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An historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations.

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This is a valuable historical overview of Kronecker function rings, Nagata rings, and related star and semistar operations. The outline of the paper is as follows: E. E. Kummer introduced the concept of ideal to re-establish the factorization theory for cyclotomic integers. R. Dedekind proved that, in the ring of integers of an algebraic number field, each ideal factors uniquely into a product of prime ideals. L. Kronecker's theory holds in a larger context, and solves a more general problem. The key point of his theory is to give an explicit description of a greatest common divisor in an extension of the original ring which mirrors the ideal structure of the original ring. Precisely, the Kronecker function ring of the ring D of algebraic integers is given by

$$\mathrm{Kr}(D)=\{f/g|\ f,g\in D[X], g\neq 0 \text{ and } c(f)\subseteq c(g)\},$$

where $c(h)$ denotes the content of a polynomial $h\in D[X]$. Then we have the following: (1) Kr(D) is a Bézout domain, and (2) let $a_0,a_1,\dots,a_n\in D$ and let $f=a_0+a_1X+\cdots+a_nX^n\in D[X]$; then $(a_0,a_1,\dots,a_n)\mathrm{Kr}(D)=f\mathrm{Kr}(D)$, $f\mathrm{Kr}(D)\cap K=(a_0,a_1,\dots,a_n)D$. Kronecker's theory led to two different major extensions: (1) W. Krull generalized the Kronecker function ring to any integrally closed domain, and (2) for any integral domain D, M. Nagata investigated the ring $D(X)=\{f/g|\ f,g\in D[X] \text{ and } c(g)=D\}$, which is called the Nagata ring of D, and is denoted by Na(D). Na(D) has some strong ideal-theoretic properties that D need not have, while maintaining a strict relation with the ideal structure of D. Let D be an integral domain with quotient field K, and let $\vec{F}(D)$ be the set of nonzero fractional ideals of D. A mapping $\star\colon \vec{F}(D)\longrightarrow\vec{F}(D)$ is called a star operation on D if, for all $0\neq z\in K$ and for all $I,J\in\vec{F}(D)$, the following properties hold: $(\star_1)$ $(zD)^\star=zD$, $(zI)^\star=zI^\star$; $(\star_2)$ $I\subseteq J$ implies $I^\star\subseteq J^\star$; $(\star_3)$ $I\subseteq I^\star$, $I^{\star\star}=I^\star$. Let D be an integrally closed domain and let $\star$ be an e.a.b. star operation on D. Krull defined the $\star$-Kronecker function ring

$$\mathrm{Kr}(D,\star)=\{f/g|\ f,g\in D[X], g\neq 0 \text{ and } c(f)^\star\subseteq c(g)^\star\}.$$

It has the following properties: (1) $\mathrm{Kr}(D,\star)$ is a Bézout domain, and (2) let $a_0,a_1,\dots,a_n\in D$ and let $f=a_0+a_1X+\cdots+a_nX^n$; then

$$(a_0,a_1,\dots,a_n)\mathrm{Kr}(D,\star)=f\mathrm{Kr}(D,\star),$$

$$(a_0,a_1,\dots,a_n)\mathrm{Kr}(D,\star)\cap K=((a_0,a_1,\dots,a_n)D)^\star.$$

A. Okabe and R. Matsuda introduced the more flexible notion of semistar operation on an integral domain D, as a natural generalization of a star operation. Let $\vec{\vec{F}}(D)$ be the set of nonzero D-submodules of K. A mapping $\star\colon\vec{\vec{F}}(D)\longrightarrow\vec{\vec{F}}(D)$ is called a semistar operation on D if, for all

$0 \neq z \in K$ and for all $E, F \in \overrightarrow{\overline{F}}(D)$, the following properties hold: $(\star_1)$ $(zE)^\star = zE^\star$; $(\star_2)$ $E \subseteq F$ implies $E^\star \subseteq F^\star$; $(\star_3)$ $E \subseteq E^\star$, $E^{\star\star} = E^\star$. For a semistar operation $\star$ on D, we define the semistar Nagata ring $\mathrm{Na}(D, \star) = \{f/g |\ f, g \in D[X], g \neq 0, c(g)^\star = D^\star\}$. Some results on $\mathrm{Na}(D)$ were generalized to the semistar setting (by B. G. Kang, Fontana and Loper): Set $N(\star) = \{h \in D[X] |\ c(h)^\star = D^\star\}$, set $\mathcal{M} = \{M \cap D |\ M \in \mathrm{Max}(\mathrm{Na}(D, \star))\}$, and set $E^{\tilde{\star}} = \bigcap\{ED_Q |\ Q \in \mathcal{M}\}$ for each $E \in \overrightarrow{\overline{F}}(D)$. Then (1) $\mathrm{Na}(D, \star) = D[X]_{N(\star)}$; (2) $\mathrm{Max}(\mathrm{Na}(D, \star)) = \{Q[X]_{N(\star)} |\ Q \in \mathcal{M}\}$; (3) $E\mathrm{Na}(D, \star) = \bigcap\{ED_Q(X) |\ Q \in \mathcal{M}\}$. The construction of a Kronecker function ring for any integral domain was considered independently by F. Halter-Koch and by Fontana and Loper. Halter-Koch's approach was axiomatic. Fontana and Loper's treatment was based on semistar operations. Halter-Koch gave the following definition. Let K be a field, and let R be a subring of $K(X)$ with $D_0 = R \cap K$. If X is a unit of R, and if $f(0) \in fR$ for each $f \in K[X]$, then R is called a K-function ring of D_0. He proved the following theorem: Let R be a K-function ring of D_0, then (1) R is a Bézout domain with quotient field $K(X)$; (2) D is integrally closed in K; (3) for each polynomial $f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$, we have $(a_0, a_1, \ldots, a_n)R = fR$. If $\star$ is a semistar operation on an integral domain D, then we define the Kronecker function ring of D with respect to $\star$ by $\mathrm{Kr}(D, \star) = \{f/g |\ f, g \in D[X], g \neq 0$, and there exists $h \in D[X] \setminus \{0\}$ with $(c(f)c(h))^\star \subseteq (c(g)c(h))^\star\}$. For any semistar operation $\star$ on D, we can define the e.a.b. semistar operation of finite type $\star_a$ on D: $E^{\star_a} = E\mathrm{Kr}(D, \star) \cap K$ for each $E \in \overrightarrow{\overline{F}}(D)$. Fontana and Loper proved the following theorem: (1) V is a $\star$-valuation overring of D if and only if $V(X)$ is a valuation overring of $\mathrm{Kr}(D, \star)$; (2) $\mathrm{Kr}(D, \star) = \mathrm{Kr}(D, \star_a)$ is a Bézout domain; (3) $\mathrm{Kr}(D, \star)$ is a K-function ring of $D^{\star_a}$. Let $\star$ be a semistar operation on D. If, for each finitely generated $F \in \vec{F}(D)$, there is a finitely generated $F' \in \vec{F}(D)$ such that $(FF')^\star = D^\star$, then D is called a Prüfer $\star$-multiplication domain (or a P$\star$MD). Theorem (Fontana, P. Jara, and E. Santos): For a semistar operation $\star$ on D, the following conditions are equivalent: (1) D is a P$\star$MD; (2) $\mathrm{Na}(D, \star)$ is a Prüfer domain; (3) $\mathrm{Na}(D, \star) = \mathrm{Kr}(D, \star)$; (4) $\tilde{\star} = \star_a$.

The authors refer to the following many other contributors to Kronecker function rings, Nagata rings, and related star and semistar operations: D. D. Anderson, D. F. Anderson, J. Arnold, K. E. Aubert, S. El Baghdadi, J. Brewer, H. S. Butts, G. W. Chang, S. J. Cook, D. E. Dobbs, H. M. Edwards, J. M. Garcia, R. Gilmer, S. Glaz, M. Griffin, J. R. Hedstrom, W. Heinzer, E. G. Houston, J. Huckaba, P. Jaffard, P. Lorenzen, T. Lucas, S. B. Malik, R. L. McCasland, J. L. Mott, J. Ohm, J. Park, G. Picozza, H. Prüfer, P. Quartaro Jr., P. Samuel, I. Sato, T. Sugatani, W. Vasconcelos, Fanggui Wang, H. Weyl, M. Zafrullah.

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