On Certain Distinguished Spectral Sets (*).

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Sunto. – *Ogni insieme parzialmente ordinato* $(X, \leqslant)$ *ammette varie* T_0*-topologie* $\mathscr{T}$ « *compatibili* » con $\leqslant$, *nel senso che* $\leqslant$ *coincide con l'ordine indotto da* $\mathscr{T}$ *su* X (*cf. Hochster* [30]). *Tra tali topologie la meno fine è la* COP(= *closures of points*)*-topologia, cioè la topologia meno fine per la quale l'insieme* $\{y \in X : x \leqslant y\}$ *è chiuso per ogni* $x \in X$. *La più fine è* L(= *left*)*-topologia discreta di Alexandroff,* X^L, *avente come base per gli aperti gli insiemi* $\{y \in X : y \leqslant x\}$ *al variare di* x *in* X. *In questo lavoro sono date condizioni su* X *affinchè questo abbia una struttura topologica spettrale noetheriana. Inoltre, vengono caratterizzati gli insiemi parzialmente ordinati* X *per i quali* X^L *è uno spazio spettrale; vengono anche caratterizzati gli spazi spettrali* Y *la cui topologia coincide con la* L*-topologia associata all'ordine indotto. La topologia dell'* « *ordine opposto* » (opposite-order topology *di Hochster, op. cit. Prop.* 8) *determina una* « *dualità* » *tra le* L*-topologie spettrali e le topologie spettrali noetheriane. Se* $Y = \mathrm{Spec}(A)$ *con la topologia di Zariski, allora* Y^H *coincide con la* O(= *ordine*)*-topologia di G. Picavet.*

1. – Introduction and summary.

After identifying two necessary conditions for an ordered set to be a spectral set (that is, order-isomorphic to the prime spectrum of a ring), KAPLANSKY [18], p. 7 raises the question of giving an order-theoretic characterization of the set of prime ideals of a ring. In its full generality, this problem is still open, although numerous properties of spectral sets are now known, particularly for totally ordered sets or trees ([21], [22]) and for ordered sets of dimension $\leqslant 1$ ([5], [22]). The analogous topological question of characterizing spectral spaces, (that is, topological spaces homeomorphic to the prime spectrum of a ring endowed with the Zariski topology [3]) has been completely settled by HOCHSTER ([16]); cf. [19], Appendice. As described below, this paper is concerned with both the order-theoretic and the topological aspects of the subject, pursuing the study begun in [8].

Although there are close connections between the topological structures and

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the order-theoretic structures on a set, it is known that, in general, order conveys less information than does topology (cf. [11], [20], [22, Example 2.2]). However, the distinction between topology and order ceases in some remarkable cases, for instance for Noetherian spectral spaces (cf. [16, Proposition 14], [26], [27], [22], [15]). Accordingly, order plays a central role in our approach. Our intent is to deepen the study of certain classes of spectral spaces whose topology is determined by the order-theoretic data. The classes of interest are « extreme » in the sense described below.

One may introduce, onto any given ordered set X, several topologies which are compatible with the given order (in a sense made precise at the beginning of section 2). In the class of such topologies compatible with the order on X; we may identify the « extreme » members, that is the coarsest and the finest ones. The coarsest is the COP-topology (« closures of points » topology [22]), which is the coarsest topology in which $x^{\uparrow} = \{y \in X | x \leqslant y\}$ is closed, for each $x \in X$, we shall let X^{COP} denote X endowed with the COP-topology. The finest topology on X, which is compatible with the given order is the left-topology (or L-topology [4, Ex. 2, p. 89]), that is, the topology with basis for open sets consisting of the sets $x^{\downarrow} = \{y \in X | y \leqslant x\}$; let X^L denote X equipped with the left topology. Clearly, if X_{opp} denotes X with its order reversed, then $(X_{\mathrm{opp}})^L = X^R$, where X^R denotes X endowed with the right-(or R-)topology, that is the topology with a basis, for open sets consisting of the sets $x^{\uparrow}$ (as x ranges over X).

Much of this paper is devoted to the study of the L-topology and the COP-topology for the case of spectral spaces. Along the way, we also study the « opposite-order » topology and the constructible topology on arbitrary spectral spaces. Corollary 4.3 and Proposition 3.1 are typical of results connecting these notions. Other results to be noted include Theorem 2.4, characterizing those ordered sets which are spectral spaces with respect to the L-topology; and Theorem 3.3, describing a « duality » of sorts between topology of the spectral L-topologies and the Noetherian spectral topologies. As a consequence of these results, we obtain in Remark 4.2 (a) another proof of the characterization of Noetherian spectral spaces [27].

If the ordered set X is the set Spec (A) of all prime ideals of a ring A (commutative with 1), then we shall denote X_Z (resp., X_F) the set X endowed with the Zariski topology (resp. the flat topology [8], that is, the topology having as closed the subsets of the form im (f^*), where f^* is the application canonically associated to a ring-homomorphism $f\colon A \to B$ (with domain A), inducing an A-flat structure on B).

2. – Spectral L-sets.

We begin with some terminology and notation which is to be used throughout this paper.

Let X be a topological space. For each $Y \subset X$, let $C\ell(Y)$ denote the closure of Y in X; if Y is a singleton set $\{y\}$, we simply write $C\ell(y)$ instead of $C\ell(\{y\})$. If X is a

T_0-space, then [16], p. 53 constructs a (partial) order on X as follows:

$$x \leqslant y \Leftrightarrow y \in C\ell(x) \quad \text{for } x, y \in X .$$

Let X_{ord} denote X equipped with this order.

On the other hand, if X is an ordered set then for each subset Y of X, let $Y^{\uparrow}$ (resp., $Y^{\downarrow}$) denote the set of elements $x \in X$ for which there exists $y \in Y$ such that $x \geqslant y$ (resp., $x \leqslant y$). The set $Y^{\uparrow}$ (resp., $Y^{\downarrow}$) is called the set of *specializations* (resp., *generalizations*) of elements of Y. If $Y = \{y\}$ is singleton, write $y^{\uparrow}$ (resp., $y^{\downarrow}$) instead of $\{y\}^{\uparrow}$ (resp., $\{y\}^{\downarrow}$). Clearly,

$$Y^{\uparrow} = \bigcup_{y \in Y} y^{\uparrow} \quad \text{and} \quad Y^{\downarrow} = \bigcup_{y \in Y} y^{\downarrow} .$$

On an ordered set X, there are several ways to construct a *topology compatible with the given order*, that is, a topology whose induced order (in the sense of the preceding paragraph) coincides with the given order on X. Indeed, it is easy to see that a topology $\mathscr{T}$, defined on an ordered set X, is compatible with the given order on X if and only if the following two conditions hold:

(*a*) For each $x \in X$, the set $x^{\uparrow}$ is closed in $\mathscr{T}$;

(*b*) Each closed subset of $\mathscr{T}$ is stable under specializations.

We then see, with the aid of Lemma 2.1 below, that as asserted in the introduction, X^L (resp., X^{COP}) is indeed the finest (resp., coarsest) topology compatible with the given order on X.

We may now define this section's object of study. An ordered set X is said to be *L-spectral* (resp., *R-spectral*) in case the topological space X^L (resp., X^R) is a spectral space, in the sense of [16].

LEMMA 2.1. – *Let X be an ordered set. Then*:

(*a*) $Y \mapsto Y^{\uparrow}$ *is a Kuratowski operator* (*on the set of subsets of* X).

(*b*) *For each* $Y \subset X$, $C\ell^L(Y) = Y^{\uparrow}$.

(*c*) *A subset U of X is open in X^L if and only if $U = U^{\downarrow}$.*

PROOF. – For (*a*), one may directly verify the conditions stated in [4], Ex. 9, p. 90. Moreover, (*c*) follows since $U^{\downarrow} = \bigcup_{x \in U} x^{\downarrow}$. For (*b*), first note using (*c*) that $C\ell^L(y) \supset y^{\uparrow}$ for each $y \in X$, so that $C\ell^L(Y) \supset Y^{\uparrow} (\supset Y)$. To complete the proof, it is enough to show that $Y^{\uparrow}$ is closed in X^L, and this holds since each element $x \in X \setminus Y^{\uparrow}$ is contained in the X^L-open neighborhood $x^{\downarrow}$ which is disjoint from $Y^{\uparrow}$.

For an ordered set X, let Max (X) and Min (X) denote the sets of maximal elements of X and of minimal elements of X, respectively. We then have the following direct consequence of Lemma 2.1.

COROLLARY 2.2. – *Let X be an ordered set. Then:*

(*a*) *x is a closed point of X^L if and only if $x \in \text{Max}(X)$.*

(*b*) *x is an open point of X^L if and only if $x \in \text{Min}(X)$.*

(*c*) *X^L is a T_1-topological space if and only if X^L is a discrete space.*

We pause to recall that a topological space X is called a *T_D-space* in case, for each $Y \subset X$, the set of accumulation points of Y is closed (cf. [7]; [2], Definition 3.1). Any (T_0) discrete Alexandroff space (4) is a T_D-space [2], Theorem 5.2. Following [14], p. 49, we call a space X *sober* if each irreducible closed subspace of X has a unique generic point.

PROPOSITION 2.3. – *Let X be an ordered set. Then:*

(*a*) *X^L is a T_0, discrete Alexandroff space, and hence a T_D-space.*

(*b*) *If U is a nonempty open subset of X^L, then the following are equivalent:*

(i) *U is quasi-compact;*

(ii) *There exist $x_1, \ldots, x_n \in U$ with $n = n(U) \geqslant 1$ such that $U = \bigcup_{i=1}^{n} x_i^{\downarrow}$;*

(iii) Card $(\text{Max}(U)) < \infty$, *and each chain in U has an upper bound in U.*

(*c*) *If X satisfies the following condition:*

(*filtr.^{L}*) *each nonempty lower-directed subset Y of X has a greatest lower bound $y = \inf(Y)$ such that $y^{\uparrow} = Y^{\uparrow}$,*

then X^L is a sober space.

PROOF. – Assertion (*a*) is a direct consequence of the definitions. Moreover, (*c*) follows by remarking that each irreducible closed subspace of X^L must be a lower-directed set. As for (*b*), note first that $x^{\downarrow}$ is quasi-compact and open for each $x \in X$, so that any finite union of generalizations of elements of X is also quasi-compact and open. Conversely, any quasi-compact open V in X^L may be expressed as such a finite union, since $V = \bigcup_{x \in V} x^{\downarrow}$. Moreover, if n is the minimum number of elements needed to express the given U as $U = x_1^{\downarrow} \cup \ldots \cup x_n^{\downarrow}$, it is clear that then (and only then) one has Max $(U) = \{x_1, \ldots, x_n\}$. These considerations lead to (*b*), completing the proof.

(4) A discrete Alexandroff space is a T_0 topological space X such that, for each $Y \subset X$, the closure of Y is the union of the closures of the points of Y, [1, p. 28].

THEOREM 2.4. – *Let X be an ordered set. Then X is an L-spectral set if and only if the following four properties hold*:

(α) *Each nonempty totally ordered subset of X has a sup*;

(β) *X satisfies the condition* (*filtr.*^{L});

(γ) Card $(\mathrm{Max}\,(X)) < \infty$;

(δ) *For each pair of distinct elements x and y of X, there exist at most finitely many elements of X which are maximal in the set of common lower bounds of x and y.*

PROOF. – First of all, let us prove necessity of the conditions (α)-(δ). Any spectral space satisfies (α): see [18], Theorem 9. Necessity of (β) is a consequence of [16], Proposition 5, and Lemma 2.1 (*b*). Necessity of (γ) follows by recalling from [11], Corollary of Proposition 1, section 3 that (γ) holds in any discrete Alexandroff space. Finally, Proposition 2.3 (*b*) leads to the necessity of (δ), since the intersection of two quasi-compact open subspaces of a spectral space is itself quasi-compact and open. The sufficiency of (α)-(δ) results by verifying the conditions in the theorem of HOCHSTER [16], Proposition 4 characterizing spectral spaces. Indeed, we know that X^L is a T_0-space (by Proposition 2.3 (*a*)); X^L is quasi-compact by virtue of the combined effect of (α) and (γ) (cf. Proposition 2.3 (*b*)); by (δ), the collection of quasi-compact open subsets of X^L is closed under finite intersections, and moreover forms a basis for the open sets in X^L; and, finally, (β) guarantees that X^L is sober, by Proposition 2.3 (*c*), to complete the proof.

REMARK 2.5. – (*a*) It is not difficult to verify that, in the statement of Theorem 2.4, condition (β) may be replaced by the requirement, « each decreasing sequence of elements of X stabilizes »: cf. [6], Proposition 5.9, p. 33.

(*b*) Recall from [8, Sec. 2] that a spectral space is discrete Alexandroff if and only if it is homeomorphic to the prime spectrum of a g-ring. One therefore immediately deduces that an ordered set X is L-spectral if and only if X is isomorphic as an ordered set to the spectrum of a g-ring.

We intend next to see how the passage from an ordered set X to the topological space X^L is reflected at the level of morphisms. The proof of the following easy lemma is omitted.

LEMMA 2.6. – *Let $f\colon X \to Y$ be a morphisms of ordered sets* (*that is, an order-preserving function*). *Then*:

(*a*) $f^L = f\colon X^L \to Y^L$ *is a continuous map. Moreover, f^L is a homeomorphism if and only if f is an isomorphism of ordered sets.*

(*b*) *For each $X_1 \subset X$, one has $f(X_1^{\uparrow}) \subset f(X_1)^{\uparrow}$ and $f(X_1^{\downarrow}) \subset f(X_1)^{\downarrow}$.*

(*c*) *For each $Y_1 \subset Y$, one has $f^{-1}(Y_1)^{\uparrow} \subset f^{-1}(Y_1^{\uparrow})$ and $f^{-1}(Y_1)^{\downarrow} \subset f^{-1}(Y_1^{\downarrow})$.*

PROPOSITION 2.7. – *For $f\colon X \to Y$, a morphism of ordered sets, the following are equivalent*:

(i) *$f\colon X^L \to Y^L$ is a closed map*;

(ii) *For each $X_1 \subset X$, one has $f(X_1^{\uparrow}) = f(X_1)^{\uparrow}$*;

(iii) *For each $x \in X$, one has $f(x^{\uparrow}) = f(x)^{\uparrow}$*;

(iv) *For each $y_1, y \in Y$ and $x_1 \in X$ such that $y_1 \leqslant y$ and $f(x_1) = y_1$, there exists $x \in X$ such that $x_1 \leqslant x$ and $f(x) = y$.*

PROOF. – By virtue of Lemma 2.6 (*b*), the conclusions (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are immediate. Finally, to prove (iv) $\Rightarrow$ (ii), let $y \in f(X_1)^{\uparrow}$, so that $f(x_1) = y_1 \leqslant y$ for some $x_1 \in X_1$. By (iv), $y = f(x)$ for some $x \in X$ such that $x_1 \leqslant x$, whence $y \in f(X_1^{\uparrow})$, and so $f(X_1)^{\uparrow} \subset f(X_1^{\uparrow})$. As the reverse inclusion is valid in general, (ii) holds, completing the proof.

COROLLARY 2.8. – *For $f\colon X \to Y$, a morphism of ordered sets, the following are equivalent*:

(i) *$f\colon X^L \to Y^L$ is an open map*;

(ii) *For each $X_1 \subset X$, one has $f(X_1^{\downarrow}) = f(X_1)^{\downarrow}$*;

(iii) *For each $x \in X$, one has $f(x^{\downarrow}) = f(x)^{\downarrow}$*;

(iv) *For each $y_1, y \in X$ and $x \in X$ such that $y_1 \leqslant y$ and $f(x) = y$, there exists $x_1 \in X$ such that $x_1 \leqslant x$ and $f(x_1) = y_1$.*

PROOF. – Observe that $f\colon X_{\mathrm{opp}} \to Y_{\mathrm{opp}}$ is a morphism of ordered sets. Since $f\colon X^L \to Y^L$ is an open map if and only if $f\colon X^R \to Y^R$ is a closed map, an application of Proposition 2.7 completes the proof.

For the applications already seen and those described briefly in the introduction, it is important to know the relation between a fixed T_0-topology on a set X and the left topology, denoted X^L instead of $(X_{\mathrm{ord}})^L$, arising from the order induced by the given topology.

PROPOSITION 2.9. – (*a*) *If $f\colon X \to Y$ is a continuous (resp., homeomorphic) map of T_0-topological spaces, then $f^L = f\colon X^L \to Y^L$ is also continuous (resp., homeomorphic).*

(*b*) *For each T_0-space X, the identity map $\mathrm{id}_X\colon X^L \to X$ is continuous.*

(*c*) *The full subcategory of the category of topological spaces and continuous maps whose objects are the discrete Alexandroff topological spaces is a coreflective subcategory.*

PROOF. – Assertion (*a*) follows from Lemma 2.6 (*a*), since $f_{\text{ord}} = f\colon X_{\text{ord}} \to Y_{\text{ord}}$ is a morphism of ordered sets. For (*b*), it suffices to observe that $C\ell(x) = x^{\uparrow} = C\ell^L(x)$ for each $x \in X$; and that each closed subset of X is stable under specializations. As for (*c*), we must show that the inclusion functor has a right adjoint [17], p. 140. Clearly, a T_0-space Y is discrete Alexandroff if and only if $Y^L = Y$. Therefore, for each continuous map $f\colon Y \to X$ of T_0-spaces such that Y is discrete Alexandroff, there is a unique continuous map $f^L = f\colon Y = Y^L \to X^L$ such that $f = \text{id}_X \circ f^L$. Hence, the required adjoint exists, and the proof is complete.

3. – Spectral spaces and L-topologies.

In the preceding section, we saw that for each T_0-topological space, there is a canonically associated discrete Alexandroff topological space having the same underlying set. In this section, we are interested more specifically in the relations between X and X^L in case X is a spectral space.

Let X be a spectral space. Denote by X_C the set X endowed with the constructible topology [14], (7.2.11), p. 337, that is the coarsest topology in which all the closed subsets of X and all the quasi-compact open subsets of X are closed. Let X^H denote X endowed with the « Hochster topology » (or the « opposite-order topology ») [16], Proposition 8, that is the topology having a basis for its closed sets consisting of the quasi-compact open subsets of X. We may take as known the following results:

(A) *X_C and X^H are spectral spaces*;

(B) $(X^H)_{\text{ord}} = (X_{\text{ord}})_{\text{opp}}$. *Hence, in particular,*

(B_1) $(X^H)^L = X^R$;

(B_2) *For each $x \in X$, $C\ell^H(x) = x^{\downarrow}$ is an irreducible closed subset of X^H*;

(B_3) *The space X is irreducible if and only if X^H has a unique closed point*;

(B_4) *X has a unique closed point if and only if X^H is irreducible*;

(C) $X_C = (X^H)_C$.

The next proposition records the relations between the closure operators of X, X^H, X_C, X^L and X^R (denoted by $C\ell$, $C\ell^H$, $C\ell_C$, $C\ell^L$ and $C\ell^R$, respectively). Since $((\text{Spec}\, A)_Z)^H = (\text{Spec}\, A)_F$, one may regard parts (*a*) and (*b*) of Proposition 3.1 as a modest improvement upon Lemma 2.5 of [8].

PROPOSITION 3.1. – *Let X be a spectral space. Then*:

(*a*) *For each $Y \subset X$, $C\ell(Y) = C\ell^L(C\ell_C(Y))$*;

(*b*) *For each $Y \subset X$, $C\ell^H(Y) = C\ell^R(C\ell_C(Y))$*;

(*c*) $(X^H)^H = X$.

PROOF. – Statement (*a*) follows easily from the facts that the constructible topology and the L-topology are finer than the given topology on X; and that each quasi-compact open subset of X is closed in the compact space X_C. Then (*b*) follows from (*a*), in view of (*A*), (*B*) and (*C*). Finally, (*c*) is an immediate consequence of (*a*) and (*b*).

REMARK 3.2. – It is not difficult to see that, in general, $C\ell_C(C\ell^L(Y)) \subset C\ell^L(C\ell_C(Y))$ and $C\ell_C(C\ell^R(Y)) \subset C\ell^R(C\ell_C(Y))$. In [9], (1.2), examples are given in which the inclusion is strict.

THEOREM 3.3. – *For a spectral space X, the following are equivalent*:

(i) $X^L = X$;

(ii) *For each* $x \in X$, $x^{\downarrow}$ *is a quasi-compact open subset of* X;

(iii) *For each family* $\{U_i | i \in I\}$ *of quasi-compact open subsets of* X, *the set* $\bigcap_{i \in I} U_i$ *is quasi-compact and open in* X;

(iv) X *is a* T_D-*space; and for each family* $\{U_i | i \in I\}$ *of quasi-compact open subsets of* X, Card $(\text{Max}\,(\bigcap U_i))$ *is finite*;

(v) *Each increasing sequence of irreducible closed subsets of* X *stabilizes; and for each family* $\{U_i | i \in I\}$ *of quasi-compact open subsets of* X, Card $(\text{Max}\,(\bigcap U_i))$ *is finite*;

(vi) *Each decreasing sequence of irreducible closed subsets of* X^H *stabilizes; and for each closed subset* F *of* X^H, *there exist* $y_1, y_2, \ldots, y_n \in F$ *such that* $F = y_1^{\downarrow} \cup y_2^{\downarrow} \cup \ldots \cup y_n^{\downarrow}$;

(vii) *Each open subset of* X^H *is the complement of a quasi-compact open subset of* X;

(viii) X^H *is a Noetherian space.*

PROOF. – In view of Remark 2.5 (*b*), the equivalences (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (viii) follow easily from [25], Propositions 1 and 4 of section 5. As for (viii) $\Leftrightarrow$ (vi), it is not difficult to see that the prime spectrum of a ring is a Noetherian space if and only if each increasing sequence of prime ideals of the ring stabilizes and each radical ideal is the intersection of finitely many prime ideals (cf. [18], Exercise 25, p. 65; [3], Proposition 8, p. 123 and Proposition 10, p. 124). Next, since each open subset of X^H is a union of a family of complements of quasi-compact open subsets of X, the implication (viii) $\Rightarrow$ (vii) follows by recalling that a topological space is Noetherian if and only if each of its open subsets is quasi-compact. To show (vii) $\Rightarrow$ (iii), note in the present situation that, for each family $\{U_i | i \in I\}$ of quasi-compact open sets of X, there exists a quasi-compact open subset U of X such that $X \setminus U = \bigcup (X \setminus U_i)$, whence $U = \bigcap U_i$: As for (iii) $\Rightarrow$ (i), each intersection of open subsets of X is open since the quasi-compact open subsets form a basis for the open subsets of X. To prove (iii) $\Rightarrow$ (iv), note that the above yields, in the present situation, that X is a discrete Alexandroff space, and hence a T_D-space, while the second assertion in (iv) follows from Proposition 2.3 (*b*). Next, (iv) $\Rightarrow$ (v) since, in a spectral space which is T_D, the infimum of any nonempty family of points coincides with one of those points (cf. [25], Proposition 2, section 1 or [23], Corollary 2.5). As for (v) $\Rightarrow$ (vi), it is easy to see that the first assertion in (v) is equivalent to the first assetion in (vi); moreover, the second part of (vi) follows immediately from the second part of (v) since each closed subset of X^H is an intersection of a family of quasi-compact open subsets of X. The proof is complete.

COROLLARY 3.4. – *For a spectral space X, the following are equivalent*:

(i) X *is a Noetherian space*;

(ii) X^H *is a discrete Alexandroff space*;

(iii) $X^H = X^R$.

PROOF. – This is an immediate consequence of Proposition 3.1 (*c*), Theorem 3.3 and assertion (B_1).

The equivalence (iii) $\Leftrightarrow$ (iv) in the next result recaptures the equivalence (*b*) $\Leftrightarrow$ (*c*) in [11], Proposition 3.

COROLLARY 3.5. – *For a spectral space X, the following are equivalent*:

(i) X *is a Noetherian discrete Alexandroff space*;

(ii) X^H *is a Noetherian discrete Alexandroff space*;

(iii) Card (X) *is finite*;

(iv) X *is a Noetherian T_D-space*;

(v) X^H *is a Noetherian T_D-space.*

PROOF. – The implications (iii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iv) are trivial. The equivalence (i) $\Leftrightarrow$ (ii) follows easily from Theorem 2.3 and Corollary 2.4, while [11], Proposition 3, section 1 establishes the equivalences (v) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv).

4. – N-spectral sets and COP-topologies.

In this section, we intend to deepen the study of the « duality » already seen to exist (in Corollary 3.4, for instance) between those spectral spaces which are discrete Alexandroff and those which are Noetherian, by expanding and making precise the connections even at the level of ordered sets.

PROPOSITION 4.1. – *Let X be an ordered set. Then one may define a (unique) topology on X which is compatible with the given order in such a way that X becomes a Noetherian spectral space if and only if the following conditions hold:*

(α') *Each nonempty totally ordered subset of X has an inf;*

(β') = (*filtr.*^{R}) *Each nonempty subset Y of X which is upper-directed has upper bound $y = \sup(Y)$ satisfying $y^{\downarrow} = Y^{\downarrow}$;*

(γ') Card $(\mathrm{Min}(X)) < \infty$;

(δ') *For each pair of distinct elements x and y of X, there exist at most finitely many elements of X which are minimal in the set of common upper bounds of x and y.*

Moreover, a topology satisfying the above conditions must be the COP-*topology.*

PROOF. – In view of Theorem 2.4, the ordered set X_{opp} is L-spectral if and only if X_{opp} satisfies conditions (α')-(δ'). Since $(X_{\mathrm{opp}})^L = X^R$, we conclude that $(X^R)^H$ is a Noetherian spectral space if and only if X satisfies (α')-(δ') (by Theorem 3.3). Moreover, it is clear that the topology of $(X^R)^H$ is compatible with the fixed order on X and has the finite unions of specializations of points as a basis for its closed sets. This topology is the unique spectral Noetherian topology compatible with the given order for, in a spectral Noetherian space, each closed set is a finite union of specializations of points.

An ordered set X satisfying the above conditions (α')-(δ') will be call an *N-spectral set.*

REMARK 4.2. – *a*)) In the statement of Proposition 4.1, condition (β') can be replaced by the condition « each increasing sequence of elements of X stabilizes » (Cf. Remark 2.5 (*a*).) Therefore, as a corollary of Theorems 2.4 and 3.3, we obtain a new proof of a result of R. and S. WIEGAND [27], Proposition 2, section 5.

(*b*) The matter of uniqueness of a Noetherian spectral topology, compatible with a given order, was raised earlier by HOCHSTER [16], Proposition 14.

COROLLARY 4.3. – *Let X be a spectral space. Then X^R (resp., X^L) is a spectral space if and only if X is a Noetherian space (resp., a discrete Alexandroff space). In this case, one has $X = X^{\mathrm{COP}} = (X^R)^H$ $\big($resp., $X^H = (X^H)^{\mathrm{COP}} = (X^L)^H = (X_{\mathrm{opp}})^{\mathrm{COP}}$, that is, $X^L = ((X^H)^{\mathrm{COP}})^H\big)$.*

PROOF. – One may infer the statement from the proof of Proposition 4.1, bearing in mind that X^H is a spectral space such that $(X^H)^R = X^L$. (Cf. also Theorem 3.3 and Corollary 3.4.).

In view of the important role which the COP-topologies play in regard to N-spectral sets, we are led to a more detailed study of these topologies. We begin with an easy proposition, whose proof is omitted.

PROPOSITION 4.4. – (*a*) *If X is an ordered set, then X^{COP} is a T_0-space, $(X^{\mathrm{COP}})_{\mathrm{ord}} = X$, and the identity map $\mathrm{id}_X\colon X^L \to X^{\mathrm{COP}}$ is continuous.*

(*b*) *Let $f\colon X \to Y$ be a morphism of ordered sets. Then the mapping $f^{\mathrm{COP}} = f\colon X^{\mathrm{COP}} \to Y^{\mathrm{COP}}$ is continuous if and only if, for each $y \in Y$, one has:*

$$f^{-1}(y^{\uparrow}) = \bigcap_{i\in I} (x_{i,1}^{\uparrow} \cup x_{i,2}^{\uparrow} \cup \ldots \cup x_{i,n_i}^{\uparrow})$$

with $x_{i,j} \in X$, $1 \leqslant j \leqslant n_i$.

(*c*) *Let $f\colon X \to Y$ be a morphism of ordered sets. Then f^{COP} is a homeomorphism if and only if f is an isomorphism.*

A morphism of ordered sets which satisfies the condition in the statement of Proposition 4.4 (*b*) will be called a COP-*morphism*.

PROPOSITION 4.5. – (*a*) *Let X be a T_0-space. Then [writing simply X^{COP} instead of $(X_{\mathrm{ord}})^{\mathrm{COP}}$] one has that the identity map $\mathrm{id}_X\colon X \to X^{\mathrm{COP}}$ is continuous.*

(*b*) *If $f\colon X \to Y$ is a homeomorphism, then $f^{\mathrm{COP}} = f\colon X^{\mathrm{COP}} \to Y^{\mathrm{COP}}$ is also a homeomorphism.*

(*c*) *The category whose objects are the* COP-*spaces (that is, the T_0-spaces X such that $X = X^{\mathrm{COP}}$) and whose morphisms are those continuous maps which are* COP-*morphisms (with respect to the underlying ordered sets) is a reflective subcategory of the category of all T_0-spaces and all the continuous maps which are* COP-*morphisms.*

PROOF. – Parts (*a*) and (*b*) are immediate. For (*c*), one need only observe that, if $f\colon X \to Y$ is a continuous map which is also a COP-morphism, then $f^{\mathrm{COP}}\colon X^{\mathrm{COP}} \to Y^{\mathrm{COP}} = Y$ is the unique continuous map such that $f^{\mathrm{COP}} \circ \mathrm{id}_X = f$.

REMARK 4.6. – (*a*) From Proposition 4.1 and Corollary 3.4 (resp., (4.1) and Theorem 3.3), one sees readily that the prime spectrum $X = \mathrm{Spec}(A)$ of a ring is an N-spectral (resp., L-spectral) set if and only if X_z is a Noetherian (resp., discrete Alexandroff) space.

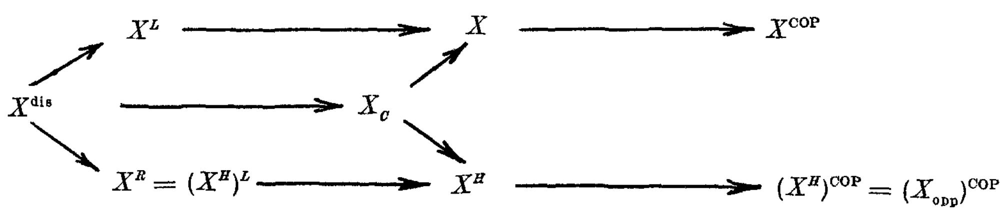
(*b*) Let $\varphi^*\colon Y_z = \mathrm{Spec}(B) \to X_z = \mathrm{Spec}(A)$ be the continuous map associated to a ring-homomorphism $\varphi\colon A \to B$. A sufficient condition for φ^* to be a COP-morphism is that B satisfy the « finite component » condition (*FC*) [as in [24], this means that each ideal of B has only finitely many minimal prime ideals]; this holds if Y_z is a Noetherian space, for example if B is Laskerian (cf. [13], Theorem 4).

(c) We stress that although each Noetherian topological space is a COP-space, the converse fails even for the case of spectral spaces. It suffices to consider the totally ordered spectral set

$$X = \{x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots < x_\infty\},$$

which is isomorphic to the spectrum of a suitable valuation domain ([12], p. 95; [11], Proposition 3).

We would like next to summarize and make precise the results comparing the different topologies introduced on a spectral space X. First of all, *the identity map* id_X *is continuous in the following diagram*:



(where X^{dis} denotes the set X endowed with the discrete topology): cf. Propositions 2.9 and 4.5; *moreover, in case* X^R (resp., X^L) *is a spectral space, then* $X^{\mathrm{COP}} = (X^R)^H$ *(resp.,* $(X_{\mathrm{opp}})^{\mathrm{COP}} = (X^L)^H$*)*: cf. Corollary 4.3.

One readily sees that the following four conditions are equivalent:

(i) $X_C = X$,

(ii) $X_C = X^H$,

(iii) $X = X^H$,

(iv) $X^R = X^{\mathrm{dis}} = X^L$;

similarly, the following five conditions are equivalent:

(I) $X^L = X_C$,

(II) $X^{\mathrm{dis}} = X_C$,

(III) $X^R = X_C$,

(IV) X *is a finite space,*

(V) *Both* X^H *and* X *are Noetherian spaces.*

It is an open question to characterize the spectral spaces X such that $X^L = X^{\mathrm{COP}}$. Note, for any such X, that both $\mathrm{Min}(X)$ and $\mathrm{Max}(X)$ are finite.

REMARK 4.7. – In order to study the properties of those prime ideals of a ring which are G-ideals in the sense of [18], p. 16, another topology on a spectral space was introduced in [25], section 3. This « Goldman topology » has a basis of open sets consisting of the subspaces which are locally closed in the original topology of X. For each subset Z on X, let $C\ell_{\text{Gold}}(Z)$ denote the closure of Z in the Goldman topology; and let X^{Gold} denote the set X endowed with the Goldman topology. Then one readily sees that:

(*a*) $C\ell_{\text{Gold}}(Z) = \{x \in X | x^{\dagger} = C\ell(Z \cap x^{\dagger})\}$. *Hence, in particular, each point of* X *is closed in* X^{Gold}; *and each subset of* X *which is stable under generalization is closed in* X^{Gold};

(*b*) *The identity map* $\text{id}_X \colon X^{\text{Gold}} \to X_C$ *is continuous*;

(*c*) X *is a* T_D*-space if and only if* $X^{\text{dis}} = X^{\text{Gold}}$;

(*d*) $X^{\text{Gold}} = X_C$ *if and only if* X *is Noetherian*;

(*e*) $X^{\text{Gold}} = X$ *if and only if* X *is a discrete space.*

Finally, with respect to morphisms, we collect and complete the results as follows. *If* X *and* Y *are spectral spaces and* $f \colon X \to Y$ *is a spectral map* (that is, the inverse image under f of each quasi-compact open subset is itself a quasi-compact open set), *then the following five statements are equivalent*:

(i) f *is homeomorphism*;

(ii) $f^L = f \colon X^L \to Y^L$ *is a homeomorphism*;

(iii) $f_{\text{ord}} = f \colon X_{\text{ord}} \to Y_{\text{ord}}$ *is an isomorphism*;

(iv) $f^R = f \colon X^R \to Y^R$ *is a homeomorphism*;

(v) $f^{\text{COP}} = f \colon X^{\text{COP}} \to Y^{\text{COP}}$ *is a homeomorphism.*

In view of Proposition 2.9, 3.1 and 4.5, a *proof* results from noticing that each continuous bijection from X_C to Y_C is a homeomorphism.

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