Available online at www.sciencedirect.com
SCIENCE DIRECT®
Journal of Algebra 274 (2004) 387–421
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

Star operations and pullbacks

Marco Fontana [a,*,1] and Mi Hee Park [b]

[a] *Dipartimento di Matematica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo, 1, 00146 Roma, Italy*
[b] *School of Mathematics, Korea Institute for Advanced Study, Seoul 130-722, South Korea*

Received 16 February 2003
Communicated by Paul Roberts

Abstract

In this paper we study the star operations on a pullback of integral domains. In particular, we characterize the star operations of a domain arising from a pullback of "a general type" by introducing new techniques for "projecting" and "lifting" star operations under surjective homomorphisms of integral domains. We study the transfer in a pullback (or with respect to a surjective homomorphism) of some relevant classes or distinguished properties of star operations such as $v-$, $t-$, $w-$, $b-$, $d-$, finite type, e.a.b., stable, and spectral operations. We apply part of the theory developed here to give a complete positive answer to a problem posed by D.F. Anderson in 1992 concerning the star operations on the "$D + M$" constructions.

1. Introduction and preliminary results

The theory of ideal systems and star operations was developed by W. Krull, H. Prüfer, and E. Noether around 1930, and is a powerful tool for characterizing several relevant classes of integral domains, for studying their mutual relations and for introducing the Kronecker function rings in a very general ring-theoretical setting. A modern treatment of various aspects of this theory can be found in the volumes by P. Jaffard [32], O. Zariski and P. Samuel [47, Appendix 4], R. Gilmer [26], M.D. Larsen and P.J. McCarthy [34], and F. Halter-Koch [28].

* Corresponding author.
E-mail addresses: fontana@mat.uniroma3.it (M. Fontana), mhpark@kias.re.kr (M.H. Park).
1 Supported in part by research grants MIUR 2001/2002 and 2003/2004.

0021-8693/$ – see front matter © 2004 Published by Elsevier Inc.
doi:10.1016/j.jalgebra.2003.10.002

Pullbacks were considered in [19] for providing an appropriate unified setting for several important "composite-type" constructions introduced in various contexts of commutative ring theory in order to construct examples and counter-examples with different pathologies: for instance, Seidenberg's constructions for (polynomial) dimensional sequences [43], Nagata's composition of valuation domains and "$K + J(R)$" constructions [39, p. 35 and Appendix A1, Example 2], Akiba's AV-domains or Dobbs' divided domains [1,16], Gilmer's "$D + M$" constructions [26], Traverso's glueings for a constructive approach to the seminormalization [44], Vasconcelos' umbrella rings and Greenberg's F-domains [27,45], Boisen–Sheldon's CPI-extensions [13], Hedstrom–Houston's pseudo-valuation domains [29], "$D + XD_S[X]$" rings and more generally, the "$A + XB[X]$" rings considered by many authors (see the recent excellent survey papers by T. Lucas [35] and M. Zafrullah [46], which contain ample and updated bibliographies on this subject).

It was natural at this stage of knowledge to investigate the behaviour of the star operations in a general pullback setting and with respect to surjective homomorphisms of integral domains, after various different results concerning distinguished star operations (like the v–, the t–, or the w– operation) and particular "composite-type" constructions were obtained by different authors (cf., for instance, [3–5,7,11,12,15,17,20,24,33,38,42], and the survey papers [10,25]).

The present work was stimulated by the papers by D.D. Anderson and D.F. Anderson on star operations, and more precisely, by the study initiated by D.F. Anderson concerning the star operations on the "$D + M$" constructions [9].

In Section 2, after introducing an operation of "glueing" of star operations in a pullback of integral domains, we will characterize the star operations of a domain arising from a pullback of "a general type." For this purpose we will introduce new techniques for "projecting" and "lifting" star operations under surjective homomorphisms of integral domains. Section 3 is devoted to the study of the transfer in a pullback (or with respect to a surjective homomorphism) of some relevant properties or classes of star operations such as v–, t–, w–, b–, d–, finite type, e.a.b., stable, and spectral operations.

We will apply part of the theory developed here to give a complete positive answer to a problem posed by D.F. Anderson in 1992 [9] concerning the star operations on the "$D + M$" constructions.

Let D be an integral domain with quotient field L. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D-submodules of L and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, i.e., all $E \in \overline{\boldsymbol{F}}(D)$ such that there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of L. Then obviously, $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

For each pair of nonzero fractional ideals E, F of D, we denote as usual by $(E :_L F)$ the fractional ideal of D given by $\{y \in L \mid yF \subseteq E\}$; in particular, for each nonzero fractional ideal I of D, we set $I^{-1} := (D :_L I)$.

We recall that a mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D),\ E \mapsto E^\star$, is called a *semistar operation on D* if the following properties hold for all $0 \neq x \in L$ and $E, F \in \overline{\boldsymbol{F}}(D)$ (cf. for instance [21,22,36,37,40,41]):

$(\star_1)$ $(xE)^\star = xE^\star$;
$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;
$(\star_3)$ $E \subseteq E^\star$ and $E^\star = (E^\star)^\star =: E^{\star\star}$.

Example 1.1.

(a) If $\star$ is a semistar operation on D such that $D^\star = D$, then the map (still denoted by) $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $E \mapsto E^\star$, is called a *star operation on D*. Recall [26, (32.1)] that a star operation $\star$ satisfies the properties $(\star_2)$, $(\star_3)$ for all $E, F \in \boldsymbol{F}(D)$; moreover, for each $0 \neq x \in L$ and $E \in \boldsymbol{F}(D)$, a star operation $\star$ satisfies the following:

$$(\star\star_1) \quad (xD)^\star = xD; \quad (xE)^\star = xE^\star.$$

A semistar operation on D such that $D \subsetneq D^\star$ is called a *proper semistar operation on D*.

(b) *The trivial semistar operation e_D on D* (simply denoted by e) is the semistar operation constant onto L, i.e., the semistar operation defined by $E^{e_D} := L$ for each $E \in \overline{\boldsymbol{F}}(D)$. Note that $\star$ is the trivial semistar operation on D if and only if $D^\star = L$.

(c) Another trivial semistar (in fact, star) operation is the *identity star operation d_D on D* (simply denoted by d) defined by $E^{d_D} := E$ for each $E \in \overline{\boldsymbol{F}}(D)$.

(d) For each $E \in \overline{\boldsymbol{F}}(D)$, set $E^{\star_f} := \bigcup\{F^\star \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}$. Then $\star_f$ is also a semistar operation on D, which is called *the semistar operation of finite type associated to $\star$*. Obviously, $F^\star = F^{\star_f}$ for each $F \in \boldsymbol{f}(D)$; moreover, if $\star$ is a star operation, then $\star_f$ is also a star operation. If $\star = \star_f$, then the semistar (respectively the star) operation $\star$ is called a *semistar* (respectively *star*) *operation of finite type* [22, Example 2.5(4)].

Note that, in general, $\star_f \leqslant \star$, i.e., $E^{\star_f} \subseteq E^\star$ for each $E \in \overline{\boldsymbol{F}}(D)$. Thus, in particular, if $E = E^\star$, then $E = E^{\star_f}$. Note also that $\star_f = (\star_f)_f$.

There are several examples of nontrivial semistar operations of finite type; the best known is probably the t-operation. Indeed, we start from the v_D *star operation* on an integral domain D (simply denoted by v), which is defined by

$$E^{v_D} := \left(E^{-1}\right)^{-1} = \left(D :_L (D :_L E)\right)$$

for any $E \in \boldsymbol{F}(D)$, and we set $t_D := (v_D)_f$ (or simply, $t = v_f$).

Other relevant examples of semistar operations of finite type will be constructed later.

A semistar operation $\star$ on D is called an *e.a.b.* (*endlich arithmetisch brauchbar*) (respectively *a.b.* (*arithmetisch brauchbar*)) *semistar operation* if

$$(EF)^\star \subseteq (EG)^\star \quad \Rightarrow \quad F^\star \subseteq G^\star$$

for each $E \in \boldsymbol{f}(D)$ and all $F, G \in \boldsymbol{f}(D)$ (respectively $F, G \in \overline{\boldsymbol{F}}(D)$) [22, Definition 2.3, Lemma 2.7].

If $\star$ is a star operation on D, then the definition of e.a.b. (respectively a.b.) operation is analogous (for an a.b. star operation, F, G are taken in $\boldsymbol{F}(D)$).

Example 1.2. Let $\iota : R \hookrightarrow T$ be an embedding of integral domains with the same field of quotients K and let $*$ be a semistar operation on R. Define $*_\iota : \overline{\boldsymbol{F}}(T) \to \overline{\boldsymbol{F}}(T)$ by setting

$$E^{*_\iota} := E^* \quad \text{for each } E \in \overline{\boldsymbol{F}}(T) \ \big(\subseteq \overline{\boldsymbol{F}}(R)\big).$$

Then we know [22, Proposition 2.8]:

(a) *If ι is not the identity map, then $*_\iota$ is a semistar, possibly non-star, operation on T, even if $*$ is a star operation on R.*
Note that when $*$ is a star operation on R and $(R :_K T) = (0)$, a fractional ideal E of T is not necessarily a fractional ideal of R, hence $*_\iota$ is not defined as a star operation on T.
(b) *If $*$ is of finite type on R, then $*_\iota$ is also of finite type on T.*
(c) *When $T := R^*$, then $*_\iota$ defines a star operation on T.*
(d) *If $*$ is e.a.b.* (*respectively a.b.*) *on R and if $T := R^*$, then $*_\iota$ is e.a.b.* (*respectively a.b.*) *on T.*

Conversely, let $\star$ be a semistar operation on the overring T of R. Define $\star^\iota : \overline{\boldsymbol{F}}(R) \to \overline{\boldsymbol{F}}(R)$ by setting

$$E^{\star^\iota} := (ET)^\star \quad \text{for each } E \in \overline{\boldsymbol{F}}(R).$$

Then we know [22, Proposition 2.9, Corollary 2.10]:

(e) *$\star^\iota$ is a semistar operation on R.*
(f) *If $\star := d_T$, then $(d_T)^\iota$ is a semistar operation of finite type on R*, which is denoted also by $\star_{\{T\}}$ (i.e., it is the semistar operation on R defined by $E^{\star_{\{T\}}} := ET$ for each $E \in \overline{\boldsymbol{F}}(R)$).
In particular, if $T = R$, then $\star_{\{R\}} = d_R$ and, if $T = K$, then $\star_{\{K\}} = e_R$. Note that if $R \subsetneq T$, then $\star_{\{T\}}$ is a proper semistar operation on R.
(g) *If $\star$ is e.a.b.* (*respectively a.b.*) *on T, then $\star^\iota$ is e.a.b.* (*respectively a.b.*) *on R.*
(h) *For each semistar operation $\star$ on T, we have $(\star^\iota)_\iota = \star$.*
(i) *For each semistar operation $*$ on R, we have $(*_\iota)^\iota \geqslant *$* (since $E^{(*_\iota)^\iota} = (ET)^{*_\iota} = (ET)^* \supseteq E^*$ for each $E \in \overline{\boldsymbol{F}}(R)$).

Other relevant classes of examples are recalled next.

Example 1.3. Let Δ be a nonempty set of prime ideals of an integral domain R with quotient field K. Set

$$E^{\star_\Delta} := \bigcap \{ER_P \mid P \in \Delta\} \quad \text{for each nonzero } R\text{-submodule } E \text{ of } K.$$

If Δ is the empty set, then we set $\star_\emptyset := e_R$. The mapping $E \mapsto E^{\star_\Delta}$, for each $E \in \overline{\boldsymbol{F}}(R)$, defines a semistar operation on R. Moreover [21, Lemma 4.1],

(a) *For each $E \in \overline{\boldsymbol{F}}(R)$ and for each $P \in \Delta$, $ER_P = E^{\star_\Delta} R_P$.*
(b) *The semistar operation $\star_\Delta$ is* stable (with respect to the finite intersections), *i.e., for all $E, F \in \overline{\boldsymbol{F}}(R)$ we have $(E \cap F)^{\star_\Delta} = E^{\star_\Delta} \cap F^{\star_\Delta}$.*
(c) *For each $P \in \Delta$, $P^{\star_\Delta} \cap R = P$.*
(d) *For each nonzero integral ideal I of R such that $I^{\star_\Delta} \cap R \neq R$, there exists a prime ideal $P \in \Delta$ such that $I \subseteq P$.*

A *semistar operation $*$ on R* is called *spectral* if there exists a subset Δ of $\mathrm{Spec}(R)$ such that $* = \star_\Delta$; in this case, we say that $*$ is *the spectral semistar operation associated with Δ.*

We say that $*$ is a *quasi-spectral semistar operation* (or that $*$ *possesses enough primes*) if, for each nonzero integral ideal I of R such that $I^* \cap R \neq R$, there exists a prime ideal P of R such that $I \subseteq P$ and $P^* \cap R = P$. For instance, it is easy to see that *if $*$ is a semistar operation of finite type, then $*$ is quasi-spectral.*

From (c) and (d), we deduce that *each spectral semistar operation is quasi-spectral.*

Given a semistar operation $*$ on R, assume that the set

$$\Pi^* := \left\{ P \in \mathrm{Spec}(R) \;\middle|\; P \neq 0 \text{ and } P^* \cap R \neq R \right\}$$

is nonempty. Then the spectral semistar operation of R defined by $*_{\mathrm{sp}} := \star_{\Pi^*}$ is called *the spectral semistar operation associated to $*$.* Note that if $*$ is quasi-spectral such that $R^* \neq K$, then Π^* is nonempty and $*_{\mathrm{sp}} \leqslant *$ [21, Proposition 4.8, Remark 4.9].

It is easy to see that $*$ *is spectral if and only if* $* = *_{\mathrm{sp}}$.

For each semistar operation $*$ on R, we can consider

$$\tilde{*} := (*_f)_{\mathrm{sp}}.$$

Then we know [21, Propositions 3.6(b), 4.23(1)]:

(e) *$\tilde{*}$ is a spectral semistar operation of finite type on R, and* if $\mathcal{M}(*_f)$ denotes the set of all the maximal elements in the set $\{I$ nonzero integral ideal of $R \mid I^{*_f} \cap R \neq R\}$, then

$$\tilde{*} = \star_{\mathcal{M}(*_f)}.$$

It is also known [21, p. 185] that for each $E \in \overline{\boldsymbol{F}}(R)$,

$$E^{\tilde{*}} = \bigcup \left\{ (E :_K F) \;\middle|\; F \in \boldsymbol{f}(R),\ F^* = R^* \right\}.$$

(f) *If $*$ is a star operation on R, then $\tilde{*}$ is a* (*spectral*) *star operation* (*of finite type*) *on R and $\tilde{*} \leqslant *$.*

If $* := v_R$, using the notation introduced by Wang Fanggui and R.L. McCasland [18], we will denote by w_R (or simply by w) the star operation $\widetilde{v_R} = (t_R)_{\mathrm{sp}}$ (cf. also [6,30]).

The construction of a spectral semistar operation associated to a set of prime ideal can be generalized as follows.

Example 1.4. Let $\mathcal{R} := \{R_\lambda \mid \lambda \in \Lambda\}$ be a nonempty family of overrings of R and define $\star_{\mathcal{R}} : \overline{\boldsymbol{F}}(R) \to \overline{\boldsymbol{F}}(R)$ by setting

$$E^{\star_{\mathcal{R}}} := \bigcap \{ER_\lambda \mid \lambda \in \Lambda\} \quad \text{for each } E \in \overline{\boldsymbol{F}}(D).$$

Then we know [22, Lemma 2.4(3), Example 2.5(6), Corollary 3.8]:

(a) *The operation $\star_{\mathcal{R}}$ is a semistar operation on R. Moreover, if $\mathcal{R} = \{R_P \mid P \in \Delta\}$, then $\star_{\mathcal{R}} = \star_\Delta$.*
(b) *$E^{\star_{\mathcal{R}}} R_\lambda = ER_\lambda$ for each $E \in \overline{\boldsymbol{F}}(R)$ and for each $\lambda \in \Lambda$.*
(c) *If $\mathcal{R} = \mathcal{W}$ is a family of valuation overrings of R, then $\star_{\mathcal{W}}$ is an a.b. semistar operation on D.*

We say that *two semistar operations on D*, $\star_1$ and $\star_2$, are *equivalent* if $(\star_1)_f = (\star_2)_f$. Then we know ([23, Proposition 3.4] and [26, Theorem 32.12]):

(d) *Each e.a.b. semistar (respectively star) operation on R is equivalent to a semistar (respectively star) operation of the type $\star_{\mathcal{W}}$ for some family $\mathcal{W}$ of valuation overrings of R (respectively for some family $\mathcal{W}$ of valuation overrings of R such that $R = \bigcap \{W \mid W \in \mathcal{W}\}$).*

If $\mathcal{W}$ is the family of *all* the valuation overrings of R, then $\star_{\mathcal{W}}$ is called *the b_R-semistar operation* (or simply *the b-semistar operation on R*). Moreover, if R is integrally closed, then $R^{b_R} = R$ [26, Theorem 19.8], and thus the operation b defines a star operation on R, which is called *the b-star operation* [26, p. 398].

Example 1.5. If $\{*_\lambda \mid \lambda \in \Lambda\}$ is a family of semistar (respectively star) operations on R, then $\bigwedge_\lambda \{*_\lambda \mid \lambda \in \Lambda\}$ (denoted simply by $\bigwedge *_\lambda$), defined by

$$E^{\bigwedge *_\lambda} := \bigcap \bigl\{E^{*_\lambda} \mid \lambda \in A\bigr\} \quad \text{for each } E \in \overline{\boldsymbol{F}}(R) \ \bigl(\text{respectively } E \in \boldsymbol{F}(R)\bigr),$$

is a semistar (respectively star) operation on R. This type of semistar operation generalizes the semistar (respectively star) operation of type $\star_{\mathcal{R}}$ (where $\mathcal{R} := \{R_\lambda \mid \lambda \in \Lambda\}$ is a nonempty family of overrings of R; Example 1.4), since

$$\star_{\mathcal{R}} = \bigwedge \star_{\{R_\lambda\}},$$

where $\star_{\{R_\lambda\}}$ is the semistar operation on R considered in Example 1.2(f).

Note the following observations:

(a) *If at least one of the semistar operations in the family* $\{*_\lambda \mid \lambda \in \Lambda\}$ *is a star operation on* R*, then* $\bigwedge *_\lambda$ *is still a star operation on* R.

(b) *Let* $\iota : R \hookrightarrow T$ *be an embedding of integral domains with the same field of quotients* K *and let* $\{*_\lambda \mid \lambda \in \Lambda\}$ *be a family of semistar operations on* R*. Then*

$$\Big(\bigwedge *_\lambda\Big)_\iota = \bigwedge (*_\lambda)_\iota.$$

(c) *Let* $\iota : R \hookrightarrow T$ *be an embedding of integral domains with the same field of quotients* K *and let* $\{\star_\lambda \mid \lambda \in \Lambda\}$ *be a family of semistar operations on* T*, then*

$$\Big(\bigwedge \star_\lambda\Big)^\iota = \bigwedge (\star_\lambda)^\iota.$$

2. Star operations and pullbacks

For the duration of this paper we will mainly consider the following situations:

(þ) *T represents an integral domain, M an ideal of T, k the factor ring T/M, D an integral domain subring of k and $\varphi : T \to T/M =: k$ the canonical projection. Set* $R := \varphi^{-1}(D) =: T \times_k D$ the pullback of D inside T with respect to φ, *hence R is an integral domain (subring of T). Let K denote the field of quotients of R.*

(þ$^+$) *Let L be the field of quotients of D. In the situation* **(þ)***, we assume, moreover, that $L \subseteq k$, and denote by* $S := \varphi^{-1}(L) =: T \times_k L$ the pullback of L inside T with respect to φ. *Then S is an integral domain with field of quotients equal to K. In this situation, M, which is a prime ideal in R, is a maximal ideal in S. Moreover, if $M \neq (0)$ and $D \subsetneq k$, then M is a divisorial ideal of R, actually, $M = (R : T)$.*

Let $\star_D$ (respectively $\star_T$) be a star operation on the integral domain D (respectively T). Our first goal is to define in a natural way a star operation on R, which we will denote by $\diamond$, associated to the given star operations on D and T. More precisely, if we denote by $\mathrm{Star}(A)$ the set of all the star operations on an integral domain A, then we want to define a map

$$\Phi : \mathrm{Star}(D) \times \mathrm{Star}(T) \to \mathrm{Star}(R), \quad (\star_D, \star_T) \mapsto \diamond.$$

For each nonzero fractional ideal I of R, set

$$I^\diamond := \bigcap \left\{ x^{-1}\varphi^{-1}\left(\left(\frac{xI + M}{M}\right)^{\star_D}\right) \,\middle|\, x \in I^{-1},\ x \neq 0 \right\} \cap (IT)^{\star_T},$$

where if $(xI + M)/M$ is the zero ideal of D (i.e., if $xI + M \subseteq M$), then we set

$$\varphi^{-1}\left(\left(\frac{xI + M}{M}\right)^{\star_D}\right) = M.$$

Proposition 2.1. *Keeping the notation and hypotheses introduced in* **(þ)**, *then* $\diamond$ *defines a star operation on the integral domain* R $(= T \times_k D)$.

Proof.

Claim 1. *For each nonzero fractional ideal* I *of* R, $I \subseteq I^{\diamond}$.

We have

$$\begin{aligned} I^{\diamond} &\supseteq \bigcap \left\{ x^{-1}\varphi^{-1}\left(\frac{xI + M}{M}\right) \,\Big|\, x \in I^{-1},\ x \neq 0 \right\} \cap IT \\ &= \bigcap \left\{ x^{-1}(xI + M) \mid x \in I^{-1},\ x \neq 0 \right\} \cap IT \\ &= \bigcap \left\{ I + x^{-1}M \mid x \in I^{-1},\ x \neq 0 \right\} \cap IT \supseteq I. \end{aligned}$$

Claim 2. *For each nonzero element* z *of* K, $(zR)^{\diamond} = zR$ (*in particular*, $R^{\diamond} = R$).

We have

$$\begin{aligned} (zR)^{\diamond} &= \bigcap \left\{ x^{-1}\varphi^{-1}\left(\left(\frac{xzR + M}{M}\right)^{\star_D}\right) \,\Big|\, x \in z^{-1}R,\ x \neq 0 \right\} \cap (zT)^{\star_T} \\ &\subseteq z\left(\varphi^{-1}\left(\left(\frac{R + M}{M}\right)^{\star_D}\right)\right) \cap zT \\ &= z\left(\varphi^{-1}\left(\frac{R}{M}\right)\right) \cap zT = zR \cap zT = zR. \end{aligned}$$

Therefore, by Claim 1, we deduce that $(zR)^{\diamond} = zR$.

Claim 3. *For each nonzero element* z *of* K *and for each nonzero fractional ideal* I *of* R, $(zI)^{\diamond} = zI^{\diamond}$.

Note that given $0 \neq z \in K$, for each nonzero $x \in I^{-1}$ there exists a unique $y \in (zI)^{-1}$ such that $x = yz$. Therefore, we have

$$\begin{aligned} I^{\diamond} &= \bigcap \left\{ x^{-1}\varphi^{-1}\left(\left(\frac{xI + M}{M}\right)^{\star_D}\right) \,\Big|\, x \in I^{-1},\ x \neq 0 \right\} \cap (IT)^{\star_T} \\ &= \bigcap \left\{ (yz)^{-1}\varphi^{-1}\left(\left(\frac{yzI + M}{M}\right)^{\star_D}\right) \,\Big|\, yz \in I^{-1},\ y \neq 0 \right\} \cap (IT)^{\star_T} \\ &= \bigcap \left\{ z^{-1}y^{-1}\varphi^{-1}\left(\left(\frac{yzI + M}{M}\right)^{\star_D}\right) \,\Big|\, y \in (zI)^{-1},\ y \neq 0 \right\} \cap (IT)^{\star_T} \\ &= z^{-1}\left(\bigcap \left\{ y^{-1}\varphi^{-1}\left(\left(\frac{yzI + M}{M}\right)^{\star_D}\right) \,\Big|\, y \in (zI)^{-1},\ y \neq 0 \right\} \cap (zIT)^{\star_T} \right) \\ &= z^{-1}(zI)^{\diamond}. \end{aligned}$$

Thus, we immediately conclude that $(zI)^{\diamond} = zI^{\diamond}$.

Claim 4. *For each pair of nonzero fractional ideals $I \subseteq J$ of R, $I^\diamond \subseteq J^\diamond$.*

Since $J^{-1} \subseteq I^{-1}$, we have

$$\begin{aligned} J^\diamond &= \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xJ+M}{M}\right)^{\star_D}\right) \,\middle|\, x \in J^{-1},\ x \neq 0\right\} \cap (JT)^{\star_T} \\ &\supseteq \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right) \,\middle|\, x \in J^{-1},\ x \neq 0\right\} \cap (IT)^{\star_T} \\ &\supseteq \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right) \,\middle|\, x \in I^{-1},\ x \neq 0\right\} \cap (IT)^{\star_T} = I^\diamond. \end{aligned}$$

Claim 5. *For each nonzero fractional ideal I of R, $I \subseteq I^\diamond \subseteq I^v$, and hence $(I^\diamond)^{-1} = I^{-1}$.*

Since $I^v = \bigcap\{zR \mid I \subseteq zR,\ z \in K\}$, by Claim 2, we deduce that

$$I \subseteq zR \quad \Rightarrow \quad I^\diamond \subseteq (zR)^\diamond = zR;$$

hence $I^\diamond \subseteq I^v$.

Claim 6. *For each nonzero fractional ideal I of R, $(I^\diamond)^\diamond = I^\diamond$.*

Since $(I^\diamond)^{-1} = I^{-1}$ for each nonzero ideal I of R, we have

$$\begin{aligned} \left(I^\diamond\right)^\diamond &= \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI^\diamond+M}{M}\right)^{\star_D}\right) \,\middle|\, x \in \left(I^\diamond\right)^{-1},\ x \neq 0\right\} \cap \left(I^\diamond T\right)^{\star_T} \\ &= \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI^\diamond+M}{M}\right)^{\star_D}\right) \,\middle|\, x \in I^{-1},\ x \neq 0\right\} \cap \left(I^\diamond T\right)^{\star_T}. \end{aligned}$$

Note that for $0 \neq x \in I^{-1}$ with $xI \subseteq M$, we have

- $xI^\diamond \subseteq M \quad \left(\text{and so} \quad x^{-1}\varphi^{-1}\left(\left(\frac{xI^\diamond+M}{M}\right)^{\star_D}\right) = x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right)\right),$

since

$$I^\diamond \subseteq x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right) = x^{-1}M.$$

Now for $0 \neq x \in I^{-1}$ with $xI \not\subseteq M$, we have

- $\left(\frac{xI^\diamond+M}{M}\right)^{\star_D} \subseteq \left(\frac{xI+M}{M}\right)^{\star_D} \quad \left(\text{and so} \quad \left(\frac{xI^\diamond+M}{M}\right)^{\star_D} = \left(\frac{xI+M}{M}\right)^{\star_D}\right),$

since

$$\begin{aligned}
&I^\diamond \subseteq x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right)\\
&\quad\Rightarrow\quad xI^\diamond \subseteq \varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right)\\
&\quad\Rightarrow\quad \frac{xI^\diamond+M}{M} = \varphi\big(xI^\diamond\big) \subseteq \varphi\left(\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right)\right) = \left(\frac{xI+M}{M}\right)^{\star_D}\\
&\quad\Rightarrow\quad \left(\frac{xI^\diamond+M}{M}\right)^{\star_D} \subseteq \left(\frac{xI+M}{M}\right)^{\star_D}.
\end{aligned}$$

Lastly,

- $\big(I^\diamond T\big)^{\star_T} \subseteq (IT)^{\star_T}$ $\quad\big($and so $\big(I^\diamond T\big)^{\star_T} = (IT)^{\star_T}\big)$,

since

$$I^\diamond \subseteq (IT)^{\star_T} \quad\Rightarrow\quad I^\diamond T \subseteq (IT)^{\star_T} \quad\Rightarrow\quad \big(I^\diamond T\big)^{\star_T} \subseteq (IT)^{\star_T}.$$

Therefore, we can easily conclude

$$\begin{aligned}
\big(I^\diamond\big)^\diamond &= \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI^\diamond+M}{M}\right)^{\star_D}\right) \,\middle|\, x\in I^{-1},\ x\neq 0\right\} \cap \big(I^\diamond T\big)^{\star_T}\\
&= \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right) \,\middle|\, x\in I^{-1},\ x\neq 0\right\} \cap (IT)^{\star_T} = I^\diamond.
\end{aligned}$$

The previous argument shows that $\diamond$ is a (well-defined) star operation on the integral domain R. $\square$

Remark 2.2.

(a) Note that in the proof of Proposition 2.1, M is possibly a nonmaximal ideal of T (and R), even though we assume that M $(= M\cap R)$ is a prime ideal of R.

(b) In the pullback setting **(þ)**, for each nonzero ideal I of R with $I\subseteq M$, $I^\diamond\subseteq M$, because

$$I^\diamond \subseteq x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star_D}\right)$$

for each $x\in I^{-1}\setminus(0)$ and $1\in R\subseteq I^{-1}$, thus

$$I^\diamond \subseteq \varphi^{-1}\left(\left(\frac{I+M}{M}\right)^{\star_D}\right) = M.$$

In particular, if $M\neq(0)$, then $M = M^\diamond$.

(c) If M is the ideal (0), then $T = k$ and $R = D$. In this extreme situation, we have $\diamond = \star_D \wedge (\star_T)^\iota$, where $\iota : R = D \hookrightarrow T = k$ is the canonical inclusion.
Note that it can happen that $\diamond = \star_D \wedge (\star_T)^\iota \lneq \star_D$. For instance, let R be a Krull domain of dimension $\geqslant 2$, P a prime ideal of R with $\mathrm{ht}(P) \geqslant 2$, $T := R_P$, and $M := (0)$ (hence, $R = D$ and $T = k$). Set $\star_D := v_D$ and $\star_T := d_T$. Then $P^{\star_D} = P^{v_D} = P^{v_R} = R$, but $P^\diamond = P^{v_D} \cap (PT)^{\star_T} = R \cap PR_P = P$.

(d) Let $M \neq (0)$. If $D = L = k$ (in particular, M must be a nonzero maximal ideal of T, and necessarily, $\star_D$ is the (unique) star operation d_D of $D = L = k$), then $R = T$. In this extreme situation, we have that $\diamond$ and $\star_T$ are two star operations on T (with $\diamond \leqslant \star_T$) that are possibly different. For instance, if T is an integral domain with a nonzero nondivisorial maximal ideal M (e.g., $T := k[X, Y]$, $M := (X, Y)$) and if $\star_T := v_T$, then $M^\diamond = M$ by (b), but $M^{\star_T} = M^{v_T} = T$.
If $D = k$, but $D \subsetneq L$, then it is not difficult to see that $\diamond = \star_T$ if and only if, for each nonzero ideal I of $R = T$ with $I \not\subseteq M$,

$$\frac{I^{\star_T} + M}{M} \subseteq \left(\frac{I + M}{M}\right)^{\star_D}.$$

Our next example will explicitly show the behaviour of the star operation $\diamond$ in some special cases of the pullback construction **(þ)**.

Example 2.3. *With the notation and hypotheses introduced in* **(þ)**, *assume, moreover, that T is local with nonzero maximal ideal M and $D = L$ is a proper subfield of k.* In this special case of the situation **(þ$^+$)**, $\star_D = d_D = e_D$ is the unique star operation on D. *Let I be a nonzero fractional ideal of R.*

(a) *If $II^{-1} = R$, then $I^\diamond = I = I^{v_R}$.*
(b) *If $II^{-1} \subsetneq R$, then $I^\diamond = I^{v_R} \cap (IT)^{\star_T}$. Moreover, if $(IT)^{\star_T} = x^{-1}T$ for some nonzero $x \in I^{-1}$, then $I^\diamond = I^{v_R}$ $(\subsetneq (IT)^{\star_T})$. If $(IT)^{\star_T} \neq x^{-1}T$ for all $x \in I^{-1}$, then $I^\diamond = (IT)^{\star_T}$.*
(c) *If $[k : L] > 2$ and if $T \subsetneq M^{-1} = (R :_K M)$, then $d_R \neq \diamond \neq v_R$ for all the star operations $\star_T$ on T.*
(d) *Let $[k : L] = 2$. If T is (local but) not a valuation domain, then $d_R \neq \diamond$ for all the star operations $\star_T$ on T. If $T = (R :_K M)$ and if $\star_T = v_T$, then $\diamond = v_R$.*

(a) is obvious, because I is invertible, hence I is divisorial (in fact, I is principal, since R is also local) and so $I = I^\diamond = I^{v_R} (\subseteq (IT)^{\star_T})$.

(b) Note that for each nonzero ideal I of R with the property that $II^{-1} \subsetneq R$, we have necessarily that $II^{-1} \subseteq M$. Moreover, for each nonzero $x \in I^{-1}$, from $xI \subseteq M$, we deduce that $I \subseteq x^{-1}M$ and so we have that $I^{v_R} = \bigcap\{x^{-1}M \mid x \in I^{-1},\ x \neq 0\}$. Therefore,

$$I^\diamond = \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI + M}{M}\right)^{\star_D}\right) \,\middle|\, x \in I^{-1},\ x \neq 0\right\} \cap (IT)^{\star_T}$$

$$= \bigcap \left\{ x^{-1}\varphi^{-1}\left(\left(\frac{M}{M}\right)^{\star_D}\right) \,\middle|\, x \in I^{-1},\ x \neq 0 \right\} \cap (IT)^{\star_T}$$
$$= \bigcap \left\{ x^{-1}M \mid x \in I^{-1},\ x \neq 0 \right\} \cap (IT)^{\star_T} = I^{v_R} \cap (IT)^{\star_T}.$$

In order to prove the second part of (b), note that in this case, for each $0 \neq x \in I^{-1}$, we have

$$I \subsetneq x^{-1}R \quad \Rightarrow \quad I \subseteq x^{-1}M \quad \Rightarrow \quad IT \subseteq x^{-1}MT = x^{-1}M$$
$$\Rightarrow \quad (IT)^{\star_T} \subseteq \left(x^{-1}M\right)^{\star_T} = x^{-1}M^{\star_T} \subseteq x^{-1}T.$$

Therefore, if $(IT)^{\star_T} = x^{-1}T$ for some nonzero $x \in I^{-1}$, then $I^{v_R} \subseteq x^{-1}R \subseteq x^{-1}T = (IT)^{\star_T}$. Thus, in this case, $I^{\diamond} = I^{v_R}$. Assume that $(IT)^{\star_T} \subsetneq x^{-1}T$ for all $x \in I^{-1}$. Then $(IT)^{\star_T} \subseteq x^{-1}M$ and thus

$$(IT)^{\star_T} \subseteq \bigcap \left\{ x^{-1}M \mid x \in I^{-1},\ x \neq 0 \right\} = I^{v_R},$$

hence $I^{\diamond} = (IT)^{\star_T}$.

(c) Let $0 \neq a \in M$, and let $z \in T \setminus R$. Set $I := (a, az)R$. Then obviously $IT = aT$ (since z is invertible in T), thus $(IT)^{\star_T} = aT = IT$.

Note that, in this case, $(IT)^{\star_T} = aT \subsetneq x^{-1}T$ for all $x \in I^{-1}$. As a matter of fact, if $aT = x^{-1}T$ for some $x \in I^{-1}$, then $ax = u$ is a unit in T and $ax \in R$ (because $a \in I$ and $x \in I^{-1}$). Hence, ax is a unit in R. Now we reach a contradiction, since we deduce that $I \subseteq x^{-1}R = aR \subseteq I$, i.e., $I = aR$.

By (b), we have that $I^{\diamond} = (IT)^{\star_T} = aT = IT \supsetneq I$, hence $\diamond \neq d_R$.

Assume also that $T \subsetneq M^{-1}$. Since $I^{\diamond} = (IT)^{\star_T} = aT$,

$$I^{v_R} = \left(I^{\diamond}\right)^{v_R} = (aT)^{v_R} = a\big(R :_K (R :_K T)\big) = a(R :_K M) \supsetneq aT = (IT)^{\star_T} = I^{\diamond}.$$

Therefore $\diamond \neq v_R$.

(d) In the present situation, we can find $a, b \in M$ such that $aT \nsubseteq bT$ and $bT \nsubseteq aT$. Set $I := (a, b)R$.

It is easy to see that I is not a principal ideal of R. (If $I = (a, b)R = cR$, then $a = cr_1, b = cr_2, c = as_1 + bs_2$ and so $1 = r_1s_1 + r_2s_2$ for some $r_1, s_1, r_2, s_2 \in R$; hence either r_1s_1 or $r_2s_2 = 1 - r_1s_1$ is a unit in the local ring R. For instance, if r_1s_1 is a unit in R, then r_1 is also a unit in R and so $cR = aR$. Thus $bR \subseteq aR$, contradicting the choice of a and b.)

Note that I is not a divisorial ideal of R. As a matter of fact, if $I = I^{v_R}$, then I should be also an ideal of T (i.e., $I = IT$) by [24, Corollary 2.10]. On the other hand, if $z \in T \setminus R$, then $az \in IT = I = (a, b)R$ and so $az = ar_1 + br_2$, i.e., $a(z - r_1) = br_2$ for some $r_1, r_2 \in R$. If $z - r_1 \in M$, then $z \in r_1 + M \subseteq R$, which contradicts the choice of z. If $z - r_1 \in T \setminus M$, then $a = br_2(z - r_1)^{-1} \in bT$, which contradicts the choice of a and b. Hence, $I \neq IT$ and so $I \neq I^{v_R}$.

If $(IT)^{\star_T} = x^{-1}T$ for some nonzero $x \in I^{-1}$, then (by (b)) $I^{\diamond} = I^{v_R} \neq I$, and so $d_R \neq \diamond$. Assume that $(IT)^{\star_T} \neq x^{-1}T$ for all $x \in I^{-1}$, then (by (b)) $I^{\diamond} = (IT)^{\star_T} \supseteq IT \supsetneq I$, and so $d_R \neq \diamond$.

Finally, suppose that $T = (R :_K M)$ and that $\star_T = v_T$. Let J be a nonzero fractional ideal of R. If J is divisorial, then obviously $J^{\diamond} = J = J^{v_R}$. Assume that J is not divisorial, then $JJ^{-1} \subsetneq R$. If $(JT)^{\star_T} = x^{-1}T$ for some nonzero $x \in J^{-1}$, then (by (b)) $J^{\diamond} = J^{v_R}$. If $(JT)^{v_T} \neq x^{-1}T$ for all $x \in J^{-1}$, then (by (b)) $J^{\diamond} = (JT)^{v_T}$. Since $T = (R :_K M) = (M :_K M)$, every divisorial ideal of T is divisorial as an ideal of R by [24, Corollary 2.9]. Therefore,

$$J^{v_R} = \left(J^{\diamond}\right)^{v_R} = \left((JT)^{v_T}\right)^{v_R} = (JT)^{v_T} = J^{\diamond},$$

hence we conclude that $\diamond = v_R$.

The previous construction of the star operation $\diamond$ on the integral domain R arising from a pullback diagram gives the idea for "lifting a star operation" with respect to a surjective ring homomorphism between two integral domains.

Corollary 2.4. *Let R be an integral domain with field of quotients K, M a prime ideal of R. Let D be the factor ring R/M and let $\varphi : R \to D$ be the canonical projection. Assume that $\star$ is a star operation on D. For each nonzero fractional ideal I of R, set*

$$\begin{aligned} I^{\star^{\varphi}} &:= \bigcap \left\{ x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star}\right) \;\middle|\; x \in I^{-1},\ x \neq 0 \right\} \\ &= \bigcap \left\{ x\varphi^{-1}\left(\left(\frac{x^{-1}I+M}{M}\right)^{\star}\right) \;\middle|\; x \in K,\ I \subseteq xR \right\}, \end{aligned}$$

where, as before, if $(zI+M)/M$ is the zero ideal of D, then we set

$$\varphi^{-1}\left(\left(\frac{zI+M}{M}\right)^{\star}\right) = M.$$

Then $\star^{\varphi}$ is a star operation on R.

Proof. *Mutatis mutandis* the arguments used in the proof of Proposition 2.1 show that $\star^{\varphi}$ is a star operation on R. $\square$

Using the notation introduced in Section 1, in particular, in Example 1.2, we immediately have the following corollary.

Corollary 2.5. *With the notation and hypotheses introduced in* **(þ)** *and Proposition* 2.1, *if we use the definition given in Corollary* 2.4, *we have*

$$\diamond = (\star_D)^{\varphi} \wedge (\star_T)^{\iota}.$$

We next examine the problem of "projecting a star operation" with respect to a surjective homomorphism of integral domains.

Proposition 2.6. *Let R, K, M, D, φ be as in Corollary 2.4 and let L be the field of quotients of D. Let $*$ be a given star operation on the integral domain R. For each nonzero fractional ideal F of D, set*

$$F^{*_\varphi} := \bigcap \bigl\{ y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F \subseteq yD \bigr\}.$$

Then $*_\varphi$ is a star operation on D.

Proof. The following claim is a straightforward consequence of the definition.

Claim 1. *For each nonzero fractional ideal F of D, $F \subseteq F^{*_\varphi}$.*

Claim 2. *For each nonzero $z \in L$, $(zD)^{*_\varphi} = zD$ (in particular, $D^{*_\varphi} = D$).*

Note that

$$\begin{aligned}(zD)^{*_\varphi} &= \bigcap \bigl\{ y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}zD\bigr)\bigr)^*\bigr) \bigm| y \in L,\ zD \subseteq yD \bigr\} \subseteq z\varphi\bigl(\bigl(\varphi^{-1}(D)\bigr)^*\bigr) \\ &= z\varphi(R^*) = z\varphi(R) = zD.\end{aligned}$$

The conclusion follows from Claim 1.

Claim 3. *For each nonzero fractional ideal F of D and for each nonzero $z \in L$, $(zF)^{*_\varphi} = zF^{*_\varphi}$.*

Given $0 \neq z \in L$, for each nonzero $y \in L$, set $w := yz \in L$. Then

$$\begin{aligned}F^{*_\varphi} &= \bigcap \bigl\{ y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F \subseteq yD \bigr\} \\ &= \bigcap \left\{ \frac{w}{z}\varphi\left(\left(\varphi^{-1}\left(\frac{z}{w}F\right)\right)^*\right) \Bigm| w \in L,\ F \subseteq \frac{w}{z}D \right\} \\ &= \bigcap \bigl\{ z^{-1}w\varphi\bigl(\bigl(\varphi^{-1}\bigl(w^{-1}zF\bigr)\bigr)^*\bigr) \bigm| w \in L,\ zF \subseteq wD \bigr\} \\ &= z^{-1}\Bigl(\bigcap \bigl\{ w\varphi\bigl(\bigl(\varphi^{-1}\bigl(w^{-1}zF\bigr)\bigr)^*\bigr) \bigm| w \in L,\ zF \subseteq wD \bigr\} \Bigr) \\ &= z^{-1}(zF)^{*_\varphi}.\end{aligned}$$

Hence, we conclude that $(zF)^{*_\varphi} = zF^{*_\varphi}$.

Claim 4. *For each pair of nonzero fractional ideals $F_1 \subseteq F_2$ of D, $(F_1)^{*_\varphi} \subseteq (F_2)^{*_\varphi}$.*

Note that if $y \in L$ and $F_2 \subseteq yD$, then obviously $F_1 \subseteq yD$, therefore

$$\begin{aligned}(F_2)^{*_\varphi} &= \bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F_2\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F_2 \subseteq yD\bigr\}\\ &\supseteq \bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F_1\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F_1 \subseteq yD\bigr\} = (F_1)^{*_\varphi}.\end{aligned}$$

Claim 5. *For each nonzero fractional ideal F of D, $(F^{*_\varphi})^{*_\varphi} = F^{*_\varphi}$.*

Note that from Claims 1, 2, and 4, if y is a nonzero element of L, we have

$$F \subseteq yD \quad\Longleftrightarrow\quad F^{*_\varphi} \subseteq (yD)^{*_\varphi} = yD,$$

therefore

$$\begin{aligned}\bigl(F^{*_\varphi}\bigr)^{*_\varphi} &= \bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F^{*_\varphi}\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F^{*_\varphi} \subseteq yD\bigr\}\\ &= \bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F^{*_\varphi}\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F \subseteq yD\bigr\}.\end{aligned}$$

On the other hand,

$$F \subseteq yD \quad\Rightarrow\quad F^{*_\varphi} \subseteq y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*\bigr) \quad\Rightarrow\quad y^{-1}F^{*_\varphi} \subseteq \varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*\bigr).$$

Therefore,

$$\varphi^{-1}\bigl(y^{-1}F^{*_\varphi}\bigr) \subseteq \varphi^{-1}\bigl(\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*\bigr)\bigr) = \bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*,$$

since

$$\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^* \supseteq \varphi^{-1}\bigl(y^{-1}F\bigr) \supseteq M = \mathrm{Ker}(\varphi).$$

Now, we can conclude:

$$\begin{aligned}\bigl(F^{*_\varphi}\bigr)^{*_\varphi} &= \bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F^{*_\varphi}\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F \subseteq yD\bigr\}\\ &\subseteq \bigcap\bigl\{y\varphi\bigl(\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*\bigr)^*\bigr) \bigm| y \in L,\ F \subseteq yD\bigr\}\\ &= \bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F \subseteq yD\bigr\} = F^{*_\varphi},\end{aligned}$$

and so, by Claim 1, $(F^{*_\varphi})^{*_\varphi} = F^{*_\varphi}$. □

In case of a pullback of type **($\mathfrak{p}^+$)** the definition of the star operation $*_\varphi$ given above is simplified as follows.

Proposition 2.7. *Let T, K, M, k, D, φ, L, S, and R be as in* ($\mathfrak{p}^+$). *Let $*$ be a given star operation on the integral domain R. For each nonzero fractional ideal F of D, we have*

$$F^{*_\varphi} = \varphi\big(\big(\varphi^{-1}(F)\big)^*\big) = \frac{(\varphi^{-1}(F))^*}{M}.$$

Proof. For the extreme cases $M = (0)$ or $D = k$, it trivially holds, so we may assume that $M \neq (0)$ and $D \subsetneq k$. We start by proving the following claim.

Claim. *Let I be a fractional ideal of R such that $M \subsetneq I \subseteq S = \varphi^{-1}(L)$ and let $s \in S \setminus M$. Then $(sI + M)^* = sI^* + M$.*

Choose $t \in S$ such that $st - 1 \in M$. Then $t(sI + M)^* = (tsI + tM)^* \subseteq (tsI + M)^* = (I + M)^* = I^*$. Therefore $st(sI + M)^* \subseteq sI^*$, so $st(sI + M)^* + M \subseteq sI^* + M \subseteq (sI + M)^*$. Put $m := st - 1$. Since $m(sI + M)^* = (msI + mM)^* \subseteq M^* = M$ (where the last equality follows from the fact that M is a divisorial ideal of R), we have $st(sI + M)^* + M = (1 + m)(sI + M)^* + M = (sI + M)^*$. Thus we can conclude that $(sI + M)^* = sI^* + M$.

Now, let F be a nonzero fractional ideal of D and let $I := \varphi^{-1}(F)$. For each element $y \in L$ such that $F \subseteq yD$, we can find $s_y, t_y \in S \setminus M$ such that $\varphi(s_y) = y$ and $\varphi(t_y) = y^{-1}$. Using the above claim, we have:

$$\begin{aligned}
F^{*_\varphi} &= \bigcap\big\{y\varphi\big(\big(\varphi^{-1}\big(y^{-1}F\big)\big)^*\big) \bigm| y \in L,\ F \subseteq yD\big\} \\
&= \bigcap\big\{y\varphi\big((t_yI + M)^*\big) \bigm| y \in L,\ F \subseteq yD\big\} \\
&= \bigcap\big\{y\varphi\big(t_yI^* + M\big) \bigm| y \in L,\ F \subseteq yD\big\} \\
&= \bigcap\big\{\varphi\big(s_y\big(t_yI^* + M\big)\big) \bigm| y \in L,\ F \subseteq yD\big\} \\
&= \bigcap\big\{\varphi\big(s_yt_yI^* + s_yM\big) \bigm| y \in L,\ F \subseteq yD\big\} \\
&= \bigcap\big\{\varphi\big(s_yt_yI^* + s_yM + M\big) \bigm| y \in L,\ F \subseteq yD\big\} \\
&= \bigcap\big\{\varphi\big(s_yt_yI^* + M\big) \bigm| y \in L,\ F \subseteq yD\big\} \\
&= \bigcap\big\{\varphi\big((s_yt_yI + M)^*\big) \bigm| y \in L,\ F \subseteq yD\big\} \\
&= \bigcap\big\{\varphi\big(I^*\big) \bigm| y \in L,\ F \subseteq yD\big\} = \varphi\big(I^*\big) = \frac{I^*}{M} = \frac{(\varphi^{-1}(F))^*}{M}. \qquad \square
\end{aligned}$$

Remark 2.8. As a consequence of Proposition 2.7 (and in the situation described in that statement) we have the following:

If I is a nonzero fractional ideal of R such that $I \subseteq S$ and $sI \subseteq R$ for some $s \in S \setminus M$, then $I^* \subseteq S$ for any star operation $*$ on R. As a matter of fact,

$$I^* \subseteq I^*S = I^*(M + sS) = I^*M + sI^*S \subseteq (IM)^* + (sI)^*S \subseteq M^* + S = M + S = S.$$

Proposition 2.9. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{p}^+$)**. *Let $\star$ be a given star operation on the integral domain D, let $* := \star^\varphi$ be the star operation on R associated to $\star$ (which is defined in Corollary 2.4) and let $*_\varphi$ $(= (\star^\varphi)_\varphi)$ be the star operation on D associated to $*$ (which is defined in Proposition 2.6). Then $\star = *_\varphi$ $(= (\star^\varphi)_\varphi)$.*

Proof. For each nonzero fractional ideal F of D and for each $y \in L$ such that $F \subseteq yD$, $J := y^{-1}F$ is a nonzero integral ideal of D. Set $I_y := \varphi^{-1}(J) = \varphi^{-1}(y^{-1}F)$ $(\subseteq R)$. Note that I_y is a nonzero ideal of R such that $M \subset I_y \subseteq R$, and so $\varphi(I_y) = I_y/M = J$ $(\subseteq D)$. Moreover, we have

$$\begin{aligned}
(I_y)^* &= \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI_y+M}{M}\right)^\star\right) \,\middle|\, x \in I_y^{-1},\ x \neq 0\right\}\\
&= \bigcap\left\{x\varphi^{-1}\left(\left(\frac{x^{-1}I_y+M}{M}\right)^\star\right) \,\middle|\, I_y \subseteq xR \subseteq K\right\}\\
&= \left(\bigcap\left\{x\varphi^{-1}\left(\left(\frac{x^{-1}I_y+M}{M}\right)^\star\right) \,\middle|\, I_y \subseteq xM,\ x \in K\right\}\right)\\
&\quad \cap \left(\bigcap\left\{x\varphi^{-1}\left(\left(\frac{x^{-1}I_y+M}{M}\right)^\star\right) \,\middle|\, I_y \subseteq xR \subseteq K,\ \text{but } I_y \nsubseteq xM\right\}\right)\\
&= \left(\bigcap\{xM \mid I_y \subseteq xM,\ x \in K\}\right)\\
&\quad \cap \left(\bigcap\left\{x\varphi^{-1}\left(\left(\frac{x^{-1}I_y+M}{M}\right)^\star\right) \,\middle|\, I_y \subseteq xR \subseteq K,\ \text{but } I_y \nsubseteq xM\right\}\right).
\end{aligned}$$

- For the first component of the previous intersection, note that since M is maximal in S and $M \subset I_y \subseteq R$, $I_yS = S$. On the other hand, $I_y \subseteq xM$, thus $\varphi^{-1}(D) = R \subseteq S = I_yS \subseteq xMS = xM$. Therefore, we have

$$\bigcap\{xM \mid I_y \subseteq xM \subseteq K\} \supseteq \varphi^{-1}(D) \supseteq \varphi^{-1}\big(\big(y^{-1}F\big)^\star\big).$$

- For the second component of the previous intersection, note that

$$\begin{aligned}
x^{-1}I_y \subseteq R \quad \text{and} \quad M \subset I_y \subseteq R \quad &\Rightarrow \quad x^{-1}I_yS \subseteq S \quad \text{and} \quad I_yS = S\\
&\Rightarrow \quad x^{-1} \in S.
\end{aligned}$$

  On the other hand, if $I_y \nsubseteq xM$ $(I_y \subseteq xR)$ and $x^{-1} \in S$, then $x^{-1} \in S \setminus M$, and so $\varphi(x^{-1}) \in \varphi(S \setminus M) = L \setminus \{0\}$. Note also that $(x^{-1}I_y + M)/M = \varphi(x^{-1})(I_y/M)$.

  Set

$$I'_y := \varphi^{-1}\big(\big(y^{-1}F\big)^\star\big) \quad \big(\supseteq \varphi^{-1}\big(y^{-1}F\big) =: I_y\big),$$

hence $I'_y/M = (y^{-1}F)^\star = (I_y/M)^\star$.

Then we have

$$
\begin{aligned}
&\bigcap\left\{x\varphi^{-1}\left(\left(\frac{x^{-1}I_y+M}{M}\right)^{\star}\right)\ \middle|\ I_y\subseteq xR\subseteq K,\ \text{but } I_y\nsubseteq xM\right\}\\
&\quad=\bigcap\left\{x\varphi^{-1}\left(\left(\varphi\bigl(x^{-1}\bigr)\frac{I_y}{M}\right)^{\star}\right)\ \middle|\ I_y\subseteq xR\subseteq K,\ \text{but } I_y\nsubseteq xM\right\}\\
&\quad=\bigcap\left\{x\varphi^{-1}\left(\varphi\bigl(x^{-1}\bigr)\left(\frac{I_y}{M}\right)^{\star}\right)\ \middle|\ I_y\subseteq xR\subseteq K,\ \text{but } I_y\nsubseteq xM\right\}\\
&\quad=\bigcap\left\{x\varphi^{-1}\left(\varphi\bigl(x^{-1}\bigr)\frac{I'_y}{M}\right)\ \middle|\ I_y\subseteq xR\subseteq K,\ \text{but } I_y\nsubseteq xM\right\}\\
&\quad=\bigcap\bigl\{x\bigl(x^{-1}I'_y+M\bigr)\ \big|\ I_y\subseteq xR\subseteq K,\ \text{but } I_y\nsubseteq xM\bigr\}\\
&\quad=\bigcap\bigl\{I'_y+xM\ \big|\ I_y\subseteq xR\subseteq K,\ \text{but } I_y\nsubseteq xM\bigr\}=I'_y=\varphi^{-1}\bigl(\bigl(y^{-1}F\bigr)^{\star}\bigr),
\end{aligned}
$$

since for $x=1$ we have $I_y\subseteq xR\subseteq K$ but $I_y\nsubseteq xM$.

Note that the first component of the intersection representing $(I_y)^*$ might not appear, but the second component necessarily appears, since at least for $x:=1$ we have that $I_y\subseteq xR\subseteq K$ but $I_y\nsubseteq xM$. Putting together the previous information about the two components of the intersection, we have

$$\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^{*}=(I_y)^{*}=\varphi^{-1}\bigl(\bigl(y^{-1}F\bigr)^{\star}\bigr).$$

Therefore we conclude that

$$
\begin{aligned}
F^{*_\varphi}&=\bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^{*}\bigr)\ \big|\ y\in L,\ F\subseteq yD\bigr\}\\
&=\bigcap\bigl\{y\varphi\bigl((I_y)^{*}\bigr)\ \big|\ y\in L,\ F\subseteq yD\bigr\}\\
&=\bigcap\bigl\{y\varphi\bigl(\varphi^{-1}\bigl(\bigl(y^{-1}F\bigr)^{\star}\bigr)\bigr)\ \big|\ y\in L,\ F\subseteq yD\bigr\}\\
&=\bigcap\bigl\{y\bigl(y^{-1}F\bigr)^{\star}\ \big|\ y\in L,\ F\subseteq yD\bigr\}\\
&=\bigcap\bigl\{yy^{-1}F^{\star}\ \big|\ y\in L,\ F\subseteq yD\bigr\}=F^{\star}. \qquad\square
\end{aligned}
$$

Remark 2.10. With the notation and hypotheses of Proposition 2.9, for each nonzero fractional ideal F of D, we have

$$F^{\star}=\varphi\bigl(\varphi^{-1}(F)^{\star^{\varphi}}\bigr).$$

As a matter of fact, by the previous proof and Proposition 2.7, we have that $F^{\star}=F^{*_\varphi}=\varphi^{-1}(F)^{\star^{\varphi}}/M$.

Corollary 2.11. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{p}^+$)**.

(a) *The map* $(-)_\varphi : \mathrm{Star}(R) \to \mathrm{Star}(D)$, $* \mapsto *_\varphi$, *is order-preserving and surjective.*
(b) *The map* $(-)^\varphi : \mathrm{Star}(D) \to \mathrm{Star}(R)$, $\star \mapsto \star^\varphi$, *is order-preserving and injective.*
(c) *Let* $\star$ *be a star operation on* D. *Then for each nonzero ideal* I *of* R *with* $M \subset I \subseteq R$,

$$I^{\star^\varphi} = \varphi^{-1}\big(\big(\varphi(I)\big)^\star\big).$$

Proof. (a) and (b) are straightforward consequences of the definitions and Proposition 2.9, since $(-)^\varphi$ is a right inverse of $(-)_\varphi$ (i.e., $(-)_\varphi \circ (-)^\varphi = \mathbf{1}_{\mathrm{Star}(D)}$).

(c) Let $* := \star^\varphi$. Then by Proposition 2.9, we know that $*_\varphi = \star$. Therefore, using Proposition 2.7, we have

$$\big(\varphi(I)\big)^\star = \big(\varphi(I)\big)^{*_\varphi} = \frac{(\varphi^{-1}(\varphi(I)))^*}{M} = \frac{I^*}{M} = \frac{I^{\star^\varphi}}{M},$$

and hence $\varphi^{-1}((\varphi(I))^\star) = I^{\star^\varphi}$. $\square$

The next result shows how the composition map

$$(-)^\varphi \circ (-)_\varphi : \mathrm{Star}(R) \to \mathrm{Star}(R)$$

compares with the identity map.

Theorem 2.12. *Let* T, K, M, k, D, φ, L, S, *and* R *be as in* **(þ$^+$)**. *Assume that* $D \subsetneq k$. *Then for each star operation* $*$ *on* R,

$$* \leqslant \big((*)_\varphi\big)^\varphi.$$

Proof. Let I be a nonzero integral ideal of R. For each nonzero $x \in I^{-1}$, if $xI \nsubseteq M$, then by Proposition 2.7,

$$\left(\frac{xI+M}{M}\right)^{*_\varphi} = \frac{(xI+M)^*}{M} \supseteq \frac{(xI)^*+M}{M}.$$

Now using the fact $M^* = M$ for $M \neq (0)$, we have

$$\begin{aligned}
I^{(*_\varphi)^\varphi} &= \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{*_\varphi}\right) \,\middle|\, x \in I^{-1},\ x \neq 0\right\} \\
&= \left(\bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{*_\varphi}\right) \,\middle|\, x \in I^{-1},\ x \neq 0,\ xI \nsubseteq M\right\}\right) \\
&\quad \cap \left(\bigcap\left\{x^{-1}M \,\middle|\, x \in I^{-1},\ x \neq 0,\ xI \subseteq M\right\}\right) \\
&\supseteq \left(\bigcap\left\{x^{-1}\varphi^{-1}\left(\frac{(xI)^*+M}{M}\right) \,\middle|\, x \in I^{-1},\ x \neq 0,\ xI \nsubseteq M\right\}\right) \\
&\quad \cap \left(\bigcap\left\{x^{-1}M^* \,\middle|\, x \in I^{-1},\ x \neq 0,\ I \subseteq x^{-1}M\right\}\right)
\end{aligned}$$

$$\supseteq \Big(\bigcap\big\{x^{-1}\big((xI)^* + M\big) \bigm| x \in I^{-1},\ x \neq 0,\ xI \not\subseteq M\big\}\Big) \cap I^*$$
$$\supseteq \Big(\bigcap\big\{x^{-1}\big((xI)^*\big) \bigm| x \in I^{-1},\ x \neq 0,\ xI \not\subseteq M\big\}\Big) \cap I^* = I^*. \qquad \square$$

In Section 3, we will show that in general $* \lneq ((*)_\varphi)^\varphi$. However, in some relevant cases, the inequality is, in fact, an equality:

Corollary 2.13. *Let T, K, M, k, D, φ, L, S, and R be as in Theorem* 2.12. *Then*

$$v_R = \big((v_R)_\varphi\big)^\varphi; \qquad (v_D)^\varphi = v_R; \qquad (v_R)_\varphi = v_D.$$

Proof. Use Proposition 2.9, Corollary 2.11(b), Theorem 2.12, and [26, Theorem 34.1(4)]. More precisely, note that $(v_R)_\varphi \leqslant v_D$, and so $v_R \leqslant ((v_R)_\varphi)^\varphi \leqslant (v_D)^\varphi \leqslant v_R$. On the other hand, if $(v_R)_\varphi \lneq v_D$, then $v_R = ((v_R)_\varphi)^\varphi \lneq (v_D)^\varphi$, which is a contradiction. $\square$

Our next goal is to apply the previous results for giving a componentwise description of the "pullback" star operation $\diamond$ considered in Proposition 2.1.

Proposition 2.14. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{b}^+$)**. *Assume that $M \neq (0)$ and $D \subsetneq k$. Let*

$$\Phi : \mathrm{Star}(D) \times \mathrm{Star}(T) \to \mathrm{Star}(R), \quad (\star_D, \star_T) \mapsto \diamond := (\star_D)^\varphi \wedge (\star_T)^\iota,$$

be the map considered in Proposition 2.1 *and Corollary* 2.5. *The following properties hold*:

(a) $\diamond_\varphi = \star_D$;
(b) $\diamond_\iota = (v_R)_\iota \wedge \star_T$ $(\in \mathrm{Star}(T))$;
(c) $\diamond = (\diamond_\varphi)^\varphi \wedge (\diamond_\iota)^\iota$.

Proof. (a) Without loss of generality, we only consider the case of integral ideals of D. Let J be a nonzero integral ideal of D and let $I := \varphi^{-1}(J)$. Since $M \subsetneq I \subseteq R$, we have $IS = S$, where $S := \varphi^{-1}(L)$, and so $IT = T$. Therefore, by Proposition 2.7 and Corollary 2.11(c),

$$J^{\diamond_\varphi} = \varphi\big(I^\diamond\big) = \varphi\big(I^{(\star_D)^\varphi} \cap I^{(\star_T)^\iota}\big) = \varphi\big(I^{(\star_D)^\varphi} \cap (IT)^{\star_T}\big) = \varphi\big(I^{(\star_D)^\varphi} \cap T\big) = \varphi\big(I^{(\star_D)^\varphi}\big)$$
$$= \varphi\big(\varphi^{-1}\big(J^{\star_D}\big)\big) = J^{\star_D}.$$

(b) Without loss of generality, we only consider the case of integral ideals of T. Let I be a nonzero ideal of T (in particular, I is a fractional ideal of R). Then for each $x \in I^{-1} = (R :_K I)$, we have $xIT = xI \subseteq R$, so $xI \subseteq (R :_K T) = M$. Therefore,

$$I^{(\star_D)^\varphi} = \bigcap\big\{x^{-1}\varphi^{-1}\big(\big(\varphi(xI)\big)^{\star_D}\big) \bigm| x \in I^{-1},\ x \neq 0\big\}$$
$$= \bigcap\big\{x^{-1}M \bigm| x \in I^{-1},\ x \neq 0\big\} = I^{v_R},$$

and so

$$I^{\diamond_\iota} = I^{\diamond} = I^{(\star_D)^\varphi} \cap I^{\star_T} = I^{v_R} \cap I^{\star_T} = I^{(v_R)_\iota} \cap I^{\star_T} = I^{(v_R)_\iota \wedge \star_T}.$$

Note that $I^{\diamond_\iota}$ $(\subseteq I^{v_R})$ is an ideal of R. Moreover, $I^{\diamond_\iota}$ is an ideal of T, because for each nonzero $x \in T$,

$$xI^{\diamond_\iota} = x\bigl(I^{v_R} \cap I^{\star_T}\bigr) = (xI)^{v_R} \cap (xI)^{\star_T} \subseteq I^{v_R} \cap I^{\star_T} = I^{\diamond_\iota}.$$

Finally, since $\star_T$ is a star operation on T, it is easy to check that $\diamond_\iota$ (restricted to $\boldsymbol{F}(T)$) belongs to $\mathrm{Star}(T)$.

(c) Since $\diamond \leqslant v_R \leqslant ((v_R)_\iota)^\iota$, (using also Example 1.5) we have that

$$\begin{aligned}\diamond &= (\star_D)^\varphi \wedge (\star_T)^\iota = (\star_D)^\varphi \wedge \bigl((v_R)_\iota\bigr)^\iota \wedge (\star_T)^\iota = (\star_D)^\varphi \wedge \bigl((v_R)_\iota \wedge \star_T\bigr)^\iota \\ &= (\diamond_\varphi)^\varphi \wedge (\diamond_\iota)^\iota. \qquad \square\end{aligned}$$

Example 2.15. *With the same notation and hypotheses of Proposition* 2.14*, we show that, in general,* $\diamond_\iota \neq \star_T$.

(1) Let $T := k[X,Y]_{(X,Y)}$ and let $M := (X,Y)T$. Then T is a 2-dimensional local UFD. Choose a subfield $D := L$ of k such that $[k:L] = 2$. In this situation we have that $T \subseteq (R:_K M) \subseteq (T:_K M)$, and $(T:_K M) = T$ because T is 2-dimensional local UFD (hence, Krull) with maximal ideal M. Therefore, $T = (R:_K M)$. By Example 2.3(d), if $\star_T := v_T$, then $\diamond = v_R$ and $M^{v_T} = T$. But $M^{\diamond_\iota} = M^{\diamond} = M^{v_R} = M \neq T = M^{v_T} = M^{\star_T}$.

(2) Note that $\diamond_\iota \neq \star_T$, even if $L = k$. It is sufficient to consider a slight modification of the previous example. Let D be any integral domain (not a field) with quotient field L. Let $T := L[X,Y]_{(X,Y)}$ and let $M := (X,Y)T$. Set $\bar{\diamond} := (v_D)^\varphi \wedge (v_T)^\iota$. Then

$$M^{\bar{\diamond}_\iota} = M^{\bar{\diamond}} = M^{(v_D)^\varphi} \cap M^{(v_T)^\iota} = M^{v_R} \cap M^{(v_T)^\iota} = M,$$

because $M^{v_R} = M$ and $M^{(v_T)^\iota} = (MT)^{v_T} = M^{v_T} = T$.

Remark 2.16.

(a) Note that, with the same notation and hypotheses of Proposition 2.14, *the map Φ is not one-to-one in general.*
This fact follows immediately from Example 2.15 and Proposition 2.14(b) and (c), since

$$(\star_D)^\varphi \wedge (\star_T)^\iota = \diamond = (\diamond_\varphi)^\varphi \wedge (\diamond_\iota)^\iota.$$

(b) In the same setting as above, *the map Φ is not onto in general.*
For instance, in the situation described in Example 2.3(d), we have that $d_R \notin \mathrm{Im}(\Phi)$. Another example, even in case $L = k$, is given next.

Example 2.17. *Let D be a* 1*-dimensional discrete valuation domain with quotient field L. Set $T := L[X^2, X^3]$, $M := X^2L[X] = XL[X] \cap T$, and $K := L(X)$. Let φ and R be as in* **($\mathfrak{p}^+$)**. *Then $v_R \notin \mathrm{Im}(\Phi)$.*

Note that, for each $\diamond \in \mathrm{Im}(\Phi)$, $\diamond \leqslant (v_D)^\varphi \wedge (v_T)^\iota \leqslant v_R$. In order to show that $v_R \notin \mathrm{Im}(\Phi)$, it suffices to prove that $(v_D)^\varphi \wedge (v_T)^\iota \neq v_R$. The fractional overring T of R is not a divisorial ideal of R, since

$$T^{v_R} = \big(R :_K (R :_K T)\big) = (R :_K M) \supseteq L[X] \supsetneq T.$$

Therefore,

$$T^{(v_D)^\varphi \wedge (v_T)^\iota} = T^{v_R \wedge (v_T)^\iota} = T^{v_R} \cap T^{(v_T)^\iota} = T^{v_R} \cap T^{v_T} = T^{v_R} \cap T = T \subsetneq T^{v_R}.$$

Theorem 2.18. *With the notation and hypotheses of Proposition* 2.14*, set*

$$\mathrm{Star}(T; v_R) := \big\{\star_T \in \mathrm{Star}(T) \bigm| \star_T \leqslant (v_R)_\iota\big\}.$$

Then

(a) $\mathrm{Star}(T; v_R) = \big\{\star_T \in \mathrm{Star}(T) \bigm| \big(v_R \wedge (\star_T)^\iota\big)_\iota = \star_T\big\} = \big\{*_\iota \bigm| * \in \mathrm{Star}(R)\big\} \cap \mathrm{Star}(T)$
$= \big\{*_\iota \bigm| * \in \mathrm{Star}(R)$ *and* $T^* = T\big\}$.

(b) *The restriction $\Phi' := \Phi|_{\mathrm{Star}(D)\times\mathrm{Star}(T;v_R)}$ is one-to-one.*

(c) $\mathrm{Im}(\Phi') = \mathrm{Star}\big(R; (\mathfrak{p}^+)\big) := \big\{* \in \mathrm{Star}(R) \bigm| T^* = T$ *and* $* = (*_\varphi)^\varphi \wedge (*_\iota)^\iota\big\}$.

Proof. (a) We start by proving the following claim.

Claim. *Let $\star_T \in \mathrm{Star}(T; v_R)$ and let $\star_D \in \mathrm{Star}(D)$ be any star operation on D. Set, as usual, $\diamond := (\star_D)^\varphi \wedge (\star_T)^\iota$. Then $\diamond_\iota = \star_T$.*

Note that, by Corollary 2.13,

$$\diamond = \Phi\big((\star_D, \star_T)\big) \leqslant \bar{\diamond} := \Phi\big((v_D, \star_T)\big) = (v_D)^\varphi \wedge (\star_T)^\iota = v_R \wedge (\star_T)^\iota \in \mathrm{Star}(R).$$

Hence, by using Theorem 2.14(b), Examples 1.2(h) and 1.5(b), we have

$$(v_R)_\iota \wedge \star_T = \diamond_\iota \leqslant \bar{\diamond}_\iota = \big(v_R \wedge (\star_T)^\iota\big)_\iota = (v_R)_\iota \wedge \big((\star_T)^\iota\big)_\iota = (v_R)_\iota \wedge \star_T,$$

thus $\diamond_\iota = \bar{\diamond}_\iota = \star_T$, because $\star_T \in \mathrm{Star}(T; v_R)$.

From the previous argument we also deduce that

$$\star_T \leqslant (v_R)_\iota \quad\Longleftrightarrow\quad \big(v_R \wedge (\star_T)^\iota\big)_\iota = \star_T.$$

Now, let $* \in \mathrm{Star}(R)$ be a star operation on R such that $*_\iota \in \mathrm{Star}(T)$. Then obviously $*_\iota \leqslant (v_R)_\iota$, whence $*_\iota \in \mathrm{Star}(T; v_R)$, and $T^* = T^{*_\iota} = T$.

If $* \in \mathrm{Star}(R)$ is such that $T^* = T$, then clearly we have $*_\iota \in \mathrm{Star}(T)$.

If $\star_T \in \mathrm{Star}(T; v_R)$, then by the claim, $\star_T = \bar{\diamond}_\iota$ with $\bar{\diamond} \in \mathrm{Star}(R)$, hence $\star_T \in \{*_\iota \mid * \in \mathrm{Star}(R)\} \cap \mathrm{Star}(T)$.

(b) is a straightforward consequence of the claim and of Proposition 2.14(a).

(c) follows from the claim and from Proposition 2.14(a) and (c). □

We next apply some of the theory developed above for answering a problem posed by D.F. Anderson in 1992.

Example 2.19 (*“$D + M$”-constructions*). Let T be an integral domain of the type $k + M$, where M is a maximal ideal of T and k is a subring of T canonically isomorphic to the field T/M, and let D be a subring of k with field of quotients L ($\subseteq k$). Set $R := D + M$. Note that R is a faithfully flat D-module.

Given a star operation $*$ on R, D.F. Anderson [8, p. 835] defined a star operation on D in the following way: for each nonzero fractional ideal F of D, set

$$F^{*_D} := (FR)^* \cap L.$$

Note that $FR = F + M$. From [8, Proposition 5.4(b)] it is known that for each nonzero fractional ideal F of D,

(1) $F^{*_D} + M = (F + M)^*$;
(2) $F^{*_D} = (F + M)^* \cap L = (F + M)^* \cap k$.

Claim. *If $\varphi : R \to D$ is the canonical projection and if $*_\varphi$ is the star operation defined in Proposition* 2.6, *then $*_D = *_\varphi$.*

In particular, by [9, Proposition 2(a), (c)], *we deduce that*

(a) $(d_R)_\varphi = d_D$, $(t_R)_\varphi = t_D$, $(v_R)_\varphi = v_D$, *and*
(b) $(*_f)_\varphi = (*_\varphi)_f$.

Note that if y is a nonzero element of the quotient field L of D, then y belongs to k, and thus, y is a unit in T and so $y^{-1}M = M$. Therefore, for each $y \in L$ such that $F \subseteq yD$, we have

$$\begin{aligned} y\varphi\big(\big(\varphi^{-1}\big(y^{-1}F\big)\big)^*\big) &= y\varphi\big(\big(y^{-1}F + M\big)^*\big) = y\varphi\big(\big(y^{-1}F + y^{-1}M\big)^*\big) \\ &= y\varphi\big(y^{-1}(F + M)^*\big) = y\varphi\big(y^{-1}\big(F^{*_D} + M\big)\big) \\ &= y\varphi\big(y^{-1}F^{*_D} + y^{-1}M\big) = y\varphi\big(y^{-1}F^{*_D} + M\big) \\ &= y\big(y^{-1}F^{*_D}\big) = F^{*_D}, \end{aligned}$$

hence (Proposition 2.6) $F^{*_\varphi} = F^{*_D}$.

By applying Proposition 2.9 and Corollary 2.11(a) to the particular case of $R = D + M$ (special case of **($\mathfrak{p}^+$)**), we know that the map

$$(-)_\varphi : \mathrm{Star}(D+M) \to \mathrm{Star}(D), \quad * \mapsto *_\varphi = *_D,$$

is surjective and order-preserving and it has the injective order-preserving map

$$(-)^\varphi : \mathrm{Star}(D) \to \mathrm{Star}(D+M), \quad \star \mapsto \star^\varphi,$$

as a right inverse. This fact gives a complete positive answer to a problem posed by D.F. Anderson (cf. [9, p. 226]).

3. Transfer of star properties

In this section we want to investigate the general problem of the transfer—in the pullback setting—of some relevant properties concerning the star operations involved. In particular, we pursue the work initiated by D.F. Anderson in [9] for the case of the "$D + M$"-constructions. We start by studying which of the properties (a) and (b) of Example 2.19 hold in a more general setting.

Proposition 3.1. *Let T, K, M, k, D, L, φ, and R be as in* **($\mathfrak{p}^+$)**.

(a) *Let $\mathcal{R} := \{R_\lambda \mid \lambda \in \Lambda\}$ be a family of overrings of R contained in T such that $\bigcap\{R_\lambda \mid \lambda \in \Lambda\} = R$, and let $\mathcal{D} := \{D_\lambda := \varphi(R_\lambda) \mid \lambda \in \Lambda\}$ be the corresponding family of subrings of k (with $\bigcap\{D_\lambda \mid \lambda \in \Lambda\} = D$), then*

$$(\star_\mathcal{R})_\varphi = \star_\mathcal{D}.$$

(b) *If $\mathcal{D} := \{D_\lambda \mid \lambda \in \Lambda\}$ is a family of overrings of D such that $\bigcap\{D_\lambda \mid \lambda \in \Lambda\} = D$ and if $\mathcal{R} := \{R_\lambda := \varphi^{-1}(D_\lambda) \mid \lambda \in \Lambda\}$ is the corresponding family of subrings of T (with $\bigcap\{R_\lambda \mid \lambda \in \Lambda\} = R$), then in general*

$$\star_\mathcal{R} \leqslant (\star_\mathcal{D})^\varphi.$$

Proof. (a) Note that in the present situation $\varphi^{-1}(D_\lambda) = R_\lambda$ for each $\lambda \in \Lambda$, $D = \bigcap\{D_\lambda \mid \lambda \in \Lambda\}$, and for each nonzero fractional ideal J of D, $J^{(\star_\mathcal{R})_\varphi} = \varphi((\varphi^{-1}(J))^{\star_\mathcal{R}})$ (Proposition 2.7). Moreover,

$$\begin{aligned}\varphi\big((\varphi^{-1}(J))^{\star_\mathcal{R}}\big) &= \varphi\Big(\bigcap\big\{\varphi^{-1}(J)R_\lambda \bigm| \lambda \in \Lambda\big\}\Big) = \varphi\Big(\bigcap\big\{\varphi^{-1}(J)\varphi^{-1}(D_\lambda) \bigm| \lambda \in \Lambda\big\}\Big)\\ &= \varphi\Big(\varphi^{-1}\Big(\bigcap\{JD_\lambda \mid \lambda \in \Lambda\}\Big)\Big) = \varphi\big(\varphi^{-1}\big(J^{\star_\mathcal{D}}\big)\big) = J^{\star_\mathcal{D}}.\end{aligned}$$

(b) Note that $\varphi(R_\lambda) = \varphi(\varphi^{-1}(D_\lambda)) = D_\lambda$ for each $\lambda \in \Lambda$. Therefore, by (a), $(\star_\mathcal{R})_\varphi = \star_\mathcal{D}$, thus $((\star_\mathcal{R})_\varphi)^\varphi = (\star_\mathcal{D})^\varphi$. If $D = k$, then $D = L$ is a field, thus $\mathcal{D} = \{D\}$ and

$\mathcal{R} = \{R\}$. So, obviously, $\star_{\mathcal{R}} = d_R \leqslant (\star_{\mathcal{D}})^{\varphi}$. If $D \subsetneq k$, then the conclusion follows from Theorem 2.12. $\square$

Proposition 3.1(a) can be generalized to a statement concerning a surjective homomorphism between two integral domains:

Proposition 3.2. *Let R, K, M, D, φ be as in Corollary* 2.4. *Let $\{*_\lambda \mid \lambda \in \Lambda\}$ be a family of star operations of R. Then*

$$\Bigl(\bigwedge *_\lambda\Bigr)_{\varphi} = \bigwedge (*_\lambda)_{\varphi}.$$

Proof. Let J be a nonzero fractional ideal of D and let y be in the quotient field L of D. Then

$$\begin{aligned} J^{\wedge(*_\lambda)_\varphi} &= \bigcap\bigl\{J^{(*_\lambda)_\varphi} \bigm| \lambda \in \Lambda\bigr\} \\ &= \bigcap\Bigl\{\Bigl(\bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}J\bigr)\bigr)^{*_\lambda}\bigr) \bigm| J \subseteq yD\bigr\}\Bigr) \Bigm| \lambda \in \Lambda\Bigr\} \\ &= \bigcap\Bigl\{y\Bigl(\bigcap\bigl\{\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}J\bigr)\bigr)^{*_\lambda}\bigr) \bigm| \lambda \in \Lambda\bigr\}\Bigr) \Bigm| J \subseteq yD\Bigr\} \\ &= \bigcap\Bigl\{y\varphi\Bigl(\bigcap\bigl\{\bigl(\varphi^{-1}\bigl(y^{-1}J\bigr)\bigr)^{*_\lambda} \bigm| \lambda \in \Lambda\bigr\}\Bigr) \Bigm| J \subseteq yD\Bigr\} \\ &= \bigcap\bigl\{y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}J\bigr)\bigr)^{\wedge *_\lambda}\bigr) \bigm| J \subseteq yD\bigr\} = J^{(\wedge *_\lambda)_\varphi}. \qquad \square \end{aligned}$$

Proposition 3.3. *Let R, K, M, D, φ be as in Corollary* 2.4. *Then*

$$(d_R)_\varphi = d_D.$$

Proof. For each nonzero fractional ideal J of D, we have

$$\begin{aligned} J^{(d_R)_\varphi} &= \bigcap\bigl\{y^{-1}\varphi\bigl(\bigl(\varphi^{-1}(yJ)\bigr)^{d_R}\bigr) \bigm| y \in J^{-1},\ y \neq 0\bigr\} \\ &= \bigcap\bigl\{y^{-1}\varphi\bigl(\bigl(\varphi^{-1}(yJ)\bigr)\bigr) \bigm| y \in J^{-1},\ y \neq 0\bigr\} \\ &= \bigcap\bigl\{y^{-1}(yJ) \bigm| y \in J^{-1},\ y \neq 0\bigr\} = J = J^{d_D}. \qquad \square \end{aligned}$$

The next couple of examples explicitly show that the inequalities in Theorem 2.12 and Proposition 3.1(b) can be strict inequalities (i.e., $* \lneq ((*)_\varphi)^\varphi$ and $\star_{\mathcal{R}} \lneq (\star_{\mathcal{D}})^\varphi$).

Example 3.4. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{p}^+$)**. *Assume, moreover, that T is local with nonzero maximal ideal M, $D = L$ is a proper subfield of k, and that $T \subsetneq M^{-1} = (R : M)$. In this situation,*

$$d_R \lneq (d_D)^\varphi = \bigl((d_R)_\varphi\bigr)^\varphi.$$

With the notation of Proposition 3.1(b), take $\mathcal{D} := \{D\}$, hence $\mathcal{R} = \{R\}$, thus $\star_{\mathcal{D}} = d_D = v_D$ and $\star_{\mathcal{R}} = d_R$. In this situation, by Corollary 2.13, $(\star_{\mathcal{D}})^{\varphi} = v_R$. Therefore, by Proposition 3.3 and Example 2.3(c) and (d),

$$(\star_{\mathcal{D}})^{\varphi} = (d_D)^{\varphi} = \big((d_R)_{\varphi}\big)^{\varphi} = v_R \gneq d_R = \star_{\mathcal{R}}.$$

Note that it is possible to give an example in which $* \lneq ((*)_{\varphi})^{\varphi}$ and $d_R \lneq (d_D)^{\varphi}$, even in the case that $D \subsetneq L = k$:

Example 3.5. *Let D be a 1-dimensional discrete valuation domain with quotient field L. Set $T := L[X^2, X^3]$, $M := (X^2, X^3)L[X] = XL[X] \cap T$, and $K := L(X)$. Let φ and R be as in* **($\mathfrak{h}^+$)** *with $L = k$. Then, $d_R \lneq v_R = ((d_R)_{\varphi})^{\varphi}$.*

Since $(d_R)_{\varphi} = d_D = v_D$ and $(v_D)^{\varphi} = v_R$ (Corollary 2.13), we have $((d_R)_{\varphi})^{\varphi} = v_R$. Now consider, for instance, the fractional ideal T of R. We know, from Example 2.17, that T is not a divisorial ideal of R, i.e., $T^{d_R} = T \lneq T^{v_R}$. Thus we have $d_R \lneq v_R = ((d_R)_{\varphi})^{\varphi}$.

The next goal is to show that $(t_R)_{\varphi} = t_D$ (but, in general, $t_R \lneq (t_D)^{\varphi} = ((t_R)_{\varphi})^{\varphi}$). We start with a more general result concerning the preservation of the "finite type" property.

Proposition 3.6. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{h}^+$)**. *Let $*$ be a given star operation on the integral domain R.*

(a) *If $*$ is a star operation of finite type on R, then $*_{\varphi}$ is a star operation of finite type on D.*
(b) *If $*$ is any star operation on R, then $(*_f)_{\varphi} = (*_{\varphi})_f$.*

Proof. (a) To prove the statement we will use the following facts:

(1) For each integral ideal I of R such that $M \subset I$,

$$\left(\frac{I}{M}\right)^{*_{\varphi}} = \big(\varphi(I)\big)^{*_{\varphi}} = \varphi\big(I^*\big) = \frac{I^*}{M} \quad \text{(Proposition 2.7)}.$$

(2) For each nonzero ideal I of R, $(I + M)^* \supseteq I^* + M$.
(3) For each nonzero ideal J of D and for each $y \in L$ with $J \subseteq yD$, if F_y is a finitely generated ideal of R such that $F_y \subseteq I_y := \varphi^{-1}(y^{-1}J)$, then $y\varphi(F_y)$ is a finitely generated ideal of D with $y\varphi(F_y) \subseteq J$.

For each nonzero ideal J of D, we have

$$\begin{aligned} J^{*_{\varphi}} &= \bigcap\big\{y\varphi\big(\big(\varphi^{-1}\big(y^{-1}J\big)\big)^*\big) \bigm| y \in L,\ J \subseteq yD\big\} \\ &= \bigcap\big\{y\varphi\big(I_y^*\big) \bigm| y \in L,\ J \subseteq yD\big\} \\ &= \bigcap\Big\{y\varphi\Big(\bigcup\big\{F_y^* \bigm| F_y \subseteq I_y,\ F_y \in \boldsymbol{f}(R)\big\}\Big) \Bigm| y \in L,\ J \subseteq yD\Big\} \end{aligned}$$

$$
\begin{aligned}
&= \bigcap\Big\{\bigcup\big\{y\varphi\big(F_y^*\big) \;\big|\; F_y \subseteq I_y,\ F_y \in \boldsymbol{f}(R)\big\} \;\Big|\; y \in L,\ J \subseteq yD\Big\}\\
&= \bigcap\Big\{\bigcup\Big\{y\frac{F_y^* + M}{M} \;\Big|\; F_y \subseteq I_y,\ F_y \in \boldsymbol{f}(R)\Big\} \;\Big|\; y \in L,\ J \subseteq yD\Big\}\\
&\subseteq \bigcap\Big\{\bigcup\Big\{y\frac{(F_y + M)^*}{M} \;\Big|\; F_y \subseteq I_y,\ F_y \in \boldsymbol{f}(R)\Big\} \;\Big|\; y \in L,\ J \subseteq yD\Big\}\\
&= \bigcap\Big\{\bigcup\Big\{y\Big(\frac{F_y + M}{M}\Big)^{*_\varphi} \;\Big|\; F_y \subseteq I_y,\ F_y \in \boldsymbol{f}(R)\Big\} \;\Big|\; y \in L,\ J \subseteq yD\Big\}\\
&= \bigcap\Big\{\bigcup\big\{y\big(\varphi(F_y)\big)^{*_\varphi} \;\big|\; F_y \subseteq I_y,\ F_y \in \boldsymbol{f}(R)\big\} \;\Big|\; y \in L,\ J \subseteq yD\Big\}\\
&= \bigcap\Big\{\bigcup\big\{\big(y\varphi(F_y)\big)^{*_\varphi} \;\big|\; F_y \subseteq I_y,\ F_y \in \boldsymbol{f}(R)\big\} \;\Big|\; y \in L,\ J \subseteq yD\Big\}\\
&\subseteq \bigcap\Big\{\bigcup\big\{G^{*_\varphi} \;\big|\; G \subseteq J,\ G \in \boldsymbol{f}(D)\big\} \;\Big|\; y \in L,\ J \subseteq yD\Big\}\\
&= \bigcup\big\{G^{*_\varphi} \;\big|\; G \subseteq J,\ G \in \boldsymbol{f}(D)\big\} \subseteq J^{*_\varphi},
\end{aligned}
$$

where we may assume each $F_y \not\subseteq M$ so that we can use Fact (1).

Thus, $J^{*_\varphi} = \bigcup\{G^{*_\varphi} \mid G \subseteq J, G \in \boldsymbol{f}(D)\}$.

(b) Since both $(*_f)_\varphi$ and $(*_\varphi)_f$ are star operations of finite type on D by (a), it suffices to show that for each nonzero finitely generated ideal J of D, $J^{(*_f)_\varphi} = J^{(*_\varphi)_f}$. Recall that if J is a nonzero finitely generated ideal of D, then $\varphi^{-1}(J)$ is a finitely generated ideal of R [20, Corollary 1.7]. Therefore,

$$
\begin{aligned}
J^{(*_\varphi)_f} = J^{*_\varphi} &= \big\{y\varphi\big(\big(\varphi^{-1}\big(y^{-1}J\big)\big)^{*}\big) \;\big|\; y \in L,\ J \subseteq yD\big\}\\
&= \big\{y\varphi\big(\big(\varphi^{-1}\big(y^{-1}J\big)\big)^{*_f}\big) \;\big|\; y \in L,\ J \subseteq yD\big\}\\
&= J^{(*_f)_\varphi}. \qquad \square
\end{aligned}
$$

Proposition 3.7. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{b}^+$)**. *Then*

$$(t_R)_\varphi = t_D.$$

Proof. Easy consequence of Corollary 2.13 and Proposition 3.6(b). $\square$

Remark 3.8. *In the same situation of Example* 3.5*, choosing D to be a Dedekind domain with infinitely many prime ideals, we have*

$$t_R \lneq (t_D)^\varphi = \big((t_R)_\varphi\big)^\varphi.$$

Using Proposition 3.7, we have $(t_D)^\varphi = ((t_R)_\varphi)^\varphi$. We claim that, in the present situation, the set of the maximal t_R-ideals of R coincides with $\mathrm{Max}(R)$.

Note first that since $\dim(T) = 1$, the contraction to R of each nonzero prime ideal of T has height 1 [19, Theorem 1.4], so it is a t_R-prime of R [32, Corollaire 3, p. 31].

Let $Q \in \mathrm{Max}(R)$. If $Q \not\supseteq M$, then Q is the contraction of a prime ideal of T, so Q is a t_R-prime. If $Q \supseteq M$, then $Q/M = (Q/M)^{v_D} = Q^{v_R}/M$ by Proposition 2.7, and hence we have $Q^{v_R} = Q$. Therefore, in this case also, Q is a t_R-prime.

Note that M is a divisorial prime ideal in R, hence in particular M is a prime t_R-ideal and it is contained in infinitely many maximal (t_R-)ideals, therefore R is not a TV-domain, i.e., $t_R \neq v_R$ [31, Theorem 1.3, Remark 2.5]. Since $((d_R)_\varphi)^\varphi = (d_D)^\varphi = (v_D)^\varphi = v_R$, automatically we have $((t_R)_\varphi)^\varphi = (t_D)^\varphi = v_R$. Thus, in this example, we have $t_R \lneq (t_D)^\varphi$.

Note also that this example shows that if $\star$ is a star operation of finite type on D, then $\star^\varphi$ is a star operation on R, which is not necessarily of finite type (e.g., take $\star := t_D = (t_R)_\varphi$).

In the pullback setting that we are considering, it is also natural to ask about the transfer of the property of being a "stable" star operation.

Proposition 3.9. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{b}^+$)** *and let $*$ be a star operation on R. Then*

$$\tilde{*}_\varphi = \widetilde{(*_\varphi)}.$$

Proof. If $D = k$, then since $D = L$ is a field, obviously we have $\tilde{*}_\varphi = \widetilde{(*_\varphi)}$. Assume that $D \subsetneq k$.

Let J be a nonzero integral ideal of D and let $I := \varphi^{-1}(J)$. We first show that $J^{\tilde{*}_\varphi} \subseteq J^{\widetilde{(*_\varphi)}}$. By Proposition 2.7, $J^{\tilde{*}_\varphi} = I^{\tilde{*}}/M$. Moreover, recall that

$$J^{\widetilde{(*_\varphi)}} = \left\{ y \in D \;\middle|\; yJ_1 \subseteq J \text{ for some finitely generated ideal } J_1 \text{ of } D \text{ such that } J_1^{*_\varphi} = D \right\}$$

(respectively, $I^{\tilde{*}} = \{x \in R \mid xI_1 \subseteq I$ for some finitely generated ideal I_1 of R such that $I_1^* = R\}$). Let $y \in J^{\tilde{*}_\varphi}$. Then $y = \varphi(x)$ for some $x \in I^{\tilde{*}}$. So $xI_1 \subseteq I$ for some finitely generated ideal I_1 of R such that $I_1^* = R$. Set $J_1 := \varphi(I_1) = (I_1 + M)/M$. Then J_1 is nonzero finitely generated, and by Proposition 2.7, $J_1^{*_\varphi} = (I_1 + M)^*/M = R/M = D$. Since $xI_1 \subseteq I$, $yJ_1 = \varphi(xI_1) \subseteq \varphi(I) = J$, and hence $y \in J^{\widetilde{(*_\varphi)}}$.

Conversely, let J be a nonzero integral ideal of D. If $y \in J^{\widetilde{(*_\varphi)}} = J^{\widetilde{(*_\varphi)_f}} = J^{\widetilde{(*_f)_\varphi}}$ (Proposition 3.6(b)), then $yJ_1 \subseteq J$ for some finitely generated ideal J_1 such that $J_1^{(*_f)_\varphi} = D$. Set $I_1 := \varphi^{-1}(J_1)$. Since $J_1^{(*_f)_\varphi} = I_1^{*_f}/M = D$ (Proposition 2.7), $I_1^{*_f} = R$. Therefore, there exists a finitely generated subideal I_0 of I_1 such that $I_0^* = R$. Write $y := \varphi(x)$ for some $x \in R$. Since $xI_0 \subseteq xI_1 \subseteq I := \varphi^{-1}(J)$, $x \in I^{\tilde{*}}$, and hence (using Proposition 2.7 again) $y \in I^{\tilde{*}}/M = J^{(\tilde{*})_\varphi}$. $\square$

Corollary 3.10. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{b}^+$)**. *Then*

$$(w_R)_\varphi = w_D.$$

Proof. Recall that $w_R = \widetilde{v_R}$ and $w_D = \widetilde{v_D}$. The conclusion follows from Proposition 3.9. $\square$

Remark 3.11. *The example considered in Remark* 3.8 *shows that we can have* $w_R \lneq ((w_R)_\varphi)^\varphi = (w_D)^\varphi$.

Since $\mathrm{Max}(R) = \mathcal{M}(t_R)$ (= the set of the maximal t_R-ideals, according to the notation in Example 1.3(e)), $w_R = \star_{\mathcal{M}(t_R)} = d_R$. In particular, $T^{w_R} = T$. On the other hand, we know that $((d_R)_\varphi)^\varphi = (d_D)^\varphi = (v_D)^\varphi = v_R$. Thus we have $((w_R)_\varphi)^\varphi = (w_D)^\varphi = v_R$. As we have already noticed (Example 3.5), T is not a divisorial ideal of R, i.e., $T^{v_R} \supsetneq T = T^{w_R}$. Thus, in this case, we have $w_R \lneq (w_D)^\varphi$.

Since the stable star operation $\tilde{*}$ is a particular type of spectral star operation, the next goal is a possible extension of Proposition 3.9 to the case of spectral star operations. We start with the following lemma.

Lemma 3.12. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{p}^+$)**. *Assume that $D \subsetneq k$.*

(a) *Let P be a prime ideal of R containing M. Set $Q := \varphi(P)$ and $R_{(P,\varphi)} := \varphi^{-1}(D_Q)$. Then $R_{(P,\varphi)} = R_P \cap T$.*

(b) *Let $\Delta(\neq \emptyset) \subseteq \mathrm{Spec}(R)$ and assume that $* := \star_\Delta \in \mathrm{Star}(R)$. Set $\Delta_1 := \{P \in \Delta \mid P \supseteq M\}$. For each nonzero integral ideal I of R containing M, we have*

$$I^* = \bigcap \{IR_{(P,\varphi)} \mid P \in \Delta_1\}.$$

(*Note that $\Delta_1 \neq \emptyset$.*)

Proof. (a) is straightforward.

(b) If $M = (0)$, then $\Delta = \Delta_1$ and $R_{(P,\varphi)} = R_P$, so it trivially holds. Assume that $M \neq (0)$. Let I be an integral ideal of R containing M. Recall that for each $P \in \Delta \setminus \Delta_1$, there exists a unique $P' \in \mathrm{Spec}(T)$ such that $P' \cap R = P$ and $R_P = T_{P'}$ [19, Theorem 1.4], hence in particular $\Delta_1 \neq \emptyset$ (otherwise $\star_\Delta$ would not be a star operation on R). We have

$$\begin{aligned} I^* &= \bigcap\{IR_P \mid P \in \Delta\} = \Bigl(\bigcap\{IR_P \mid P \in \Delta_1\}\Bigr) \cap \Bigl(\bigcap\{IR_P \mid P \in \Delta \setminus \Delta_1\}\Bigr) \\ &= \Bigl(\bigcap\{IR_P \mid P \in \Delta_1\}\Bigr) \cap \Bigl(\bigcap\{R_P \mid P \in \Delta \setminus \Delta_1\}\Bigr) \\ &\supseteq \Bigl(\bigcap\{IR_P \mid P \in \Delta_1\}\Bigr) \cap T \supseteq \bigcap\{IR_{(P,\varphi)} \mid P \in \Delta_1\}. \end{aligned}$$

Conversely, let $x \in I^*$ and let $P \in \Delta_1$ (which is nonempty). Then there exists $s \in R \setminus P$ such that $sx \in I$. Since $\varphi(s) \in D \setminus \varphi(P)$, $\varphi(s)$ is a unit element of $D_{\varphi(P)}$, and hence there exists $t \in R_{(P,\varphi)}$ such that $\varphi(t)\varphi(s) = 1$, or equivalently, $ts - 1 \in M$. Put $ts - 1 =: m \in M$, then $tsx = (1 + m)x = x + mx$. Since $tsx \in IR_{(P,\varphi)}$ and $mx \in MI^* \subseteq MR = M \subseteq I \subseteq IR_{(P,\varphi)}$, we have $x = tsx - mx \in IR_{(P,\varphi)}$. $\square$

Proposition 3.13. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{p}^+$)**. *Let Δ be a nonempty set of prime ideals of R and assume that $* := \star_\Delta \in \mathrm{Star}(R)$. Set $\Delta_\varphi := \{\varphi(P) \mid P \in \Delta, P \supseteq M\}$ $(\subseteq \mathrm{Spec}(D))$. Then*

$$(\star_\Delta)_\varphi = \star_{\Delta_\varphi}.$$

Proof. If $D = k$, then since $D = L$ is a field, we obviously have $(\star_\Delta)_\varphi = \star_{\Delta_\varphi}$. Assume that $D \subsetneq k$, then $\Delta_\varphi \neq \emptyset$. Let J be a nonzero integral ideal of D and let $I := \varphi^{-1}(J)$. Set $\Delta_1 = \{P \in \Delta \mid P \supseteq M\}$, hence $\Delta_\varphi = \{\varphi(P) \mid P \in \Delta_1\}$. Since I is an integral ideal of R containing M, $I^* = \bigcap\{IR_{(P,\varphi)} \mid P \in \Delta_1\}$ by Lemma 3.12(b), and so, using Proposition 2.7, we have $J^{*_\varphi} = \varphi(I^*) = \bigcap\{\varphi(I)D_{\varphi(P)} \mid P \in \Delta_1\} = \bigcap\{JD_{\varphi(P)} \mid P \in \Delta_1\} = \bigcap\{JD_Q \mid Q \in \Delta_\varphi\} = J^{\star_{\Delta_\varphi}}$. □

Remark 3.14.

(1) Note that from Proposition 3.13 we can deduce another proof of Proposition 3.9. As a matter of fact, for each star operation $*$ on R, $\tilde{*} = \star_\Delta$, where $\Delta := \mathcal{M}(*_f)$ (Example 1.3(e)). In the present situation, $\Delta_1 := \{P \in \mathcal{M}(*_f) \mid P \supseteq M\}$. By using Propositions 2.7 and 3.6(b), it is easy to see that

$$P \in \Delta_1 \quad \Longleftrightarrow \quad Q := \varphi(P) \in \mathcal{M}\big((*_\varphi)_f\big).$$

(2) *Note that if $\star := \star_\Delta$ is a spectral star operation on D, then $\star^\varphi$ is not necessarily a spectral star operation on R* (in particular, $(\star_\Delta)^\varphi \neq \star_{\Delta^\varphi}$, where $\Delta^\varphi := \{P \in \mathrm{Spec}(R) \mid \varphi(P) \in \Delta\}$).
To show this fact, let D be a 1-dimensional discrete valuation domain with quotient field L and maximal ideal N. Let $T := L[\![X^2, X^3]\!]$ and let $M := X^2L[\![X]\!] = XL[\![X]\!] \cap T$. Under these hypotheses, let R be the integral domain defined (as a pullback of type **($\mathfrak{p}^+$)**) from D, T and the canonical projection $\varphi : T \to L$. Then, R is a 2-dimensional non-Noetherian local domain. Let $\Delta := \mathrm{Max}(D) = \{N\}$. Then $\star := \star_\Delta = d_D = v_D$ and $\star^\varphi = (v_D)^\varphi = v_R$ (Corollary 2.13). Since $\Delta^\varphi = \mathrm{Max}(R)$, $\star_{\Delta^\varphi} = d_R$. Suppose that $\star^\varphi$ is spectral, then by Propositions 3.13 and 2.9, we have necessarily that $\star^\varphi$ coincides with $\star_{\Delta^\varphi}$, i.e., $v_R = \star^\varphi = \star_{\Delta^\varphi} = d_R$. This is a contradiction, since

$$T^{v_R} = \big(R :_K (R :_K T)\big) = (R :_K M) \supseteq L[\![X]\!] \supsetneq T = T^{d_R}.$$

Proposition 3.15. *Let T, K, M, k, D, φ, L, S, and R be as in* **($\mathfrak{p}^+$)**. *If $*$ is an a.b.* (*respectively e.a.b.*) *star operation on R, then $*_\varphi$ is an a.b.* (*respectively e.a.b.*) *star operation on D.*

Proof. Let J be a nonzero finitely generated ideal of D and let J_1, J_2 be two arbitrary nonzero ideals of D such that $(JJ_1)^{*_\varphi} \subseteq (JJ_2)^{*_\varphi}$. Set $I := \varphi^{-1}(J)$, $I_i := \varphi^{-1}(J_i)$ for $i = 1, 2$. Since J is finitely generated and $IS = S$ (because $I \supset M$ and M is a maximal ideal of S), there exists a finitely generated subideal I_0 of I such that $\varphi(I_0) = J$ and $I_0S = S$. Then, by Proposition 2.7, we have $(I_0I_1 + M)^* \subseteq (I_0I_2 + M)^*$. Note that

$I_0 I_i \supseteq I_0 M = I_0 MS = I_0 SM = SM = M$ for $i = 1, 2$, thus we have $(I_0 I_1)^* \subseteq (I_0 I_2)^*$. Since I_0 is finitely generated and $*$ is an a.b. star operation, $I_1{}^* \subseteq I_2{}^*$ and so $J_1{}^{*_\varphi} \subseteq J_2{}^{*_\varphi}$. The statement for the e.a.b. case follows from Proposition 3.6(b) and from the fact that $*$ is e.a.b. if and only if $*_f$ is a.b.. $\square$

Remark 3.16.

(1) *Under the assumption of Proposition* 3.15*, if v_R is e.a.b., then $(v_R)_\varphi = v_D$ is e.a.b.. In other words, if R is a v_R-domain, then D is a v_D-domain* [26, p. 418].

(2) *Let $\star$ be an a.b.* (*respectively e.a.b.*) *star operation on D. Then, in general, $\star^\varphi$ is not an a.b.* (*respectively e.a.b.*) *star operation on R.*
To show this fact, take D, T, and R as in Remark 3.14(2). Since D is a 1-dimensional discrete valuation domain, its unique star operation d_D $(= b_D = v_D)$ is an a.b. star operation (and hence an e.a.b. star operation). Since R is not integrally closed (because $X \in K \setminus R$ is integral over R), R has no e.a.b. star operations (and hence no a.b. star operations).
Note that it is possible to give an example of this phenomenon also with R integrally closed.

Example 3.17. *Let D be a* 1*-dimensional discrete valuation domain with quotient field L, let $T := L[X, Y]$ and $M := (X, Y)L[X, Y]$. Under these hypotheses, let $R := D + (X, Y)L[X, Y]$ be the integral domain defined* (*as a pullback of type* $(\mathfrak{b}^+)$) *from D, T and the canonical projection $\varphi : T \to L$. Then $(b_D)^\varphi$ is not e.a.b.* (*and hence not a.b.*) *on R.*

Note that M is a divisorial ideal of R of finite type, in fact, $M = I^{v_R}$, where $I := (X, Y)R$. Now, choose $a_1, a_2 \in D \setminus (0)$ such that $a_1 D \not\subseteq a_2 D$ (e.g., put $a_1 := a$, $a_2 := a^2$, where a is a nonzero nonunit element in D). Set $I_1 := a_1 R$ and $I_2 := a_2 R$. Then $(II_i)^{v_R} = (a_i I)^{v_R} = a_i I^{v_R} = a_i M = M$ (where the last equality holds because a_i is a unit in T) for each $i = 1, 2$. Thus we have $(II_1)^{v_R} = (II_2)^{v_R}$. On the other hand, since $(I_i)^{v_R} = I_i = a_i R = a_i(D + M) = a_i D + M$ for each $i = 1, 2$, and $a_1 D \not\subseteq a_2 D$, we have that $(I_1)^{v_R} \not\subseteq (I_2)^{v_R}$. Therefore, v_R is not an e.a.b. operation. Since D is a 1-dimensional discrete valuation domain, the unique star operation $d_D = b_D = v_D$ on D is an a.b. star operation (and hence an e.a.b. star operation), but $v_R = (v_D)^\varphi$ (Corollary 2.13) is not e.a.b. (and hence not a.b.).

Recall that given an integral domain T, *the paravaluation subrings of T*, in Bourbaki's sense [14, Chapter 6, §1, Exercise 8], are the subrings of T obtained as an intersection of T with a valuation domain having the same quotient field as T. It is easy to see that if R is a subring of T then the integral closure of R in T coincides with the intersection of all the paravaluation subrings of T containing R [14, Chapter 6, §1, Exercise 9].

Lemma 3.18. *Let T, K, M, k, D, φ, L, S, and R be as in* $(\mathfrak{b}^+)$*. Assume that $D \subsetneq L = k$. Assume, moreover, that D is integrally closed* (*or equivalently, that R is integrally closed in T*)*. Let $\mathcal{P} := \mathcal{P}(R, T)$* (*respectively $\mathcal{V}$, $\mathcal{V}_1$, $\mathcal{W}$*) *be the set of all the paravaluation subrings of T containing R* (*respectively the set of all valuation overrings of R*; *the set* *of all valuation overrings (V_1, N_1) of R such that $N_1 \cap R \supseteq M$; the set of all the valuation overrings of D). Set $b_{R,T} := \star_{\mathcal{P}}$ (respectively $b_R := \star_{\mathcal{V}}$, $\star_1 := \star_{\mathcal{V}_1}$, $b_D := \star_{\mathcal{W}}$). Then*

(a) *$b_{R,T}$ (respectively b_D) is a star operation on R (respectively on D); b_R and $\star_1$ are semistar operations on R. Moreover,*

$$b_{R,T} \leqslant \star_1 \wedge \star_{\{T\}} \leqslant b_R.$$

(b) $(b_{R,T})_\varphi = b_D$.

(c) *If R is integrally closed (which happens if T is integrally closed), then $\star_1 \wedge \star_{\{T\}}$ and b_R are star operations on R. Moreover, $(b_R)_\varphi = b_D$ and $b_R \leqslant (b_D)^\varphi$.*

(d) *If $T := V$ is a valuation domain, then $b_{R,T} = \star_1 = \star_1 \wedge \star_{\{T\}} = b_R$.*

Proof. Note that if $(V_2, N_2) \in \mathcal{V} \setminus \mathcal{V}_1$, then $N_2 \cap R \not\supseteq M$, and so there exists a unique prime ideal Q_2 in T such that $R_{N_2 \cap R} = T_{Q_2}$ [19, Theorem 1.4]. Therefore, $V_2 \supseteq R_{N_2 \cap R} = T_{Q_2} \supseteq T$.

(a) The first part of this statement is an obvious consequence of the definitions and the assumption that R is integrally closed in T (and equivalently, D is integrally closed [19, Corollary 1.5]). For each $I \in \overline{\boldsymbol{F}}(R)$, we have

$$\begin{aligned} I^{b_R} &= \bigcap\{IV \mid V \in \mathcal{V}\} = \left(\bigcap\{IV_1 \mid V_1 \in \mathcal{V}_1\}\right) \cap \left(\bigcap\{IV_2 \mid V_2 \in \mathcal{V} \setminus \mathcal{V}_1\}\right) \\ &\supseteq \left(\bigcap\{IV_1 \mid V_1 \in \mathcal{V}_1\}\right) \cap IT = I^{\star_1} \cap I^{\star_{\{T\}}} \supseteq \left(\bigcap\big\{I(V_1 \cap T) \bigm| V_1 \in \mathcal{V}_1\big\}\right) \\ &\supseteq \left(\bigcap\big\{I(V \cap T) \bigm| V \in \mathcal{V}\big\}\right) = I^{b_{R,T}}. \end{aligned}$$

(b) Note that since L is a field, the paravaluation subrings of L containing D coincide with the valuation rings in L containing D [14, Chapter 6, §1, Exercise 8(d)]. Moreover, if W is a valuation overring of D, then $\varphi^{-1}(W)$ is a paravaluation subring of T containing R [14, Chapter 6, §1, Exercise 8(c)]. On the other hand, if $V' \cap T$ is a paravaluation subring of T (where V' is a valuation domain in the field K, quotient field of R), then necessarily $\varphi(V' \cap T)$ is a paravaluation subring of $\varphi(T) = L$, i.e., it is a valuation domain in L containing D [14, Chapter 6, §1, Exercise 8(d)]. Therefore, for each $J \in \boldsymbol{F}(D)$, $\varphi^{-1}(J^{b_D}) = (\varphi^{-1}(J))^{b_{R,T}}$. Now, we can conclude, since we know that for each $J \in \boldsymbol{F}(D)$, $J^{(b_{R,T})_\varphi} = ((\varphi^{-1}(J))^{b_{R,T}})/M$ (Proposition 2.7).

(c) If R is integrally closed, then b_R is a star operation on R [26, Corollary 32.8], and so by (a) it follows that $\star_1 \wedge \star_{\{T\}}$ is also a star operation on R.

Let $\mathcal{W} = \{W_\lambda \mid \lambda \in \Lambda\}$. For each $\lambda \in \Lambda$, let $R_\lambda := \varphi^{-1}(W_\lambda)$. Then, by the argument used in the proof of (b), we have $\mathcal{P} = \{R_\lambda \mid \lambda \in \Lambda\}$. Denote by A' the integral closure of an integral domain A. Since R_λ is integrally closed in T, $R_\lambda = {R_\lambda}' \cap T$. Let $\iota'_\lambda : R_\lambda \hookrightarrow {R_\lambda}'$ and $\iota_\lambda : R_\lambda \hookrightarrow T$ be the canonical embeddings, and set $*_\lambda := (b_{{R_\lambda}'})^{\iota'_\lambda} \wedge (d_T)^{\iota_\lambda}$ for each $\lambda \in \Lambda$ (note that $(d_T)^{\iota_\lambda}$ coincides with the semistar operation $\star_{\{T\}}$ on R_λ). Then $*_\lambda$ is a star operation on R_λ (see also [2, Theorem 2]).

Claim 1. *Let I be an integral ideal of R properly containing M. Then $(IR_\lambda)^{*_\lambda} = IR_\lambda$.*

Let Q_λ be the maximal ideal of the valuation domain W_λ. If $\varphi(IR_\lambda) = (IR_\lambda)/M \neq yQ_\lambda$ for all $y \in L \setminus (0)$, then since $\varphi(IR_\lambda)$ is a divisorial ideal of the valuation domain W_λ, $\varphi(IR_\lambda) = \varphi((IR_\lambda)^{*_\lambda})$ (and hence, $(IR_\lambda)^{*_\lambda} = IR_\lambda$) by Proposition 2.7. Assume that $\varphi(IR_\lambda) = (IR_\lambda)/M = yQ_\lambda$ for some $y \in L \setminus (0)$. Choose $s \in S \setminus M$ such that $\varphi(s) = y$ and let $P_\lambda := \varphi^{-1}(Q_\lambda) \subsetneq R_\lambda$. Then $IR_\lambda = sP_\lambda + M$, and by the claim in the proof of Proposition 2.7, we have $(IR_\lambda)^{*_\lambda} = sP_\lambda{}^{*_\lambda} + M$. By (b), $R_\lambda = V_\lambda \cap T$ for some valuation overring V_λ of R, which has center P_λ on R_λ, thus $P_\lambda{}^{*_\lambda} = (P_\lambda R_\lambda{}')^{b_{R_\lambda{}'}} \cap P_\lambda T \subseteq P_\lambda V_\lambda \cap T = P_\lambda$. Therefore, in either case, we have $(IR_\lambda)^{*_\lambda} = IR_\lambda$.

Claim 2. $(b_R)_\varphi \leqslant (b_{R,T})_\varphi$ $(= b_D$ *by* (b)).

It suffices to show that for each nonzero integral ideal J of D, $J^{(b_R)_\varphi} \subseteq J^{(b_{R,T})_\varphi}$, i.e., for each integral ideal I of R properly containing M, $I^{b_R} \subseteq I^{b_{R,T}}$. Let I be such an ideal. Then

$$\begin{aligned}
I^{b_{R,T}} &= \bigcap \{IR_\lambda \mid \lambda \in \Lambda\} = \bigcap \bigl\{(IR_\lambda)^{*_\lambda} \bigm| \lambda \in \Lambda\bigr\} = \bigcap \bigl\{\bigl(IR'_\lambda\bigr)^{b_{R'_\lambda}} \cap IT \bigm| \lambda \in \Lambda\bigr\} \\
&= \bigcap \bigl\{\bigl(IR'_\lambda\bigr)^{b_{R'_\lambda}} \cap T \bigm| \lambda \in \Lambda\bigr\} = \bigcap \bigl\{\bigl(IR'_\lambda\bigr)^{b_{R'_\lambda}} \bigm| \lambda \in \Lambda\bigr\} \cap T \\
&= \bigcap \Bigl\{\bigcap \bigl\{IV \bigm| V \in \mathcal{V}_\lambda := \bigl\{\text{valuation overrings of } R'_\lambda\bigr\}\bigr\} \Bigm| \lambda \in \Lambda\Bigr\} \cap T \\
&\supseteq \bigcap \{IV \mid V \in \mathcal{V}\} = I^{b_R}.
\end{aligned}$$

Therefore, by Claim 2, (a), and the first part of (c), we conclude that $(b_R)_\varphi = b_D$. Finally, by Theorem 2.12, we have $b_R \leqslant ((b_R)_\varphi)^\varphi = (b_D)^\varphi$.

(d) If $T := V$ is a valuation domain, then each valuation overring of R is comparable with V. As a matter of fact, if V' is a valuation overring of R and $V' \nsubseteq V$, then there exists $y \in V' \setminus V$, hence $y^{-1} \in M$, thus for each $v \in V$, we have $v = v(y^{-1}y) = (vy^{-1})y \in MV' \subseteq V'$. Therefore, $V \subseteq V'$. From this observation, we immediately deduce that when T is a valuation domain, $b_{R,T} = \star_1 = \star_1 \wedge \star_{\{T\}} = b_R$. □

Remark 3.19. In a pullback situation of type $(\mathfrak{b}^+)$, when D is integrally closed, we have already noticed that if R is not integrally closed, then there is no hope that $(b_D)^\varphi = b_R$ (Remark 3.16(2)). More explicitly, Example 3.17 shows that we can have $b_R \lneq (b_D)^\varphi$, even when R is integrally closed. The next example shows that $b_R \lneq (b_D)^\varphi$ is possible even under the hypotheses of Lemma 3.18(d).

Example 3.20. *Let $T := V$ be a valuation domain with maximal ideal M and let $\varphi : V \to V/M =: k$ be the canonical projection. Let D be a Dedekind domain with infinitely many prime ideals and with quotient field $L = k$. Set $R := \varphi^{-1}(D)$. Then $b_R \lneq (b_D)^\varphi$.*

By the same argument as in Remark 3.8, we can see that R is not a TV-domain, i.e., $t_R \neq v_R$. Meanwhile, since R is a Prüfer domain, $b_R = d_R = t_R$, and since D is a Dedekind domain, $b_D = v_D$ and so $(b_D)^\varphi = (v_D)^\varphi = v_R$ (Corollary 2.13). Therefore, we have $b_R = t_R \lneq v_R = (b_D)^\varphi$.

References

[1] T. Akiba, A note on AV-domains, Bull. Kyoto Univ. Ed. Ser. B 31 (1967) 1–3.
[2] D.D. Anderson, Star operations induced by overrings, Comm. Algebra 16 (1988) 2535–2553.
[3] D.D. Anderson, D.F. Anderson, Some remarks on star operations and the class group, J. Pure Appl. Algebra 51 (1988) 27–33.
[4] D.D. Anderson, D.F. Anderson, Examples of star operations, Comm. Algebra 18 (1990) 1621–1643.
[5] D.D. Anderson, D.F. Anderson, M. Zafrullah, Splitting the t-class group, J. Pure Appl. Algebra 74 (1991) 17–37.
[6] D.D. Anderson, S.J. Cook, Two star-operations and their induced lattices, Comm. Algebra 28 (2000) 2461–2475.
[7] D.D. Anderson, E. Houston, M. Zafrullah, t-linked extensions, the t-class group and Nagata's Theorem, J. Pure Appl. Algebra 86 (1993) 109–124.
[8] D.F. Anderson, A general theory of class groups, Comm. Algebra 16 (1988) 805–847.
[9] D.F. Anderson, Star operations and the $D + M$ construction, Rend. Circ. Mat. Palermo (2) Suppl. 41 (1992) 221–230.
[10] D.F. Anderson, The class group and the local class group of an integral domain, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, in: Math. Appl., vol. 520, Kluwer Academic, Dordrecht, 2000, pp. 33–55.
[11] D.F. Anderson, S. El Baghdadi, S.-E. Kabbaj, On the class group of $A + XB[X]$ domains, in: D. Dobbs, M. Fontana, S.-E. Kabbaj (Eds.), Advances in Commutative Ring Theory, in: Lecture Notes in Pure and Appl. Math., vol. 205, Dekker, New York, 1999, pp. 73–85.
[12] E. Bastida, R. Gilmer, Overrings and divisorial ideals of the form $D + M$, Michigan Math. J. 20 (1973) 79–95.
[13] M. Boisen, P. Sheldon, CPI-extensions: overrings of integral domains with special prime spectrum, Canad. J. Math. 24 (1977) 722–737.
[14] N. Bourbaki, Algèbre Commutative, Hermann, Paris, 1961.
[15] J. Brewer, E. Rutter, $D + M$ constructions with general overrings, Michigan Math. J. 23 (1976) 33–41.
[16] D. Dobbs, Divided rings and going-down, Pacific J. Math. 67 (1976) 353–363.
[17] D. Dobbs, E. Houston, T. Lucas, M. Zafrullah, t-linked overrings and Prüfer v-multiplication domains, Comm. Algebra 17 (1989) 2835–2852.
[18] W. Fanggui, R.L. McCasland, On w-modules over strong Mori domains, Comm. Algebra 25 (1997) 1285–1306.
[19] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. (4) 123 (1980) 331–355.
[20] M. Fontana, S. Gabelli, On the class group and the local class group of a pullback, J. Algebra 181 (1996) 803–835.
[21] M. Fontana, J. Huckaba, Localizing systems and semistar operations, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, Kluwer Academic, Dordrecht, 2000, pp. 169–187, Chapter 8.
[22] M. Fontana, K.A. Loper, Kronecker function rings: a general approach, in: D.D. Anderson, I.J. Papick (Eds.), Ideal Theoretic Methods in Commutative Algebra, in: Lecture Notes in Pure and Appl. Math., vol. 220, Dekker, 2001, pp. 189–205.
[23] M. Fontana, K.A. Loper, A Krull-type theorem for the semistar integral closure of an integral domain, Commutative Algebra, Arab. J. Sci. Eng. Sect. C Theme Issues 26 (2001) 89–95.
[24] S. Gabelli, E. Houston, Coherentlike conditions in pullbacks, Michigan Math. J. 44 (1997) 99–123.
[25] S. Gabelli, E. Houston, Ideal theory in pullbacks, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, in: Math. Appl., vol. 520, Kluwer Academic, Dordrecht, 2000, pp. 199–227.
[26] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.
[27] B.V. Greenberg, Coherence in polynomial rings, J. Algebra 50 (1978) 12–25.
[28] F. Halter-Koch, Ideal Systems: An Introduction to Multiplicative Ideal Theory, Dekker, New York, 1998.
[29] J.R. Hedstrom, E.G. Houston, Pseudo-valuation domains, Pacific J. Math. 75 (1978) 137–147.
[30] J.R. Hedstrom, E.G. Houston, Some remarks on star-operations, J. Pure Appl. Algebra 18 (1980) 37–44.
[31] E.G. Houston, M. Zafrullah, Integral domains in which each t-ideal is divisorial, Michigan Math. J. 35 (2) (1988) 291–300.

[32] P. Jaffard, Les Systèmes d'Idéaux, Dunod, Paris, 1960.
[33] B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J. Algebra 123 (1989) 151–170.
[34] M.D. Larsen, P.J. McCarthy, Multiplicative Ideal Theory, Academic Press, New York, 1971.
[35] T. Lucas, Examples built with $D+M$, $A+XB[X]$ and other pullback constructions, in: S. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, in: Math. Appl., vol. 520, Kluwer Academic, Dordrecht, 2000, pp. 341–368.
[36] R. Matsuda, I. Sato, Note on star operations and semistar operations, Bull. Fac. Sci. Ibaraki Univ. Ser. A 28 (1996) 5–22.
[37] R. Matsuda, T. Sugatani, Semistar operations on integral domains, II, Math. J. Toyama Univ. 18 (1995) 155–161.
[38] A. Mimouni, TW-domains and strong Mori domains, J. Pure Appl. Algebra 177 (2003) 79–93.
[39] M. Nagata, Local Rings, Interscience, New York, 1972.
[40] A. Okabe, R. Matsuda, Star operations and generalized integral closures, Bull. Fac. Sci. Ibaraki Univ. Ser. A 24 (1992) 7–13.
[41] A. Okabe, R. Matsuda, Semistar operations on integral domains, Math. J. Toyama Univ. 17 (1994) 1–21.
[42] M.H. Park, Group rings and semigroup rings over strong Mori domains, J. Pure Appl. Algebra 163 (2001) 301–318.
[43] A. Seidenberg, A note on dimension theory of rings, Pacific J. Math. 3 (1953) 505–512; Part II, Pacific J. Math. 4 (1954) 603–614.
[44] C. Traverso, Seminormality and Picard group, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (2) 24 (1970) 585–595.
[45] W. Vasconcelos, The local rings of global dimension two, Proc. Amer. Math. Soc. 35 (1972) 381–386.
[46] M. Zafrullah, Various facets of rings between $D[X]$ and $K[X]$, Comm. Algebra 31 (5) (2003) 2497–2540.
[47] O. Zariski, P. Samuel, Commutative Algebra, vol. II, Van Nostrand, Princeton, 1960.