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Direct Limits of Jaffard Domains and S-Domains

by

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Abstract. It is proved under mild assumptions that the class of Jaffard domains and the class of S-domains are each stable under direct limit. New examples of Jaffard domains obtained thereby include the factorial domain of Fujita, and Nagata rings in arbitrarily many indeterminates over a Jaffard domain. New examples of S-domains are the polynomial rings in arbitrarily many indeterminates over any domain. Also, any locally finite-dimensional directed union of universally catenarian going-down domains is itself a universally catenarian going-down domain. However, many related types of rings (such as [stably] strong S-domains or [universally] catenarian domains) are not preserved by direct limit. Numerous examples illustrate the need for various hypotheses, the failure of various converses, etc., as well as the sharpness of bounds that we give for the dimension and the valuative dimension of a direct limit.

1. Introduction

A well known and useful result [12, Proposition 22.6] states that any directed union of Prüfer domains is a Prüfer domain. This was generalized to direct limits of Prüfer domains in [8]. One purpose of this paper is to develop similar stability results for other related classes of (commutative integral) domains. A natural class to consider is that of the (not necessarily Noetherian) universally catenarian domains introduced in [3], since each locally finite-dimensional Prüfer domain is universally catenarian (cf. [3, Theorem 6.2]). Unfortunately, we show in Example 2.15 that universal catenarity is not stable under direct limit. However, a satisfactory analogue of the motivating result is given in Corollary 2.10: any locally finite-dimensional directed union of universally catenarian going-down domains is itself a universally catenarian going-down domain. (Going-down domains were introduced in [6]; each Prüfer domain is a going-down domain.)

Corollary 2.10 follows ultimately from our first main result, Theorem 2.3. This asserts that, under mild restrictions, direct limit preserves Jaffard domains. (Recall from [1] that a domain A of finite (Krull) dimension n is a Jaffard domain if its

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valuative dimension, $\dim_v(A)$, is also n; each finite-dimensional universally catenarian domain is a Jaffard domain.) New examples of Jaffard domains obtained as applications of Theorem 2.3 include the factorial domain constructed by Fujita [11] (see Corollary 2.4) and the Nagata ring in arbitrarily many indeterminates over a Jaffard domain (Corollary 2.5). We assume familiarity with Nagata rings, as in [12, section 33].

Theorem 2.3 is preceded by two lemmas giving inequalities describing how dimension and valuative dimension behave under direct limit. Equality holds for certain directed unions (Corollary 2.8). However, examples in section 3, especially Example 3.5, show emphatically that the inequality results are best-possible. In fact, section 3 is devoted to examples that illuminate the results in section 2 by showing that certain hypotheses cannot be deleted, certain converses fail, etc. Much of section 3 depends on "gluing" ideas, as in [9, Theorem 1.4], with which we assume familiarity.

It was shown in [15, Theorem 3.5], essentially via a result of Nagata [16], that any Prüfer domain is a (stably) strong S-domain. Moreover, each finite-dimensional strong S-domain is a Jaffard domain. It now seems natural to ask whether direct limit preserves (stably) strong S-domains. Unfortunately, Example 2.15 is a counterexample to this too. However, there is a positive result: Theorem 2.12 establishes that, with mild restrictions, S-domains are stable under direct limit. One consequence (Corollary 2.13) is that polynomial rings in arbitrarily many indeterminates over any domain must be S-domains.

What are the appropriate transition maps for our directed systems of rings? A clue comes via Noetherian rings. (A Noetherian domain is perhaps the most important example of an S-domain and, in the finite-dimensional case, of a Jaffard domain.) It is known [2a, Exercise 12 (e), page 44] that any directed system of Noetherian rings with flat transition maps has a coherent direct limit. Since flat ring-homomorphisms satisfy going-down (cf. [14, Exercise 37, page 44]), we often consider transition maps satisfying going-down. Occasionally, we consider ones with going-up. Both are tractable because direct limit preserves going-down [8] and going-up [7].

All rings considered are commutative, with unit; and all ring-homomorphisms are unital. Suitable background on direct limits is [13, pages 128–130]. Any nonreferenced material is standard, as in [12], [14].

2. Stability theorems for Jaffard domains and S-domains

To avoid unnecessary repetition, let us fix notation for much of sections 2 and 3. Data will consist of a directed system (A_j, f_{jk}) of rings indexed by a directed set $(I, \leq)$; and its direct limit, $A = \varinjlim A_j$, together with the canonical maps $f_j\colon A_j \to A$. Put $d_j = \dim(A_j)$ and $d = \dim(A)$. The case in which A is a directed union of A_j's corresponds to the f_{jk}'s being inclusion maps; thus, directed unions can be treated by assuming all f_{jk} to be monomorphisms. Finally, notice that if A_j is a domain for each $j \in I$, then A is also a domain.

Before giving a stability result for Jaffard domains, we give two lemmas describing how dimension and valuative dimension behave under direct limit. The statement of Lemma 2.1 is part of [2b, Exercice 11, page VIII. 82]. We include a proof for the sake of completeness.

LEMMA 2.1. *With the above notation,* $d \leq \sup(d_j)$.

Proof. Pick a chain $P_0 \subset P_1 \subset \cdots \subset P_e$ of $e+1$ distinct prime ideals in A. (Take $e=d$ if $d<\infty$.) Choose $y_i \in P_i \backslash P_{i-1}$ for $i=1, \cdots, e$. Since I is directed, there exist $j \in I$ and $x_i \in A_j$ such that $f_j(x_i)=y_i$ (for $i=1, \cdots, e$). Then, thanks to the existence of the x_i's, $\{f_j^{-1}(P_i): 0 \leq i \leq e\}$ is a chain of $e+1$ distinct primes in A_j, whence $e \leq d_j$, completing the proof.

It is well known that *Spec*(A), with the Zariski topology, maps homeomorphically onto $\varprojlim$ *Spec*(A_j). (This follows, for instance, from [13, Proposition 6.1.2, page 128].) The induced order-isomorphism readily leads to another proof of Lemma 2.1.

We next give the analogue of Lemma 2.1 for valuative dimension.

LEMMA 2.2. *Suppose that* A_j *is a domain for each* $j \in I$. *Then* $\dim_v(A) \leq \sup(\dim_v(A_j))$.

Proof. Essentially by definition, $\dim_v(A)=\sup(\{\dim(B)$: B an overring of $A\})$. Now, if B is an overring of A, it follows from [8, Lemma 2.6] that $B=\varinjlim B_j$, where B_j is a suitable overring of A_j. Then, using Lemma 2.1, we have

$$\dim(B) \leq \sup(\dim(B_j)) \leq \sup(\dim_v(A_j))$$

completing the proof.

THEOREM 2.3. *Suppose there exists* $j \in I$ *such that* A_k *is a Jaffard domain whenever* $j \leq k$ *in* I. *If* $d=\sup(d_i)<\infty$, *then* A *is a Jaffard domain.*

Proof. Let $J=\{k \in I: j \leq k\}$. Since J is confinal in I, A is canonically the direct limit of the directed system (A_k, f_{kn}) indexed by J. Moreover, the assumptions are preserved if we replace I with J. Indeed

$$d \leq \sup\{d_k : k \in J\} \leq \sup\{d_i : i \in I\} = d$$

where the first inequality follows from Lemma 2.1 and the second is trivial. Thus, without loss of generality, A_k is a Jaffard domain for each $k \in I$.

Now, using Lemma 2.2 and the fact that $\dim_v(A_k)=d_k$(since A_k is Jaffard), we have

$$\dim(A) \leq \dim_v(A) \leq \sup(\dim_v(A_k)) = \sup(d_k) = d = \dim(A) .$$

Thus, $\dim_v(A)=\dim(A)<\infty$, completing the proof.

The assumption that $d<\infty$ was made in Theorem 2.3 in order to avoid non-Jaffard (indeed, infinite-dimensional) examples such as $R[X_1, X_2, \cdots]=$

$\varinjlim R[X_1, \cdots, X_n]$, where R is any Noetherian ring. Similar assumptions in subsequent results are made for similar reasons.

The next seven results are applications of Theorem 2.3. The first two are about specific rings; the remaining five are more general. We begin these applications by considering a three-dimensional non-Noetherian UFD constructed by Fujita [11]. (Some errors have been found in [11] but, according to a private communication from Fujita, the main conclusions are correct.) Since it is an open problem to compute the valuative dimension of a UFD, the next result is of some interest. It answers affirmatively a conjecture of Alain Bouvier.

COROLLARY 2.4. *The example of Fujita is a Jaffard domain.*

Proof. Let us recall the construction from [11]. Let $Y_1, Y_2, Y_3, X_1, X_2, \cdots$ be denumerably many indeterminates over a field k. Put $A_0 = k[[Y_1, Y_2, Y_3]]$; and let A_1 be the Nagata ring $A_0(X_1)$. Next, put

$$A_2 = A_1[f_1/Y_1]_{(Y_1, f_1/Y_1, Y_3)}, \quad \text{where } f_1 = Y_3X_1 + Y_2; \quad \text{and } A_3 = A_2(X_2).$$

For each positive integer j, put

$A_{2j+1} = A_{2j}(X_{j+1})$, with maximal ideal (Y_1, f_{j+1}, Y_3), where
$f_{j+1} = Y_3X_{j+1} + f_j/Y_1$ and
$A_{2j+2} = A_{2j+1}[f_{j+1}/Y_1]_{(Y_1, f_{j+1}/Y_1, Y_3)}$.

It was shown in [11] that $A = UA_j$ is a three-dimensional quasilocal UFD. Notice next that for each j, A_j is a three-dimensional regular local ring. In particular, A_j is Noetherian and, hence, a Jaffard domain. Viewing the directed union UA_j as a direct limit, we see via Theorem 2.3 that A is a Jaffard domain, completing the proof.

It was shown in [1, Proposition 1.21 and Corollary 1.23 (a)] that if $X_1, \cdots, X_n$ are finitely many indeterminates over a Jaffard domain A, then the Nagata ring $A(X_1, \cdots, X_n)$ is also a Jaffard domain, having the same dimension as A. We next extend this result to any number of indeterminates.

COROLLARY 2.5. *Let $\{X_i\}$ be a set of (arbitrarily many) algebraically independent indeterminates over a d-dimensional Jaffard domain A. Let B be the Nagata ring $A(\{X_i\})$. Then B is a d-dimensional Jaffard domain.*

Proof. It is easy to see that B is a directed union of the Nagata rings of the form $A(X_{i_1}, \cdots, X_{i_n})$. According to the result recalled above from [1], each $A(X_{i_1}, \cdots, X_{i_n})$ is a d-dimensional Jaffard domain. By Theorem 2.3, it therefore suffices to show that $\dim(B) = d$. By Lemma 2.1, $\dim(B) \leq d$. For the reverse inclusion, one need only remark via [12, Proposition 33.1 (4)] that if $0 \neq P_0 \subset \cdots \subset P_d$ is a chain of $d+1$ distinct primes in A, then $\{P_kA(\{X_i\}) : 0 \leq k \leq d\}$ is a chain of distinct nonzero primes in B. The proof is complete.

We turn now to more general considerations. The next result is stated for motivational purposes. It is an immediate consequence of the observations that direct limits preserve integrality; and that if $D \subset E$ is an integral extension of domains, then D is a Jaffard domain if and only if E is a Jaffard domain [1, Proposition 1.1].

PROPOSITION 2.6. *Suppose that A_j is a domain for each $j \in I$ and that f_{jk} is an integral monomorphism whenever $j \leq k$ in I. Then the following conditions are equivalent:*

(1) *A_j is a Jaffard domain for some $j \in I$;*
(2) *There exists $j \in I$ such that A_k is a Jaffard domain whenever $j \leq k$ in I;*
(3) *A_j is a Jaffard domain for all $j \in I$;*
(4) *A is a Jaffard domain.*

Since integral maps satisfy going-up, the next result generalizes the implication $(2) \Rightarrow (4)$ in Proposition 2.6. Note that it is a corollary of Theorem 2.3, not of Proposition 2.6.

COROLLARY 2.7. *Suppose that A_j is a domain for each $j \in I$ and that f_{jk} is a monomorphism satisfying going-up whenever $j \leq k$ in I. Suppose also that $e = \sup(d_j) < \infty$. Then $d = e$. (Thus, if there exists $j \in I$ such that A_k is a Jaffard domain whenever $j \leq k$ in I, then A is a Jaffard domain.)*

Proof. We have $d \leq e$ by Lemma 2.1. For the reverse inequality, it suffices to prove that $\dim(A_j) \leq d$ for each $j \in I$. Since monomorphisms satisfying going-up must also satisfy lying-over [14, Theorem 42], it suffices to show that (the monomorphism $A_j \to A$) f_j satisfies going-up. This, in turn, holds since direct limits preserve going-up [7, Theorem 2.1 (b)]. Finally, the parenthetical assertion now follows from Theorem 2.3. The proof is complete.

Many important examples arise as directed unions of valuation domains. We next analyze the dimensions of such, generalizing the context as well. Among the examples in section 3 that illuminate the results of this section, we note that Examples 3.5 and 3.6 show the need for the "$f_{jk}(M_j) \subset M_k$" hypothesis in Corollary 2.8. (Corollary 2.8 may be viewed as the "going-down" analogue of Corollary 2.7.)

COROLLARY 2.8. *Suppose that (A_j, M_j) is a quasilocal domain for each $j \in I$ and that f_{jk} is a local monomorphism satisfying going-down whenever $j \leq k$ in I. ("Local" here means that $f_{jk}(M_j) \subset M_k$.) Then $d = \sup(d_j)$. (Hence if $d < \infty$ and each A_j is a Jaffard domain, A is also a Jaffard domain.)*

Proof. A is a quasilocal domain whose maximal ideal M satisfies $f_j^{-1}(M) = M_j$ for each $j \in I$ (cf. [13, Proposition 6.1.4, page 129]). Moreover, f_j: $A_j \to A$ satisfies going-down since direct limits preserve going-down [8, Theorem 2.1]. Hence $d_j \leq d$ for each $j \in I$. It follows that $\sup(d_j) \leq d$. Since Lemma 2.1 gives the reverse inequality, we have the asserted equality. Finally, Theorem 2.3 now gives the parenthetical assertion, to complete the proof.

The next two applications concern some classes of rings that were of special interest in [1] and [3].

COROLLARY 2.9. *Suppose, for each $j \in I$, that A_j is a locally Jaffard domain and that, for each $P \in Spec(A)$, one has $ht(P) = \sup(ht(f_j^{-1}(P))) < \infty$. Then A is a locally Jaffard domain.*

Proof. Put $P_j = f_j^{-1}(P)$. By [13, Proposition 6.1.6 (ii), page 130], A_P is canonically isomorphic to $\varinjlim (A_j)_{P_j}$, which is a direct limit of Jaffard domains. By Theorem 2.3, A_P is a Jaffard domain, completing the proof.

COROLLARY 2.10. *Let $A = UA_j$ be a directed union of universally catenarian going-down domains A_j. (So f_{jk} is an inclusion map whenever $j \leq k$ in I.) Suppose, for each $P \in Spec(A)$, that $\sup(ht(f_j^{-1}(P))) < \infty$. Then A is also a universally catenarian going-down domain. (Hence, if $\dim(A) < \infty$, then A is a Jaffard domain.)*

Proof. The hypothesis on P, together with Lemma 2.1 and the isomorphism noted in the proof of Corollary 2.9, yields that A is locally finite-dimensional. Also, A is a going-down domain since direct limits preserve going-down domains [8, Corollary 2.7]. Thus, essentially by [3, Theorem 6.2], it suffices to show that A' is a Prüfer domain. (As usual, D' denotes the integral closure of a domain D.) Now, A'_j is a Prüfer domain, essentially by [3, Theorem 6.2]. Hence, the integral closure of A_j in the quotient field of A (call this B_j) is also a Prüfer domain (cf. [12, Theorem 22.3]). However, it is clear that A' is the directed union UB_j of Prüfer domains. Hence, by [12, Proposition 22.6], A' is a Prüfer domain, as required. Finally, the parenthetical assertion follows from [3, Corollary 3.3], to complete the proof.

What about stability results for related classes of Jaffard domains? Here is one such result. Let $A = UA_j$ be a directed union of locally finite-dimensional going-down strong S-domains A_j such that $\sup(ht(f_j^{-1}(P))) < \infty$ for each $P \in Spec(A)$; then A is also a locally finite-dimensional going-down strong S-domain. (In view of [4, Theorem 1], this result is just a translation of Corollary 2.10.) This raises the question whether direct limits preserve strong S-domains. As we shall see in Examples 2.15 and 2.16, the answer is negative; these examples show that several related questions also have negative answers. So, it is of some interest to give a positive stability result for S-domains. We do so in Theorem 2.12 and then give two applications. First, we give the following useful result.

LEMMA 2.11. *A chain $P_0 \subset \cdots \subset P_m$ of $m+1$ distinct primes in A is saturated if the chain $\{f_j^{-1}(P_i) : 0 \leq i \leq m\}$ consists of $m+1$ distinct primes and is saturated in A_j for each $j \in I$.*

Proof. Without loss of generality, $m = 1$. Put $P = P_0$, $Q = P_1$, $P_j = f_j^{-1}(P)$, and $Q_j = f_j^{-1}(Q)$. (No confusion with the notation P_0 should arise: just arrange $0 \notin I$.) If the result fails, there exists $W \in Spec(A)$ lying strictly between P and Q; put $W_j = f_j^{-1}(W)$. By the "saturated" hypothesis, for each j, W_j is either P_j or Q_j. Let $J = \{j \in I : W_j = P_j\}$ and $K = \{j \in I : W_j = Q_j\}$. Now, if $j \leq k$ in I, we have $f_k \circ f_{jk} = f_j$, and so $f_{jk}^{-1} \circ f_k^{-1} = f_j^{-1}$. It follows that if $j \in J$ (resp., $j \in K$) and $j \leq k$ in I, then $k \in J$ (resp., $k \in K$). Since I is directed and J is disjoint from K, either J or K coincides with I.

Without loss of generality, $J=I$. Then, by [13, Proposition 6.1.2 (ii), page 128], $P=\varinjlim P_j=\varinjlim W_j=W$, the desired contradiction, to complete the proof.

THEOREM 2.12. *Suppose that A_j is an S-domain for each $j\in I$ and that f_{jk} is a monomorphism satisfying going-down whenever $j\leq k$ in I. Then A is an S-domain.*

Proof. Let P be a height 1 prime ideal of A. Put $P_j=f_j^{-1}(P)$ for each $j\in I$. Since f_j satisfies going-down by [8, Remark 2.2 (a)], we have $ht(P_j)\leq 1$. Since $P=\varinjlim P_j\neq 0$, there exists $i\in I$ such that $ht(P_k)=1$ whenever $i\leq k$ in I. Now, consider $A[X]=\varinjlim A_k[X]$ where k ranges over the indexes satisfying $i\leq k$ in I. Since A_k is an S-domain, $ht(P_k[X])=1$; that is, $0\subset P_k[X]$ is saturated for each k. By Lemma 2.11, it follows that $0\subset P[X]$ is saturated in $A[X]$. In other words, $ht(P[X])=1$, to complete the proof.

Just as with the applications of the eariler theorem, we shall discuss the specific before the general. Corollary 2.13 (a) generalizes the fact that if $\{X_i\}$ are indeterminates over a UFD A, then $A[\{X_i\}]$ is (a UFD and hence) an S-domain. For Corollary 2.13 (b), note that by definition, $A[[\{X_i\}]]=\varinjlim A[[X_{i_1},\cdots,X_{i_n}]]$.

COROLLARY 2.13. *Let $\{X_i\}$ be a nonempty set of (arbitrarily many) indeterminates over a domain A. Then:*

(a) *$A[\{X_i\}]$ is an S-domain,*
(b) *If A is Noetherian, then $A[[\{X_i\}]]$ is a coherent S-domain.*

Proof. (a) $A[\{X_i\}]$ is a directed union of the domains of the form $A[X_{i_1},\cdots,x_{i_n}]$, $n\geq 1$. Each of the latter domains is an S-domain, by [10, Proposition 2.1]. Each transition map in this directed system is flat (indeed, induces a free module), hence satisfies going-down. The assertion now follows from Theorem 2.12.

(b) View $B=A[[\{X_i\}]]$ as a direct limit of the (Noetherian) domains $A[[X_{i_1},\cdots,X_{i_n}]]$. Each transition map is flat, hence satisfies going-down. (The point is that if D is a Noetherian ring, then $D[[X]]\cong\Pi D$ is D-flat: cf. [5, Theorem 2.1].) The coherence assertion follows via [2a, Exercise 12 (e), page 44]; the S-domain assertion, via Theorem 2.12. The proof is complete.

The next application is in the spirit of Proposition 2.6. Note that its implication (3) $\Rightarrow$ (4) follows directly from Theorem 2.12.

COROLLARY 2.14. *Suppose that A_j is a domain for each $j\in I$ and that f_{jk} is an integral monomorphism satisfying going-down whenever $j\leq k$ in I. Then the following conditions are equivalent:*

(1) *A_j is an S-domain for some $j\in I$;*
(2) *There exists $j\in I$ such that A_k is an S-domain whenever $j\leq k$ in I;*
(3) *A_j is an S-domain for all $j\in I$;*
(4) *A is an S-doamin.*

Proof. (4) ⇒ (3) by [15, proof of Theorem 4.6] since $f_j: A_j \to A$ is an integral monomorphism for each $j \in I$; (3) ⇒ (2) ⇒ (1) trivially; and (1) ⇒ (4) by [15, Theorem 4.9] since f_j also satisfies going-down (cf. [8, Remark 2.2 (a)]). The proof is complete.

We close this section with two examples which, in contrast to Theorems 2.3 and 2.12, show that several relevant properties are not stable under direct limit.

EXAMPLE 2.15. Direct limits do not preserve any of the following four properties: stably strong S-domain, strong S-domain, catenarity, universal catenarity. Indeed, there is a directed union $A = \bigcup A_j$ of denumerably many universally catenarian (hence catenarian and [stably] strong S-) domains A_j such that the inclusion map $A_j \to A_k$ satisfies going-down whenever $j \leq k$, although A is neither catenarian nor a [stably] strong S-domain. (*A fortiori*, A is not universally catenarian; by Theorem 2.12, any such A is an S-domain.)

In detail, put $A_j = \boldsymbol{Q}[X_1, \cdots, X_j]$, with $A = \varinjlim A_j = \boldsymbol{Q}[X_1, X_2, \cdots]$. As in the proof of Corollary 2.13 (a), each transition map satisfies going-down. Moreover, each A_j is universally catenarian (since $\boldsymbol{Q}$, being trivially Cohen-Macaulay, is universally catenarian). However, [3, Proposition 2.1] yields that A is not a [stably] strong S-domain. Since $A = B[X_1]$ where $B = \boldsymbol{Q}[X_2, X_3, \cdots]$, and $B \cong A$ is not a strong S-domain, it follows from [3, Lemma 2.3] that A is not catenarian. The verification is complete.

EXAMPLE 2.16. Let d be a positive integer. Then there exist a domain A and denumerably many indeterminates $X_1, X_2, \cdots$ over A such that:

(a) For any subset $\{X_{i_j}\}$ of $\{X_i\}$, the ring $A(\{X_{i_j}\})$ is a d-dimensional strong S-domain;

(b) $A[X_1, \cdots, X_n]$ is a strong S-domain for each nonnegative integer n; and

(c) $A[\{X_i\}]$ is (an S-domain but) not a strong S-domain.

In detail, take A to be a d-dimensional denumerable valuation domain, say

$$A = \boldsymbol{Q} + Y_1 \boldsymbol{Q}[Y_1]_{(Y_1)} + \cdots + Y_d \boldsymbol{Q}(Y_1, \cdots, Y_{d-1})[Y_d]_{(Y_d)}.$$

Indeed, since A is a Jaffard domain, the first assertion in (a) follows from Corollary 2.5; the second, since each $A(\{X_{i_j}\})$ is a valuation domain (cf. [12, Proposition 33.1 and Theorem 33.4]). Next, (b) is a consequence of [10, Proposition 2.1]. Finally, as for (c), [3, Proposition 2.1] yields that $B = A[\{X_i\}]$ is not a strong S-domain, while Corollary 2.13 (a) assures that B is an S-domain. The verification is complete.

3. Additional examples

The examples in this section show that various bounds in section 2 are best-possible, various hypotheses in section 2 cannot be deleted, etc. Following each name of an example in this section, we list between braces $\{\cdots\}$ the relevant results from section 2. The examples are *increasing unions* $A = \bigcup A_j$, that is, directed unions of denumerably many rings $A_0 \subset A_1 \subset A_2 \subset \cdots$.

EXAMPLE 3.1 {Lemma 2.1, Lemma 2.2}. Let e be a nonnegative integer. Then there exists an increasing union $A=UA_j$ of Noetherian domains A_j such that A is Noetherian, $\dim(A_j)=\dim_v(A_j)=j+e$ for each $j\geq 0$, and $\dim(A)=\dim_v(A)=e$.

In detail, let $X_1, \cdots, X_e, Y_1, Y_2, \cdots$ be infinitely many indeterminates over a field k. Let $A_0=k[X_1, \cdots, X_e]$; this means $A_0=k$ if $e=0$. For $j\geq 1$, put

$$A_j=k(Y_1, Y_2, \cdots, Y_{j(j-1)/2})[X_1, \cdots, X_e, Y_{j(j-1)/2+1}, \cdots, Y_{j(j+1)/2}].$$

Then $A=UA_j$ is an increasing union since

$$\begin{aligned} A_j &\subset k(Y_1, \cdots, Y_{j(j+1)/2})[X_1, \cdots, X_e] \\ &\subset k(Y_1, \cdots, Y_{j(j+1)/2})[X_1, \cdots, X_e, Y_{j(j+1)/2+1}, \cdots, Y_{(j+1)(j+2)/2}]=A_{j+1}. \end{aligned}$$

By the Hilbert Basis Theorem, each A_j is Noetherian; its (valuative) dimension is $j(j+1)/2-j(j-1)/2+e=j+e$. (Cf. [12, Theorem 30.5 and Corollary 30.10].) The remaining assertions follow easily since $A=k(Y_1, Y_2, \cdots)[X_1, \cdots, X_e]$.

Note that the cse $d=0$, $e=1$ of Example 3.2 is handled with a different example in [2b, Exercise 11,page VIII. 82].

EXAMPLE 3.2 {Lemma 2.1, Lemma 2.2}. Let $0\leq d<e$ be integers. Then there exists an increasing union $A=UA_j$ of Noetherian domains A_j such that A is Noetherian, $\dim(A_j)=\dim_v(A_j)=e$ for each $j\geq 0$, and $\dim(A)=\dim_v(A)=d$.

In detail, consider indeterminates $Y_1, \cdots, Y_d; X_{01}, \cdots, X_{0(e-d)}; \cdots; X_{j1}, \cdots, X_{j(e-d)}; \cdots$ over a field k. Put

$$A_0=k[X_{01}, \cdots, X_{0(e-d)}, Y_1, \cdots, Y_d];$$

$$A_1=k(X_{01}, \cdots, X_{0(e-d)})[X_{11}, \cdots, X_{1(e-d)}, Y_1, \cdots, Y_d];$$

and, for any $j\geq 1$,

$$A_j=k(X_{01}, \cdots, X_{0(e-d)}, \cdots, X_{(j-1)1}, \cdots, X_{(j-1)(e-d)})[X_{j1}, \cdots, X_{j(e-d)}, Y_1, \cdots, Y_d].$$

It is clear that $A=UA_j$ is an increasing union of Noetherian domains, and that $\dim(A_j)=\dim_v(A_j)=(e-d)+d=e$. The assertions about A follow since one may verify that $A=k(\{X_{ji}: 0\leq j, 1\leq i\leq e-d\})[Y_1, \cdots, Y_d]$.

EXAMPLE 3.3 {Theorem 2.3}. Let d be a positive integer. Then there exists an increasing union $A=UA_j$ of d-dimensional non-Jaffard domains A_j such that A is a d-dimensional Jaffard domain.

In detail, let $X_1, X_2, \cdots$ be denumerably many indeterminates over a field k, and let $(V, M)=k(\{X_i\})+M$ be a d-dimensional valuation domain. Put $A_0=k+M$ and, for each $j\geq 1$, put

$$A_j=k(X_1, \cdots, X_j)+M=V\times_{k(\{X_i\})}k(X_1, \cdots, X_j).$$

Now, $Spec(A_j)=Spec(V)$ as sets (cf. [9, Theorem 1.4]), whence $\dim(A_j)=\dim(V)=d$. Moreover A is just the d-dimensional valuation domain V, and so A is a Jaffard domain. However, applying [1, Proposition 2.5] to the above pullback description of A_j yields that A_j is not a Jaffard domain, since $k(\{X_i\})$ is not algebraic over $k(X_1, \cdots, X_j)$. (In fact, the cited result yields that $\dim_v(A_j)=\infty$.) The verification is complete.

Before giving the final two examples, we need the following technical facts.

LEMMA 3.4. *Let V be a valuation domain with quotient field K; $\underline{X}=\{X_1, \cdots, X_n\}$ a finite set of indeterminates over K; $P\in Spec(V)$; $k=V_P/P$; and W a nontrivial valuation ring of $k(\underline{X})$ containing k. Let V^* be the "composite" valuation ring*

$$V^*=V_P(X)\times_{k(X)}W\,.$$

Then:

(a) $V^*\cap K=V_P$.

(b) The conductor of V^* in $V_P(\underline{X})$ is $I=PV_P(\underline{X})$. If $Q\in Spec(V^*)$ contains I, then $Q\cap V_P=Q\cap V=P$.

Proof. (a) It is well known that V^* is a valuation domain (cf. [17, (11.4)]). Consider the commutative diagram

$$\begin{tikzcd}
V_P \arrow[rr, hookrightarrow] \arrow[dd, twoheadrightarrow] & & V_P(X) \arrow[d, equal] \\
& V^* \arrow[r, hookrightarrow] \arrow[d, twoheadrightarrow] & V_P(\underline{X}) \arrow[d, twoheadrightarrow] \\
k \arrow[r, hookrightarrow] & W \arrow[r, hookrightarrow] & k(X)
\end{tikzcd}$$

By the universal mapping property of pullback, there is a ring-homomorphism $V_P\to V^*$, necessarily an injection, making the induced diagram commute. Hence $V_P\subset V^*\cap K$. For the reverse inclusion, notice that $V^*\cap K\subset V_P(X)\cap K=V_P$, to complete the proof of (a).

(b) The first assertion is immediate since $W\neq k(\underline{X})$. Next, since P is a common ideal of V and V_P, it suffices to show that $Q\cap V_P$ contains (and hence equals) P. For this, note that

$$P=PV_P(\underline{X})\cap V_P\subset Q\cap V_P$$

thus completing the proof.

EXAMPLE 3.5 {Lemma 2.1, Corollary 2.8; cf. also Examples 3.2 and 3.6}. Let $0\leq d<e$ be integers. Then there exists an increasing union $A=UA_j$ of valuation (hence quasilocal) domains A_j such that $\dim(A_j)=e$ for each $j\geq 0$ and $\dim(A)=d$.

In detail, let V be an e-dimensional valuation domain with quotient field K. Let P be the height d prime ideal of V; put $V^*=V_P$ and $k=V_P/P$. Consider indeterminates $X_{11}, \cdots, X_{1(e-d)}$ over K and k. Let K_1 (resp., k_1) be the field resulting by adjoining these indeterminates to K (resp., k). Next, take W_1 as any $(e-d)$-dimensional valuation ring of k_1 containing k. Let V_1 be the "composite" valuation ring

$$V_1 = V^*(X_{11}, \cdots, X_{1(e-d)}) \times_{k_1} W_1 .$$

It is clear, via [9, Theorem 1.4] and Lemma 3.4, that $\dim(V_1) = \dim(V^*(\{X_{1i}\})) + \dim(W_1) = d + (e-d) = e$; that $V_1 \cap K = V^*$; and that each prime of V_1 with height at least d lies over P.

We iterate the construction. Here is the next step. Let P_1 be the height d prime of V_1; K_2 (resp., k_2) result from K_1 (resp., k_1) by adjoining indeterminates $X_{21}, \cdots, X_{2(e-d)}$; W_2 be an $(e-d)$-dimensional valuation ring of k_2 containing k_1; and V_2 be the "composite" valuation ring

$$V_2 = (V_1)_{P_1}(X_{21}, \cdots, X_{2(e-d)}) \times_{k_2} W_2 .$$

Continuing in this way, we obtain e-dimensional valuation domains $V_1 \subset V_2 \subset \cdots$. Put $A = UV_j$. We claim that $\dim(A) = d$.

Let $W \in Spec(A)$. Put $W_j = W \cap V_j$ for each $j \geq 1$. Let P_j denote the height d prime of V_j. For each j, either $W_j \subset P_j$ or $P_j \subset W_j$. If $P_j \subset W_j$ for some j, we have $W_{j+1} \cap V_j = P_j$ by Lemma 3.4 (b); as $W_{j+1} \cap V_j = W \cap V_j = W_j$, it follows that $P_j = W_j$. Hence, $W_j \subset P_j$ for all j; that is, $Q = UP_j$ contains W. Thus, to prove the claim, it suffices to show that $Q \in Spec(A)$ and $ht_A(Q) = d$.

It is clear from the pullback construction of the V_j's that $P_1 \subset P_2 \subset \cdots$. Hence $Q \in Spec(A)$. Moreover, $B = U(V_j)_{P_j}$ is an increasing union to which Corollary 2.8 applies. (The required going-down property holds for the transition maps because any valuation domain is a going-down domain.) Hence $\dim(B) = \sup(\dim((V_j)_{P_j})) = d$. But (B, Q) is quasilocal by [13, Proposition 6.1.4, page 129], whence $ht_B(Q) = \dim(B) = d$. Since A is a Prüfer (indeed, valuation) domain by [12, Proposition 22.6], the inclusion map $A \to B$ satisfies going-down, whence $ht_A(Q) \leq ht_B(Q) = d$. But it is clear by applying [9, Theorem 1.4] to the construction of the V_j's that the saturated chain of primes leading down from P_j in V_j gives a compatible family in the sense of [13, Proposition 6.1.2 (i), page 128]. The upshot in the direct limit is a chain of $d+1$ distinct primes inside Q; that is, $ht_A(Q) \geq d$, proving the claim. Therefore, by defining $A_j = V_{j-1}$ for each $j \geq 0$, the assertion follows.

EXAMPLE 3.6 {Lemma 2.1, Corollary 2.8; cf. also Examples 3.2 and 3.5}. Let d be a positive integer. Then there exists an increasing union $A = UA_j$ of valuation (hence quasilocal) domains A_j such that $\dim(A_j) = \infty$ for each $j \geq 0$ and $\dim(A) = d$.

The details are somewhat like those of Example 3.5, and so we only sketch them. Let V be an infinite-dimensional valuation domain, with quotient field K, such that $Spec(V)$, as a partially ordered set, looks like

$$0 \subset Q_1 \subset Q_2 \subset \cdots \subset Q_d = Q \subset \cdots \subset P_j \subset P_{j-1} \subset \cdots \subset P_1 \subset M .$$

(The notation means, i.a., that $ht_V(Q) = d$; Q has no immediate successor; and P_j has "coheight" j, in the sense that $\dim(V/P_j) = j$.) Put $V^j = V_{P_j}$ and $k_j = V^j/P_j$ for each

$j \geq 1$. Let W_1 be a one-dimensional valuation ring of $k_1(X_1)$ containing k_1. Let (V_1, M_1) be the "composite" valuation ring

$$V_1 = V^1(X_1) \times_{k_1(X_1)} W_1 .$$

Reasoning as in the preceding example, we see that $\dim(V_1) = \infty$; $V_1 \cap K = V^1$; the conductor of V_1 in $V^1(X_1)$ is $P_{1,1} = P_1 V^1(X_1)$; and M_1 and $P_{1,1}$ each lie over P_1.

Here is the next step. Let W_2 be a two-dimensional valuation domain of $k_2(X_1, X_2, Y_1)$ containing $k_2(X_1)$. Let (V_2, M_2) be the "composite" valuation ring

$$V_2 = V^2(X_1, X_2, Y_1) \times_{k_2(X_1, X_2, Y_1)} W_2 .$$

Reasoning as above, we have $\dim(V_2) = \infty$; $V_2 \cap K = V^2$; the conductor of V_2 in $V^2(X_1, X_2, Y_1)$ is $P_{2,2} = P_2 V^2(X_1, X_2)$; and M_2, $P_{2,2}$, and the prime of V_2 with coheight 1 each lie over P_2.

The pattern is clear. For instance,

$$V_3 = V^3(X_1, X_2, Y_1, X_3, Y_2, Y_3) \times_{k(X_1, X_2, Y_1, X_3, Y_2, Y_3)} W_3$$

where W_3 is a three-dimensional valuation domain of $k(X_1, X_2, Y_1, X_3, Y_2, Y_3)$ containing $k_3(X_1, X_2, Y_1)$. Notice, as above, that the "top" four prime ideals of W_3 each lie over P_3. Continuing in this way, we obtain a sequence of infinite-dimensional valuation domains $V_1 \subset V_2 \subset \cdots$; let A denote their union. It will suffice to show that $\dim(A) = d$.

The verification proceeds nearly as in Example 3.5. Here is one difference. If $W \in Spec(A)$ and Q_j is the prime of V_j corresponding to Q, we must show that $W_j = W \cap V_j$ is contained in Q_j. If this fails, Q_j is properly contained in W_j and there exists $k > j$ such that no prime of V_k lies over W_j; this contradicts the existence of $W_k = W \cap V_k$. Hence, each prime of A is contained in $Q^* = UQ_j$.

As in the proof of Example 3.5, we see via Corollary 2.8 that $B = U(V_j)_{Qj}$ is d-dimensional; via [13, Proposition 6.1.4, page 129] and going-down considerations that $ht_A(Q^*) \leq ht_B(Q^*) = d$; and via [9, Theorem 1.4] and [13, Proposition 6.1.2 (i), page 128] that $ht_A(Q^*) \geq d$. The verification is complete.

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