

New Distinguished Classes of Spectral Spaces: A Survey

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Abstract In the present survey paper, we present several new classes of Hochster’s spectral spaces “occurring in nature,” actually in multiplicative ideal theory, and not linked to or realized in an explicit way by prime spectra of rings. The general setting is the space of the semistar operations (of finite type), endowed with a Zariski-like topology, which turns out to be a natural topological extension of the space of the overrings of an integral domain, endowed with a topology introduced by Zariski. One of the key tool is a recent characterization of spectral spaces, based on the ultrafilter topology, given in Finocchiaro, *Commun Algebra*, 42:1496–1508, 2014, [15]. Several applications are also discussed.

Keywords Spectral space and spectral map · Star and semistar operations · Zariski, constructible, inverse and ultrafilter topologies · Gabriel-Popescu localizing system · Riemann–Zariski space of valuation domains · Kronecker function ring

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1 Introduction and Preliminaries

Let X be a topological space. According to [35], X is called a *spectral space* if there exists a ring R such that $\text{Spec}(R)$, with the Zariski topology, is homeomorphic to X . Spectral spaces can be characterized in a purely topological way: a topological space X is spectral if and only if X is T_0 (this means that for every pair of distinct points of X , at least one of them has an open neighborhood not containing the other), quasi-compact, admits a basis of quasi-compact open subspaces that is closed under finite intersections, and every irreducible closed subspace C of X has a (unique) generic point (i.e., there exists one point $x_C \in C$ such that C coincides with the closure of this point) [35, Proposition 4].

In the present survey paper, we present several new classes of spectral spaces occurring naturally in multiplicative ideal theory. Before doing this, we introduce, for convenience of the reader, some background material.

1.1 Semistar Operations

Let D be an integral domain with quotient field K . Let $\overline{F}(D)$ [respectively, $F(D)$; $f(D)$] be the set of all nonzero D -submodules of K [respectively, nonzero fractional ideals; nonzero finitely generated fractional ideals] of D (thus, $f(D) \subseteq F(D) \subseteq \overline{F}(D)$).

A *semistar operation* on D is a map $\star : \overline{F}(D) \rightarrow \overline{F}(D)$, $E \mapsto E^\star$, such that, for every $z \in K$, $z \neq 0$, and for every $E, F \in \overline{F}(D)$, the following properties hold: (\star_1) $E \subseteq E^\star$; (\star_2) $E \subseteq F$ implies $E^\star \subseteq F^\star$; (\star_3) $(E^\star)^\star = E^\star$; (\star_4) $(zE)^\star = z \cdot E^\star$. If $D = D^\star$, then the map $\star|_{F(D)} : F(D) \rightarrow F(D)$ is called a *star operation* on D .

Semistar operations were introduced by Okabe and Matsuda in 1994 [46] (although this kind of operations were considered by J. Huckaba in 1988, in the setting of rings with zero divisors [38, Sect. 20]), producing a more general and flexible concept than the earlier notion of a star operations which in turn were defined by Krull [40–42] and used, among others, by Gilmer [32, Sect. 32].

A star operation, in Krull's original terminology, was called "prime operation" (*Strich-Operation* or *'-Operation*, in German [40, 41]). The notion of semiprime operation and the relation with that of semistar operation has been investigated in [14] (see also [51]). Semiprime operations include various examples of specific closures, used mainly in the Noetherian setting, the most important of which is probably tight closure, originally defined in [36]. (See [13] for a survey on closure operations.)

1.2 Riemann–Zariski Spaces

Let K be a field and let A be any subring of K . Let $\text{Zar}(K|A)$ denote the set of all the valuation domains of K that contain A as a subring. In the special case where $A := D$ is an integral domain with quotient field K , we simply set

$$\text{Zar}(D) := \text{Zar}(K|D) = \{V \mid V \text{ is a valuation domain overring of } D\}.$$

O. Zariski [52] introduced a topological structure on the set $Z := \text{Zar}(K|A)$ by taking, as a basis for the open sets, the subsets $B_F := \{V \in Z \mid V \supseteq A[F]\}$, for F varying in the family of all finite subsets of K (see also [53, Chap. VI, Sect. 17, p. 110]). This topology is called *the Zariski topology on Z* and the set Z , equipped with this topology (denoted also by Z^{zar}), is usually called *the Riemann–Zariski space of $K|A$* (sometimes also called abstract Riemann surface or generalized Riemann manifold of $K|A$).

In 1944, Zariski [52] proved a general result that implies the quasi-compactness of Z^{zar} , and later it was proven that Z^{zar} is a spectral space, in the sense of M. Hochster [35] (for the case of the space $\text{Zar}(D)$ see [12, Theorem 4.1]). More precisely, in [11, Theorem 2] (respectively, in [17, Corollary 3.4]) the authors provide explicitly a ring R_D (respectively, $R_{K|A}$) having the property that $\text{Spec}(R_D)$ (respectively, $\text{Spec}(R_{K|A})$) is canonically homeomorphic to $\text{Zar}(D)$ (respectively, to $\text{Zar}(K|A)$), both endowed with the Zariski topology (see also [37]).

Recently in [21] the Zariski topology on $\text{Zar}(D)$ was explicitly extended on the larger space $\text{Overr}(D)$ of all overrings of D , by taking, as a basis of open sets the collection of the sets of the type $\text{Overr}(D[F])$, for F varying in the family of all finite subsets of K (see also [53, p. 115]). Clearly, in this way, $\text{Zar}(D)$ becomes a subspace of $\text{Overr}(D)$.

1.3 The Inverse Topology on a Spectral Space

Let X be a topological space and let Y be any subset of X . We denote by $\text{Cl}(Y)$ the closure of Y in the topological space X . Recall that the topology on X induces a natural preorder \leq_X on X (simply denoted by \leq , if no confusion can arise), defined by setting $x \leq_X y$ if $y \in \text{Cl}(\{x\})$. It is straightforward that \leq_X is a partial order if and only if X is a T_0 space (e.g., this holds when X is spectral). The set $Y^{\text{gen}} := \{x \in X \mid y \in \text{Cl}(\{x\}), \text{ for some } y \in Y\}$ is called *closure under generalizations of Y* . Similarly, using the opposite order, the set $Y^{\text{sp}} := \{x \in X \mid x \in \text{Cl}(\{y\}), \text{ for some } y \in Y\}$ is called *closure under specializations of Y* . We say that Y is *closed under generalizations* (respectively, *closed under specializations*) if $Y = Y^{\text{gen}}$ (respectively, $Y = Y^{\text{sp}}$). For two elements x, y in a spectral space X , we have:

$$x \leq y \iff \{x\}^{\text{gen}} \subseteq \{y\}^{\text{gen}} \iff \{x\}^{\text{sp}} \supseteq \{y\}^{\text{sp}}.$$

Suppose that X is a spectral space; then, X can be endowed with another topology, introduced by Hochster [35, Proposition 8], whose basis of closed sets is the collection of all the quasi-compact open subspaces of X . This topology is called *the inverse topology on X* . For a subset Y of X , let $\text{Cl}^{\text{inv}}(Y)$ be the closure of Y , in the inverse topology of X ; we denote by X^{inv} the set X , equipped with the inverse topology. The name given to this new topology is due to the fact that, given $x, y \in X$, $x \in \text{Cl}^{\text{inv}}(\{y\})$ if and only if $y \in \text{Cl}(\{x\})$, i.e., the partial order induced by the inverse topology is the opposite order of the partial order induced by the given spectral topology [35, Proposition 8].

By definition, for any subset Y of X , we have

$$\text{Cl}^{\text{inv}}(Y) = \bigcap \{U \mid U \text{ open and quasi-compact in } X, U \supseteq Y\}.$$

In particular, keeping in mind that the inverse topology reverses the order of the given spectral topology, it follows [35, Proposition 8] that the closure under generizations $\{x\}^{\text{gen}}$ of a singleton is closed in the inverse topology of X , since

$$\{x\}^{\text{gen}} = \text{Cl}^{\text{inv}}(\{x\}) = \bigcap \{U \mid U \subseteq X \text{ quasi-compact and open, } x \in U\}.$$

On the other hand, it is trivial, by the definition, that the closure under specializations of a singleton $\{x\}^{\text{sp}}$ is closed in the given topology of X , since $\{x\}^{\text{sp}} = \text{Cl}(\{x\})$.

2 Ultrafilter Topology and Spectral Spaces

The characterization of spectral spaces given in [35, Proposition 4] is often not easy to handle. In particular, it might be arduous to verify that a space is spectral using direct arguments involving irreducible closed subspaces.

The main result of the present section (Theorem 2.8) provides a criterion for deciding when a topological space is spectral, based on the use of ultrafilters. To introduce this statement, we need some basic and preliminary results on various topological structures that can be considered on the prime spectrum of a ring.

It is well known that the prime spectrum of a commutative ring endowed with the Zariski topology is always T_0 , but almost never T_2 nor T_1 (it is T_2 or Hausdorff only in the zero-dimensional case, cf. for instance [45, Théorème 1.3]). Thus, in the general case, it is natural to look for a Hausdorff topology \mathcal{T} on $\text{Spec}(R)$ such that the following properties are satisfied at the same time:

- \mathcal{T} is finer than the Zariski topology;
- $(\text{Spec}(R), \mathcal{T})$ is compact (i.e., quasi-compact and T_2 , using the terminology of [33]).

A classical answer to the previous question is given in [33, (7.2.11)], even in the more general setting of the underlying topological space of a scheme, by considering

the *constructible topology* (see [10], [4, Chap. 3, Exercises 27, 28 and 30]) or the *patch topology* [35].

As in [49], we introduce the *constructible topology* by a Kuratowski closure operator: if X is a spectral space, we set, for each subset Y of X ,

$$\mathrm{Cl}^{\mathrm{cons}}(Y) := \bigcap \{U \cup (X \setminus V) \mid U \text{ and } V \text{ open and quasi-compact in } X, \\ U \cup (X \setminus V) \supseteq Y\}.$$

We denote by X^{cons} the set X , equipped with the constructible topology. For Noetherian spectral spaces, the clopen subsets of the constructible topology are precisely the constructible subsets after C. Chevalley [10], i.e., the finite unions of locally closed subspaces. It is straightforward that the constructible topology is a refinement of the given topology (it is the coarsest topology on X for which the quasi-compact open subspaces are clopen) and it is always Hausdorff. Finally, by [17, Remark 2.2], we have $\mathrm{Cl}^{\mathrm{inv}}(Y) = (\mathrm{Cl}^{\mathrm{cons}}(Y))^{\mathrm{gen}}$. It follows that each closed set in the inverse topology is closed under generizations and, from [17, Proposition 2.6], that a quasi-compact subspace Y of X closed for generizations is inverse closed. On the other hand, the closure of a subset Y in the given topology of X , $\mathrm{Cl}(Y)$, coincides with $(\mathrm{Cl}^{\mathrm{cons}}(Y))^{\mathrm{sp}}$ [17, Remark 2.2].

In the following result we collect some well-known classical properties of $\mathrm{Spec}(R)$, equipped with the constructible topology.

Theorem 2.1 (cf. [4, Chap. 3, Exercises 27, 28 and 30], [26, Proposition 5], [45, Théorème 2.2], [47, Proposition 5] and [48]) *Let R be a ring. We denote by $\mathrm{Spec}(R)^{\mathrm{zar}}$ (respectively, $\mathrm{Spec}(R)^{\mathrm{cons}}$) the set $\mathrm{Spec}(R)$, endowed with the Zariski topology (respectively, the constructible topology). The following properties hold.*

- (1) $\mathrm{Spec}(R)^{\mathrm{cons}}$ is compact, Hausdorff and totally disconnected (and, by definition, the topology is finer than the Zariski topology).
- (2) $\mathrm{Spec}(R)^{\mathrm{cons}} = \mathrm{Spec}(R)^{\mathrm{zar}}$ if and only if R is zero-dimensional.
- (3) Assume that $\mathrm{Spec}(R)^{\mathrm{zar}}$ is a Noetherian space. Then, a subset of $\mathrm{Spec}(R)$ is clopen in $\mathrm{Spec}(R)^{\mathrm{cons}}$ if and only if it is constructible, according to Chevalley (see [9, 10] and [33, (2.3.11) and (2.4.1)]) (i.e., it is a finite union of locally closed subsets of $\mathrm{Spec}(R)^{\mathrm{zar}}$).
- (4) Let $\{\mathbb{X}_f \mid f \in R\}$ be a collection of algebraically independent indeterminates over R , let I be the ideal of the polynomial ring $R[\{\mathbb{X}_f \mid f \in R\}]$ generated by the set $\{f^2\mathbb{X}_f - f; f\mathbb{X}_f^2 - \mathbb{X}_f \mid f \in R\}$, and consider the ring $\mathrm{T}(R) := R[\{\mathbb{X}_f \mid f \in R\}]/I$. Then, the following statements hold.
 - (4.a) $\mathrm{T}(R)$ is absolutely flat (or, von Neumann regular, i.e., for each $a \in \mathrm{T}(R)$ there exists $x \in \mathrm{T}(R)$ such that $ax^2 = a$), called the absolutely flat cover of R .
 - (4.b) The canonical embedding $\iota : R \rightarrow \mathrm{T}(R)$ is an epimorphism in the category of rings. Furthermore, ι is an isomorphism if and only if R is absolutely flat.

(4.c) *The canonical continuous map $\iota^a : \text{Spec}(\mathbf{T}(R))^{\text{zar}} \rightarrow \text{Spec}(R)^{\text{cons}}$, induced by ι , is an homeomorphism. In particular, the topological space $\text{Spec}(R)^{\text{cons}}$ is spectral.*

In [26] a new description of $\text{Spec}(R)^{\text{cons}}$ is presented, by using a new tool: *convergence by ultrafilters*.

For the reader's convenience, we recall now some basic facts about ultrafilters (for further properties see, for example, [43]). Let \mathbf{X} be a nonempty set. A nonempty collection \mathcal{U} of nonempty subsets of \mathbf{X} is called *an ultrafilter on \mathbf{X}* if the following axioms hold:

- If $Y, Z \in \mathcal{U}$, then $Y \cap Z \in \mathcal{U}$.
- If $Y \in \mathcal{U}$ and $Y \subseteq Z \subseteq \mathbf{X}$, then $Z \in \mathcal{U}$.
- If $Y \subseteq \mathbf{X}$ then either $Y \in \mathcal{U}$ or $\mathbf{X} \setminus Y \in \mathcal{U}$.

It is easy to see that, for each $x \in \mathbf{X}$, the collection $\mathcal{U}_x := \{Y \subseteq \mathbf{X} \mid x \in Y\}$ is an ultrafilter on \mathbf{X} , called *the trivial (or principal) ultrafilter generated by x* . Every finite set admits only trivial ultrafilters. The existence of nontrivial ultrafilters on infinite sets is guaranteed by the Axiom of Choice. Precisely, it is proved under ZFC that, if \mathcal{F} is a nonempty collection of subsets of \mathbf{X} with the finite intersection property, then there exists an ultrafilter \mathcal{U} on \mathbf{X} such that $\mathcal{F} \subseteq \mathcal{U}$.

Now, let R be a ring, let Y be a nonempty subset of $\text{Spec}(R)$ and let \mathcal{U} be an ultrafilter on Y . For each $f \in R$ we set $\mathbb{V}(f) := \{P \in \text{Spec}(R) \mid f \in P\}$. It is easy to show that the set $P_{Y, \mathcal{U}} := P_{\mathcal{U}} := \{f \in R \mid \mathbb{V}(f) \cap Y \in \mathcal{U}\}$ is a prime ideal of R [8, Lemma 2.4], called *the ultrafilter limit point of Y , with respect to \mathcal{U}* . According to [26, Definition 1], a nonempty subset Y of $\text{Spec}(R)$ is *ultrafilter closed* if, for any ultrafilter \mathcal{U} on Y , we have $P_{\mathcal{U}} \in Y$. We assume that the empty set is ultrafilter closed. The following result relates the constructible topology and the convergence by ultrafilters.

Theorem 2.2 (cf. [26, Theorem 8]) *Let R be a ring and let $Y \subseteq \text{Spec}(R)$. Then, the following conditions are equivalent.*

- (i) *Y is closed, with respect to the constructible topology.*
- (ii) *Y is ultrafilter closed.*

In [15, Sect. 2], the convergence by ultrafilters, presented in [26], is extended in a more general setting. Precisely, let \mathbf{X} be a nonempty set and \mathcal{F} be a nonempty collection of subsets of \mathbf{X} . If Y is a nonempty subset of \mathbf{X} and \mathcal{U} is an ultrafilter on Y , we define

$$Y_{\mathcal{F}}(\mathcal{U}) := \{x \in \mathbf{X} \mid [\forall F \in \mathcal{F}, x \in F \iff F \cap Y \in \mathcal{U}]\}$$

and call it *the \mathcal{F} -ultrafilter limit set of Y , with respect to \mathcal{U}* .

Example 2.3 (cf. [15, Example 2.1(2)]) Let R be a ring, let \mathcal{P} denote the collection of the principal open subset of $\text{Spec}(R)$, i.e.,

$$\mathcal{P} := \{D(f) := \{P \in \text{Spec}(R) \mid f \notin P\} \mid f \in R\}.$$

If \mathcal{U} is an ultrafilter on a subset Y of $\text{Spec}(R)$, then $Y_{\mathcal{P}}(\mathcal{U}) = \{P_{\mathcal{U}}\}$, where $P_{\mathcal{U}}$ denotes, as before, the ultrafilter limit point of Y , with respect to \mathcal{U} .

Example 2.4 Let K be a field and let A be any subring of K . In the space $\text{Zar}(K|A)$, let

$$\mathcal{B} := \{B_F := \text{Zar}(K|A[F]) \mid F \subseteq K, F \text{ finite}\},$$

denote the standard basis for the open sets for the Zariski topology on $\text{Zar}(K|A)$. If Z is a nonempty subset of $\text{Zar}(K|D)$ and \mathcal{U} is an ultrafilter on Z , it is easy to show that the subset

$$Z_{\mathcal{U}} := \{x \in K \mid \text{Zar}(K|A[x]) \cap Z \in \mathcal{U}\}$$

is still a valuation domain of K (cf. [8, Lemma 2.9] and [16, Proposition 3.1]), called *the ultrafilter limit point of Z , with respect to \mathcal{U}* . Then we have $Z_{\mathcal{B}}(\mathcal{U}) = \{Z_{\mathcal{U}}\}$.

The next goal is to extend the notion of ultrafilter closure given for the prime spectrum of a ring in a general setting.

Let \mathbf{X} be a nonempty set, \mathcal{F} a nonempty collection of subsets of \mathbf{X} , and fix a nonempty subset Y of \mathbf{X} . We say that Y is *\mathcal{F} -stable under ultrafilters* if, for any ultrafilter \mathcal{U} on Y , we have $Y_{\mathcal{F}}(\mathcal{U}) \subseteq Y$.

Let \mathcal{P} be as in Example 2.3. It is easily seen that a subset of the prime spectrum of a ring is \mathcal{P} -stable under ultrafilters if and only if it is ultrafilter closed, that is, it is closed in the constructible topology (by Theorem 2.2).

Proposition 2.5 (cf. [15, Propositions 2.6, 2.11, 2.13 and Theorem 2.14]) *Let \mathbf{X} be a nonempty set, \mathcal{F} be a nonempty collection of subsets of \mathbf{X} . Then, the following properties hold.*

- (1) *The collection of all the subsets of \mathbf{X} that are stable under ultrafilters is the family of the closed sets for a topology on \mathbf{X} , called the \mathcal{F} -ultrafilter topology. We will denote by $\mathbf{X}^{\mathcal{F}\text{-ultra}}$ the set \mathbf{X} , equipped with the \mathcal{F} -ultrafilter topology.*
- (2) *If \mathcal{B} is the Boolean subalgebra of the power set of \mathbf{X} generated by \mathcal{F} , then \mathcal{B} is a collection of clopen subsets of $\mathbf{X}^{\mathcal{F}\text{-ultra}}$.*
- (3) *For each subset Y of \mathbf{X} , the closure of Y in $\mathbf{X}^{\mathcal{F}\text{-ultra}}$ is the set*

$$\bigcup \{Y_{\mathcal{F}}(\mathcal{U}) \mid \mathcal{U} \text{ ultrafilter on } Y\}.$$

- (4) *The following conditions are equivalent.*

- (i) *$\mathbf{X}^{\mathcal{F}\text{-ultra}}$ is quasi-compact.*
- (ii) *For any ultrafilter \mathcal{U} on \mathbf{X} , the ultrafilter limit set $\mathbf{X}_{\mathcal{F}}(\mathcal{U})$ is nonempty.*

Example 2.6 (cf. [15, Remark 2.7]) Let \mathbf{X} be a nonempty set.

- (1) If $\mathcal{B}(\mathbf{X})$ denotes the power set of \mathbf{X} , the $\mathcal{B}(\mathbf{X})$ -ultrafilter topology is the discrete topology.
- (2) The $\{\mathbf{X}\}$ -ultrafilter topology is the chaotic topology (i.e., the open sets are just X and \emptyset).
- (3) Let R be a ring, $\mathbf{X} := \text{Spec}(R)$ and \mathcal{P} be as in Example 2.3. Then, the \mathcal{P} -ultrafilter topology is the constructible topology on \mathbf{X} by [15, Corollary 2.17].

We apply the previous construction when the given set is a topological space and the collection of subsets \mathcal{F} is a basis for the topology.

Proposition 2.7 (cf. [15, Proposition 3.1]) *Let (X, \mathcal{T}) be a nonempty topological space and \mathcal{B} be a basis of open sets of X . Then, the following statements hold.*

- (1) *The \mathcal{B} -ultrafilter topology is finer than or equal to the topology \mathcal{T} .*
- (2) *If (X, \mathcal{T}) is a T_0 space, then $X^{\mathcal{B}\text{-ultra}}$ is a Hausdorff and totally disconnected space.*
- (3) *Assume now that (X, \mathcal{T}) is T_0 and that $X^{\mathcal{B}\text{-ultra}}$ is compact. Then, the \mathcal{B} -ultrafilter topology is the coarsest topology for which \mathcal{B} is a family of clopen sets. Moreover, (X, \mathcal{T}) is a spectral space and the constructible topology on (X, \mathcal{T}) is precisely the \mathcal{B} -ultrafilter topology.*

Note that part (3) of the previous proposition generalizes [26, Theorem 8] and [16, Theorem 3.4].

By using Propositions 2.5(4), 2.7(3) and keeping in mind [35, Corollary to Proposition 7], we can deduce new characterizations of spectral spaces and hence new criteria, based on ultrafilters, to decide if a given topological space is spectral.

Theorem 2.8 (cf. [15, Corollary 3.3]) *For a nonempty topological space X , the following conditions are equivalent:*

- (i) *X is a spectral space.*
- (ii) *There exists a basis \mathcal{B} for the open sets of X such that $X^{\mathcal{B}\text{-ultra}}$ is a compact and Hausdorff space.*
- (iii) *X is a T_0 space and there is a basis \mathcal{B} for the open sets of X such that, for any ultrafilter \mathcal{U} on X , the ultrafilter limit set $X_{\mathcal{B}}(\mathcal{U})$ is nonempty.*
- (iv) *X is a T_0 space and there is a subbasis \mathcal{S} for the open sets of X such that, for any ultrafilter \mathcal{U} on X , the ultrafilter limit set $X_{\mathcal{S}}(\mathcal{U})$ is nonempty.*

The proof of Theorem 2.8 is not constructive, since it is based on the Axiom of Choice and some of its consequences.

As an application of Theorem 2.8, we now determine some new classes of spectral spaces. The key point of the proofs resides on the existence of ultrafilter limit points.

Example 2.9 (cf. [15, Proposition 3.5]) Let $A \subseteq B$ be a ring extension, and let $X := \mathbf{R}(B|A)$ denote the collection of all the intermediate rings between A and B . We can make X a topological space, by generalizing the Zariski topology introduced on the

space of the overrings on an integral domain (see Sect. 1.2) and taking as a subbasis of open sets the collection

$$\mathcal{S} := \{\mathbf{R}(B|A[x]) \mid x \in B\}.$$

We claim that X is a spectral space. It is easily seen that X is T_0 because, if $C \neq D \in X$, we can assume, without loss of generality, that there is an element $c \in C \setminus D$, and then the open set $\mathbf{R}(B|A[c])$ contains C and does not contain D . By Theorem 2.8, we have to show that, if \mathcal{U} is an ultrafilter on X , then the ultrafilter limit set $X_{\mathcal{S}}(\mathcal{U})$ is nonempty. Consider the subset

$$A_{\mathcal{U}} := \{x \in B \mid \mathbf{R}(B|A[x]) \in \mathcal{U}\}$$

of B . We claim that $A_{\mathcal{U}}$ is a subring of B .

This follows immediately from the definition of an ultrafilter, since, if $x, y \in A_{\mathcal{U}}$ then each of the sets $\mathbf{R}(B|A[x-y])$, $\mathbf{R}(B|A[xy])$ contain $\mathbf{R}(B|A[x]) \cap \mathbf{R}(B|A[y]) \in \mathcal{U}$, and thus $\mathbf{R}(B|A[x-y])$, $\mathbf{R}(B|A[xy]) \in \mathcal{U}$, that is, $x-y, xy \in A_{\mathcal{U}}$. Furthermore, $A_{\mathcal{U}}$ contains A because, for each $a \in A$, $\mathbf{R}(B|A[a]) = X \in \mathcal{U}$. Therefore, $A_{\mathcal{U}}$ is an element of X . The fact that $A_{\mathcal{U}} \in X_{\mathcal{S}}(\mathcal{U})$ follows immediately from the definition of $A_{\mathcal{U}}$ and thus, by Theorem 2.8, X is a spectral space.

In particular, if $A := D$ is an integral domain and $B := K$ is the quotient field of D , we deduce from the previous example that:

Corollary 2.10 *The space $\text{Overr}(D)$ of the overrings of an integral domain D , endowed with the Zariski topology, is a spectral space.*

Example 2.11 (cf. [15, Proposition 3.6]) Let A, B and X be as in the previous example, and let $X' := \mathbf{R}'(B|A)$ be the subset of X consisting of all the subrings of B that are integrally closed in B . We claim that, with the subspace topology induced by that of X , the topological space X' is spectral.

It is obvious that a subbasis of open sets for the topology of X' is given by the family $\mathcal{S}' := \{\mathbf{R}'(B|A[x]) \mid x \in B\}$. As in the previous example, the key fact is the existence in X' of ultrafilter limit points, with respect to every ultrafilter \mathcal{U} on X' . Indeed, it is not difficult to show that

$$A'_{\mathcal{U}} := \{x \in B \mid \mathbf{R}'(B|A[x]) \in \mathcal{U}\}$$

is a subring of B containing A that is integrally closed in B . Thus, again by definition, the ultrafilter limit set $X'_{\mathcal{S}'}(\mathcal{U})$ is nonempty, containing $A'_{\mathcal{U}}$. Again, by Theorem 2.8, we conclude that X' is a spectral space.

In particular, if $A := D$ is an integral domain and $B := K$ is the quotient field of D , we deduce from the previous example that:

Corollary 2.12 *The subspace $\text{Overr}_{ic}(D)$ of $\text{Overr}(D)$, consisting of the integrally closed overrings of an integral domain D , endowed with the Zariski topology, is a spectral space.*

Example 2.13 We preserve the notation of Example 2.9, and let $X'' := \mathbf{L}(B|A)$ be the (possibly empty) subspace of $\mathbf{R}(B|A)$ consisting of all the local rings T such that $A \subseteq T \subseteq B$. A subbasis for the open sets of X'' is clearly the family

$$S'' := \{\mathbf{L}(B|A[x]) \mid x \in B\}$$

We claim that, if X'' is nonempty, then it is spectral. Again, we need to prove that, for any ultrafilter \mathcal{U} on X'' the ultrafilter limit set $X''_{S''}(\mathcal{U})$ is nonempty. As before, it is easy to infer that $A''_{\mathcal{U}} := \{x \in B \mid \mathbf{L}(B|A[x]) \in \mathcal{U}\} \in \mathbf{R}(B|A)$. It will be immediate to conclude that $A''_{\mathcal{U}} \in X''_{S''}(\mathcal{U})$ if we show that $A''_{\mathcal{U}}$ is a local ring. We claim that the unique maximal ideal of $A''_{\mathcal{U}}$ is

$$M := \{x \in B \mid \{T \in X'' \mid x \in T \setminus U(T)\} \in \mathcal{U}\}$$

where, as usual, $U(T)$ denotes the set of units of a ring T . Thus it suffices to note that $U(A''_{\mathcal{U}}) = A''(\mathcal{U}) \setminus M$ (this follows easily from definitions).

In particular, if $A := D$ is an integral domain and $B := K$ is the quotient field of D , we deduce from the previous example that:

Corollary 2.14 *The subspace $\text{Overr}_{\text{loc}}(D)$ of $\text{Overr}(D)$, consisting of the local overrings of an integral domain D , endowed with the Zariski topology, is a spectral space.*

3 Spaces of Semistar Operations

Let D be an integral domain with quotient field K . As in the star operation setting, to each semistar operation \star can be associated a map $\star_f : \overline{F}(D) \rightarrow \overline{F}(D)$ defined by

$$E^{\star_f} := \bigcup \{F^{\star} \mid F \subseteq E, F \in f(D)\},$$

for every $E \in \overline{F}(D)$. The map \star_f is again a semistar operation, which coincides with \star on finitely generated modules; moreover, $(\star_f)_f = \star_f$. If $\star = \star_f$, we say that \star is a *semistar operation of finite type*. We call \star_f the *finite-type semistar operation associated to \star* .

For each $T \in \text{Overr}(D)$, the map $\wedge_{\{T\}} : \overline{F}(D) \rightarrow \overline{F}(D)$, defined by $E^{\wedge_{\{T\}}} := ET$, for each $E \in \overline{F}(D)$, is an example of semistar operation of finite type on D , called the *semistar extension to T* .

We denote by $\text{SStar}(D)$ (respectively, $\text{SStar}_f(D)$) the set of all semistar operations (respectively, semistar operations of finite type) on D . The set $\text{SStar}(D)$ can be endowed with a natural partial order \preceq which turns it into a complete lattice: if \star_1, \star_2 are two semistar operations, say that $\star_1 \preceq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for every

$E \in \overline{F}(D)$. In particular, $\star_f \preceq \star$, and \star_f is the biggest semistar operation of finite type smaller than \star .

The infimum $\wedge_{\mathcal{S}}$ of a nonempty family \mathcal{S} of semistar operations can be written explicitly as follows:

$$E^{\wedge_{\mathcal{S}}} = \bigcap \{E^{\star} \mid \star \in \mathcal{S}\}, \quad \text{for each } E \in \overline{F}(D).$$

In particular, if \mathcal{T} is a nonempty family of overrings of D , then the infimum of the family of semistar operations $\{\wedge_{\{T\}} \mid T \in \mathcal{T}\}$ is denoted by $\wedge_{\mathcal{T}}$.

On the other hand, there is not a general explicit formula for the supremum $\vee_{\mathcal{S}} := \bigwedge \{\sigma \in \text{SStar}(D) \mid \star \preceq \sigma \text{ for all } \star \in \mathcal{S}\}$, although, if $\mathcal{S} \subseteq \text{SStar}_f(D)$, then

$$E^{\vee_{\mathcal{S}}} = \bigcup \{E^{\star_1 \circ \star_2 \circ \dots \circ \star_n} \mid \star_1, \dots, \star_n \in \mathcal{S}\} \quad (1)$$

where $\star_1 \circ \star_2 \circ \dots \circ \star_n$ denotes the usual composition of functions (see [3, p.1628] and [21, Lemma 2.12]).

A nonzero ideal I of D is called a *quasi- \star -ideal* if $I = I^{\star} \cap D$. A *quasi- \star -prime* is a quasi- \star -ideal which is also a prime ideal; the set of all quasi- \star -prime ideals of D is denoted by $\text{QSpec}^{\star}(D)$. The set of maximal elements in the set of proper quasi- \star -ideals of D (ordered by set-theoretic inclusion) is denoted by $\text{QMax}^{\star}(D)$, and it is a subset of $\text{QSpec}^{\star}(D)$. By Zorn's Lemma, it is easy to show that if \star is a semistar operation of finite type then $\text{QMax}^{\star}(D) \neq \emptyset$. If every quasi- \star -ideal is contained in a quasi- \star -prime, then \star is said to be *quasi-spectral* or *semifinite*. Every operation of finite type is not only quasi-spectral, but it has the stronger property that every quasi- \star -ideal is contained in a maximal quasi- \star -ideal. Note that a semistar operation \star may be quasi-spectral even if $\text{QMax}^{\star}(D)$ is empty (see [21, Remark 5.6] for an example).

A semistar operation \star is called *spectral* if there is a nonempty subset $Y \subseteq \text{Spec}(D)$ such that $\star = \wedge_{\mathcal{L}(Y)}$, where $\mathcal{L}(Y) := \{D_P \mid P \in Y\}$. We set $s_Y := \wedge_{\mathcal{L}(Y)}$ and we call s_Y the *spectral semistar operation associated to* $Y \subseteq \text{Spec}(D)$.

A *semistar operation* \star is called *stable* if $(E \cap F)^{\star} = E^{\star} \cap F^{\star}$ for every pair $E, F \in \overline{F}(D)$.

Remark 3.1 Every spectral semistar operation is quasi-spectral (or semifinite) by [22, Lemma 1.4(5)] and every spectral semistar operation, or more generally every operation induced by a family of D -flat overrings of D , is stable. However, the converse does not hold in general [34, Sect. 3, p. 441], but if \star is a stable semistar operation then \star is spectral if and only if it is quasi-spectral (see [1, Theorem 4] and [22, Theorem 4.12(3)]). In particular, a stable semistar operation of finite type is spectral.

In [21], the set $\text{SStar}(D)$ was endowed with a topology (called the *Zariski topology*) by declaring open the sets of the form

$$\vee_E := \{\star \in \text{SStar}(D) \mid 1 \in E^{\star}\},$$

for $E \in \overline{F}(D)$. This topology makes $\text{SStar}(D)$ into a quasi-compact, T_0 space with a unique closed point (the identity semistar operation d_D) and a generic point (the trivial semistar extension $\wedge_{\{K\}}$). In particular, $\text{SStar}(D)$ is never T_1 (nor T_2) unless $D = K$.

Proposition 3.2 *Let D be an integral domain, let $\text{Overr}(D)$ and $\text{SStar}_f(D)$ be endowed with their Zariski topologies, and let $\iota : \text{Overr}(D) \rightarrow \text{SStar}_f(D)$ be the injective map defined by $\iota(T) := \wedge_{\{T\}}$, for each $T \in \text{Overr}(D)$. Then, the following statements hold.*

- (1) *The map ι is a topological embedding [21, Proposition 2.5].*
- (2) *The mapping $\pi : \text{SStar}_f(D) \rightarrow \text{Overr}(D)$, defined by $\pi(\star) := D^\star$, for each $\star \in \text{SStar}_f(D)$, is a continuous surjection such that $\pi \circ \iota$ is the identity of $\text{Overr}(D)$. In other words, π is a topological retraction.*

Note that part (2) of the previous proposition follows from the fact that, for each subbasic open set $B_x := \text{Overr}(D[x])$ of $\text{Overr}(D)$, we have $\pi^{-1}(B_x) = \{\star \in \text{SStar}_f(D) \mid D[x] \subseteq D^\star\} = \{\star \in \text{SStar}_f(D) \mid 1 \subseteq (x^{-1}D)^\star\} = \vee_{x^{-1}D}$.

The following result relates the quasi-compactness of a collection of semistar operations on the same integral domain with the finite type property of their infimum.

Proposition 3.3 (cf. [21, Proposition 2.7]) *Let D be an integral domain and let \mathcal{S} be a quasi-compact subspace of $\text{SStar}_f(D)$. Then, $\wedge_{\mathcal{S}}$ is of finite type.*

Remark 3.4 Let \mathcal{S} be a subset of $\text{SStar}(D)$ and set $\mathcal{S}_f := \{\star_f \mid \star \in \mathcal{S}\}$. Consider the following properties:

- (a) \mathcal{S} is quasi-compact in $\text{SStar}(D)$;
- (b) \mathcal{S}_f is quasi-compact in $\text{SStar}_f(D)$;
- (c) $\wedge_{\mathcal{S}_f}$ is a semistar operation of finite type;
- (d) $\wedge_{\mathcal{S}_f} = (\wedge_{\mathcal{S}})_f$.

Then (a) \Rightarrow (b) \Rightarrow (c) \Leftrightarrow (d).

In fact, it is straightforward that (a) \Rightarrow (b) (see also Proposition 3.10). By Proposition 3.3, (b) \Rightarrow (c). For (c) \Rightarrow (d), note that in general $\wedge_{\mathcal{S}_f} \leq \wedge_{\mathcal{S}}$ and $(\wedge_{\mathcal{S}})_f \leq \wedge_{\mathcal{S}_f}$. The conclusion follows from the fact that, under (c), $(\wedge_{\mathcal{S}_f})_f = \wedge_{\mathcal{S}_f}$. Finally, (d) \Rightarrow (c) is trivial.

Since, for each overring T of an integral domain D , the semistar operation $\wedge_{\{T\}}$ is of finite type, we get the following result, just by applying Propositions 3.2 and 3.3.

Corollary 3.5 (cf. [21, Corollary 2.8]) *Let D be an integral domain and let \mathcal{T} be a quasi-compact subspace of $\text{Overr}(D)$. Then $\wedge_{\mathcal{T}}$ is of finite type.*

In particular, the previous corollary applies when \mathcal{T} is *locally finite*, i.e., if every nonzero element of D is nonunit in finitely many overrings of the family \mathcal{T} [21, Corollary 2.10]. However, the finite type property of a semistar operation $\wedge_{\mathcal{T}}$, induced by

a collection \mathcal{T} of overrings, does not imply the quasi-compactness of \mathcal{T} , as the following example shows. This example provides a negative answer to the Conjecture in [21, p. 214].

Example 3.6 Let k be a field, let \mathbb{X} be an indeterminate over k , let $D := k[[\mathbb{X}^4, \mathbb{X}^5, \mathbb{X}^6, \mathbb{X}^7]] = k + \mathbb{X}^4 k[[\mathbb{X}]]$ and let $K := k((\mathbb{X}))$. Since D is Noetherian and a conductive domain (i.e., $(D : T) \neq (0)$ for each $T \in \text{Overr}(D)$ with $T \neq K$, see [5, Theorem 1]), $\overline{F}(D) = F(D) \cup \{K\} = f(D) \cup \{K\}$, and thus every semistar operation on D is of finite type. For every $\alpha \in K$, consider the ring $T_\alpha := D[\mathbb{X}^2 + \alpha\mathbb{X}^3] = k + (\mathbb{X}^2 + \alpha\mathbb{X}^3)k + \mathbb{X}^4 k[[\mathbb{X}]]$, and, for every $A \subseteq k$, let $\mathcal{T}_A := \{T_\alpha \mid \alpha \in A\}$. Then, as observed above, the semistar operation $\wedge_{\mathcal{T}_A}$ is of finite type. However, if A is infinite (so, for example, if k is infinite and $A = k$), then \mathcal{T}_A is not quasi-compact. Indeed, the open cover $\{\text{Overr}(T_\alpha) \mid \alpha \in A\}$ of \mathcal{T}_A in $\text{Overr}(D)$ has no finite subcovers, since $\text{Overr}(T_\alpha) \cap \mathcal{T}_A = \{T_\alpha\}$.

The following example shows how to use Corollary 3.5 for establishing the failure of quasi-compactness for some distinguished subspaces of $\text{Overr}(D)$.

Example 3.7 Let D be a Noetherian domain of dimension ≥ 2 , and let \mathcal{D} be the set of Noetherian valuation overrings of D , i.e., the union of $\{K\}$ with the set of discrete valuation overrings of D . If I is a proper ideal of D , then $I^{\wedge_{\mathcal{D}}} = I^b$, where $b := \wedge_{\text{Zar}(D)}$ (see, for example, [39, Proposition 6.8.4], after noting that the terminology used therein is slightly different). In particular, the same holds for every $F \in f(D)$, so that $(\wedge_{\mathcal{D}})_f = b$. However, if $W \in \text{Zar}(D) \setminus \mathcal{D}$ (for example, if $\dim(W) \geq 2$, where the existence of such a W is guaranteed by [32, Corollary 19.7]), then W is contained in (at most) one element V of \mathcal{D} , so that $WV = V$, while $WV' = K$ for each $V' \in \mathcal{D}$, $V' \neq V$. Hence, $W^{\wedge_{\mathcal{D}}} \neq W$, while $W^b = W$ and thus, $\wedge_{\mathcal{D}} \neq b$. Therefore, $\wedge_{\mathcal{D}}$ is not of finite type, and so \mathcal{D} is not a quasi-compact subset of $\text{Overr}(D)$ (or of $\text{Zar}(D)$).

Theorem 3.8 (cf. [21, Theorem 2.13]) *Let D be an integral domain. Then, $\text{SStar}_f(D)$ is a spectral space.*

The proof uses Theorem 2.8, so it is not constructive. However, if A is a ring such that $\text{Spec}(A) \simeq \text{SStar}_f(D)$, we can assume that:

- (a) A_{red} (the reduced ring associated to A) is an integral domain (since $\text{SStar}_f(D)$ has a unique generic point),
- (b) A_{red} (and A) is local (since $\text{SStar}_f(D)$ has a unique closed point), and
- (c) $\dim(A) = \dim(A_{\text{red}}) \geq |\text{Spec}(D)| - 1$ (see the following Propositions 4.3 and 4.6).

On the other hand, since the proof of Theorem 2.8 uses in a crucial way the characterization (1) of the supremum of a family of finite-type semistar operations, it cannot readily be adapted to $\text{SStar}(D)$ and so, up to now, we do not know whether $\text{SStar}(D)$ is a spectral space.

We denote by $\overline{\text{SStar}}(D)$ (respectively, $\widetilde{\text{SStar}}(D)$) the subset of $\text{SStar}(D)$ consisting of all stable semistar operations (respectively, all stable semistar operations of finite type).

Remark 3.9 (a) If we set $\text{SStar}_{sp}(D) := \{\star \in \text{SStar}(D) \mid \star \text{ is spectral}\}$ (respectively, $\text{SStar}_{f,sp}(D) := \{\star \in \text{SStar}_f(D) \mid \star \text{ is spectral}\}$), then by Remark 3.1 $\text{SStar}_{sp}(D) \subseteq \overline{\text{SStar}}(D)$, and the inclusion might be proper. However, in the finite type case, we have equality [22, Proposition 4.23(2)], i.e.,

$$\text{SStar}_{f,sp}(D) = \overline{\text{SStar}}(D) \cap \text{SStar}_f(D) = \widetilde{\text{SStar}}(D).$$

(b) Let $\text{Loc}(D)$ and $\text{Overr}_{flat}(D)$ be, respectively, the set of localizations of D and the set of D -flat overrings of D (and so $\text{Loc}(D) \subseteq \text{Overr}_{flat}(D)$). We observe that the topological embedding $\iota : \text{Overr}(D) \hookrightarrow \text{SStar}_f(D)$, considered in Proposition 3.2(1), restricts to a topological embedding $\iota_{\text{Loc}} : \text{Loc}(D) \hookrightarrow \overline{\text{SStar}}(D)$ (or to a topological embedding $\iota_{flat} : \text{Overr}_{flat}(D) \hookrightarrow \widetilde{\text{SStar}}(D)$).

On the opposite side, the map $\pi : \text{SStar}_f(D) \rightarrow \text{Overr}(D)$ (Proposition 3.2(2)) does not always restrict to a map $\widetilde{\text{SStar}}(D) \rightarrow \text{Overr}_{flat}(D)$, since not all intersection of localizations of D are D -flat (see for instance [34, Sect. 3, p. 441]).

Given a semistar operation \star on D , we can always associate to \star two semistar operations $\bar{\star}$ and $\tilde{\star}$ on D defined as follows: for each $E \in \overline{F}(D)$,

$$\begin{aligned} E^{\bar{\star}} &:= \bigcup \{(E : I) \mid I \text{ nonzero ideal of } D \text{ such that } I^{\star} = D^{\star}\}, \\ E^{\tilde{\star}} &:= \bigcup \{(E : J) \mid J \text{ nonzero finitely generated ideal of } D \\ &\quad \text{such that } J^{\star} = D^{\star}\}. \end{aligned}$$

It is easy to see that $\tilde{\star} \preceq \bar{\star} \preceq \star$ and, moreover, that $\bar{\star}$ (respectively, $\tilde{\star}$) is the largest stable (respectively, stable of finite type) semistar operation that precedes \star , called *the stable* (respectively, *the finite type stable*) *semistar operation associated to \star* . Therefore, \star is stable (respectively, stable of finite type) if and only if $\star = \bar{\star}$ (respectively, $\star = \tilde{\star}$) [22, Proposition 3.7, Corollary 3.9]. Note that, for each semistar operation \star , we always have $\tilde{\star} = s_Y$, where $Y = \text{QMax}^{\star_f}(D)$ (cf. [22, p. 182, Proposition 4.3], [24, Proposition 3.4(4)], [25, Remark 10] and, for the star operation case, [2, Corollary 2.10]).

Proposition 3.10 (cf. [18, Proposition 4.1] and [21, Proposition 2.4]) *Let $\Phi_f : \text{SStar}(D) \rightarrow \text{SStar}(D)$ (respectively, $\bar{\Phi} : \text{SStar}(D) \rightarrow \text{SStar}(D)$); $\tilde{\Phi} : \text{SStar}(D) \rightarrow \text{SStar}(D)$) be the map defined by $\star \mapsto \star_f$ (respectively, $\star \mapsto \bar{\star}$; $\star \mapsto \tilde{\star}$). Then:*

- (1) *The images of Φ_f , $\bar{\Phi}$ and $\tilde{\Phi}$ are, respectively, $\text{SStar}_f(D)$, $\overline{\text{SStar}}(D)$ and $\widetilde{\text{SStar}}(D)$.*
- (2) *The maps Φ_f , $\bar{\Phi}$ and $\tilde{\Phi}$ are continuous in the Zariski topology.*
- (3) *The maps Φ_f , $\bar{\Phi}$ and $\tilde{\Phi}$ are topological retractions of $\text{SStar}(D)$ onto their respective images.*

Another point of similarity between finite type, stable, and spectral operations is given by the open sets needed to generate the Zariski topology, induced by the Zariski topology of $\text{Overr}(D)$. Indeed, if \star is of finite type, let $E \in \overline{f}(D)$, and let $\star \in V_E$, that is, $1 \in E^\star$, then there is a finitely generated submodule $F \subseteq E$ such that $1 \in F^\star$, so that $\star \in V_F$; it follows that

$$V_E \cap \text{SStar}_f(D) = \bigcup \{V_F \cap \text{SStar}_f(D) \mid F \subseteq E, F \in f(D)\}$$

and thus $\{V_F \cap \text{SStar}_f(D) \mid F \in f(D)\}$ is a subbasis for the Zariski topology on $\text{SStar}_f(D)$. Similarly, if \star is stable, then $1 \in E^\star$ if and only if $1 \in E^\star \cap D^\star = (E \cap D)^\star$. Therefore, the Zariski topology on $\overline{\text{SStar}}(D)$ is generated by the $V_I \cap \overline{\text{SStar}}(D)$, as I ranges among the integral ideals of D . The same reasoning shows that $\{V_J \cap \widetilde{\text{SStar}}(D) \mid J \subseteq D, J \in f(D)\}$ is a subbasis for the Zariski topology on $\widetilde{\text{SStar}}(D)$. This implies that stable semistar operations are completely determined by their action inside the ring. In particular, if $\ast : F(D) \rightarrow F(D)$ is a stable *star* operation, then there is a unique stable *semistar* operation $\hat{\ast} : \overline{F}(D) \rightarrow \overline{F}(D)$ such that $\hat{\ast}|_{F(D)} = \ast$.

Remark 3.11 Note that the subbasic open sets $\overline{U}_I := V_I \cap \overline{\text{SStar}}(D) = \{\star \in \text{SStar}(D) \mid 1 \in I^\star\} \cap \overline{\text{SStar}}(D)$ (respectively, $\widetilde{U}_I := V_I \cap \widetilde{\text{SStar}}(D) = \{\star \in \text{SStar}(D) \mid 1 \in I^\star\} \cap \widetilde{\text{SStar}}(D)$) of $\overline{\text{SStar}}(D)$ (respectively, of $\widetilde{\text{SStar}}(D)$), where I is an ideal of D , form a basis of $\overline{\text{SStar}}(D)$ (respectively, $\widetilde{\text{SStar}}(D)$), since $\overline{U}_{I'} \cap \overline{U}_{I''} = \overline{U}_{I' \cap I''}$ (respectively, $\widetilde{U}_{I'} \cap \widetilde{U}_{I''} = \widetilde{U}_{I' \cap I''}$), for all I' and I'' ideals of D .

On the other hand, when considering finitely generated ideals J of D , in general the \widetilde{U}_J 's do not form a basis for the open sets in $\widetilde{\text{SStar}}(D)$, since $\widetilde{U}_{J'} \cap \widetilde{U}_{J''} = \widetilde{U}_{J' \cap J''}$, and $J' \cap J''$ is not necessarily finitely generated, even if J' and J'' are finitely generated ideals of D .

Besides the Zariski topology, we can also endow $\text{SStar}(D)$ with possibly weaker topologies induced by the sets considered in the above paragraph.

Proposition 3.12 (cf. [21, Proposition 2.1 and Remark 2.2]) *Preserve the notation of Proposition 3.10, and endow $\text{SStar}(D)$ with the topology generated by $\{V_F \mid F \in f(D)\}$ (respectively, $\{V_I \mid I \text{ ideal in } D\}$; $\{V_J \mid J \subseteq D, J \in f(D)\}$). Then, Φ_f (respectively, $\overline{\Phi}$; $\widetilde{\Phi}$) is the Kolmogoroff quotient of $\text{SStar}(D)$ onto $\text{SStar}_f(D)$ (respectively, $\overline{\text{SStar}}(D)$; $\widetilde{\text{SStar}}(D)$), i.e., it is the canonical map to the quotient by the equivalence relation of “topological indistinguishability” (where two points of a topological space are topologically indistinguishable if they have exactly the same neighborhoods).*

Let $Y \subseteq \text{Spec}(D)$ be a nonempty set defining a spectral semistar operation. Then its closure, in the inverse topology (denoted by $\text{Cl}^{\text{inv}}(Y)$, see Sect. 1.3), provides some useful information about s_Y .

Proposition 3.13 (cf. [21, Corollaries 4.4 and 5.2, Proposition 5.1] and [22, Lemma 4.2 and Remark 4.5]) *Let D be an integral domain and let Y and Z be two nonempty subsets of $\text{Spec}(D)$. The following statements hold.*

- (1) $s_Y = s_Z$ if and only if $Y^{\text{gen}} = Z^{\text{gen}}$.
- (2) s_Y is of finite type if and only if Y is quasi-compact.
- (3) $\widetilde{s}_Y = \widetilde{s}_Z$ if and only if $\text{Cl}^{\text{inv}}(Y) = \text{Cl}^{\text{inv}}(Z)$.
- (4) $\widetilde{s}_Y = s_{\text{Cl}^{\text{inv}}(Y)}$.

Note that, in general, $(s_Y)_f$ is quasi-spectral but not spectral, and it is spectral if and only if $(s_Y)_f$ is stable [22, Proposition 4.23(2)]. In other words, it is possible that $\widetilde{s}_Y \preceq (s_Y)_f$ (see [21, Remark 5.3] and [2, p. 2466]) and thus it is not true in general that $(s_Y)_f = s_{\text{Cl}^{\text{inv}}(Y)}$.

The following result provides control of the infimum and the supremum of a family of spectral operations:

Lemma 3.14 (cf. [18, Lemma 4.3]) *Let \mathcal{D} be a nonempty set of spectral semistar operations on an integral domain D . For each spectral semistar operation \star , set $\Delta(\star) := \text{QSpec}^\star(D)$. Then, the following statements hold.*

- (1) $\wedge_{\mathcal{D}}$ is spectral with $\Delta(\wedge_{\mathcal{D}}) = \bigcup\{\Delta(\star) \mid \star \in \mathcal{D}\}$.
- (2) If $\vee_{\mathcal{D}}$ is quasi-spectral, then it is spectral with $\Delta(\vee_{\mathcal{D}}) = \bigcap\{\Delta(\star) \mid \star \in \mathcal{D}\}$.

Note that the hypothesis that $\vee_{\mathcal{D}}$ be quasi-spectral in point (2) is necessary: for example, if \mathbb{A} is the ring of all algebraic integers, $\star_P := s_{\text{Max}(\mathbb{A}) \setminus \{P\}}$ and $\mathcal{D} := \{\star_P \mid P \in \text{Max}(\mathbb{A})\} \subseteq \text{SStar}_{\text{sp}}(\mathbb{A})$, then $\vee_{\mathcal{D}}$ is a semistar operation that closes \mathbb{A} and thus closes every principal ideal of \mathbb{A} , while $\text{QSpec}^{\vee_{\mathcal{D}}}(D) = \{(0)\}$, hence $\vee_{\mathcal{D}}$ is not quasi-spectral. (See [18, Example 4.4] for more details.)

Lemma 3.14(2) provides useful information on the supremum of a family of spectral semistar operations of finite type, allowing one to prove that the space of all stable semistar operations of finite type is spectral. The proof of the following theorem follows closely the one of Theorem 3.8.

Theorem 3.15 (cf. [18, Theorem 4.5]) *Let D be an integral domain. Then, $\widetilde{\text{SStar}}(D)$ is a spectral space.*

Stable semistar operations are closely related to the concept of *localizing systems*, in the sense of Gabriel-Popescu (cf. for instance [6, Chap. II], [7, 30, 44, 50]). Recall that a localizing system on D is a subset \mathcal{F} of ideals of D such that:

- if $I \in \mathcal{F}$ and J is an ideal of D such that $I \subseteq J$, then $J \in \mathcal{F}$;
- if $I \in \mathcal{F}$ and J is an ideal of D such that, for each $i \in I$, $(J :_D iD) \in \mathcal{F}$, then $J \in \mathcal{F}$.

A localizing system \mathcal{F} is said to be *of finite type* if for each $I \in \mathcal{F}$ there exists a nonzero finitely generated ideal $J \in \mathcal{F}$ such that $J \subseteq I$. For instance, if T is an overring of R , $\mathcal{F}(T) := \{I \mid I \text{ ideal of } D, IT = T\}$ is a localizing system of finite type, while, if V is a valuation domain with a nonzero idempotent prime ideal P , then $\widetilde{\mathcal{F}}(P) := \{I \mid I \text{ ideal of } V \text{ and } I \supseteq P\}$ is a localizing system of V which is not of finite type [28, Proposition 5.1.12 and Remark 5.1.13]. We denote by $\text{LS}(D)$ (respectively, $\text{LS}_{\mathcal{F}}(D)$) the set of all localizing systems (respectively, localizing systems of

finite type) on D . We can introduce on these sets a natural topology, that we still call the *Zariski topology*, whose subbasic open sets are the $W_I := \{\mathcal{F} \in \text{LS}(D) \mid I \in \mathcal{F}\}$, as I varies among the ideals in D .

Theorem 3.16 (cf. [18, Proposition 3.5, Proposition 4.1(5) and Corollary 4.6]) *Let D be an integral domain. The map $\lambda : \text{LS}(D) \rightarrow \overline{\text{SStar}}(D)$ (respectively, the map $\lambda_f : \text{LS}_f(D) \rightarrow \widetilde{\text{SStar}}(D)$), defined by $\mathcal{F} \mapsto \star_{\mathcal{F}}$, establishes a homeomorphism between spaces endowed with the Zariski topologies (respectively, the induced topologies from the Zariski topologies). In particular, by Theorem 3.15, $\text{LS}_f(D)$ is a spectral space.*

4 The Space of Inverse-Closed Subsets of a Spectral Space

Let D be an integral domain. By the results in the previous sections, the spaces $\text{Overr}(D)$, $\widetilde{\text{SStar}}(D)$ and $\text{SStar}_f(D)$ are spectral spaces. Since $\text{Spec}(D)$ can be embedded in each of these spaces, they can be seen as peculiar “spectral extensions” of $\text{Spec}(D)$.

In particular, in this section we focus on the canonical embedding $\text{Spec}(D) \hookrightarrow \widetilde{\text{SStar}}(D)$, in order to generalize this spectral extension to arbitrary rings or to arbitrary spectral spaces. For this purpose, we need some preliminaries, including the notions and properties of Sect. 1.3.

We start by observing that the natural injection $s : \text{Spec}(D) \rightarrow \widetilde{\text{SStar}}(D)$, defined by $s(P) := s_{\{P\}} = \wedge_{\{D_P\}}$, is a topological embedding of topological (spectral) spaces (both endowed with the Zariski topology). Indeed, if J is a finitely generated ideal of D and $\widetilde{U}_J := \bigvee_J \cap \widetilde{\text{SStar}}(D) = \{\star \in \text{SStar}(D) \mid 1 \in J^*\} \cap \widetilde{\text{SStar}}(D)$ is a generic subbasic open set of $\widetilde{\text{SStar}}(D)$, then

$$s^{-1}(\widetilde{U}_J) = \{P \in \text{Spec}(D) \mid 1 \in JD_P\} = D(J).$$

Remark 4.1 The map $s : \text{Spec}(D) \rightarrow \widetilde{\text{SStar}}(D)$ is the composition of the homeomorphism $\ell : \text{Spec}(D) \rightarrow \text{Loc}(D)$, defined by $\ell(P) := D_P$, for each $P \in \text{Spec}(D)$ and the topological embedding $\iota_{\text{Loc}} : \text{Loc}(D) \hookrightarrow \widetilde{\text{SStar}}(D)$ (defined in Remark 3.9(b)). Note also that the homeomorphism ℓ induces an isomorphism of partially ordered sets (with the ordering induced by the topologies), however the ordering in $\text{Loc}(D)$, induced by the Zariski topology, is the opposite order of the set-theoretic inclusion.

Given a spectral space X , let $\mathcal{X}(X) := \{Y \subseteq X \mid Y \neq \emptyset, Y = \text{Cl}^{\text{inv}}(Y)\}$. If $X = \text{Spec}(R)$ for some ring R , we write for short $\mathcal{X}(R)$ instead of $\mathcal{X}(\text{Spec}(R))$.

We define a *Zariski topology* on $\mathcal{X}(X)$ by taking, as subbasis of open sets, the sets of the form

$$\mathcal{U}(\Omega) := \{Y \in \mathcal{X}(X) \mid Y \subseteq \Omega\},$$

where Ω varies among the quasi-compact open subspaces of X . Note that the previous subbasis is in fact a basis, since $\mathcal{U}(\Omega) \cap \mathcal{U}(\Omega') = \mathcal{U}(\Omega \cap \Omega')$ and $\Omega \cap \Omega'$ is a quasi-compact open subspace of X , for any pair Ω, Ω' of quasi-compact open subspaces of X . Moreover, $\Omega \in \mathcal{U}(\Omega)$, since a quasi-compact open subset Ω of X is a closed set in the inverse topology of X . Note also that, when $X = \text{Spec}(R)$, for some ring R , a generic basic open set of the Zariski topology on $\mathcal{X}(R)$ is of the form

$$\mathcal{U}(\text{D}(J)) = \{Y \in \mathcal{X}(R) \mid Y \subseteq \text{D}(J)\}$$

where J is any finitely generated ideal of R .

The main result in this setting is the following, which provides a description of the space $\mathcal{X}(X)$ (see [19]).

Theorem 4.2 *Let X be a spectral space.*

- (1) *The space $\mathcal{X}(X)$, endowed with the Zariski topology, is a spectral space.*
- (2) *Let $Y_1, Y_2 \in \mathcal{X}(X)$. Then, $Y_1 \subseteq Y_2$ if and only if $Y_1 \leq_{\mathcal{X}(X)} Y_2$.*
- (3) *The canonical map $\varphi : X \rightarrow \mathcal{X}(X)$, defined by $\varphi(x) := \{x\}^{\text{gen}}$, for each $x \in X$, is a spectral embedding (which is also an order-preserving embedding between ordered sets, with the ordering induced by the Zariski topologies).*
- (4) *$\mathcal{X}(X)$ has a unique maximal point (i.e., X).*
- (5) *Let Z be another spectral space and let $\varphi : X \rightarrow \mathcal{X}(X)$ be the spectral embedding defined in (3). Consider a spectral map $\lambda : X \rightarrow Z$ satisfying the following condition:*

(sup-completion) *For each nonempty quasi-compact subspace Y of X , there exists $z_Y := \sup\{\lambda(y) \mid y \in Y\}$ (where \sup is taken with respect to the ordering induced by the topology of Z) and if Y' is another nonempty quasi-compact subspace of X , with $\text{Cl}_X^{\text{inv}}(Y') \neq \text{Cl}_X^{\text{inv}}(Y)$, then $z_{Y'} \neq z_Y$. Moreover, if \mathcal{W} denotes the set of all nonempty quasi-compact open subspaces Ω of X , then $\mathcal{B} := \{\{z_\Omega\}^{\text{gen}} \mid \Omega \in \mathcal{W}\}$ is a subbasis for the open sets of Z .*

Then, the following properties hold.

- (5.a) *There exists a spectral embedding $\lambda^\# : \mathcal{X}(X) \rightarrow Z$ such that $\lambda^\# \circ \varphi = \lambda$.*
- (5.b) *If, furthermore, $z = \sup_Z\{\lambda(x) \mid x \in \lambda^{-1}(\{z\}^{\text{gen}})\}$ for each $z \in Z$, then $\lambda^\# : \mathcal{X}(X) \rightarrow Z$ is the unique spectral embedding (in fact, homeomorphism) such that $\lambda^\# \circ \varphi = \lambda$.*

Let X be a spectral space and let $\hat{\mathcal{X}}(X) := \{Y \subseteq X \mid Y = \text{Cl}_X^{\text{inv}}(Y)\} = \mathcal{X}(X) \cup \{\emptyset\}$. The techniques used for proving Theorem 4.2(1) allow also to show that $\hat{\mathcal{X}}(X)$ (endowed with an obvious extension of the topology of $\mathcal{X}(X)$) is a spectral space. Moreover, since $\mathcal{U}(\emptyset) = \{\emptyset\}$ is open in $\hat{\mathcal{X}}(X)$, then we deduce that $\mathcal{X}(X)$ is a closed (spectral) subspace of $\hat{\mathcal{X}}(X)$.

As a consequence of the previous theorem, it is possible to compare the dimensions of X and $\mathcal{X}(X)$ with the cardinality $|X|$ of the spectral space X (see [19]).

Proposition 4.3 *Let X be a spectral space and let $\varphi : X \rightarrow \mathcal{X}(X)$ be the topological embedding defined in Theorem 4.2(2). Then,*

- (1) $\varphi(X) = \mathcal{X}(X)$ if and only if (X, \leq) is linearly ordered.
- (2) $\dim(\mathcal{X}(X)) = |X| - 1 \geq \dim(X)$. Moreover, in the finite dimensional case, $\dim(\mathcal{X}(X)) = \dim(X)$ if and only if X is linearly ordered.

While the inequality $|X| - 1 \geq \dim(X)$ is sharp, the more noncomparable elements the set X contains, the smaller $\dim(X)$ is with respect to $|X|$. For example, if X is homeomorphic to the prime spectrum of the direct product of $n + 1$ fields, $n \geq 1$, then $\dim(X) = 0$ while $|X| - 1 = n$.

Furthermore, if $\dim(X)$ is not finite, then clearly $\dim(\mathcal{X}(X)) = \dim(X)$, but we can easily choose X to be not totally ordered.

We also note that, if $\phi : X \rightarrow Y$ is a spectral map of spectral space, the map $\mathcal{X}(\phi) : \mathcal{X}(X) \rightarrow \mathcal{X}(Y)$ defined by $\mathcal{X}(\phi)(C) := \phi(C)^{\text{gen}}$ for every inverse-closed subset C of X is again a spectral map. It follows that the assignment $X \mapsto \mathcal{X}(X)$, $\phi \mapsto \mathcal{X}(\phi)$ is a (covariant) functor from the category of spectral spaces into itself (see [19] for details).

We show next that the map $\lambda^\# : \mathcal{X}(X) \rightarrow Z$ (Theorem 4.2(5.a)) is not unique. The following example shows in fact that it is possible that there exist two different spectral maps (with at most one non-injective) $\Lambda_1, \Lambda_2 : \mathcal{X}(X) \rightarrow Z$, $\Lambda_1 \neq \Lambda_2$, such that $\Lambda_1 \circ \varphi = \lambda = \Lambda_2 \circ \varphi$.

Example 4.4 Consider the spectral space $X := \{0, a, b, c\}$, with $0 < a, b, c$ and a, b, c not comparable. Let $\Lambda : \mathcal{X}(X) \rightarrow \mathcal{X}(X)$ be the function defined by

$$\Lambda(C) := \begin{cases} C & \text{if } C \neq \{a, b\}^{\text{gen}}, \\ X & \text{if } C = \{a, b\}^{\text{gen}}. \end{cases}$$

The unique basic open set of $\mathcal{X}(X)$ containing $\{a, b\}^{\text{gen}}$ is $\mathcal{U}(\{a, b\}^{\text{gen}})$, and clearly we have $\Lambda^{-1}(\mathcal{U}(\{a, b\}^{\text{gen}})) = \mathcal{U}(\{a\}^{\text{gen}}) \cup \mathcal{U}(\{b\}^{\text{gen}})$. For any other basic open set \mathcal{U} of $\mathcal{X}(X)$, we have $\Lambda^{-1}(\mathcal{U}) = \mathcal{U}$. This shows that Λ is a nontrivial spectral map, $\Lambda \neq \text{id}_{\mathcal{X}(X)}$, such that $\Lambda(\{x\}^{\text{gen}}) = \{x\}^{\text{gen}}$, for each $x \in X$.

The following statement provides an explicit characterization of the space $\mathcal{X}(X)$ and follows immediately from Theorem 4.2(5).

Corollary 4.5 *Let $\lambda : X \rightarrow Z$ be a spectral embedding of spectral spaces. Then, the following conditions are equivalent.*

- (i) Z is a partially ordered set (under the ordering induced by the topology), for each $z \in Z$, $z = \sup_Z \{\lambda(x) \mid x \in \lambda^{-1}(\{z\}^{\text{gen}})\}$, and λ satisfies the condition (sup-completion).
- (ii) Z is homeomorphic to $\mathcal{X}(X)$, via a unique homeomorphism $\Lambda : \mathcal{X}(X) \rightarrow Z$ such that $\Lambda \circ \varphi = \lambda$.

In the special case where $X = \text{Spec}(D)$ for some integral domain D , the spectral space $\mathcal{X}(D) := \{Y \subseteq \text{Spec}(D) \mid \emptyset \neq Y = \text{Cl}^{\text{inv}}(Y)\}$ can be interpreted in terms of stable semistar operations of finite type (see [19]).

Proposition 4.6 *Let D be an integral domain and let $\mathcal{X}(D) := \{Y \subseteq \text{Spec}(D) \mid \emptyset \neq Y = \text{Cl}^{\text{inv}}(Y)\}$. The map $s^\# : \mathcal{X}(D) \rightarrow \widetilde{\text{SStar}}(D)$, defined by $s^\#(Y) := s_Y$ for each $Y \in \mathcal{X}(D)$, is a homeomorphism with inverse map $\Delta : \widetilde{\text{SStar}}(D) \rightarrow \mathcal{X}(D)$, defined by $\Delta(\star) := \text{QSpec}^*(D)$ for each \star stable semistar operation of finite type on D . Moreover, if $\varphi : \text{Spec}(D) \rightarrow \mathcal{X}(D)$ is the topological embedding defined in Theorem 4.2(3) and $s : \text{Spec}(D) \rightarrow \widetilde{\text{SStar}}(D)$ is the topological embedding defined by $P \mapsto s_{\{P\}}$, for each prime ideal P of D , then $s^\# \circ \varphi = s$.*

As a consequence of the previous proposition and Theorem 4.2(1) we reobtain immediately Theorem 3.15, that is, the space of all stable semistar operations of finite type on an integral domain is a spectral space.

5 A Topological Version of Hilbert's Nullstellensatz

As a first application of the general construction considered in the previous section, we give now a topological version of Hilbert's Nullstellensatz.

Given a ring R , consider the set $\text{Rd}(R) := \{I \mid I \text{ ideal of } R \text{ and } I = \text{rad}(I)\}$ of radical ideals of R and, more generally, the set $\text{Id}(R) := \{I \mid I \text{ ideal of } R\}$, endowed with the *hull–kernel topology*, defined by taking as a basis for the open sets the subsets

$$U(x_1, x_2, \dots, x_n) := \{I \in \text{Id}(R) \mid x_i \notin I \text{ for some } i, 1 \leq i \leq n\},$$

where $x_1, x_2, \dots, x_n \in R$. We denote by $\text{Id}(R)^{\text{hk}}$ (respectively, $\text{Rd}(R)^{\text{hk}}$) the set of all the ideals of R (respectively, of all the radical ideals of R), endowed with the hull–kernel topology (respectively, with the induced topology from the hull–kernel topology of $\text{Id}(R)$). In this situation, the inclusion maps $\text{Spec}(R) \subseteq \text{Rd}(R) \subseteq \text{Id}(R)$ become topological embeddings; in other words the hull–kernel topology induced on $\text{Spec}(R)$ coincides with the Zariski topology.

For deepening the study of the topological space $\text{Rd}(R)^{\text{hk}}$ we introduce an analogue, in the inverse topology, of the space $\mathcal{X}(R)$ (Sect. 4).

Let X be a spectral space and let $\text{Cl}(Y)$ denote the closure of a subspace Y in the given topology of X . For the sake of simplicity, we denote by X' the spectral space X^{inv} , i.e., the set X endowed with the inverse topology [35, Proposition 8]. We set $\mathcal{X}'(X) := \{Y \subseteq X \mid Y \neq \emptyset, Y = \text{Cl}(Y)\}$ and, for each quasi-compact open subspaces Ω of X , we set $\mathcal{U}'(\Omega) := \{Y \in \mathcal{X}'(X) \mid Y \cap \Omega = \emptyset\} = \mathcal{U}(\Omega')$, where $\Omega' := X \setminus \Omega$.

It is well known that $(X^{\text{inv}})^{\text{inv}}$ coincides with X (with the given spectral topology) [35, Proposition 8] hence, *mutatis mutandis*, we can now apply Theorem 4.2, since $\mathcal{X}'(X) = \mathcal{X}(X')$, and we easily get the following.

Proposition 5.1 *Let X be a spectral space and let $X' := X^{\text{inv}}$.*

- (1) *The space $\mathcal{X}'(X) := \{Y \subseteq X \mid Y \neq \emptyset, Y = \text{Cl}(Y)\}$ is a spectral space, when endowed with the topology, called the Zariski topology, having as a basis of open sets, the sets of the form $\mathcal{U}'(\Omega)$, where Ω varies among the quasi-compact open subspaces of X .*
- (2) *The canonical map $\varphi' : X' \rightarrow \mathcal{X}'(X)$, defined by $\varphi'(x) := \{x\}^{\text{sp}}$, for each $x \in X$, is a spectral embedding between spectral spaces.*

Suppose now that $X := \text{Spec}(R)$ is the prime spectrum of a commutative ring R , endowed with the Zariski topology. We recall that a basis of open sets of X^{inv} is the collection of sets $\{\text{V}(J) \mid J \text{ is a finitely generated ideal of } R\}$ which makes X^{inv} a spectral space [35, Proposition 8].

Remark 5.2 With the notation introduced above, let $\varphi' : X' = \text{Spec}(R)^{\text{inv}} \hookrightarrow \mathcal{X}'(X)^{\text{zar}} = \mathcal{X}'(X)^{\text{zar}}$ be the canonical topological embedding defined by $\varphi'(x) := \{x\}^{\text{sp}}$. Then, it is easy to see that the map $\psi := (\varphi')^{\text{inv}} : X = (\text{Spec}(R)^{\text{inv}})^{\text{inv}} \hookrightarrow \mathcal{X}'(X)^{\text{inv}}$ defined by $\psi(x) := \{x\}^{\text{gen}}$ is a topological embedding (acting like φ as a set-theoretic map).

The next result provides a topological version of Hilbert Nullstellensatz (see [20]).

Theorem 5.3 *Let R be a ring and let $\mathcal{X}'(R) := \mathcal{X}'(\text{Spec}(R))$ be the spectral space of the nonempty Zariski closed subspaces of $\text{Spec}(R)$ (Proposition 5.1). We can also consider the space $\mathcal{X}'(R)$ as a spectral space endowed with the inverse topology [35, Proposition 8]. Then, for each $C \in \mathcal{X}'(R)$, the map:*

$$\mathcal{J} : \mathcal{X}'(R)^{\text{inv}} \rightarrow \text{Rd}(R)^{\text{hk}} \text{ defined by } \mathcal{J}(C) := \bigcap \{P \in \text{Spec}(R) \mid P \in C\},$$

is a homeomorphism.

Related to the previous Theorem 5.3, it is possible to prove, with a standard argument based on Theorem 2.8, that the set of all ideals of a ring is also a spectral space. More precisely:

Proposition 5.4 (cf. [20]) *Let $\text{Id}(R)$ be the space of all ideals of a ring R , endowed with the hull–kernel topology. Then, $\text{Id}(R)$ is a spectral space, having $\text{Rd}(R)$ (endowed with the hull–kernel topology) as a spectral subspace.*

The following Hasse diagram summarizes some of the results proved above.

$$\begin{array}{ccccc}
 & & \text{Id}(R)^{\text{hk}} & & \\
 & & \uparrow & & \\
 \mathcal{X}'(\text{Spec}(R))^{\text{inv}} & \simeq & \text{Rd}(R)^{\text{hk}} & & \mathcal{X}'(\text{Spec}(R))^{\text{zar}} \\
 \uparrow & & \uparrow & & \uparrow \\
 \text{Spec}(R)^{\text{zar}} & \xlongequal{\quad} & \text{Spec}(R)^{\text{hk}} & \xlongequal{\quad} & \text{Spec}(R)^{\text{zar}}
 \end{array}$$

6 The Space of eab Semistar Operations of Finite Type

In the present section, we give another application of Theorem 4.2. More precisely, we apply the construction of the space $\mathcal{X}(X)$ to the case of the Riemann–Zariski spectral space $X := \text{Zar}(D)$ of all valuation overrings of an integral domain D (endowed with the Zariski topology, see Sect. 1.2).

Let \star be a semistar operation on an integral domain D . We say that \star is an *eab semistar operation* (respectively, an *ab semistar operation*) if, for every $F, G, H \in \mathbf{f}(D)$ (respectively, for every $F \in \mathbf{f}(D)$, $G, H \in \overline{\mathbf{F}}(D)$) the inclusion $(FG)^\star \subseteq (FH)^\star$ implies $G^\star \subseteq H^\star$. Note that, if \star is eab, then \star_f is also eab, since \star and \star_f agree on finitely generated fractional ideals. The concepts of eab and ab operations coincide on finite-type operations, but not in general [27, 29].

It is easy to see that a *valuative semistar operation*, i.e., a semistar operation of the type $\wedge_{\mathcal{W}}$, where $\mathcal{W} \subseteq \text{Zar}(D)$, is an eab semistar operation. In particular, the *b-operation*, where $b := \wedge_{\text{Zar}(D)}$, is an eab semistar operation of finite type, since $\text{Zar}(D)$ is quasi-compact (Corollary 3.5).

To every semistar operation $\star \in \text{SStar}(D)$ we can associate a map \star_a defined by

$$F^{\star_a} := \bigcup \{((FG)^\star : G^\star) \mid G \in \mathbf{f}(D)\}$$

for every $F \in \mathbf{f}(D)$, and then we can extend it to arbitrary D -modules $E \in \overline{\mathbf{F}}(D)$ by setting $E^{\star_a} := \bigcup \{F^{\star_a} \mid F \subseteq E, F \in \mathbf{f}(D)\}$. The map \star_a is always an eab semistar operation of finite type on D . Moreover, $\star = \star_a$ if and only if \star is an eab semistar operation of finite type and, if \star is an eab semistar operation, then $\star_a = \star_f$ [23, Proposition 4.5].

Remark 6.1 (a) Let T be an overring of D , and let \star_T be a semistar operation on T . Then, we can define a semistar operation \star on D by $\star := \star_T \circ \wedge_{\{T\}}$, i.e., $E^\star := (ET)^{\star_T}$ for every $E \in \overline{\mathbf{F}}(D)$. If now $F \in \mathbf{f}(T)$, then

$$\begin{aligned} F^{\star_a} &= \bigcup \{((FG)^\star : G^\star) \mid G \in \mathbf{f}(D)\} = \bigcup \{((FGT)^{\star_T} : (GT)^{\star_T}) \mid G \in \mathbf{f}(D)\} = \\ &= \bigcup \{((FTH)^{\star_T} : H^{\star_T}) \mid H \in \mathbf{f}(T)\} = (FT)^{(\star_T)_a} = F^{(\star_T)_a}. \end{aligned}$$

Hence, for every $E \in \overline{\mathbf{F}}(D)$, $E^{\star_a} = (ET)^{(\star_T)_a}$, that is, $\star_a = (\star_T)_a \circ \wedge_{\{T\}}$.

(b) W. Krull only considered the concept of an “arithmetisch brauchbar” operation (for short ab-operation, as above) [41]. He did not consider the concept of “endlich arithmetisch brauchbar” operation (or, more simply, eab-operation as above). This concept stems from the original version of Gilmer’s book [31].

(c) Denote by $\text{SStar}_{\text{val}}(D)$ (respectively, $\text{SStar}_{\text{eab}}(D)$; $\text{SStar}_{f, \text{eab}}(D)$) the set of valutive (respectively, eab; eab of finite type) semistar operations on D . Every valutive operation is eab, but not every eab operation is valutive; however, the two definitions agree on finite-type operations, i.e.,

$$\text{SStar}_{\text{eab}}(D) \cap \text{SStar}_f(D) =: \text{SStar}_{f, \text{eab}}(D) = \text{SStar}_{\text{val}}(D) \cap \text{SStar}_f(D),$$

(see, for instance, [23, Corollary 5.2]). A similar relationship holds between spectral and stable semistar operations, with the valutive operations corresponding to the spectral ones and the eab operations to the stable ones, i.e., every spectral semistar operation is stable but not every stable semistar operation is spectral, however $\text{SStar}_{f, \text{sp}}(D) = \overline{\text{SStar}}(D) \cap \text{SStar}_f(D) = \widetilde{\text{SStar}}(D)$ (Remark 3.9(a)).

Recall also that there are examples of eab semistar operations which are quasi-spectral but not valutive [27, Example 15].

It is not hard to prove the following statement, which is a companion to Proposition 3.10.

Proposition 6.2 (cf. [18, Proposition 5.2]) *Let D be an integral domain and let $\Phi_a : \text{SStar}(D) \rightarrow \text{SStar}(D)$ be the map defined by $\star \mapsto \star_a$. Then:*

- (1) *The image of Φ_a coincides with $\text{SStar}_{f, \text{eab}}(D)$.*
- (2) *The map Φ_a is continuous in the Zariski topology.*
- (3) *The map Φ_a is a topological retraction of $\text{SStar}(D)$ onto $\text{SStar}_{f, \text{eab}}(D)$.*

The relation between valutive operations and subsets of $\text{Zar}(D)$ behaves very similarly to the relation between spectral operations and subsets of $\text{Spec}(D)$ (Proposition 3.13).

Proposition 6.3 *Let D be an integral domain and let Y and Z be two nonempty subsets of $\text{Zar}(D)$. Then, the following statements hold.*

- (1) *$\wedge_Y = \wedge_Z$ if and only if $Y^{\text{gen}} = Z^{\text{gen}}$.*
- (2) *\wedge_Y is of finite type if and only if Y is quasi-compact [21, Proposition 4.5].*
- (3) *$(\wedge_Y)_f = (\wedge_Z)_f$ if and only if $\text{Cl}^{\text{inv}}(Y) = \text{Cl}^{\text{inv}}(Z)$ [17, Theorem 4.9].*
- (4) *$(\wedge_Y)_f = \wedge_{\text{Cl}^{\text{inv}}(Y)}$ [17, Corollary 4.17].*

Note that $Y^{\text{gen}} = \{V \in \text{Zar}(D) \mid V \supseteq V_0, \text{ for some } V_0 \in Y\}$. For the statement (1), assume first that $\wedge_Y = \wedge_Z$. Let V be a valuation domain such that $V \in Y^{\text{gen}} \setminus Z^{\text{gen}}$. Then, for any $W \in Z$, we can pick an element $x_W \in W \setminus V$. It follows that $I := (x_W^{-1} \mid W \in Z) \subseteq M_V$, where M_V is the maximal ideal of V . Thus, if $V_0 \in Y$ is such that $V_0 \subseteq V$ (such a V_0 exists since $V \in Y^{\text{gen}}$), we have $IV_0 \subseteq M_{V_0}$ and, in particular, $1 \notin I^{\wedge_Y}$. On the other hand, clearly $1 \in I^{\wedge_Z}$, a contradiction. The converse it is straightforward since, for each $Y \subseteq \text{Zar}(D)$, $\wedge_Y = \wedge_{Y^{\text{gen}}}$.

Remark 6.4 Since $b = \wedge_{\text{Zar}(D)}$ is a semistar operation of finite type (and this can be proved completely independently from the topological point of view, see [39, Proposition 6.8.2] and [21, Remark 4.6]), from Proposition 6.3 we get a new proof of the fact that $\text{Zar}(D)$ is a quasi-compact space (this is a special case of Zariski's theorem [53, Theorem 40, p. 113]).

The embedding $\iota : \text{Overr}(D) \rightarrow \text{SStar}_f(D)$ (Proposition 3.2) restricts to an embedding $\text{Zar}(D) \hookrightarrow \text{SStar}_{f, \text{eab}}(D)$, while the image of the restriction $\pi|_{\text{SStar}_{f, \text{eab}}(D)}$ of the canonical map $\pi : \text{SStar}_f(D) \rightarrow \text{Overr}(D)$ (defined by

$\star \mapsto D^\star$) coincides with $\text{Overr}_{i_c}(D)$, i.e., with the space of the overrings of D that are integrally closed in K (since, by a well known Krull's theorem, every integrally closed ring can be represented as an intersection of valuation rings [53, Theorem 6, p. 15]).

Using the b -operation, we can introduce a general version of the classical Kronecker function ring, introduced by L. Kronecker in the case of Dedekind domains. Let \mathbb{X} be an indeterminate over D and let $\mathfrak{c}(h)$ be the content of a polynomial $h \in D[\mathbb{X}]$ (i.e., the ideal of D generated by the coefficients of h). Then, we set:

$$\begin{aligned} \text{Kr}(D) &:= \text{Kr}(D, b) := \{f/g \mid f, g \in D[\mathbb{X}], g \neq 0, \text{ with } \mathfrak{c}(f)^b \subseteq \mathfrak{c}(g)^b\} \\ &= \bigcap \{V(\mathbb{X}) \mid V \in \text{Zar}(D)\}, \end{aligned}$$

where $V(\mathbb{X})$ denotes the Gaussian (or trivial) extension of V to $K(\mathbb{X})$, i.e., $V(\mathbb{X}) := V[\mathbb{X}]_{(MV[\mathbb{X}])}$. This is a Bézout domain with quotient field $K(\mathbb{X})$, called *the b -Kronecker function ring of D* (see [23, Definition 3.2, Corollary 3.4(2) and Theorem 5.1], [25, Theorem 14] and [32, Theorem 32.11]). It follows immediately that the localization map $\text{Spec}(\text{Kr}(D)) \rightarrow \text{Zar}(\text{Kr}(D))$ (defined by $P \mapsto \text{Kr}(D)_P$) is actually an homeomorphism. Moreover, the map $\Psi : \text{Zar}(D) \rightarrow \text{Zar}(\text{Kr}(D))$ (defined by $V \mapsto V(\mathbb{X})$) is a homeomorphism [17, Propositions 3.1 and 3.3], so that $\text{Spec}(\text{Kr}(D))$ realizes $\text{Zar}(D)$ as a spectral space [11, Theorem 2].

In particular, the homeomorphism (and so the isomorphism of partially ordered sets) that we denote by θ , from $\text{Spec}(\text{Kr}(D))$ to $\text{Zar}(D)$ induces a 1-1 correspondence Θ_0 between the set $\{Y \subseteq \text{Spec}(\text{Kr}(D)) \mid Y = Y^\downarrow\}$ (where $Y^\downarrow := \{z \in \text{Spec}(\text{Kr}(D)) \mid z \leq y, \text{ for some } y \in Y\} = Y^{\text{gen}}$) and the set $\{\mathcal{W} \subseteq \text{Zar}(D) \mid \mathcal{W} = \mathcal{W}^\uparrow\}$ (where $\mathcal{W}^\uparrow := \{W' \in \text{Zar}(D) \mid W' \supseteq W, \text{ for some } W \in \mathcal{W}\} = \mathcal{W}^{\text{gen}}$). Therefore Θ_0 induces a bijection $\Theta : \text{SStar}_{sp}(\text{Kr}(D)) \rightarrow \text{SStar}_{val}(D)$ defined by $\Theta(s_Y) := \wedge_{\Theta_0(Y)}$, where $\Theta_0(Y) = \{V \in \text{Zar}(D) \mid M(\mathbb{X}) \cap \text{Kr}(D) \in Y\} =: \mathcal{V}(Y)$ and $M(\mathbb{X})$ is the maximal ideal of $V(\mathbb{X})$.

Theorem 6.5 (cf. [18, Theorem 5.11]) *Let D be an integral domain. Then, the bijection Θ , restricted to $\text{SStar}_f(D)$, induces a homeomorphism between $\widetilde{\text{SStar}}(\text{Kr}(D))$ and $\text{SStar}_{f, eab}(D)$. In particular, $\text{SStar}_{f, eab}(D)$ is a spectral space.*

Another interpretation of the previous theorem can be given by considering the spectral space $\mathcal{X}(X)$, when X coincides with $\text{Zar}(D)$. This point of view sheds new light on the analogies between the spectral spaces $\widetilde{\text{SStar}}(D)$ ($= \text{SStar}_{f, sp}(D)$), by Remark 3.9(a) and $\text{SStar}_{f, eab}(D)$, after recalling that $\mathcal{X}(D) := \mathcal{X}(\text{Spec}(D))$ is canonically homeomorphic to $\widetilde{\text{SStar}}(D)$ (Proposition 4.6).

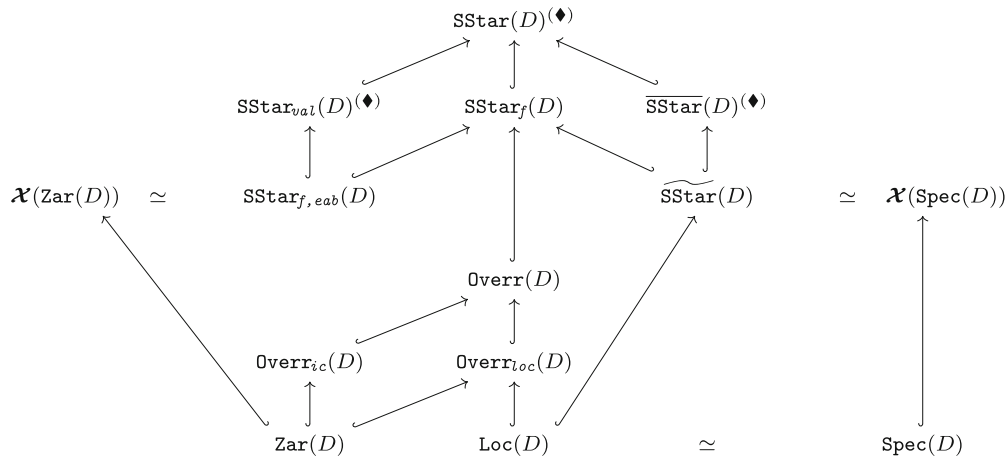
Corollary 6.6 *Let D be an integral domain. The map*

$$\Lambda : \mathcal{X}(\text{Zar}(D)) \rightarrow \text{SStar}_{f, eab}(D), \text{ defined by } \Lambda(\mathcal{Y}) := \wedge_{\mathcal{Y}},$$

for each inverse-closed subset \mathcal{Y} of $\text{Zar}(D)$, is a homeomorphism.

Proof (Sketch) The proof is based on the following key facts. The space $\mathcal{X}(\text{Zar}(D))$ is canonically homeomorphic to $\mathcal{X}(\text{Kr}(D))$ [19]. By Proposition 4.6, $\mathcal{X}(\text{Kr}(D)) \simeq \widetilde{\text{SStar}}(\text{Kr}(D)) (= \text{SStar}_{f, sp}(\text{Kr}(D)))$ and finally that the map Θ_f , restriction of Θ to $\text{SStar}_{f, sp}(\text{Kr}(D))$, from $\text{SStar}_{f, sp}(\text{Kr}(D))$ onto $\text{SStar}_{f, eab}(D)$, is a homeomorphism (for more details [18, Theorem 5.11(2)]). \square

The following Hasse diagram summarizes the topological embeddings of some of the spaces considered in the present paper. All spaces are spectral except possibly the three spaces denoted by (\blacklozenge) .



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