)}(t) \) the \( i \)-th derivative of a power series \( h(t) \in k[[t]] \), it is easy to see that

\[
R \bowtie I = \{(f(t), g(t)) \mid f(t), g(t) \in R, \ f^{(i)}(0) = g^{(i)}(0) \ \forall \ i = 0, \ldots, n - 1\}.
\]

(b) Let \( R := k[x, y] \) and \( I := xR \). In this case

\[
R \bowtie I = \{(f(x, y), g(x, y)) \mid f(x, y), g(x, y) \in R, \ f(0, y) = g(0, y)\}.
\]

Setting \( Y = \text{Spec}(R \bowtie I) \) and \( X = \text{Spec}(R) \), by Proposition 3.13, \( V_Y(\mathcal{O}_1) \cong \text{Spec}(k[x, y]) \). On the other hand, by Theorem 3.8, \( V_Y(\mathcal{O}_1) \cap V_Y(\mathcal{O}_2) = V_Y(\langle xR \times xR \rangle) \cong V_X(xR) \cong \text{Spec}(k[y]) \). Hence the ring \( R \bowtie I \) is the coordinate ring of two affine planes with a common line. Note that we can present \( R \bowtie I \) as quotient of a polynomial ring in the following way: consider the homomorphism \( \lambda : k[x, y, z] \longrightarrow R \times R \), defined by \( \lambda(x) := (x, x) \), \( \lambda(y) := (y, y) \) and \( \lambda(z) := (0, x) \). It is not difficult to see that \( \text{Im}(\lambda) = R \bowtie I \) and \( \text{Ker}(\lambda) = (zx - z^2)k[x, y, z] \).

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References


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