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Factoring ideals in Prüfer domains

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Abstract

We show that in certain Prüfer domains, each nonzero ideal I can be factored as $I = I^v\Pi$, where I^v is the divisorial closure of I and Π is a product of maximal ideals. This is always possible when the Prüfer domain is h-local, and in this case such factorizations have certain uniqueness properties. This leads to new characterizations of the h-local property in Prüfer domains. We also explore consequences of these factorizations and give illustrative examples.

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Let R be a Prüfer domain. Recall that R has finite character if each nonzero element of R is contained in only finitely many maximal ideals of R and that R is h-local if it has finite character and each nonzero prime ideal of R is contained in a unique maximal ideal of R. It follows from [1, Theorem 4.12] that if R is h-local, then each nonzero ideal I of R factors as $I = I^v\Pi$, where I^v denotes the divisorial closure of I and Π is a product of maximal ideals. Part of the first section of this work may be viewed as an elaboration of this result. We observe that, for a nonzero ideal I of an h-local Prüfer domain, we have $I = I^v M_1 \cdots M_n$, where the M_i are precisely the nondivisorial maximal ideals M of R which contain I and for which IR_M remains nondivisorial in R_M (and where we take the empty product of maximal ideals to be R itself); moreover, this factorization is unique in the sense that no M_i can be deleted. On the other hand, we show that in certain almost Dedekind domains, one can have a weaker factorization property: each nonzero ideal I factors as $I = I^v\Pi$, where Π is a product of (not necessarily distinct) maximal ideals. We show (Proposition 1.7) that in a Prüfer domain with this weak factorization property each nonmaximal prime ideal is divisorial, each branched nonmaximal prime ideal is the radical of a finitely generated ideal, and each branched idempotent maximal ideal is sharp. (Relevant definitions are reviewed in the sequel.) If, in addition to possessing the weak factorization property, the Prüfer domain R has finite character, then R is h-local (Theorem 1.13). Moreover, a Prüfer domain is h-local if and only if it has the strong factorization property (Theorem 1.12). Another interesting property of h-local Prüfer domains is that a nonzero ideal of such a domain is divisorial if and only if it is locally divisorial (at maximal ideals). In fact, we show in Theorem 1.12 that a Prüfer domain with this property is h-local.

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In Section 2, we explore in h-local Prüfer domains how a given factorization of an ideal I affects that of rad I and II^{-1} and how factorizations of ideals I and J affect those of such related ideals as IJ, $I \cap J$, and $I + J$.

Section 3 is devoted to examples. As has already been mentioned, it is possible for an almost Dedekind domain to possess the weak factorization property; in Example 3.2 we show that this can happen even in an almost Dedekind domain with infinitely many nondivisorial maximal ideals. While in a Prüfer domain with the strong factorization property, the sum of two divisorial ideals must be again divisorial, we show in Example 3.3 that an almost Dedekind domain may have the weak factorization property and still possess divisorial ideals I and J with $I + J$ not divisorial. We also give an example (Example 3.5) of a one-dimensional Bezout domain R which does not have the weak factorization property, and we observe that in this example, there is a divisorial ideal J and a maximal ideal M with JR_M not divisorial.

1. The strong and weak factorization properties

We begin by recalling some facts which we shall use frequently and without further reference. Let V be a valuation domain with maximal ideal M. If M is divisorial, then M is principal and every nonzero ideal of V is divisorial by [10, Lemma 5.2]. On the other hand, if M is not divisorial, then by [3, Lemma 4.2] a nonzero ideal I of V is nondivisorial if and only if $I = xM$ for some element $x \in V$.

Theorem 1.1. *Let R be an h-local Prüfer domain. Then*

(1) *For each divisorial ideal I of R, if $M \supseteq I$ with M a nondivisorial maximal ideal of R, then IR_M is divisorial in R_M, and IR_M is properly contained in MR_M.*
(2) *For each nonzero nondivisorial ideal I of R, I factors as a product $BM_1M_2 \cdots M_n$ where B is a divisorial ideal and the M_i are distinct nondivisorial maximal ideals of R that contain I for which IR_{M_i} is not a divisorial ideal of R_{M_i}. Moreover, this factorization is unique in the sense that $B = I^v$ and the M_i include all maximal ideals that contain I where IR_{M_i} is not divisorial.*

Proof. Let A be a nonzero ideal of R. Since R is h-local, $(AR_M)^{-1} = A^{-1}R_M = (A^vR_M)^{-1}$ for each maximal ideal M ([2, Lemma 2.3] or [16, Theorem 3.10]). Moreover, $A^vR_M = (AR_M)^v$. In particular, if M is nondivisorial, then $(MR_M)^v = M^vR_M = R_M$, so that MR_M is not divisorial, while if I is divisorial, then IR_M is also divisorial. This proves (1).

If AR_M is not divisorial, then it must be of the form xMR_M for some $x \in R$. In this case, we have $A^vR_M = (AR_M)^v = xR_M$ and $AR_M = A^vMR_M$.

Now let I be a nonzero nondivisorial ideal of R. Let $M_1, M_2, \ldots, M_n$ be the nondivisorial maximal ideals that contain I where IR_{M_i} is not divisorial. (It will follow from the rest of the proof that $n > 0$, but for the moment we take the empty product to be R.) Consider the ideal $J = I^vM_1M_2 \cdots M_n$. For each M_i, it is clear that $JR_{M_i} = IR_{M_i}$ from the argument above. Let M be any other maximal ideal. If M does not contain I, then $JR_M = R_M = IR_M$. On the other hand if M contains I, we must have that $(IR_M)^v = IR_M$. As $I^vR_M = (IR_M)^v$, we obtain $IR_M = I^vR_M = JR_M$. Hence $I = J$.

Now suppose $I = BN_1N_2 \cdots N_m$ with B divisorial and the N_i distinct members of $\{M_1, M_2, \ldots, M_n\}$. Since for each i, BR_{M_i} is divisorial (but perhaps trivial) and IR_{M_i} is not, checking locally at M_i shows that some N_j must equal M_i. Hence $m = n$ and each M_i is needed in the factorization. Rewriting, we have $I = BM_1M_2 \cdots M_n$. Thus, since the M_i are nondivisorial (and since for a maximal ideal M, we have M nondivisorial if and only if $M^{-1} = M^v = R$), $I^v = (BM_1 \cdots M_n)^v = B^v = B$. □

Definition 1.2. A Prüfer domain R has the *strong factorization property* if for each nonzero ideal I of R, we have (1) $I = I^vM_1 \cdots M_n$ where $M_1, \ldots, M_n$ are precisely the nondivisorial maximal ideals of R which contain I for which IR_M is nondivisorial and (2) this factorization is unique in the sense that no M_i can be omitted.

Remark 1.3. In Definition 1.2, we take the empty product of maximal ideals to be R; thus, if IR_M is divisorial for each maximal ideal M, then $I = I^v$ (that is, I is divisorial).

Thus, according to Theorem 1.1, h-local Prüfer domains possess the strong factorization property. In Theorem 1.12 below, we show that the converse holds.

Remark 1.4. Let I be a nonzero ideal of the Prüfer domain R, denote by $\mathrm{Max}(R, I)$ the set of maximal ideals of R containing I, and set

$$\mathfrak{M}(I) := \{M \in \mathrm{Max}(R, I) \mid M^v = R, IR_M \neq (IR_M)^v\}$$
$$\mathfrak{M}'(I) := \{M' \in \mathrm{Max}(R, I) \mid M'^v = R, IR_{M'} = (IR_{M'})^v\}$$
$$\mathfrak{N}(I) := \{N \in \mathrm{Max}(R, I) \mid N = N^v\}.$$

Then Definition 1.2 requires that $\mathfrak{M}(I)$ be finite (possibly empty), that $I = I^v \prod_{M \in \mathfrak{M}(I)} M$, and that this factorization be irredundant. We say nothing about the possible finiteness of $\mathfrak{M}'(I)$ or $\mathfrak{N}(I)$. It is also possible that I could have a different factorization involving some of the maximal ideals in $\mathfrak{M}'(I) \cup \mathfrak{N}(I)$. For example, let (V, M) be a valuation domain containing a nonprincipal divisorial ideal I. Then $\mathfrak{M}(I)$ is empty, and the factorization of I is just $I = I^v$. However, since I is not principal, we also have $I = IM (= I^v M)$. (The fact that I not principal implies that $I = IM$ is probably well known, but here is a proof: Begin with an element $x \in I$. Since I is not principal, we may then choose $y \in I \setminus Vx$ so that $x/y \in M$ and $x = y(x/y) \in IM$.) By constructing V appropriately, we may have M divisorial or not, that is, $\mathfrak{N}(I) = \{M\}$ or $\mathfrak{M}'(I) = \{M\}$.

Remark 1.5. Using the notation in Remark 1.4 and following the proof of [1, Theorem 4.12], we have for any nonzero ideal I in an h-local Prüfer domain a decomposition of I^v as follows. Set $I' = \bigcap_{M' \in \mathfrak{M}'(I)} (IR_{M'} \cap R)$ and $J_I = \bigcap_{N \in \mathfrak{N}(I)} (IR_N \cap R)$. For each $M \in \mathfrak{M}(I)$, there is an invertible ideal L of R with $IR_M \cap R = LM$; set L_I equal to the product of these L's. Then $I^v = L_I I' J_I$.

We now introduce our second factorization property.

Definition 1.6. A Prüfer domain R has the *weak factorization property* if each nonzero ideal I can be written as $I = I^v \Pi$, where Π is a (finite) product of (not necessarily distinct) maximal ideals (and where, again, the empty product of maximal ideals is taken to be R).

Before stating our next few results, we need some terminology. Recall that a domain R satisfies the *trace property* if, for each nonzero ideal I of R, we have that II^{-1} is equal either to R or to a prime ideal of R. The domain R satisfies the *radical trace property* if each nonzero ideal I of R satisfies $II^{-1} = R$ or $II^{-1} = \mathrm{rad}\,(II^{-1})$. Finally, R satisfies the *weak trace property for primary ideals* if, for each nonzero, nonmaximal prime ideal P and each P-primary ideal Q, we have $QQ^{-1} = P$. For information about the trace and radical trace properties, the reader is referred to [6, 14]. Now recall from [7] that a domain R is said to be a #-*domain* if $\bigcap_{M \in \mathcal{M}} R_M \neq \bigcap_{N \in \mathcal{N}} R_N$ for each pair of distinct nonempty subsets $\mathcal{M}$ and $\mathcal{N}$ of the set of maximal ideals of R, equivalently, if for each maximal ideal M of R, R_M does not contain $\bigcap R_N$, where the intersection is taken over those maximal ideal N with $N \neq M$. This was extended to focus on a single maximal ideal in [13]: a maximal ideal is *sharp* if R_M does not contain $\bigcap_{N \neq M} R_N$. By [9, Corollary 2] a maximal ideal M of a Prüfer domain R is sharp if and only if there is a finitely generated ideal of R which is contained in M and no other maximal ideal of R. Finally, a domain R is a ##-*domain* if each overring of R is a #-domain (see [9]).

Proposition 1.7. *Let R be a Prüfer domain with the weak factorization property. Then*

(1) *each ideal which is primary to a nonmaximal ideal of R is divisorial (in particular, each nonmaximal prime is divisorial),*
(2) *if M is an idempotent maximal ideal of R and I is a nondivisorial M-primary ideal, then $I = I^v M$,*
(3) *each branched maximal idempotent ideal of R is sharp,*
(4) *R has the weak trace property for primary ideals, and*
(5) *each branched nonmaximal prime ideal of R is the radical of a finitely generated ideal.*

Proof. (1) Let Q be a P-primary ideal of R with P nonmaximal. Write $Q = Q^v \Pi$, where Π is a product of maximal ideals. Then $\Pi \nsubseteq P$, whence $Q^v \subseteq Q$, and so Q is divisorial.

(2) This is clear.

(3) Let M be a branched idempotent maximal ideal of R. Since M is branched, there is an M-primary ideal I with $I \neq M$. If $I^v \nsubseteq M$, then $I^v = R$, and $I = I^v M$ by (2). But this yields $I = M$, a contradiction. Hence $I^v \subseteq M$, and M is sharp by [16, Proposition 2.2].

(4) Let Q be a proper P-primary ideal with P not maximal. Then Q is divisorial by (1). We shall show that $QQ^{-1} = P$. By [6, Corollary 3.1.8 and Theorem 3.1.2] $P^{-1} = \bigcap R_M \cap R_P$, where the intersection $\bigcap R_M$ is taken over those maximal ideals which do not contain P. For $x \in Q^{-1}$, we have $(R :_R x) \nsubseteq M$, since $Q \nsubseteq M$; thus $x \in R_M$. Hence $Q^{-1} \subseteq \bigcap R_M$. The same argument shows that $Q^{-1} \subseteq \Omega(P) := \bigcap R_N$, where N ranges over the prime ideals of R which do not contain P.

For $y \in P^{-1}$, we have $y \in R_P$, whence $ay \in R$ for some $a \notin P$. Then $ayQ \subseteq Q$ yields $yQ \subseteq Q$ (since it is clear that $yQ \subseteq R$). Thus $P^{-1}Q \subseteq Q$. Therefore, $(QQ^{-1})^{-1} = (QQ^{-1} : QQ^{-1}) \supseteq P^{-1}$, and we have $QQ^{-1} \subseteq P^v = P$ by (1). We also have that $P^{-1} \subseteq (QQ^{-1})^{-1} \subseteq Q^{-1} \subseteq \Omega(P)$ with Q^{-1} a ring. Since R is a Prüfer domain, this yields $(QQ^{-1})^{-1} = P^{-1}$ [6, Theorem 3.3.7], whence $(QQ^{-1})^v = P$ (again by (1)). If QQ^{-1} is not divisorial, then $QQ^{-1} = (QQ^{-1})^v\Pi = P\Pi$, for some product Π of maximal ideals each of which necessarily contains P (since each contains Q). A routine local check then shows that $P\Pi = P$, so that $QQ^{-1} = P$, as desired.

(5) This follows from (1) and [5, Proposition 2.9]. □

Next, we give some consequences of the strong factorization property.

Theorem 1.8. *Let R be a Prüfer domain with the strong factorization property. Then*

(1) *If I is a nonzero ideal of R, then I is divisorial if and only if IR_M is divisorial for each maximal ideal M of R.*
(2) *If M is a maximal ideal of R and A is a divisorial ideal of R_M, then $A \cap R$ is divisorial in R.*
(3) *For each maximal ideal M, if M is not divisorial, then MR_M is not divisorial. Thus the nondivisorial maximal ideals are those that are idempotent.*
(4) *For each nonzero ideal I of R and each maximal ideal M of R, we have $(IR_M)^v = I^vR_M$.*
(5) *If I is an ideal contained in no nondivisorial maximal ideals, then I is divisorial.*
(6) *For each fractional ideal F, $F = F^vM_1M_2\cdots M_n$ where the M_i are the maximal ideals that contain some particular fixed nonzero principal multiple xF of F with xFR_{M_i} not divisorial. Moreover, the factorization is unique.*
(7) *If R has finite character, and I is an ideal for which IR_M is divisorial only in the trivial case $IR_M = R_M$, then I^v is invertible.*

Proof. (1) Let I be a nonzero ideal of R, and let M be a maximal ideal. Suppose that I is divisorial. If M is nondivisorial, then IR_M is divisorial by Definition 1.2. If M is divisorial, then it is invertible; hence MR_M is principal, and every ideal of R_M is divisorial. For the converse, see Remark 1.3.

(2) Let M be maximal, and let A be a divisorial ideal of R_M. Set $I = A \cap R$, and write $I = I^vM_1\cdots M_n$ as in Definition 1.2. Since $IR_M = A$ is divisorial, $M \notin \{M_1, \cdots M_n\}$. We then have $IR_M = I^vM_1\cdots M_nR_M = I^vR_M$. Hence $I^v \subseteq I^vR_M \cap R = IR_M \cap R = I$, as desired.

(3) From (1) if M is a nondivisorial maximal ideal, then MR_M is also nondivisorial and hence idempotent. Since idempotence is a local property, M is itself idempotent.

(4) Let the factorization of I be $I = I^vM_1\cdots M_n$, and let M be a maximal ideal of R. If $M \notin \{M_i\}$, then $(IR_M)^v = (I^vM_1\cdots M_nR_M)^v = (I^vR_M)^v = I^vR_M$, with the last equality following from (1). If $M = M_i$ for some i, then $(IR_M)^v = (I^vM_1\cdots M_nR_M)^v = (I^vMR_M)^v = (I^vR_M)^v = I^vR_M$, with the penultimate equality following from (2) and the last equality following from (1).

(5) This is immediate from the definition.

(6) Let F be a fractional ideal and let $x \in R \setminus \{0\}$ be such that $xF \subseteq R$. Then we can factor xF uniquely as $(xF)^vM_1M_2\cdots M_n$ where the M_i are the nondivisorial maximal ideals that contain xF where xFR_{M_i} is not divisorial. Of course, $(xF)^v = xF^v$, so we can cancel the x to obtain $F = F^vM_1M_2\cdots M_n$. For any nonzero $y \in (R : F)$, we obtain a (possibly different) factorization $F = F^vN_1N_2\cdots N_k$ where the N_j are such that yFR_{N_j} is not divisorial. If these two factorizations were actually different, we would have two distinct factorizations of xyF, one as $xyF^vM_1M_2\cdots M_n$ and the other as $xyF^vN_1N_2\cdots N_k$. Thus we must have a unique factorization for F.

(7) Let I be as indicated. Then $I = I^vM_1\cdots M_n$, where the M_i are precisely the maximal ideals which contain I. For each i, IR_M not divisorial yields an element $x_i \in I^v$ with $IR_{M_i} = x_iM_iR_{M_i}$ and $I^vR_{M_i} = x_iR_{M_i}$. Let $A = (x_1, x_2, \ldots, x_n)$. At most finitely many maximal ideals contain A, say $N_1, N_2, \ldots, N_k$. For those N_j that are not among the M_is, we may choose an element $y_j \in I \setminus N_j$. Let B be the ideal generated by A and the y_j. Obviously, $B \subseteq I^v$. Now consider the ideal $J = BM_1M_2\cdots M_n$ and let M be a maximal ideal of R. If $M = M_i$ for some i, then $JR_{M_i} = BM_iR_{M_i}$. Thus $JR_{M_i} = I^vM_iR_{M_i} = IR_{M_i}$ since $x_iR_{M_i} = I^vR_{M_i}$ and $x_i \in B \subseteq I^v$. If M is a maximal

ideal not among the M_i, then $B \not\subseteq M$, and we have $JR_M = BR_M = R_M = IR_M$ since no other maximal ideals contain B. Hence $J = I$. As B is divisorial and factorizations are unique, we must have $B = I^v$. Therefore, I^v is invertible. $\square$

We observe that, in view of Theorem 1.12 below, part (5) of Proposition 1.8 is [11, Proposition 6.5 (a)] and part (7) may be viewed as a generalization of [11, Proposition 6.5 (b)].

We need a couple of general results before proving that statement (1) in Theorem 1.8 is equivalent to the h-local property. Our next lemma provides a way to prove statement (2) of Theorem 1.8 using only the assumption that each locally divisorial of the Prüfer domain R is divisorial.

Lemma 1.9. *Let I be a nonzero ideal of a Prüfer domain R and let M a maximal ideal that contains I. For $J = IR_M \bigcap R$, JR_N is a divisorial ideal of R_N for each maximal ideal $N \neq M$.*

Proof. Let N be a maximal ideal of R with $N \neq M$. Then $JR_N = (IR_M \bigcap R)R_N = IR_M R_N \bigcap R_N = IR_P \bigcap R_N$ where P is the largest prime contained in $M \cap N$. If JR_N is not divisorial, then $JR_N = xNR_N$ for some $x \in R$. This yields $JR_P = xR_P$, and we then have $x \in JR_P \bigcap R_N = IR_P \bigcap R_N = JR_N = xNR_N$, a contradiction. Hence JR_N is divisorial. $\square$

Theorem 1.10. *Let R be a Prüfer domain and let P be a nonzero nonmaximal prime that is the radical of a finitely generated ideal. If I is a finitely generated ideal whose radical is P and M is a maximal ideal that contains P, then the ideal $J = IR_M \bigcap R$ is divisorial if and only if M is the only maximal ideal that contains P.*

Proof. Let $J = IR_M \bigcap R$ where M is a maximal ideal that contains P. It is clear that if M is the only maximal ideal that contains P, then $J^v = J = I$.

For the remainder of the proof, we assume that M is not the only maximal ideal that contains P. Denote by P' the largest prime ideal contained in all the maximal ideals which contain I. Then P' is properly contained in M. We shall show that $J^{-1} = P'I^{-1}$.

We check the inclusion $P'I^{-1} \subseteq J^{-1}$ locally. At M we have $I^{-1}P'JR_M = I^{-1}P'IR_M \subseteq R_M$. For $N \in \mathrm{Max}(R, I) \setminus \{M\}$, we have $I^{-1}P'JR_N = I^{-1}P'(IR_{P'} \cap R_N) \subseteq I^{-1}P'IR_{P'} = I^{-1}IP'R_N \subseteq R_N$. Finally, for $L \notin \mathrm{Max}(R, I)$, we have $I^{-1}P'JR_L = I^{-1}R_L = (IR_L)^{-1} = R_L$. Thus $P'I^{-1} \subseteq J^{-1}$.

For the reverse inclusion, let $t \in J^{-1}$. Choose any $N \in \mathrm{Max}(R, I)\setminus\{M\}$, and then choose $a \in NR_N \setminus P'R_N$. Then $a^{-1}I \subseteq IR_{P'} \cap R_N = JR_N$. Hence $ta^{-1}I \subseteq tJR_N \subseteq R_N$, yielding $tI \subseteq aR_N$. It follows that $tI \subseteq P'R_N \cap R = P'$. Thus $J^{-1}I \subseteq P'$, and we have $J^{-1} \subseteq I^{-1}P'$, as desired.

Finally, we show that J is not divisorial. Suppose, on the contrary, that $J = J^v = IP'^{-1}$. Then $I^{-1}J = P'^{-1}$. Now choose $m \in M \setminus P'$, and then choose $u \in (I, m)^{-1} \setminus R_M$. Then $(R :_R u) \not\subseteq P'$ and $(R :_R u) \not\subseteq L$ for each maximal ideal L with $L \notin \mathrm{Max}(R, I)$. It follows that $u \in R_{P'} \cap \left(\bigcap\{R_L \mid L \notin \mathrm{Max}(R, P')\}\right) = P'^{-1}$ [6, Theorem 3.1.2 and Corollary 3.1.8]. Hence $u \in P'^{-1}R_M = I^{-1}JR_M = R_M$, a contradiction. $\square$

Lemma 1.11. *Let R be a Prüfer domain. If R has the property that an ideal I of R is divisorial whenever IR_M is divisorial for each maximal ideal M, then R has the radical trace property.*

Proof. Assume that R has the property that each locally divisorial ideal is divisorial. By [14, Theorem 23], to show that R has the radical trace property, it suffices to show if Q is a P-primary ideal such that Q^{-1} is a ring, then $Q = P$. To this end, let Q be a proper P-primary ideal. Since R is integrally closed, Q^{-1} is a ring if and only if $Q^{-1} = P^{-1} = (P : P)$ [6, Proposition 3.1.16].

If P is not maximal, then QR_M is divisorial for each maximal ideal M (see the argument that JR_N is divisorial in Lemma 1.9 above). Hence Q is divisorial and we have $P^{-1} \subsetneq Q^{-1}$. Thus Q^{-1} is not a ring.

If P is maximal and Q is divisorial, then we again have that Q^{-1} is not a ring. The only other case is when $QR_P = xPR_P$ with P idempotent and x some nonzero element of P. Then $Q' = xR_P \bigcap R$ is a proper P-primary ideal which is divisorial since it is divisorial in each R_N. Hence we have $P^{-1} \subsetneq Q'^{-1} \subseteq Q^{-1}$, and again Q^{-1} is not a ring. $\square$

Theorem 1.12. *The following are equivalent for a Prüfer domain R.*

(1) *R is h-local.*
(2) *R has the strong factorization property.*

(3) *For each nonzero ideal I of R, I is divisorial if and only if IR_M is divisorial in R_M for each maximal ideal M of R.*
(4) *For each nonzero ideal I of R, if IR_M is divisorial for each maximal ideal M, then I is divisorial.*

Proof. Observe that (1) implies (2) by Theorem 1.1 (2), (2) implies (3) by Theorem 1.8 (1), and (3) implies (4) is trivial. Assume that R is a Prüfer domain with the property that each locally divisorial ideal is divisorial. Then it has the radical trace property by Lemma 1.11.

Now let P be a nonzero nonmaximal branched prime. Since R has the radical trace property, P is the radical of a finitely generated ideal I by [14, Theorem 23]. If M is a maximal ideal that contains P, then $J = IR_M \bigcap R$ is locally divisorial by Lemma 1.9. Hence by Theorem 1.10, M is the unique maximal ideal that contains P.

Since each unbranched prime must contain a nonzero branched prime, each nonzero prime is contained in a unique maximal ideal. Thus R is h-local by [16, Proposition 3.4]. □

Our next result adds another equivalence to the h-local property for Prüfer domains.

Theorem 1.13. *Let R be a Prüfer domain with finite character, and suppose that R has the weak factorization property. Then R is h-local.*

Proof. We shall make frequent use of the fact, which follows easily from [9, Theorem 1], that a Prüfer domain with finite character satisfies both the #- and ##-properties. To show that R is h-local, it suffices to show that each nonzero prime ideal is contained in a unique maximal ideal. Suppose to the contrary that R has a prime ideal P contained in more than one maximal ideal. Since R has finite character, P is contained in only finitely many maximal ideals, say $M_1, \ldots, M_n$, $n > 1$. Let $\{P_\alpha\}$ denote the set of prime ideals of R which contain P and are contained in $M_1 \cap (\bigcup_{j=2}^n M_j)$. This is a chain of prime ideals, and so $P_1 = \bigcup_\alpha P_\alpha$ is a prime ideal; moreover, $P_1 \subseteq M_1$, and, by prime avoidance, $P_1 \subseteq M_i$ for some $i > 1$. One sees easily that P_1 is maximal with respect to being contained in M_1 and at least one other maximal ideal. Hence we may as well assume that P has this property.

Denote by $\{N_\alpha\}$ the set of maximal ideals of R which do not contain P. Set $T = \bigcap_{j>1} R_{M_j} \cap (\bigcap_\alpha R_{N_\alpha})$. Since R has finite character, we may find a finitely generated ideal I with the property that M_1 is the only maximal ideal containing I. For $x \in I^{-1}$, we have $I \subseteq (R :_R x)$, so that $(R :_R x)$ is contained in M_1 but no other maximal ideal of R. It follows that $x \in T$. Hence $I^{-1} \subseteq T$, and since I is invertible, $I \supseteq T^{-1}$. In particular, $M_1 \supseteq T^{-1}$.

By [6, Corollary 3.1.8 and Theorem 3.1.2], $P^{-1} = R_P \cap (\bigcap_\alpha R_{N_\alpha})$. In particular $P^{-1} \supseteq T$. By Proposition 1.7, P is divisorial. Hence $P \subseteq T^{-1}$. We claim, in fact, that $P = T^{-1}$. Suppose not. Then shrink M_1 to a prime ideal Q minimal over T^{-1}. By the maximality property of P and the fact that R has the ##-property, we may choose a finitely generated ideal J contained in Q such that M_1 is the only maximal ideal of R containing J. As in the preceding paragraph, we have $T^{-1} \subseteq J$. In fact, $T^{-1} \subseteq J^n$ for each positive integer n. Hence in R_{M_1}, we have that $T^{-1}R_{M_1}$ is contained in the prime ideal $\bigcap_{n \geq 1} J^n R_{M_1}$ of R_{M_1}. This prime ideal is $Q_0 R_{M_1}$ for some prime ideal Q_0 of R, and we must have $P \subseteq T^{-1} \subseteq Q_0 \subsetneqq Q$, a contradiction. Thus $P = T^{-1}$, as claimed.

We next claim that T is a fractional ideal of R which is not divisorial. Otherwise, the fact that $P = T^{-1}$ implies that $P^{-1} = T$. However, observe that $T \subseteq R_{M_2}$, and so it suffices to show that $P^{-1} \nsubseteq R_{M_2}$. To see this, observe by the #-property, $R_{M_1} \cap (\bigcap_\alpha R_{N_\alpha}) \nsubseteq R_{M_2}$. Since $P^{-1} = R_P \cap (\bigcap_\alpha R_{N_\alpha}) \supseteq R_{M_1} \cap (\bigcap_\alpha R_{N_\alpha})$, we also have $P^{-1} \nsubseteq R_{M_2}$. Thus T is not divisorial. Note that $P^{-1} = T^v \neq T$.

Now consider a possible factorization of T: $T = T^v \cdot \Pi$, where Π is a product of maximal ideals. Then $T = P^{-1}\Pi$. Since $P^{-1} \subseteq R_{N_\alpha}$, we have $N_\alpha P^{-1} \neq P^{-1}$ (note that P^{-1} is a ring). If N_α appears as part of Π, then $1 \in T = P^{-1}\Pi \subseteq P^{-1}N_\alpha$, a contradiction. Hence no N_α appears in Π. On the other hand, we claim that $M_i P^{-1} = P^{-1}$ for each i. Otherwise, P^{-1} contains a prime ideal L contracting to M_i in R, from which it follows that the valuation domains $(P^{-1})_L$ and R_{M_i} must coincide. However, the argument in the preceding paragraph showing that $P^{-1} \nsubseteq R_{M_2}$ can easily be adapted to show that $P^{-1} \nsubseteq R_{M_i}$. Hence the claim is true, and we have $T = P^{-1}\Pi = P^{-1}$, a contradiction. This completes the proof. □

The situation with respect to the weak factorization property is dramatically different. Suppose that R is an almost Dedekind domain with exactly one nondivisorial maximal ideal—see [8, Example 42.6]. Then R is certainly not h-local, but Theorem 1.15 below implies that R has the weak factorization property.

Lemma 1.14. *Let R be an almost Dedekind domain, let P be an invertible maximal ideal of R, and let I be a nonzero ideal of R. Then $I^v R_P = IR_P$.*

Proof. Since P is invertible, so is P^i for each $i = 1, 2, \ldots$. Hence $I \subseteq P^i$ if and only if $I^v \subseteq P^i$. Since R_P is a rank one discrete valuation domain, we have $IR_P = P^n R_P$ for some $n \geq 0$. Since P^n is primary, we then have $I \subseteq IR_P \cap R \subseteq P^n R_P \cap R = P^n$. Note that $I \not\subseteq P^{n+1}$. It follows that $I^v \subseteq P^n$ and hence that $I^v R_P = P^n R_P = IR_P$. $\square$

Theorem 1.15. *Let R be an almost Dedekind domain, and let I be a nonzero ideal of R which is contained in only finitely many nondivisorial maximal ideals of R. Then $I = I^v \cdot \Pi$, where Π is a product of maximal ideals. Thus, if R is an almost Dedekind domain in which each nonzero ideal is contained in only finitely many nondivisorial maximal ideals, then R has the weak factorization property.*

Proof. Denote by $M_1, \ldots, M_n$ the noninvertible maximal ideals which contain I. For $M \in \{M_i\}$, we have $IR_M = M^r R_M$ and $I^v R_M = M^s R_M$ for integers r, s with $0 \leq s \leq r$. Hence $IR_M = I^v M^{r-s} R_M$. Therefore, for each $i = 1, \ldots, n$, we have a nonnegative integer t_i with $IR_{M_i} = I^v M^{t_i} R_{M_i}$. We claim that $I = I^v \cdot \prod_{i=1}^n M_i^{t_i}$. We verify this locally. Let P be a maximal ideal of R. If $P = M_j$ for some j, then

$$IR_P = IR_{M_j} = I^v M_j^{t_j} R_{M_j} = I^v \cdot \left(\prod_{i=1}^n M_i^{t_i} R_{M_j}\right) = \left(I^v \cdot \prod_{i=1}^n M_i^{t_i}\right) R_P.$$

If $P \notin \{M_i\}$ and P is invertible, then, applying Lemma 1.14, we have

$$IR_P = I^v R_P = \left(I^v \cdot \prod_{i=1}^n M_i^{t_i}\right) R_P.$$

Finally, if $P \notin \{M_i\}$ and P is noninvertible, then $I \not\subseteq P$, so that

$$IR_P = R_P = I^v R_P = \left(I^v \cdot \prod_{i=1}^n M_i^{t_i}\right) R_P. \quad \square$$

Thus any almost Dedekind domain with only finitely many nondivisorial maximal ideals has the weak factorization property by Theorem 1.15. In fact, it is possible to give examples of almost Dedekind domains which have infinitely many nondivisorial maximal ideals but in which each nonzero ideal is nonetheless contained in only finitely many nondivisorial maximal ideals—see Example 3.2 below.

The next result shows that the integers t_i in the proof of Theorem 1.15 cannot be "controlled".

Proposition 1.16. *Let R be an almost Dedekind domain, let $M_1 \ldots, M_n$ be distinct noninvertible maximal ideals of R, and let $r_1, \ldots, r_n, s_1, \ldots, s_n$ be integers with $0 \leq s_i \leq r_i$. Then there is a nonzero ideal I of R such that $I = I^v \cdot \prod_{i=1}^n M_i^{r_i - s_i}$, and for each j, $IR_{M_j} = M_j^{r_j} R_{M_j}$ and $I^v R_{M_j} = M_j^{s_j} R_{M_j}$.*

Proof. Note that $M_i \neq M_i^2$ for each i (since this is true locally). Hence by "extended" prime avoidance [12, Theorem 81], we may pick $a_i \in M_i \setminus (\bigcup_{j \neq i} M_j \cup M_i^2)$. Note that we then have $M_i R_{M_i} = a_i R_{M_i}$. Set $I = \prod_{i=1}^n a_i^{s_i} M_i^{r_i - s_i}$. Since the M_i are nondivisorial, we have $I^v = \prod_{i=1}^n a_i^{s_i} R$ and hence $I = I^v \cdot \prod_{i=1}^n M_i^{r_i - s_i}$. Moreover, for each j, $IR_{M_j} = a_j^{s_j} M_j^{r_j - s_j} R_{M_j} = M_j^{r_j} R_{M_j}$, and $I^v R_{M_j} = a_j^{s_j} R_{M_j} = M_j^{s_j} R_{M_j}$. $\square$

2. Effects of the strong factorization property

Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D-submodules of K, and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, i.e., $E \in \boldsymbol{F}(D)$ if $E \in \overline{\boldsymbol{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of K. Then, obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$. A *semistar operation on* D is a map $* : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, such that, for each nonzero element $x \in K$ and for each $E, F \in \boldsymbol{F}(D)$, we have:

(1) $(xE) = (xE)^*$,
(2) $E^* \subseteq F^*$ whenever $E \subseteq F$, and
(3) $E \subseteq E^*$ and $(E^*)^* = E^*$.

The semistar operation $*$ on D is called a *(semi)star operation on D* if $D^* = D$. (The use of the term "(semi)star" is due to the fact that, when $D = D^*$, $*$ is not really a star operation since it remains defined on the D-submodules of K and not only on the fractional ideals.)

A *localizing system on D* is a set $\mathcal{F}$ of ideals of D such that:

(1) if $I \in \mathcal{F}$ and J is an ideal of D with $I \subseteq J$, then $J \in \mathcal{F}$, and
(2) if $I \in \mathcal{F}$ and J is an ideal of D with $(J :_D a) \in \mathcal{F}$ for each $a \in I$, then $J \in \mathcal{F}$.

It is easily seen that a localizing system $\mathcal{F}$ is a multiplicative system of ideals and that $D_{\mathcal{F}} := \{x \in K \mid xI \subseteq D \text{ for some } I \in \mathcal{F}\}$ is an overring of D. For background on localizing systems, see [4], and for background on semistar operations, see [15,4].

Now set

$$\mathcal{F}^v := \{I \mid I \text{ ideal of } D, I^v = D\},$$
$$\Pi^v := \{Q \in \mathrm{Spec}(D) \mid Q^v \neq D \text{ and } Q \neq 0\},$$
$$\mathcal{F}(\Pi^v) := \{I \mid I \text{ ideal of } D, I \not\subseteq Q, \text{ for each } Q \in \Pi^v\}.$$

Lemma 2.1. (1) *$\mathcal{F}^v$ is a localizing system of D (called the localizing system associated with the v-operation).*
(2) *The operation $\overline{v} := *_{\mathcal{F}^v}$ defined, for each $E \in \overline{\boldsymbol{F}}(D)$, as follows:*

$$E^{\overline{v}} := \bigcup\{(E : I) \mid I \in \mathcal{F}^v\},$$

is a (semi)star operation defined on D which is stable (i.e. $(E \cap F)^{\overline{v}} = E^{\overline{v}} \cap F^{\overline{v}}$, for all $E, F \in \overline{\boldsymbol{F}}(D)$), and it is the largest stable (semi)star operation on D.
(3) *The operation $v_{\mathrm{sp}} := *_{\Pi^v}$ defined, for each $E \in \overline{\boldsymbol{F}}(D)$, as follows:*

$$E^{v_{\mathrm{sp}}} := \bigcap\{ED_Q \mid Q \in \Pi^v\},$$

is a semistar operation defined on D (called the spectral semistar operation associated with the v-operation) and $\overline{v} \leq v_{\mathrm{sp}}$.
(4)

$$\mathcal{F}^{v_{\mathrm{sp}}} := \{I \mid I \text{ ideal of } D, I^{v_{\mathrm{sp}}} = D\} = \mathcal{F}(\Pi^v).$$

(5) *The following are equivalent:*
 (i) *v_{sp} is a (semi)star operation on D;*
 (ii) *$v_{\mathrm{sp}} \leq v$;*
 (iii) *$D = \bigcap\{D_Q \mid Q \in \Pi^v\}$.*

Proof. Statements (1)–(3) follow from [4, Proposition 2.8, Theorem 2.10 (B), Proposition 3.7 (1), and Proposition 4.11 (2)]. Statements (4) and (5) are easy consequences of the definitions. □

Remark 2.2. Note, with respect to Lemma 2.1 (2), that $\overline{v} \leq v$ and so $D^{\overline{v}} = D^v = D$; hence $\overline{v}$ is a (semi)star operation on D. As a matter of fact, if $x \in E^{\overline{v}} = \bigcup\{(E : I) \mid I \in \mathcal{F}^v\}$ then, for some $I \in \mathcal{F}^v$, we have that $I \subseteq (E :_D xD)$, thus $(E :_D xD) \in \mathcal{F}^v$. Therefore, $D = (E :_D xD)^v \subseteq (E^v :_D xD)$, and hence necessarily $1 \in (E^v :_D xD)$, thus $x \in E^v$.

Proposition 2.3. *Assume that D is an h-local Prüfer domain. Then:*

(1) *$\overline{v} = v$.*
(2) *The following statements are equivalent:*
 (i) *The v-operation is quasi-spectral (i.e. for each nonzero ideal I of D, with $I^v \neq D$, there exists a prime ideal Q of D such that $I \subseteq Q$ and $Q = Q^v$);*
 (ii) *$v_{\mathrm{sp}} \leq v$;*
 (iii) *$D = \bigcap\{D_Q \mid Q \in \mathrm{Spec}(D), Q = Q^v\}$;*
 (iv) *$\overline{v} = v_{\mathrm{sp}} = v$;*
 (v) *$\mathcal{F}^v = \mathcal{F}(\Pi^v)$.*

Proof. (1) It is easy to see that $M^{\overline{v}} = D$ for each nondivisorial maximal ideal M. Hence if I is a nonzero ideal of D, the factorization $I = I^v M_1 \cdots M_n$ yields $I^{\overline{v}} = (I^v M_1 \cdots M_n)^{\overline{v}} = (I^v)^{\overline{v}} = I^v$.

(2) These equivalences follow from (1), Theorem 1.8, Lemma 2.1(5), and [4, Proposition 4.8 and Theorem 4.12 (2)]. □

Remark 2.4. If V is a valuation domain whose maximal ideal N is idempotent but branched, then V does not satisfy any of the (equivalent) conditions in Proposition 2.3(2). On the other hand, if D is an h-local Prüfer domain with non-idempotent maximal ideals, then each nonzero ideal of D is divisorial [10, Theorem 5.1]; in this case, the (equivalent) conditions in Proposition 2.3(2) hold trivially.

We next study how factorization of an ideal I affects the factorization of its radical and how factorization of ideals I and J affect the factorization of IJ, $I \cap J$, and $I + J$.

Proposition 2.5. *Let R be an h-local Prüfer domain, and let I, J be nonzero ideals of R. Suppose that I, J have the following factorizations as in Definition 1.2:*

$$I = I^v M_1 \cdots M_k M_{k+1} \cdots M_m H_1 \cdots H_r \quad \textit{and}$$
$$J = J^v N_1 \cdots N_l N_{l+1} \cdots N_n H_1 \cdots H_r,$$

where the H_i are the nondivisorial maximal ideals which contain $I + J$ and for which both IR_{H_i} and JR_{H_i} are nondivisorial, JR_{M_i} is principal (including the possibility that $JR_{M_i} = R_{M_i}$) for $i = 1, \ldots, k$, and divisorial but not principal for $i = k+1, \ldots, m$, and IR_{N_i} is principal for $i = 1, \ldots, l$ and divisorial but not principal for $i = l+1, \ldots, n$. Further assume that $P_1, \ldots, P_u$ are the nondivisorial maximal ideals for which IR_{P_i} and JR_{P_i} are divisorial but IJR_{P_i} is not divisorial for each i. Then the canonical factorizations of IJ and I^vJ^v are as follows:

$$IJ = (IJ)^v M_1 \cdots M_k N_1 \cdots N_l H_1 \cdots H_r P_1 \cdots P_u \tag{1}$$
$$I^v J^v = (IJ)^v P_1 \cdots P_u. \tag{2}$$

Proof. (1) For each $i = 1, \ldots, k$, we have elements $x_i, y_i \in R$ with $IR_{M_i} = x_i M_i R_{M_i}$ and $JR_{M_i} = y_i R_{M_i}$, so that $IJR_{M_i} = x_i y_i M_i R_{M_i}$. Thus IJR_{M_i} is not divisorial, and each of these M_i must appear in the factorization of IJ. Similarly, $N_1, \ldots, N_l$ must appear. For $i = k+1, \ldots, m$, there is an element $z_i \in R$ with $IJR_{M_i} = z_i M_i JR_{M_i} = z_i JR_{M_i}$; the second equality follows from the fact that in a valuation domain with maximal ideal Q a nonprincipal ideal K satisfies $K = KQ$ (see Remark 1.4). In this case, IJR_{M_i} is divisorial, and so M_i does not appear in the factorization of IJ. Similarly, $N_{l+1}, \ldots, N_n$ do not appear. For $H \in \{H_i\}_{i=1}^r$, since both IR_H and JR_H are nondivisorial, there are elements x, y with $IJR_H = xHyHR_H = xyHR_H$ (note that H is idempotent by Theorem 1.8(2)); this is not divisorial, so each H_i must appear. Finally, it is clear that the P_i must appear and that no other maximal ideals can appear.

(2) First, observe that if Q is a nondivisorial maximal ideal for which IR_Q, JR_Q, and IJR_Q are all divisorial, then by [2, Lemma 2.3], $I^vJ^vR_Q = (IR_Q)^v(JR_Q)^v = IJR_Q$, which is divisorial. Hence no such Q appears in the factorization of I^vJ^v. Let $M \in \{M_i\}_{i=1}^m$. Then there is an element $x \in R$ with $I^vJ^vR_M = I^vJR_M = J(IR_M)^v = J(xMR_M)^v = JxR_M$, which is divisorial. Thus no M_i appears; similarly, no N_i appears. For $H \in \{H_i\}_{i=1}^r$, we have an element $y \in R$ with $I^vJ^vR_H = I^v(JR_H)^v = I^v(yMR_H)^v = I^v yR_H$, which is divisorial. Thus no H_i appears. □

Proposition 2.6. *Let R be an h-local Prüfer domain, and let I, J be nonzero ideals of R. Suppose that I, J have the following factorizations as in Definition 1.2:*

$$I = I^v M_1 \cdots M_k M_{k+1} \cdots M_m H_1 \cdots H_r \quad \textit{and}$$
$$J = J^v N_1 \cdots N_l N_{l+1} \cdots N_n H_1 \cdots H_r,$$

where the H_i are the nondivisorial maximal ideals which contain $I + J$ and for which both IR_{H_i} and JR_{H_i} are nondivisorial, $IR_{M_i} \subseteq JR_{M_i}$ for $i = 1, \ldots, k$, $IR_{M_i} \not\subseteq JR_{M_i}$ for $i = k+1, \ldots, m$, $JR_{N_i} \subseteq IR_{N_i}$ for $i = 1, \ldots, l$, and $JR_{N_i} \not\subseteq IR_{N_i}$ for $i = l+1, \ldots, n$. Then $I \cap J$ has the following factorization

$$I \cap J = (I \cap J)^v M_1 \cdots M_k N_1 \cdots N_l H_1 \cdots H_r.$$

Proof. For $i = 1, \ldots, k$, $(I \cap J)R_{M_i} = IR_{M_i}$, and so M_i appears in the factorization of $I \cap J$. Moreover, for $j > k$, $(I \cap J)R_{M_j} = JR_{M_j}$; since JR_{M_j} is divisorial, M_j does not appear. The N_i are handled similarly. Finally, it is straightforward to show that the H_i appear and that no other maximal ideals can appear. □

Proposition 2.7. *Let R be an h-local Prüfer domain, and let I, J be nonzero ideals of R. Then:*

(1) *If I and J are divisorial, then $I + J$ is divisorial.*
(2) *In general, $(I + J)^v = I^v + J^v$.*
(3) *Let I and J have the following factorizations as in Definition 1.2:*

$$I = I^v M_1 \cdots M_k M_{k+1} \cdots M_m H_1 \cdots H_r$$
$$J = J^v N_1 \cdots N_l N_{l+1} \cdots N_n H_1 \cdots H_r,$$

where the H_i are the nondivisorial maximal ideals which contain $I + J$ and for which both IR_{H_i} and JR_{H_i} are nondivisorial, $IR_{M_i} \subseteq JR_{M_i}$ for $i = 1, \ldots, k$, $IR_{M_i} \nsubseteq JR_{M_i}$ for $i = k+1, \ldots, m$, $JR_{N_i} \subseteq IR_{N_i}$ for $i = 1, \ldots, l$, and $JR_{N_i} \nsubseteq IR_{N_i}$ for $i = l+1, \ldots, n$. Then the factorization of $I + J$ is

$$I + J = (I + J)^v M_{k+1} \cdots M_m N_{l+1} \cdots N_n H_1 \cdots H_r.$$

Proof. (1) Let M be a maximal ideal of R. By Theorem 1.8, both IR_M and JR_M are divisorial. Since $(I + J)R_M$ is equal to one of these, it is divisorial. Hence $I + J$ is divisorial, again by Theorem 1.8.

(2) Using (1), we have $(I + J)^v = (I^v + J^v)^v = I^v + J^v$.

(3) Using the same reasoning as in the proof of Proposition 2.5 (1), we see easily that each H_i must appear in the factorization of $I + J$. Similarly, for $M \in \{M_i\}_{i=k+1}^m$, we have $(I + J)R_M = IR_M$, so these M_i must appear. Each N_i, $i = l+1, \ldots, n$, must also appear. The same reasoning shows that none of the other M_i or N_j can appear, and it is clear that no other maximal ideals can appear. □

Proposition 2.8. *Let R be an h-local Prüfer domain, and let I be an ideal of R with factorization (as in Definition 1.2)*

$$I = I^v M_1 \cdots M_l M_{l+1} \cdots M_k M_{k+1} \cdots M_n,$$

where $M_1, \ldots, M_k$ are minimal over I, $I^v \subseteq M_i$ for $i = 1, \ldots, l$, $I^v \nsubseteq M_i$ for $i = l+1, \ldots, k$, and M_i is not minimal over I for $i = k+1, \ldots, n$. Let $\{N_1, \ldots, N_r\}$ denote the (possibly empty) set of nondivisorial maximal ideals that are minimal over I and are such that IR_{N_i} is divisorial. Then:

(1) *The factorization of* $\operatorname{rad} I$ *is* $\operatorname{rad} I = (\operatorname{rad} I)^v M_1 \cdots M_k N_1 \cdots N_r$.
(2) $(\operatorname{rad} I)^v = (\operatorname{rad} I^v)^v$.
(3) *The factorization of* $\operatorname{rad} I^v$ *is* $\operatorname{rad} I^v = (\operatorname{rad} I)^v M_1 \cdots M_l N_1 \cdots N_r$.

Proof. (1) For $i = 1, \ldots, k$, $(\operatorname{rad} I)R_{M_i} = M_i R_{M_i}$, so M_i must appear in the factorization of $\operatorname{rad} I$. Also, since $(\operatorname{rad} I)R_{N_i} = N_i R_{N_i}$, each N_i must appear. For any other nondivisorial maximal ideal P containing I, P is not minimal over I, whence $(\operatorname{rad} I)R_P$ is a nonmaximal, and hence divisorial, prime ideal of R_P.

(3) We have $(\operatorname{rad} I)^v = (\operatorname{rad}(I^v \prod_{i=1}^n M_i))^v = (\operatorname{rad} I^v \cap \prod_{i=1}^n M_i)^v = (\operatorname{rad} I^v)^v$, with the last equality following from the fact that the v-operation is stable in the presence of strong factorization (Proposition 2.3).

(4) For $Q \in \{M_i\}_{i=1}^l$, it is clear that Q is minimal over I^v. For $Q \in \{N_i\}_{i=1}^r$, use the fact that R is h-local to obtain $IR_Q = I^v R_Q$. Since $I \subseteq Q$, we must have $I^v \subseteq Q$, and, again, Q is minimal over I^v. In either case, we therefore have $(\operatorname{rad} I^v)R_Q = QR_Q$, which is nondivisorial, whence Q must appear in the factorization of $\operatorname{rad} I^v$. It is clear that no other maximal ideals can appear. □

Proposition 2.9. *Let R be an h-local Prüfer domain. Let I be a nonzero ideal of R, and suppose that the factorization of I (as in Definition 1.2) is $I = I^v M_1 \cdots M_n$. Let $P_1, \ldots, P_u$ be the nondivisorial maximal ideals containing II^{-1} for which IR_{P_i} is divisorial but $II^{-1}R_{P_i}$ is nondivisorial. Then the factorization of II^{-1} is $II^{-1} = (II^{-1})^v M_1 \cdots M_n P_1 \cdots P_u$.*

Proof. For $M \in \{M_i\}_{i=1}^n$, there is an element $x \in R$ with

$$II^{-1}R_M = xMI^{-1}R_M = xM(IR_M)^{-1} = xM(xMR_M)^{-1} = MR_M,$$

where the second equality follows from the fact that R is h-local [2, Lemma 2.3]. Hence each M_i must appear. It is clear that each P_i must appear and that no other maximal ideals can appear. □

We observe that the P_i in Propositions 2.5 and 2.9 can actually occur—see Example 3.4 below.

We end this section with a result which contains more information related to Propositions 2.6 and 2.9.

Proposition 2.10. *Let R be an h-local Prüfer domain. If I is a nondivisorial ideal of R with factorization $I = I^v M_1 \cdots M_n$ (as in Definition 1.2), then*

(1) *for each $i = 1, \ldots, n$, $I^v I^{-1} \not\subseteq M_i$, and $I^v R_{M_i}$ is principal;*
(2) *$II^{-1} = I^v I^{-1} M_1 \cdots M_n$, and for each $i = 1, \ldots, n$ M_i is minimal over II^{-1} and $II^{-1}R_{M_i} = M_i R_{M_i}$;*
(3) *there is a finitely generated ideal $J \subseteq I^v$ with $I + J = I^v$, and, for any such J, $(I \cap J)^v = J$; and*
(4) *for each nonzero ideal $B \subseteq I^v$, $(I \cap B)^v = B^v$.*

Proof. Let $M \in \{M_i\}$. Write $I^v M R_M = IR_M = xMR_M$, where (we may assume) $x \in I^v$. Then $I^v R_M = (IR_M)^v = xR_M$, and by [16, Theorem 3.10] $I^{-1}R_M = (IR_M)^{-1} = x^{-1}R_M$. It follows that $I^v I^{-1} \not\subseteq M$. In particular, $I^v R_M$ is invertible, and hence principal, in R_M, proving (1).

For (2), from what was just proved, we have $II^{-1} = I^v I^{-1} M_1 \cdots M_n$ with $I^v I^{-1}$ and $M_1 \cdots M_n$ comaximal. Thus $II^{-1}R_{M_i} = M_i R_{M_i}$, as desired.

Now let $J = (x_1, \ldots, x_n) \subseteq I^v$ be such that $x_i R_{M_i} = I^v R_{M_i}$ for each i. Then for $M \in \{M_i\}$, we have $I^v R_M = JR_M = (I + J)R_M$. On the other hand, if N is a maximal ideal with $N \notin \{M_i\}$, then $IR_N = (IR_N)^v = I^v R_N$, from which it follows easily that $(I + J)R_N = I^v R_N$. Therefore, $I + J = I^v$. Using Proposition 2.3(1), we also obtain $(I \cap J)^v = I^v \cap J^v = J^v = J$, proving (3). Statement (4) also follows from Proposition 2.3(1). $\square$

3. Examples

We begin with a lemma which is probably known but for which we have no convenient reference.

Lemma 3.1. *For any nonempty set of indeterminates $\mathcal{Z} = \{Z_\alpha\}$ and any field F, the ring $D = \bigcap F[\mathcal{Z}]_{(Z_\alpha)}$ is a PID with* $\mathrm{Max}(D) = \{Z_\alpha D \mid Z_\alpha \in \mathcal{Z}\}$.

Proof. Let $f \in F[\mathcal{Z}]$. If no Z_α divides f in $F[\mathcal{Z}]$, then f^{-1} is in each localization $F[\mathcal{Z}]_{(Z_\alpha)}$. Thus a reduced rational expression g/f from the quotient field of $F[\mathcal{Z}]$ is in D if and only if no Z_α divides f. Thus each element of D has the reduced form g/f where no Z_α divides f. Clearly g/f is a unit of D if and only if no Z_α divides g. It follows that each nonzero prime ideal of D is principal of the form $Z_\alpha D$ for some (unique) Z_α. Hence D is a PID. $\square$

Example 3.2. An example of an almost Dedekind domain D with infinitely many nondivisorial maximal ideals such that D has the weak factorization property.

Notation:

(1) For each $n \geq 1$, let $\mathcal{X}_n = \prod_{i>0} X_{n,i}$ where $\{X_{n,i} \mid 1 \leq i,\ 1 \leq n\}$ is a countably infinite set of algebraically independent indeterminates.
(2) For each n and each $k \geq 0$, let $\mathcal{X}_{n,k} = \prod_{i>k} X_{n,i}$ (so $\mathcal{X}_{n,0} = \mathcal{X}_n$).
(3) Let $E_0 = K[\{\mathcal{X}_n \mid 1 \leq n\}]$ and for each n, let $Q_{n,0} = (\mathcal{X}_n)E_0$.
(4) For each $k \geq 1$, let $E_k = K[\{X_{n,j} \mid 1 \leq j \leq k, 1 \leq n\}, \{\mathcal{X}_{n,k} \mid 1 \leq n\}]$, $P_{n,j} = (X_{n,j})E_k$ for $j \leq k$ and $Q_{n,k} = (\mathcal{X}_{n,k})E_k$.
(5) Let $D_0 = \bigcap (E_0)_{Q_{n,0}}$ and for $k \geq 1$, let $D_k = \left(\bigcap (E_k)_{Q_{n,k}}\right) \cap \left(\bigcap (E_k)_{P_{n,j}}\right)$.
(6) Finally let $D = \bigcup D_k$.

Then

(1) D is an almost Dedekind domain which is also a Bezout domain.
(2) Each nonzero ideal is contained in at most finitely many nondivisorial maximal ideals.
(3) D has the weak factorization property.

Proof. Each D_k is a PID. Also it is clear that each maximal ideal M of D_k contracts to a maximal ideal N_j of D_j for each $j < k$ and $N_j(D_k)_M = M(D_k)_M$. Moreover, each maximal ideal of D_k survives in D_m for each $m > k$. Thus by [13, Theorem 2.10], D is an almost Dedekind domain that is also a Bezout domain — given a finitely generated ideal I of D, $I = I_k D$ where $I_k = I \cap D_k$ for some k.

By the proof of [13, Theorem 2.10], each maximal ideal M of D is the union of its contractions to the D_k's. As in the proof of [13, Example 3.2], D has two distinct types of maximal ideals. For each $X_{n,k}$, the ideal $M_{n,k} = X_{n,k}D$ is a principal maximal ideal of D. The other maximal ideals are those of the form $M_n = \bigcup_{j\geq 0} Q_{n,j}$. For each n, we let $\mathcal{F}_n = \{M_n, M_{n,1}, M_{n,2}, \ldots\}$ and call this the family of maximal ideals centered on $\mathcal{X}_n$. These are the only maximal ideals of D that contain $\mathcal{X}_n$ (and each does). Since D is an almost Dedekind domain, some member of $\mathcal{F}_n$ is not finitely generated. The only one that is not principal is M_n. Thus M_n is not divisorial.

For a nonzero proper ideal I, recall that $\mathrm{Max}(R, I)$ is the set of maximal ideals of D that contain I; let us refer to this as the *support* of I. We will show that $\mathrm{Max}(R, I)$ is contained in a finite union of families $\mathcal{F}_n$. To this end, let f be a nonzero nonunit of D and let D_k be the smallest member of the chain that contains f. By the argument above, $f = ug/v$ with u and v units of D_k and g a finite product of monomials of the form $\mathcal{X}_{n,k}$ and $X_{m,i}$ with $i \leq k$. Since u and v are units of D, the monomials in g completely determine the families that contain the support of f. Thus $\mathrm{Max}(R, (f))$ is contained in the union of finitely many families $\mathcal{F}_n$. Hence the same is true for the support of each nonzero proper ideal. Moreover, since each family contains exactly one nondivisorial ideal, each nonzero proper ideal is contained in at most finitely many nondivisorial maximal ideals. Therefore, D has the weak factorization property by Theorem 1.15. □

Example 3.3. An example of a Prüfer domain R with the weak factorization property such that R contains ideals I, J with I and J divisorial but $I + J$ not divisorial.

We recall the construction of the domain in [13, Example 3.2].

Let $\mathcal{X} = \prod_{i>0} X_i$, where the X_i are indeterminates. Let K be a field, and for each n, let $\mathcal{X}_n = \prod_{k\geq n} X_k$ and $E_n = K[X_1, \ldots, X_{n-1}, \mathcal{X}_n]$ ($E_0 = K[\mathcal{X}]$). Set $P_{n,k} = X_k E_n$, $P_n = \mathcal{X}_n E_n$, and $D_n = \left(\bigcap_{k<n}(E_n)_{P_{n,k}}\right) \cap (E_n)_{P_n}$. Let $Q_{n,k} = P_{n,k}D_n$ and $Q_n = P_n D_n$. Then each D_n is a semilocal PID, and $D = \bigcup D_n$ is an almost Dedekind domain with a unique noninvertible maximal ideal. We also have the following.

(1) D has only countably many maximal ideals $M, M_1, M_2, \ldots$, where $M = \bigcup Q_n$, and $M_n = X_n D$. Also D has nonzero Jacobson radical, since $\mathcal{X}$ is in each maximal ideal. The maximal ideal M is nondivisorial, while the M_n's are all principal.
(2) The ideals $I = \bigcap_{k\geq 1} M_{2k}$ and $J = \bigcap_{k\geq 1} M_{2k-1}$ are (nonzero) divisorial ideals, but $I + J$ is nondivisorial.
(3) D has the weak factorization property.

Proof. Statement (1) is from [13, Example 3.2].

Since D has nonzero Jacobson radical, I and J are nonzero; they are divisorial since each M_n is divisorial. We have $I + J \subseteq M$ since each element of D which is contained in infinitely many M_n is also in M (see either Lemma 2.2 or Theorem 2.5 of [13]). In fact, we claim that $I + J = M$. Observe that $\mathcal{X}R_M = MR_M$ so that $(I + J)R_M = MR_M$. Moreover, for each positive even integer k the element $X_2X_4 \cdots X_k\mathcal{X}_{k+1}$ is in I but is a unit in $D_{M_{k-1}}$; hence $(I + J)D_{M_{k-1}} = ID_{M_{k-1}} = D_{M_{k-1}} = MD_{M_{k-1}}$. Applying the same argument to J, we obtain $(I + J)D_{M_k} = MD_{M_k}$. It follows that $I + J = M$, so $I + J$ is not divisorial. □

Example 3.4. An example of a valuation containing V containing a divisorial I for which II^{-1} is not divisorial (thus the product of divisorial ideals need not be divisorial).

Let (V, M) be an valuation domain with value group the additive rational numbers. Note that M is not principal and therefore not divisorial. Let I denote the ideal consisting of those elements of V having value greater than $\sqrt{2}$. For each positive rational number α, let x_α denote an element of V with value α. Then $I = \bigcap_{\alpha<\sqrt{2}}(x_\alpha)$. Hence I is divisorial. However, I is not (principal hence not) invertible, whence by [6, Proposition 4.2.1] II^{-1} must be a prime ideal of V. Since V is one-dimensional, we must therefore have $II^{-1} = M$, which is not divisorial.

Example 3.5. An example of a one-dimensional Bezout domain which does not have the weak factorization property.

Let $\mathcal{X} = \prod_{k\geq 0} X_k^{2^k}$ where $\{X_k\}$ is a countably infinite set of indeterminates. Let K be a field, and for each integer n, let $\mathcal{X}_n = \prod_{k\geq n} X_k^{2^{k-n}}$ and $E_n = K[X_0, X_1, \ldots, X_{n-1}, \mathcal{X}_n]$ (with $E_0 = K[\mathcal{X}]$). Let $P_{n,k} = X_k E_n$ for $k < n$, $P_n = \mathcal{X}_n E_n$, and $D_n = (\bigcap (E_n)_{P_{n,k}}) \cap (E_n)_{P_n}$. Use $Q_{n,k}$ to denote the extension of $P_{n,k}$ to D_n and Q_n to denote the extension of P_n to D_n. Each $Q_{n,k}$ is principal as is each Q_n. Also each D_n is a semilocal PID.

Let $D = \bigcup D_n$. Then D is a one-dimensional Bezout domain with nonzero Jacobson radical. Also, D has countably many maximal ideals. Of these, all but one is principal. The one that is not principal is idempotent. This maximal ideal is not the radical of a finitely generated ideal, so it is non-sharp. It follows that from Proposition 1.7(3) that D does not have the weak factorization property.

Proof. Let $I = (a_1, a_2, \ldots, a_m)$ be a finitely generated proper ideal of D. Let D_n be the smallest ring in $\{D_i\}$ that contains the set $\{a_1, a_2, \ldots, a_m\}$. Since D_n is a PID, there is an element $a \in I \cap D_n$ such that $I \cap D_n = aD_n$. In particular, each a_i is in aD_n and it follows that $I = aD$. Thus D is a Bezout domain.

For integers $0 \leq m < n$ and $0 \leq k < n$, $Q_{n,k} \bigcap D_m = Q_{m,k}$ when $k < m$ and $Q_{n,k} \bigcap D_m = Q_m$ when $m \leq k$. In the first case, $Q_{m,k}(D_m)_{Q_{n,k}} = Q_{n,k}(D_n)_{Q_{n,k}}$, and in the second, $Q_m(D_n)_{Q_{n,k}} = Q^j_{n,k}(D_n)_{Q_{n,k}}$ where $j = 2^{k-m}$.

Let f be a nonzero member of D. Since D is the union of the chain D_n and no nonunit of D_n becomes a unit in a larger D_m, f is a nonunit of D if and only if it is a nonunit in the smallest D_n that contains it. In D_n, f is a nonunit if and only if has the form ug/v where u and v are polynomials of E_n that are units of D_n and g is a finite (nonempty) product of the monomials $X_0, X_1, \ldots, X_{n-1}$ and $\mathcal{X}_n$. If the factorization of g does not include a positive power of $\mathcal{X}_n$, then for all $m > n$, $f \notin Q_m$. On the other hand, if the factorization of g does include a positive power of $\mathcal{X}_n$, then $f \in Q_m$ for all $m \geq n$. In the latter case, we also have that $f \in Q_{m,k}$ for all $m > k \geq n$ since $\mathcal{X}_n = \mathcal{X}_m^{2^{m-n}} \prod_{k=n}^{m-1} X_k^{2^{k-n}}$.

For each n the ideal $M_n = X_n D$ is a height one maximal ideal of D, being the union of the chain of primes $Q_0 \subset \cdots \subset Q_{n-1} \subset Q_{n,n} \subset Q_{n+1,n} \subset \cdots$. The only other maximal ideal of D is the ideal $M = \bigcup Q_n$, the union of the chain $\{Q_n\}_{0 \leq n}$. The height of M is also one, so D is one-dimensional. Let f be a nonzero member of M. Then there is an integer n such that f is in Q_m for each $m \geq n$. But this implies that $f \in Q_{m,k}$ for each pair $m > k \geq n$. Since D is a Bezout domain, M cannot be the radical of a finitely generated ideal. $\square$

Remark 3.6. It is perhaps worth noting that the preceding provides an example of a divisorial ideal J in a Prüfer domain such that JR_M is not divisorial for some maximal ideal M. With the notation above, let J be the intersection of the principal maximal ideals. Then J is nonzero and divisorial. We must have $J \subseteq M$. Otherwise, for $x \in J \setminus M$ we would have $(M, x) = R$. However, writing $1 = m + rx$, $m \in M$, $r \in R$ then yields that M is the only maximal ideal containing m, a contradiction. Since J is a radical ideal, we must then have $JR_M = MR_M$, which is nondivisorial.

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References

[1] V. Barucci, E. Houston, T. Lucas, I. Papick, m-Canonical ideals in integral domains II, in: D.D. Anderson, I. Papick (Eds.), Ideal Theoretic Methods in Commutative Algebra, in: Lecture Notes in Pure and Appl. Math., vol. 220, Dekker, New York, 2001, pp. 89–108.
[2] S. Bazzoni, L. Salce, Warfield domains, J. Algebra 185 (1996) 836–868.
[3] E. Bastida, R. Gilmer, Overrings and divisorial ideals of rings of the form $D + M$, Michigan Math. J. 20 (1973) 79–95.
[4] M. Fontana, J. Huckaba, Localizing systems and semistar operations, in: S.T. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, Kluwer Academic Publishers, 2000, pp. 169–198.
[5] M. Fontana, E. Houston, T. Lucas, Toward a classification of prime ideals in Prüfer domains (manuscript).
[6] M. Fontana, J. Huckaba, I. Papick, Prüfer Domains, Marcel Dekker Inc., New York, 1997.
[7] R. Gilmer, Overrings of Prüfer domains, J. Algebra 4 (1966) 331–340.
[8] R. Gilmer, Multiplicative Ideal Theory, M. Dekker, New York, 1972.
[9] R. Gilmer, W. Heinzer, Overrings of Prüfer domains. II, J. Algebra 7 (1967) 281–302.
[10] W. Heinzer, Integral domains in which each non-zero ideal is divisorial, Mathematika 15 (1968) 164–170.
[11] W. Heinzer, J. Huckaba, I. Papick, m-canonical ideals in integral domains, Comm. Algebra 26 (1998) 3021–3043.
[12] I. Kaplansky, Commutative Rings, Allyn and Bacon, Inc., Boston, 1970.
[13] K.A. Loper, T. Lucas, Factoring ideals in almost Dedekind domains, J. Reine Angew. Math. 565 (2003) 61–78.
[14] T. Lucas, The radical trace property and primary ideals, J. Algebra 184 (1996) 1093–1112.
[15] A. Okabe, R. Matsuda, Semistar-operations on integral domains, Math. J. Toyama Univ. 17 (1994) 1–21.
[16] B. Olberding, Globalizing local properties of Prüfer domains, J. Algebra 205 (1998) 480–504.