Direct Limits and Going-down

by

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1. Introduction

In the course of studying the flat spectral topology and a related discrete Alexandroff topology in [8], we had occasion to prove that direct limits of certain types of directed systems of GD (going-down)-homomorphisms were also GD-homomorphisms [8, Lemma 2.14]. In the present note, we establish a generalization of this result which is valid for all directed systems of GD-homomorphisms (Theorem 2.1). Analogous results are obtained for, i.e., the INC-property (Proposition 2.3). The major portion of this article is devoted to showing that various classes of going-down rings (in the sense of [4]) are closed under direct limit (cf. Corollary 2.7). Results of this sort are motivated in part by the well-known fact that directed unions of Prüfer domains are themselves Prüfer domains; indeed, the present work includes three proofs of a direct limit generalization of this fact.

Throughout, rings are assumed commutative, with 1; a subring must contain the 1 of the larger ring; and ring-homomorphisms are assumed unital, that is, send 1 to 1.

2. Main results

We begin with the promised sharpening of [8. Lemma 2.14].

THEOREM 2.1. Let $(I, \leq)$ be a directed set, and let $(A_i, f_{ij})$ and $(B_i, g_{ij})$ each be direct systems of rings indexed by $I$. For each $i \in I$, let $h_i : A_i \rightarrow B_i$ be a ring-homomorphism satisfying GD such that, whenever $i \leq j$ in $I$, then $g_{ij} f_{ij} : A_i \rightarrow B_j$. Set $A = \varinfty A_i$, $B = \varinfty B_i$, and $h = \varinfty h_i$. Then $h : A \rightarrow B$ also satisfies GD.

Proof. If the assertion fails, then [12], Exercise 37, p. 44 supplies $P \in \text{Spec}(A)$ and $Q \in \text{Spec}(B)$ such that $Q \not\supset h(P)B$ and...
Thus, $f_h(P)B \cap f_h(A)P(B)Q \neq \emptyset$.

Thus, $\sum_{i=1}^{n} f_h(p_i)h_i = h(a)h$ for suitable elements $p_i \in P, h_i \in B$, $a \in A, P$ and $b \in B, Q$. By the construction of direct limits (see [3], Lemma 1, p. 203), we produce an index $z \in I$ and elements $x_{a_i} \in B_z$ such that $g_{a_i}(x_{a_i}) = b_i$ for $i = 1, \ldots, n$ and $g_{a_i}(x_{a_i+1}) = b$, where $g_{a_i} : B_z \to B$ is the canonical map. Similarly, there exist $b \in I$ and $y_{a_i} \in A_b$ such that $f_{b}(y_{a_i}) = p_i$ for $i = 1, \ldots, n$ and $f_{b}(y_{a_i+1}) = a$, where $f_b : A_b \to A$ is the canonical map. Since directedness of $I$ yields an index $\gamma$ majorizing both $x$ and $b$, we may suppose that $x = b$. (In detail, $x_a$ may be replaced by $g_{a_i}(x_{a_i})$ since $g_{a_i}x_{a_i} = g_{a_i}$, etc.) As $h_{f_z} = g_{h_z}$ and $x$ is a homomorphism, it follows that

$$g_{a_i}\left(\sum_{i=1}^{n} h_{a_i}(y_{a_i}x_{a_i})\right) = g_{a_i}(h_{a_i}(y_{a_i+1}x_{a_i+1})).$$

Thus, by [3], (ii), p. 204, there is an index $k$ in $I$ such that $z \leq k$ and

$$g_{a_k}(\sum h_{a_k}(y_{a_k}x_{a_k})) = g_{a_k}(h_{a_k}(y_{a_k+1}x_{a_k+1})).$$

Since $g_{a_k}h_z = h_uf_{a_k}$ by hypothesis, we have

$$\sum h_{a_k}(f_{a_k}(y_{a_k}))g_{a_k}(x_{a_k}) = h_{a_k}(f_{a_k}(y_{a_k+1}))g_{a_k}(x_{a_k+1}).$$

To complete the proof, it suffices to show (using the preceding equation) that $h_k$ does not satisfy GD.

Indeed, in view of [12], Exercise 37, p. 44, it suffices to verify that $P_i = f_k^*(P) \in \text{Spec}(A_k)$ and $Q_i = g_k^*(Q) \in \text{Spec}(B_k)$ satisfy $Q_i \subset h_k(P_i)$; $f_{a_k}(y_{a_k}) \in P_i$ for each $i = 1, \ldots, n$; $f_{a_k}(y_{a_k+1}) \in A_k P_i$; and $g_{a_k}(x_{a_k+1}) \in B_k Q_i$. For the first of these, one need only notice that $g_{a_k}h_k(P) = h_k(f_k(P)) \subset h(P) = Q$. The second and third assertions follow readily from the above information about $p_i$ and $a_k$, since $f_{a_k}f_{a_k} = f_{a_k}$. Similarly, the final assertion reduces to requiring $b \notin Q$, and so the proof is complete.

**Remark 2.2.** (a) Some special cases of the preceding result are noted next. First, if $A_i \to B$ is a directed system of ring-homomorphisms each of which satisfies GD, then the direct limit map $\text{lim } A_i \to B$ also satisfies GD. Of course, this follows from Theorem 2.1 by setting each $B_i = B$ and $g_{i,j} = 1$. Secondly, specializing to the case $A_i = A, f_{i,j} = 1$ recovers [8, Lemma 2.14]. Thirdly, let $(B_i, g_{i,j})$ be a directed system indexed by $I$ and set $B = \text{lim } B_i$. If $k \in I$ is such that $g_{i,j} : B_i \to B_j$ satisfies GD whenever $k \leq j$, then the canonical map $g_k : B_k \to B$ also satisfies GD for $k \leq j$. For a proof, let $B' = \text{lim } B_j$, where the indexes range over those $j \in I$ such that $k \leq j$. By the preceding observation, the canonical map $g_k : B_k \to B'$ satisfies GD. However, since $I$ is directed, a cofinality argument identifies $B$ with $B'$, whence $g_k$ is identified with $g_k'$, and the assertion follows.

(b) As a special case of the second observation in (a), that is of [8, Lemma 2.14], we easily recover [13, Corollary 2, whose proof was our original inspiration for Theorem 2.1. Specifically, we have that if $A$ is a subring of $B$ such that $A \subset A[b_1, \ldots, b_n]$ satisfies GD for each finite subset $\{b_1, \ldots, b_n\}$ of $B$, then $A \subset B$ also satisfies
GD. The point, of course, is that \( B \) is the direct limit of its subrings of the form \( A[b_1, \ldots, b_n] \). If the \( A \)-subalgebras corresponding to \( n = 1 \) happen to be cofinal amongst the \( A \)-subalgebras of finite type then, to use the terminology of [13], “simple going down” for \( A \subseteq B \) implies that \( A \subseteq B \) satisfies GD. It would be of interest to characterize those extensions \( A \subseteq B \) for which such cofinality obtains.

(c) Note that analogues of Theorem 2.1 and the above consequences may fail when “satisfies GD” is replaced by “induces an open mapping of prime spectra with their Zariski topologies.” In particular, consideration of the extension \( Z \subseteq Q \) reveals the falsity of the analogue of [8, Lemma 2.14]; indeed [14], Remark 3.12 guarantees, in the terminology of [14], that \( Z \) is an FTO-domain which is not an open domain. For some positive analogues, see the next result.

First, some terminology. As in [12], p. 28, it will be convenient to let INC denote the incomparability property. Adapting from [14], p. 2, we shall say that a ring-homomorphism \( f : A \rightarrow B \) is an \( i \)-map if \( f^* : \text{Spec}(B) \rightarrow \text{Spec}(A) \) is an injection; and an integral domain \( A \) is called an \( i \)-domain if the inclusion \( A \subseteq B \) is an \( i \)-map for each overring \( B \) of \( A \).

**Proposition 2.3.** Let \((A_i, f_{ik})\) and \((B_i, g_{jk})\) be directed systems of rings, each indexed by a directed set \( (I, \leq) \). For each \( j \in I \), let \( h_j : A_j \rightarrow B_j \) be a ring-homomorphism satisfying INC (resp., which is an \( i \)-map) such that, whenever \( j < k \) in \( I \), then \( g_k h_j = h_k f_{jk} \). Then \( h = \lim_{\to} h_j : A = \lim_{\to} A_j \rightarrow B = \lim_{\to} B_j \) satisfies INC (resp., is an \( i \)-map).

**Proof.** If the assertion concerning INC is assumed to fail, then there exist distinct comparable prime ideals \( Q \subseteq W \) of \( B \) such that \( h^*(Q) = h^*(W) = P \in \text{Spec}(A) \). Select \( b \in W \setminus Q \). By the construction of direct limits, there exist an index \( k \in I \) and an element \( x \in B_k \) such that \( g_k(x) = b \), where \( g_k : B_k \rightarrow B \) is the canonical map. Let \( W_1 = g_k^*(W) \), \( Q_1 = g_k^*(Q) \) and \( P_1 = f_1^*(P) \), where \( f_1 : A_1 \rightarrow A \) is the canonical map. Evidently, \( Q_1 \subseteq W_1 \); are comparable prime ideals of \( B_1 \), distinct since \( x \in W_1 \setminus Q_1 \). But the condition \( g_1 h_1 = h_1 f_1 \) readily yields that \( h_1^*(Q_1) = f_1^*(P) = h_1^*(W_1) \), contradicting the assumption that \( h_1 \) satisfies INC. The preceding argument also applies, mutatis mutandis, to give the assertion about \( i \)-maps.

To avoid unnecessary repetition, let us fix notation for (2.4)-(2.10). Data will consist of a directed system \((A_i, f_{ik})\) of rings indexed by a directed set \( (I, \leq) \); and its direct limit, \( A = \lim_{\to} A_j \), together with the canonical maps \( f_j : A_j \rightarrow A \).

**Corollary 2.4.** If \( A_j \) is an \( i \)-domain for each \( j \in I \), then \( A \) is also an \( i \)-domain.

**Proof.** If not, then as in the preceding proof (also cf. [14], Proposition 2.10), there exists an element \( u \) in the quotient field of \( A \) such that \( B = A[u] \) has distinct prime ideals \( Q, W \) such that \( u \in W \setminus Q \) and \( Q \cap A = W \cap A = P \in \text{Spec}(A) \). Write \( u = ab^{-1} \) for appropriate nonzero \( a, b \in A \). By the construction of direct limits, there exists \( k \in I \) and \( c \in A_k \) such that \( f_k(c) = a \) and \( f_k(d) = b \).

We claim that there exists a ring-homomorphism \( H : D = A_k[cd^{-1}] \rightarrow B \) which restricts to \( f_k \) on \( A_k \) and sends \( cd^{-1} \) to \( u \). For this, it is enough to show that if \( h \) is the
homomorphism $A_d[X] \rightarrow B$ which restricts to $f_d$ on $A_d$ and sends $X$ to $u_d$, then $h$ vanishes on the kernel of the evaluation map $A_d[X] \rightarrow D$. Now, if

$$g = a_0 X^n + a_1 X^{n-1} + \cdots + a_n \in A_d[X]$$

is an $n$-th degree polynomial in that kernel, i.e. satisfies $g(cd^{-1}) = 0$, then,

$$0 = d^n g(cd^{-1}) = a_0 e^n + a_1 e^{n-1} d + \cdots + a_n d^n.$$

Applying $f_d$ and dividing by $b^n$ results in

$$0 = f_d(a_0) u^n + f_d(a_1) u^{n-1} + \cdots + f_d(a_n) = h(g),$$

as claimed.

Observe that $Q_1 : H^*(Q)$ and $l_4/r : H^*(m)$ are prime ideals of $D$, distinct since $cd^{-1} \in W_1 \setminus Q_1$. But the conditions satisfied by $H$ imply that $Q_1 \cap A_d = f_d^*(P) = W_1 \cap A_d$, whence the inclusion $A_d \subset A_d[cd^{-1}]$ is not an $i$-map, contradicting the assumption that $A_d$ is an $i$-domain. This completes the proof.

Corollary 2.4 is reminiscent of the well-known fact (cf. [9], Proposition 22.6) that a directed union of Prufer domains is itself a Prufer domain. Indeed, Prufer domains may be characterized as the integrally closed $i$-domains [9], Theorem 26.2. Recall that, in general, each overring of an $i$-domain is a going-down ring [14], Corollary 2.13. Accordingly, one might conjecture that the classes of Prufer domains and of going-down rings are closed under taking direct limits. We shall soon establish these conjectures, together with their analogue for locally divided domains, a type of going-down ring figuring intimately in the analysis of arbitrary going-down rings (cf. [5], Theorem 2.5 and Corollary 2.8). For completeness, we recall that an integral domain $D$ is called divided in case $P = PD_P$ for each $P \in \text{Spec}(D)$; and $D$ is said to be locally divided if $D_P$ is divided for each $P \in \text{Spec}(D)$.

**Proposition 2.5.** (a) If $A_j$ is a Prufer domain for each $j \in I$, then $A$ is also a Prufer domain.

(b) If $A_j$ is divided for each $j \in I$, then $A$ is divided.

(c) If $A_j$ is locally divided for each $j \in I$, then $A$ is locally divided.

**Proof.** In any event, $A$ is an integral domain (cf. [2], Proposition 3, p. 122).

(a) One proof proceeds by applying Corollary 2.4, since any direct limit of integrally closed integral domains is itself integrally closed. For a more direct proof, we shall use the criterion that an integral domain is a Prufer domain if and only if each of its ideals is flat. Let $J$ be any ideal of $A$ and, for each $j \in I$, set $J_j = f_j^{-1}(J)$. Since $A_j$ is a Prufer domain, $J_j$ is $A_j$-flat. Thus, by [1], Proposition 9, p. 35, $\lim J_j$ is $A$-flat. However, according to [10], Proposition 6.1.2 (ii), p. 128, $\lim J_j \cong J$, and so the assertion follows.

(b) Let $P \in \text{Spec}(A)$ and, for each $j \in I$, set $P_j = f_j^*(P) \in \text{Spec}(A_j)$. As above, $P \cong \lim P_j$. However, each $A_j$ is supposed divided, and so $P_j = P_j(A_j)_{PD_P} \cong P_j \otimes A_j(A_j)_{PD_P}$. Since tensor product commutes with direct limit, the proof
that $P = PA_P$ will be complete if $A_P \cong \lim (A_j)_{P_j}$. However, this needed isomorphism does hold, by virtue of [10]. Proposition 6.1.6 (ii), p. 130, whose applicability is a consequence of noticing that $P_j = f^*_j(P_k)$ whenever $j \leq k$ in $I$.

(c) We shall offer two proofs. First, let $P \in \text{Spec}(A)$ and, for each $j \in I$, set $P_j = f^*_j(P)$. As in the proof of (b), an appeal to [10] reveals $A_P = \lim (A_j)_{P_j}$. Since $A_j$ is assumed to be locally divided, $(A_j)_{P_j}$ is divided, and so an appeal to (b) shows that $A_P$ is divided. Since $P$ was an arbitrary prime of $A$, the assertion follows.

To sketch another proof of (c), we recall from [6], Theorem 2.4 that an integral domain $D$ is locally divided if and only if $D + QD_Q$ is $D$-flat for each $Q \in \text{Spec}(D)$. Now, let $P, P_j$ be as above. Since $A_j$ is locally divided for each $j$ and direct limit preserves flatness, it follows that $B = \lim (A_j + P(A_j)_{P_j})$ is $A$-flat. However, one may verify routinely that the canonical $D$-module epimorphism $B \rightarrow A + PA_P$ is an isomorphism, from which the assertion (again) follows.

**Lemma 2.6.** Assume that each $A_j$ is a domain. Then each overring $B$ of $A$ may be expressed as $B = \lim B_j$, where $B_j$ is an overring of $A_j$ for each $j \in I$, such that the canonical diagram

$$
\begin{array}{ccc}
A_j & \longrightarrow & B_j \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
$$

commutes whenever $j \in I$.

**Proof.** Let $K$ be the quotient field of $A$, and consider an overring $B$ of $A$ (so that $A \subset B \subset K$). For each $j \in I$, let $P_j = f^*_j(0) \in \text{Spec}(A_j)$. As in the proofs of parts (b) and (c) of Proposition 2.5, we have $\lim (A_j)_{P_j} \cong K$. In particular, $((A_j)_{P_j}, g_{jk})$ is a directed system indexed by $I$; note that $g_{jk}$ restricts to $f_{jk}$ on $A_j$. Let $g_j : (A_j)_{P_j} \rightarrow K$ be the canonical structure map, and set $B_j = g_j^{-1}(B)$. Then $A_j \subset B_j$ since $g_j$ restricts to $f_j$ on $A_j$. The idea of the proof is now to verify the following assertions:

1. $(B_j, g_{jk}|_{B_j})$ is a directed set of rings indexed by $I$;
2. Whenever $j \leq k$ in $I$, one has $(g_{jk}|_{B_j})h_j = h_jf_{jk}$, and
3. The direct limit of the system in (1) may be identified with $B$ in such a way that $\lim h_j$ becomes identified with the inclusion map $A \rightarrow B$.

Now, (1) follows readily from the condition $g_{jk}g_{jk}|_{B_j} = g_j$. The compatibility condition (2) is a consequence of the above remarks. Finally, to establish (3), observe that the direct limit of the system in (1) may be viewed as $B' = \bigcup (\text{im}(g_{jk}|_{B_j}))$, the union indexed by $I$. Evidently, $B' \subset B$, by the definition of $B_j$. For the reverse inclusion, view any given $b \in B$ inside $\lim (A_j)_{P_j}$ and use the construction of direct limits (cf. [3], Lemma 1(i), p. 204) to find $k \in I$ and $x \in (A_k)_{P_k}$ such that $g_k(x) = b$; then $x \in \cap_{j} g^{-1}_j(B) = B$, whence $b \in B'$. Thus $B = B'$. For the final assertion in (3), one has to verify that the inclusion map $A \rightarrow B$ is compatible with the composite maps $A_j \rightarrow B_j \rightarrow B$, and this holds since $g_j$ restricts to $f_j$. The proof is complete.

Lemma 2.6 is perfectly suited for our present purposes. For example, it
immediately recovers Corollary 2.4 since a direct limit of (abelian group) monomorphisms is itself a monomorphism. Similarly, Lemma 2.6 also leads to a (third) proof of Proposition 2.5(a), since Prüfer domains may be characterized as the integral domains each of whose overrings is flat [15], Theorem 4. More to the point, we now give the promised result.

**COROLLARY 2.7.** If $A_j$ is a going-down ring for each $j \in I$, then $A$ is also a going-down ring.

**Proof.** Use the criterion that an integral domain $D$ is a going-down ring if and only if the inclusion $D \subseteq E$ satisfies GD for each overring $E$ of $D$. Apply Lemma 2.6 and Theorem 2.1, to complete the proof.

Our next result concerns QR-domains. Recall that an integral domain $D$ is said to be a QR-domain in case each overring of $D$ is a quotient ring (ring of fractions) of $D$. As quotient rings are flat, any QR-domain must be a Prüfer domain and, in particular, a going-down ring.

**COROLLARY 2.8.** If $A_j$ is a QR-domain for each $j \in I$, then $A$ is also a QR-domain.

**Proof.** Let $B$ be an overring of $A$. By Lemma 2.6, $B = \lim B_j$, where $B_j$ is a suitable overring of $A_j$ for each $j \in I$. By hypothesis, $B_j = (A_j)_{T_j}$, where we may suppose that $T_j$ is a saturated multiplicative subset of $A_j$. Observe, using (2) in the proof of Lemma 2.6, that $f_{jk}(T_j) \subseteq T_k$ whenever $j \leq k$ in $I$. Letting $T$ be the multiplicative set $\bigcup \{f_j(T_j) \mid j \in I\} \subseteq A$, one readily verifies (cf. [10], Proposition 6.1.5, p. 129) that the canonical ring-homomorphism $\lim B_j \to A$ is an isomorphism, completing the proof.

Our final results concern strong extensions [7] and pseudo-valuation domains [11]. Recall that an extension $D \subseteq E$ of rings is said to be strong if, whenever $x,y \in P$ for some $x \in E, y \in E$ and $P \in \text{Spec}(D)$, then either $x$ or $y$ is in $P$; and that a domain $D$ is a pseudo-valuation domain (PVD) in case $D \subseteq K$ is strong, where $K$ is the quotient field of $D$. Any PVD is a divided ring and, hence, a going-down ring.

**PROPOSITION 2.9.** Let $(A_j, f_{jk})$ and $(B_j, g_{jk})$ be directed systems of rings, each indexed by a directed set $(I, \leq)$. For each $j \in I$, let $h_j : A_j \to B_j$ be a strong extension such that, whenever $j \leq k$ in $I$, then $g_{jk} h_j = h_j f_{jk}$. Then $h = \lim h_j : A = \lim A_j \to B = \lim B_j$ is a strong extension.

**Proof.** Suppose that $xy \in P$ for some $x \in B, y \in B, P \in \text{Spec}(B)$. By the nature of direct limits, there exists an index $j$ and elements $x, y_j$ of $B_j$ such that $g_j(x_j) = x$, $g_j(y_j) = y$ and $x y_j \in P_j = g_j^{-1}(P)$ (cf. [10], Proposition 6.1.2). Since $h_j$ is assumed strong, either $x_j$ or $y_j$ is in $P_j$ and so either $x$ or $y$ is in $P$, as desired.

**COROLLARY 2.10.** If $A_j$ is a PVD for each $j \in I$ and if $f_{jk}$ is a monomorphism whenever $j \leq k$ in $I$, then $A$ is a PVD and the quotient field of $A$ is $\lim K_j$, where $K_j$
denotes the quotient field of $A_j$ for each $j \in I$.

**Proof.** By the definition of PVD’s, $A_j \rightarrow K_j$ is a strong extension for each $j \in I$. Thus, by Proposition 2.9, $A \rightarrow \lim K_j$ is also strong. Since $\lim K_j$ is readily shown to be the quotient field of $A$, the assertions follow.

**References**