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SHARPNESS AND SEMISTAR OPERATIONS IN PRÜFER-LIKE DOMAINS

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ABSTRACT. Let $\star$ be a semistar operation on a domain D, $\star_f$ the finite-type semistar operation associated to $\star$, and D a Prüfer $\star$-multiplication domain (P$\star$MD). For the special case of a Prüfer domain (where $\star$ is equal to the identity semistar operation), we show that a nonzero prime P of D is sharp, that is, that $D_P \nsupseteq \bigcap D_M$, where the intersection is taken over the maximal ideals M of D that do not contain P, if and only if two closely related spectral semistar operations on D differ. We then give an appropriate definition of $\star_f$-sharpness for an arbitrary P$\star$MD D and show that a nonzero prime P of D is $\star_f$-sharp if and only if its extension to the $\star$-Nagata ring of D is sharp. Calling a P$\star$MD $\star_f$-sharp ($\star_f$-doublesharp) if each maximal (prime) $\star_f$-ideal of D is sharp, we also prove that such a D is $\star_f$-doublesharp if and only if each $(\star, t)$-linked overring of D is $\star_f$-sharp.

INTRODUCTION

A nonzero prime ideal P of a Prüfer domain D is said to be *sharp* if $\bigcap\{D_M \mid M \in \text{Max}(D),\ P \nsubseteq M\} \nsubseteq D_P$. In [16] Gilmer showed that an almost Dedekind domain with all maximal ideals sharp must be a Dedekind domain. Then Gilmer and Heinzer [18] made a more thorough study of sharpness in Prüfer domains, proving [18, Theorem 3] that in a Prüfer domain D all nonzero primes are sharp if and only if in each overring of D all maximal ideals are sharp. The primary goal of this work is to extend the study of sharpness to Prüfer $\star$-multiplication domains (P$\star$MDs) D, where $\star$ is an arbitrary semistar operation (definition recalled below) on D.

At this point it is helpful to recall a result of Griffin: An integral domain D is a PvMD (where v is the ordinary star operation on D and t is the canonically associated finite-type star operation associated to v) if and only if D_M is a valuation domain for each maximal prime t-ideal M of D. In Section 1, we provide background on semistar operations and P$\star$MDs and establish a few new results. In particular, we show (Lemma 1.8) that if D is a P$\star$MD (for some semistar operation $\star$ on D), then $R := D^{\tilde{\star}}$ is an "ordinary" PvMD and $D^{\tilde{\star}} = \bigcap\{D_{Q\cap R} \mid$

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Q is a maximal t-ideal of $R\}$ (where the semistar operation $\tilde{\star}$ is a spectral semistar operation naturally related to $\star$ and is defined below).

In Section 2 we give a semistar characterization of sharpness in Prüfer domains, and we show that this can be extended to the P$\star$MD setting. It is well-known that D is a P$\star$MD if and only if the associated Nagata ring, defined by $\mathrm{Na}(D,\star) := \{f/g \mid f,g \in D[X],\ g \neq 0,\ \boldsymbol{c}(g)^\star = D^\star\}$ (where X is an indeterminate and $\boldsymbol{c}(g)$ denotes the *content* of g, that is, the ideal generated by the coefficients of g) is a Prüfer domain [9, Theorem 3.1]. This provides an important tool for achieving our main results in Section 3. In our main theorem, we give a bijection between a certain set of overrings of a P$\star$MD and the entire set of overrings of the associated $\star$-Nagata ring of D, and, in Corollary 3.7, we obtain an extension of the result of Gilmer and Heinzer mentioned above to P$\star$MDs.

1. Background results

Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D–submodules of K, and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, i.e., $E \in \boldsymbol{F}(D)$ if $E \in \overline{\boldsymbol{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D–submodules of K. Then, obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

Following Okabe-Matsuda [26], a *semistar operation* on D is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D),\ E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

($\star_1$) $(xE)^\star = xE^\star$;
($\star_2$) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

A *(semi)star operation* is a semistar operation that, restricted to $\boldsymbol{F}(D)$, is a star operation (in the sense of [17, Section 32]). It is easy to see that a semistar operation $\star$ on D is a (semi)star operation if and only if $D^\star = D$.

If $\star$ is a semistar operation on D, then we can consider a map $\star_{_f} : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ defined, for each $E \in \overline{\boldsymbol{F}}(D)$, as follows:

$$E^{\star_f} := \bigcup\{F^\star \mid\ F \in \boldsymbol{f}(D) \text{ and } F \subseteq E\}.$$

It is easy to see that $\star_{_f}$ is a semistar operation on D, called *the finite type semistar operation associated to* $\star$ or *the semistar operation of finite type associated to* $\star$. Note that, for each $F \in \boldsymbol{f}(D)$, $F^\star = F^{\star_f}$. A semistar operation $\star$ is called a *semistar operation of finite type* (or, *finite type semistar operation*) if $\star = \star_{_f}$. It is easy to see that $(\star_{_f})_{_f} = \star_{_f}$ (that is, $\star_{_f}$ is of finite type).

If $\star_1$ and $\star_2$ are two semistar operations on D, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(D)$, equivalently, if $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$, for each $E \in \overline{\boldsymbol{F}}(D)$. Obviously, for each semistar operation $\star$, we have $\star_{_f} \leq \star$. Let d_D (or, simply, d) be the *identity (semi)star operation on* D. Clearly, $d \leq \star$ for all semistar operations $\star$ on D.

We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* if $I^\star \cap D = I$, a *quasi-$\star$-prime ideal* if it is a prime quasi-$\star$-ideal, and a *quasi-$\star$-maximal ideal* if it is maximal in the set of all proper quasi-$\star$-ideals. A quasi-$\star$-maximal ideal is a prime ideal. It is possible to prove that each proper quasi-$\star_f$-ideal is contained in a quasi-$\star_f$-maximal ideal. More details can be found in [12, page 4781]. We will denote by $\mathrm{QMax}^{\star}(D)$ (respectively, $\mathrm{QSpec}^{\star}(D)$) the set of all quasi-$\star$-maximal ideals (respectively, quasi-$\star$-prime ideals) of D. When $\star$ is a (semi)star operation, the notion of quasi-$\star$-ideal coincides with the "classical" notion of *$\star$-ideal* (i.e., a nonzero ideal I such that $I^\star = I$).

We say that $\star$ is *a stable semistar operation on D* if

$$(E \cap F)^\star = E^\star \cap F^\star, \quad \text{for all } E, F \in \overline{\boldsymbol{F}}(D).$$

For each $Q \in \mathrm{Spec}(D)$, let $\mathtt{s}_Q$ be the semistar operation (of finite type) on D defined as follows, for each $E \in \overline{\boldsymbol{F}}(D)$:

$$E^{\mathtt{s}_Q} := ED_Q .$$

Let Y be a subset of $\mathrm{Spec}(D)$ and let $\mathtt{s}_Y$ be the semistar operation on D defined as follows, for each $E \in \overline{\boldsymbol{F}}(D)$:

$$E^{\mathtt{s}_Y} := \bigcap \{ ED_Q \mid Q \in Y \} .$$

If $Y = \emptyset$, then $\mathtt{s}_\emptyset$ is the trivial semistar operation defined by $E^{\mathtt{s}_\emptyset} := K$, for each $E \in \overline{\boldsymbol{F}}(D)$.

A semistar operation of the type $\mathtt{s}_Y$, for some $Y \subseteq \mathrm{Spec}(D)$, is called a *spectral semistar operation* on D. As a consequence of flatness, it is easy to see that any spectral semistar operation is stable.

It is obvious that $Y' \subseteq Y''(\subseteq \mathrm{Spec}(D))$ implies $\mathtt{s}_{Y''} \leq \mathtt{s}_{Y'}$.

Set $\widetilde{\star} := \mathtt{s}_{\mathrm{QMax}^{\star_f}(D)}$. The star operation $\widetilde{\star}$ is called *the spectral semistar operation of finite type associated to $\star$*. It is known that $\widetilde{\star} \leq \star_f$ and $\mathrm{QMax}^{\star_f}(D) = \mathrm{QMax}^{\widetilde{\star}}(D)$. Note that the finite type semistar operation associated to a stable semistar operation is not necessarily stable. On the other hand, it is well known that finite type stable semistar operations coincide with finite type spectral semistar operations (see, for instance, [4, page 2952]).

In the following, we collect some of the properties concerning the relation between $Y \subseteq \mathrm{Spec}(D)$ and $\mathtt{s}_Y$. Let $Y^{\mathrm{gen}} := \{P \in \mathrm{Spec}(D) \mid P \subseteq Q, \text{ for some } Q \in Y\}$ and let $\mathtt{Cl}^{inv}(Y)$ be the closure of Y in the inverse topology of $\mathrm{Spec}(D)$, i.e., $\mathtt{Cl}^{inv}(Y) := \bigcap \{ D(J) \mid D(J) \supseteq Y,\ J \in \boldsymbol{f}(D)\}$.

Lemma 1.1. *Let D be an integral domain and let $Y, Y', Y'' \subseteq \mathrm{Spec}(D)$.*

(1) $\mathtt{s}_Y$ *is of finite type if and only if Y is quasi-compact.*
(2) $\mathtt{s}_{Y'} = \mathtt{s}_{Y''} \Leftrightarrow Y'^{\mathrm{gen}} = Y''^{\mathrm{gen}}$.
(3) $\widetilde{\mathtt{s}_{Y'}} = \widetilde{\mathtt{s}_{Y''}} \Leftrightarrow \mathtt{Cl}^{\mathrm{inv}}(Y') = \mathtt{Cl}^{\mathrm{inv}}(Y'')$.

For the proof, see [5, Corollary 4.4 and Proposition 5.1] and [4, Proposition 4.1].

A generalization of the classical Nagata ring construction was considered by Kang (1987 [23] and 1989 [24]). This construction has been generalized to the semistar setting: Given *any* integral domain D and *any* semistar operation $\star$ on D, we define *the semistar Nagata ring* as follows:

$$\mathrm{Na}(D, \star) := \{ f/g \mid f, g \in D[X],\ g \neq 0,\ \boldsymbol{c}(g)^\star = D^\star \} .$$

Note that $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \star_{_f})$. Therefore, the assumption $\star = \star_{_f}$ is not really restrictive when considering semistar Nagata rings.

If $\star = d$ is the identity (semi)star operation on D, then:

$$\mathrm{Na}(D, d) = D(X)\,.$$

Some results on *star* Nagata rings proved by Kang in 1989 are generalized to the semistar setting in the following:

Lemma 1.2. *Let $\star$ be a nontrivial semistar operation on an integral domain D. Set:*

$$N(\star) := N_D(\star) := \{h \in D[X] \mid \boldsymbol{c}(h)^\star = D^\star\}\,.$$

(1) $N(\star) = D[X] \setminus \bigcup\{Q[X] \mid Q \in \mathrm{QMax}^{\star_f}(D)\}$ *is a saturated multiplicatively closed subset of* $D[X]$ *and* $N(\star) = N(\star_{_f})$.
(2) $\mathrm{Max}(D[X]_{N(\star)}) = \{Q[X]_{N(\star)} \mid Q \in \mathrm{QMax}^{\star_f}(D)\}$.
(3) $\mathrm{Na}(D, \star) = D[X]_{N(\star)} = \bigcap\{D_Q(X) \mid Q \in \mathrm{QMax}^{\star_f}(D)\}$.
(4) $\mathrm{QMax}^{\star_f}(D)$ *coincides with the canonical image in* $\mathrm{Spec}(D)$ *of the maximal spectrum of* $\mathrm{Na}(D, \star)$; *i.e.,* $\mathrm{QMax}^{\star_f}(D) = \{M \cap D \mid M \in \mathrm{Max}(\mathrm{Na}(D, \star))\}$.
(5) *For each* $E \in \overline{\boldsymbol{F}}(D)$, $E^{\tilde{\star}} = E\mathrm{Na}(D, \star) \cap K$.

For the proof see [12, Theorem 3.1 and Proposition 3.4].

If $\star$ is *any* semistar operation on *any* integral domain D, then we define *the Kronecker function ring of D with respect to the semistar operation $\star$* by:

$$\mathrm{Kr}(D, \star) := \{f/g \mid f, g \in D[X],\ g \neq 0, \text{ and there exists } h \in D[X] \setminus \{0\} \text{ with } (\boldsymbol{c}(f)\boldsymbol{c}(h))^\star \subseteq (\boldsymbol{c}(g)\boldsymbol{c}(h))^\star\}.$$

This definition, given in [10, Theorem 5.1], leads to a natural extension of the "classical" Kronecker function ring. In order to relate this general construction with the Kronecker function ring as defined by Krull (see, for instance, [17, page 401]), we recall that it is possible to associate to an arbitrary semistar operation $\star$ an `eab` semistar operation of finite type as follows, for each $F \in \boldsymbol{f}(D)$ and for each $E \in \overline{\boldsymbol{F}}(D)$:

$$\begin{aligned} F^{\star_a} &:= \cup\{((FH)^\star : H^\star) \mid H \in \boldsymbol{f}(D)\}\,, \\ E^{\star_a} &:= \cup\{F^{\star_a} \mid F \subseteq E\,,\ F \in \boldsymbol{f}(D)\}\,, \end{aligned}$$

(for the definition of `eab` operation see [17, page 394] and [20, Definition 2.3]). The previous construction is essentially due to P. Jaffard (1960) [22, Chapitre II, §2] and F. Halter-Koch (1997, 1998) [20, Section 6], [21, Chapter 19].

Obviously, $(\star_{_f})_a = \star_a$ and so, when $\star = \star_{_f}$, then $\star$ is `eab` if and only if $\star = \star_a$ [10, Proposition 4.5].

Let D be an integral domain and set $\mathrm{Zar}(D) := \{V \mid V$ is a valuation overring of $D\}$. It is well known that $\mathrm{Zar}(D)$ can be equipped with the *Zariski topology*, i.e., the topology having, as subbasic open subspaces, the subsets $\mathrm{Zar}(D[x])$ for x varying in K. The set $\mathrm{Zar}(D)$, endowed with the Zariski topology, is often called the *Riemann-Zariski space of D* [27, Chapter VI, §17, page 110]. Since it is a spectral space, $\mathrm{Zar}(D)$ can be also endowed with the inverse topology, that is the topology having as a subbasis for the closed sets the quasi-compact open subspaces in the Zariski topology.

Given a family $\mathscr{V}$ of valuation overrings of an integral domain D, we can define a semistar operation $\wedge_{\mathscr{V}}$ on D, by setting $E^{\wedge_{\mathscr{V}}} := \bigcap\{EV \mid V \in \mathscr{V}\}$ for each $E \in \overline{\boldsymbol{F}}(D)$. Clearly, $\wedge_{\mathscr{V}}$ is an `eab` semistar operation and it is known that $\wedge_{\mathscr{V}}$ is a semistar operation of finite type if and only if $\mathscr{V}$ is a quasi-compact subspace of $\mathrm{Zar}(D)$ [5, Proposition 4.5].

We also need to recall the notion of $\star$-valuation overring, considered by P. Jaffard (1960) [22, page 46] (see also Halter-Koch (1997) [21, Chapters 15 and 18]).

For a domain D and a semistar operation $\star$ on D, we say that a valuation overring V of D is *a $\star$–valuation overring of D* provided $F^\star \subseteq FV$, for each $F \in \boldsymbol{f}(D)$. Note that, by definition the $\star$–valuation overrings coincide with the $\star_f$–valuation overrings.

We collect in the following lemma some properties needed later.

Lemma 1.3. *Let $\star$ be a semistar operation on an integral domain D with quotient field K. Then:*

(1) $\mathrm{Na}(D, \star) \subseteq \mathrm{Kr}(D, \star)$.

(2) *V is a $\star$–valuation overring of D if and only if $V(X)$ is a valuation overring of $\mathrm{Kr}(D, \star)$.*
The map $W \mapsto W \cap K$ establishes a bijection between the set of all valuation overrings of $\mathrm{Kr}(D, \star)$ and the set of all the $\star$–valuation overrings of D.

(3) $\mathrm{Kr}(D, \star) = \mathrm{Kr}(D, \star_f) = \mathrm{Kr}(D, \star_a) = \cap\{V(X) \mid V$ *is a $\star$–valuation overring of D*$\}$ *is a Bézout domain with quotient field $K(X)$.*

(4) $E^{\star_a} = E\mathrm{Kr}(D, \star) \cap K = \cap\{EV \mid V$ *is a $\star$–valuation overring of D*$\}$*, for each $E \in \overline{\boldsymbol{F}}(D)$.*

For the proof see [10, Theorem 3.11], [11, Theorem 3.5], [12, Proposition 4.1].

Let $\star$ be a semistar operation on an integral domain D. Note that, from the previous Lemma 1.3(4), it follows that $\star_a = \wedge_{\mathscr{V}^\star(D)}$, where $\mathscr{V}^\star(D)$ denotes the set of all $\star$–valuation overrings of D.

Recall that a *Prüfer $\star$–multiplication domain* (for short, a *P$\star$MD*) is an integral domain D such that $(FF^{-1})^{\star_f} = D^{\star_f}$ $(= D^\star)$ (i.e., F is *$\star_f$–invertible*) for each $F \in \boldsymbol{f}(D)$. Clearly, the notions of P$\star$MD and P$\star_f$MD coincide and, given two semistar operations $\star_1 \leq \star_2$ on D, if D is a P$\star_1$MD then D is also a P$\star_2$MD.

In the following lemma, we collect some properties of Prüfer $\star$–multiplication domains.

Lemma 1.4. *Let D be an integral domain with quotient field K and $\star$ a semistar operation on D.*

(1) *The following are equivalent:*
   (i) *D is a P$\star$MD.*
   (ii) $\mathrm{Na}(D, \star)$ *is a Prüfer domain.*
   (iii) $\mathrm{Na}(D, \star) = \mathrm{Kr}(D, \star)$.

(iv) $\tilde{\star} = \star_a$.
(v) $\star_{_f}$ *is stable and* ***eab***.
(vi) *Each ideal of* $\mathrm{Na}(D,\star)$ *is extended from an ideal of* D.
In particular, if D *is a P*$\star$*MD, then* $\star_{_f} = \tilde{\star}$ *and so* D *is a P*$\star$*MD if and only if it is a P*$\tilde{\star}$*MD.*

(2) *Assume that* D *is a P*$\star$*MD. The contraction map* $\mathrm{Spec}(\mathrm{Na}(D,\star)) \to \mathrm{QSpec}^{\star_f}(D)$*,* $Q \mapsto Q \cap D$*, is a bijection with inverse map* $P \mapsto P\mathrm{Na}(D,\star)$.

Proof. For the proof of (1) (i)-(v), cf. [9, Theorem 3.1 and Remark 3.1].

Assume that (iii) holds, let J be an ideal of $\mathrm{Na}(D,\star)$, and write $J = I\mathrm{Na}(D,\star)$ for some ideal I of $D[X]$. We claim that $J = \boldsymbol{c}(I)\mathrm{Na}(D,\star)$. To verify this, let $f \in I$. By [9, Lemma 2.5(e)], $\boldsymbol{c}(f)\mathrm{Na}(D,\star) = \boldsymbol{c}(f)\mathrm{Kr}(D,\star) = f\mathrm{Kr}(D,\star) = f\mathrm{Na}(D,\star) \subseteq J$. It follows that $\boldsymbol{c}(I)\mathrm{Na}(D,\star) \subseteq J$. The reverse inclusion is clear. Thus (iii) implies (vi).

Now assume (vi), and let A be a nonzero finitely generated ideal of $\mathrm{Na}(D,\star)$. Then by assumption we may write $A = I\mathrm{Na}(D,\star)$ for some ideal I of D, and we may assume that I is finitely generated. Choose $f \in D[X]$ with $\boldsymbol{c}(f) = I$. Again by assumption, we may write $f\mathrm{Na}(D,\star) = J\mathrm{Na}(D,\star)$ for some ideal J of D. Now let M be a maximal ideal of $\mathrm{Na}(D,\star)$; we have $M = Q\mathrm{Na}(D,\star)$ for some $Q \in \mathrm{QMax}^{\star_f}(D)$ (Lemma 1.2). Localizing at M, we obtain $fD_Q(X) = JD_Q(X)$. By [1, Theorem 1], we must then have $AD_Q(X) = \boldsymbol{c}(f)D_Q(X) = fD_Q(X)$. Hence A is locally principal and therefore invertible. It follows that $\mathrm{Na}(D,\star)$ is a Prüfer domain, and we have (vi) implies (ii).

(2) Let $Q \in \mathrm{Spec}(\mathrm{Na}(D,\star))$. Then $Q = (Q\cap D)\mathrm{Na}(D,\star)$ and $(Q\cap D)^{\star_f} \cap D = (Q\cap D)^{\tilde{\star}} \cap D = (Q\cap D)\mathrm{Na}(D,\star) \cap K \cap D = Q \cap D$. Thus $Q\cap D \in \mathrm{QSpec}^{\star_f}(D)$. Conversely, let $P \in \mathrm{QSpec}^{\star_f}(D)$. Then $P \subseteq M$ for some $M \in \mathrm{QMax}^{\star_f}(D)$ and $M\mathrm{Na}(D,\star) \in \mathrm{Max}(\mathrm{Na}(D,\star))$. Therefore, $\mathrm{Na}(D,\star)_{M\mathrm{Na}(D,\star)} = D_M(X)$ is a valuation overring of $\mathrm{Na}(D,\star)$. Since $D_P(X)$ is an overring of $D_M(X)$ and hence of $\mathrm{Na}(D,\star)$, $D_P(X) = \mathrm{Na}(D,\star)_Q$ for some prime ideal Q of $\mathrm{Na}(D,\star)$. In fact, $Q = PD_P(X) \cap \mathrm{Na}(D,\star)$. On the other hand, since $Q = (Q\cap D)\mathrm{Na}(D,\star)$, we have $Q = P\mathrm{Na}(D,\star)$, and hence $P\mathrm{Na}(D,\star) \in \mathrm{Spec}(\mathrm{Na}(D,\star))$. □

Note that the statements of the previous lemma generalize some of the classical characterizations of Prüfer v–multiplication domains (for short, PvMD's); for the appropriate references see [9].

From Lemmas 1.3 and 1.4 we obtain the following.

Corollary 1.5. *Let* D *a P*$\star$*MD with quotient field* K*. We denote by* $\mathscr{V}^\star(D)$ *the set of all the* $\star$*-valuation overrings of* D*. Then,*

(1) *the canonical map* $\mathrm{Zar}(\mathrm{Na}(D,\star)) \to \mathrm{Zar}(D)$*,* $W \mapsto W \cap K$*, is a continuous surjective map having as its image* $\mathscr{V}^\star(D)$.
(2) *the canonical continuous surjective map* $\mathrm{Zar}(D) \to \mathrm{Spec}(D)$ *restricted to* $\mathscr{V}^\star(D)$ *is a bijection with* $\mathrm{QSpec}^{\star_f}(D)$.

Proof. (1) is a direct consequence of Lemma 1.3(2).

(2) Since D is a P$\star$MD, $\mathrm{Na}(D,\star) = \mathrm{Kr}(D,\star)$ is a Prüfer domain and so the canonical map $\mathrm{Zar}(\mathrm{Na}(D,\star)) \to \mathrm{Spec}(\mathrm{Na}(D,\star))$ is a homeomorphism. The conclusion follows from (1) and Lemma 1.4(2). $\square$

We recall next some results connecting the Prüfer *semistar* multiplication case with the Prüfer *star* multiplication case.

Let $\star$ be a semistar operation on an integral domain D and let T an overring of D; we denote by $\star^{|T}$ the semistar operation on T obtained by restriction on $\overline{\boldsymbol{F}}(T)$ ($\subseteq \overline{\boldsymbol{F}}(D)$) of $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$.

Clearly, if $\star$ has finite type, then so does $\star^{|T}$ (see [10, Proposition 2.8] or [9, Example 2.1(e)]). By [9, Proposition 3.1 and Corollary 3.2], if D is a P$\star$MD, then T is a P$\star^{|T}$MD. In case of the overring T of D equal to $R := D^\star$, it is easy to see that $\star^{|R}$ induces a star operation on R, when restricted to $\boldsymbol{F}(R)$, and the following holds.

Lemma 1.6. *Let D be an integral domain and let $\star$ be a semistar operation on D. Let $R := D^{\tilde{\star}}$ and let $\tilde{\star}^{|R} : \boldsymbol{F}(R) \to \boldsymbol{F}(R)$ be the restriction of $\tilde{\star}$ to $\boldsymbol{F}(R)$ (so that $\tilde{\star}^{|R}$ is a star operation on R). Then, D is a P$\star$MD if and only if R is a P$\tilde{\star}^{|R}$MD.*

Proof. See [9, Proposition 3.3 and Theorem 3.1]. $\square$

Lemma 1.7. *Let R be an integral domain and let $*$ be a star operation on R. Then, R is a P$*$MD if and only if R is a PvMD and $*_f = t_R$.*

Proof. See [9, Proposition 3.4]. $\square$

Lemma 1.8. *Let D be an integral domain, let $\star$ be a semistar operation on D, and let $R = D^{\tilde{\star}}$. Assume that D is a P$\star$MD. Then, R is a Pv_RMD and:*

$$\mathrm{QSpec}^{\star_f}(D) = \{Q \cap D \mid Q \in \mathrm{Spec}^{t_R}(R)\}\,,$$

and $R_Q = D_{Q\cap D}$ for each $Q \in \mathrm{Spec}^{t_R}(R)$.

Proof. As above, let $* := \widetilde{\star}^{|R} : \boldsymbol{F}(R) \to \boldsymbol{F}(R)$ be the restriction of $\tilde{\star}$ to $\boldsymbol{F}(R)$. Clearly, $*$ is a star operation on R and R is a P$*$MD (Lemma 1.6). Therefore, R is a Pv_RMD and $*_f = t_R$ (Lemma 1.7). Also, since D is a P$\star$MD, $\tilde{\star} = \star_f$ (Lemma 1.4) and its restriction $*$ is of finite type. Thus we have $* = *_f = t_R$. Now let P be a quasi-$\star_f$-prime ideal of D. Then D_P is a valuation domain and $D_P = R_Q$, where $Q := PD_P \cap R$. Therefore, Q is a t_R-prime ideal of R [24, Lemma 3.17] and $P = Q \cap D$. Conversely, let Q be a t_R-prime ideal of R and let $P := Q \cap D$. Then $P \subseteq P^{\star_f} \cap D = (Q\cap D)^{\star_f} \cap D \subseteq Q^{\star_f} \cap D = Q^* \cap D = Q^{*_f} \cap D = Q^{t_R} \cap D = Q \cap D = P$. Thus P is a quasi-$\star_f$-prime ideal of D. $\square$

Corollary 1.9. *Let D be an integral domain, let $\star$ be a semistar operation on D, let $R = D^{\tilde{\star}}$, and let $*$ the star operation on R defined by restricting $\widetilde{\star}$ to $\boldsymbol{F}(R)$. Assume that D is a P$\star$MD. Then the Prüfer domain $\mathrm{Na}(D,\star)$ coincides with $\mathrm{Na}(R,*) = \mathrm{Na}(R,v_R)$.*

Proof. This is an easy consequence of Lemma 1.8, since in the present situation, $* = t_R$ and $\mathrm{Na}(D,\star) = \bigcap\{D_P(X) \mid P \in \mathrm{QSpec}^{\star_f}(D)\} = \bigcap\{R_Q(X) \mid Q \in \mathrm{Spec}^{t_R}(R)\} = \mathrm{Na}(R,t_R) = \mathrm{Na}(R,v_R)$. $\square$

2. A semistar characterization of sharpness in Prüfer domains

Given an integral domain D and a prime ideal $P \in \mathrm{Spec}(D)$, set

$$\begin{aligned}\nabla(P) &:= \{M \in \mathrm{Max}(D) \mid M \not\supseteq P\},\\ \Delta(P) &:= \nabla(P) \cup \{P\},\\ \Theta(P) &:= \bigcap\{D_M \mid M \in \nabla(P)\}.\end{aligned}$$

We use the simpler notation ∇ (respectively, Δ; Θ) when no possible confusion can arise from the omission of the prime ideal P. We say that P is *sharp* (or, has the *#-property*) if $\Theta(P) \not\subseteq D_P$ (see [16, Lemma 1] and [6, Section 1 and Proposition 2.2]).

The goal of this section is to provide a characterization of sharpness using (spectral) semistar operations, at least in some important classes of integral domains.

Clearly, for each $P \in \mathrm{Spec}(D)$, we have the following relations among the semistar operations associated to ∇ and to Δ:

$$\mathtt{s}_\Delta := \mathtt{s}_{\Delta(P)} \leq \mathtt{s}_{\nabla(P)} =: \mathtt{s}_\nabla\,,$$

$$\widetilde{\mathtt{s}_\Delta} \lneq \widetilde{\mathtt{s}_\nabla} \ \Rightarrow\ (\mathtt{s}_\Delta)_{_f} \lneq (\mathtt{s}_\nabla)_{_f} \ \Rightarrow\ \mathtt{s}_\Delta \lneq \mathtt{s}_\nabla\,.$$

Lemma 2.1. *Let D be an integral domain and let P be a nonzero prime ideal of D. If P is sharp, then $(s_{\Delta(P)})_{_f} \lneq (s_{\nabla(P)})_{_f}$. Moreover, assume that D is a finite-conductor domain, i.e., the intersection of any two principal ideals of D is finitely generated. Then P being sharp implies $\widetilde{s_{\Delta(P)}} \lneq \widetilde{s_{\nabla(P)}}$.*

Proof. The first statement follows from the fact that $D \in \boldsymbol{f}(D)$ and $\Theta(P) \not\subseteq D_P$ implies that $D^{s_\Delta} \neq D^{s_\nabla}$.

For the second statement, note first that $\Theta(P) \not\subseteq D_P$ if and only if there exists an element x in the quotient field of D such that $(D :_D xD) \subseteq P$ but $(D :_D xD) \not\subseteq M$ for all $M \in \nabla$. On the other hand, by Lemma 1.1, we have $\widetilde{s_\Delta} \neq \widetilde{s_\nabla}$ if and only if $\mathtt{Cl}^{inv}(\Delta) \neq \mathtt{Cl}^{inv}(\nabla)$, i.e., $P \notin \mathtt{Cl}^{inv}(\nabla)$. Note here that $P \notin \mathtt{Cl}^{inv}(\nabla)$ is equivalent to the existence of a finitely generated ideal J of D such that $J \subseteq P$ but $J \not\subseteq M$ for all $M \in \nabla$. Therefore, if D is a finite-conductor domain, then $(D :_D xD)$ is a finitely generated ideal and hence the conclusion follows. □

Proposition 2.2. *Let $P \in \mathrm{Spec}(D)$. Assume that D is a Prüfer domain. Then, P is sharp if and only if $\widetilde{s_{\Delta(P)}} \lneq \widetilde{s_{\nabla(P)}}$.*

Proof. It is known that, in a Prüfer domain D, $\bigcap\{D_M \mid M \in \nabla\} \not\subseteq D_P$ if and only if there exists a finitely generated ideal J of D contained in P but not contained in any $M \in \nabla$ [18, Corollary 2]. Therefore, the conclusion follows from the above lemma and its proof. □

In general, the condition $\mathtt{s}_{\Delta(P)} \lneq \mathtt{s}_{\nabla(P)}$ does not imply that P is a sharp prime ideal (even in the Prüfer domain case), as the following example shows.

Example 2.3. Let D be an almost Dedekind domain with a unique maximal ideal M_0 non-finitely generated. Then, for $P = M_0$,

$$\begin{array}{ll} \nabla := \nabla(M_0) = \mathrm{Max}(D) \setminus \{M_0\}, & \Delta := \Delta(M_0) = \mathrm{Max}(D), \\ \nabla^{\mathrm{gen}} = \mathrm{Spec}(D) \setminus \{M_0\}, & \Delta^{\mathrm{gen}} = \mathrm{Spec}(D), \\ \mathtt{Cl}^{inv}(\nabla) = \mathtt{Cl}^{inv}(\Delta) = \mathrm{Spec}(D). & \end{array}$$

Therefore, by Lemma 1.1, $\mathtt{s}_\Delta \lneq \mathtt{s}_\nabla$, but $\widetilde{\mathtt{s}_\Delta} = \widetilde{\mathtt{s}_\nabla}$ and hence M_0 is not a sharp ideal.

Note also that, in the present situation, Max(D) is quasi-compact in Spec(D) (endowed with the Zariski topology), but Max$(D) \setminus \{M_0\}$ is not. Therefore, by Lemma 1.1, $\mathtt{s}_\Delta$ is of finite type but $\mathtt{s}_\nabla$ is not. In fact, $\widetilde{\mathtt{s}_\Delta} = (\mathtt{s}_\Delta)_f = \mathtt{s}_\Delta = d = \widetilde{\mathtt{s}_\nabla} = (\mathtt{s}_\nabla)_f \lneq \mathtt{s}_\nabla$.

It is natural to ask if the condition $(\mathtt{s}_{\Delta(P)})_f \lneq (\mathtt{s}_{\nabla(P)})_f$ implies that P is sharp. The answer to this question is affirmative in the case of Prüfer domains:

Corollary 2.4. *Let $P \in \mathrm{Spec}(D)$. Assume that D is a Prüfer domain. Then, the following statements are equivalent:*

(i) *P is a sharp prime of D.*
(ii) $\widetilde{s_{\Delta(P)}} \lneq \widetilde{s_{\nabla(P)}}$.
(iii) $(s_{\Delta(P)})_f \lneq (s_{\nabla(P)})_f$.
(iv) $(s_{\Delta(P)})_a \lneq (s_{\nabla(P)})_a$. □

Proof. If D is a Prüfer domain, then for any semistar operation $\star$ on D, D is a P$\star$MD and hence by Lemma 1.4(1), $\widetilde{\star} = \star_f = \star_a$. Therefore, the conclusion follows from Proposition 2.2. □

In Proposition 2.2 (and hence in Corollary 2.4), the hypothesis that D is a Prüfer domain cannot be omitted. That is, in general, the condition $\widetilde{\mathtt{s}_{\Delta(P)}} \lneq \widetilde{\mathtt{s}_{\nabla(P)}}$ does not imply that P is a sharp prime ideal (even in the finite-conductor domain case):

Example 2.5. Let $D := K[X, Y]$, where K is a field, and let P be a maximal ideal of D. Then $\Delta := \Delta(P) = \mathrm{Max}(D)$ and hence $\widetilde{\mathtt{s}_\Delta} = (\mathtt{s}_\Delta)_f = \mathtt{s}_\Delta = d$. Since D is a Krull domain, $D = \bigcap\{D_Q \mid Q \in X^1(D)\}$, where $X^1(D)$ is the set of height one prime ideals of D. Also, since D is a Hilbert ring and every maximal ideal of D has height 2, each $Q \in X^1(D)$ is contained in infinitely many maximal ideals of D by [25, Theorem 147] (or by [25, Theorem 30]). Thus, we have $Q \subset N$ for some $N \in \nabla := \nabla(P) = \mathrm{Max}(D) \setminus \{P\}$. Therefore, $\bigcap\{D_N \mid N \in \nabla\} \subseteq \bigcap\{D_Q \mid Q \in X^1(D)\} = D$, i.e., $\bigcap\{D_N \mid N \in \nabla\} \subseteq D_P$. Thus, P is not sharp. Also, since P is finitely generated,

$$P^{(\mathtt{s}_\nabla)_f} = P^{\mathtt{s}_\nabla} = \bigcap\{PD_N \mid N \in \nabla\} = \bigcap\{D_N \mid N \in \nabla\} = D.$$

This implies that $d \neq (\mathtt{s}_\nabla)_f$ and $P \notin \mathrm{QSpec}^{(\mathtt{s}_\nabla)_f}(D)$. It is easy to check that $\nabla = \mathrm{QMax}^{(\mathtt{s}_\nabla)_f}(D)$. Then $\widetilde{\mathtt{s}_\nabla} = s_{\mathrm{QMax}^{(\mathtt{s}_\nabla)_f}(D)} = s_\nabla$. Thus, we have $\widetilde{\mathtt{s}_\nabla} = (\mathtt{s}_\nabla)_f = s_\nabla$, and hence $\widetilde{\mathtt{s}_\Delta} \lneq \widetilde{\mathtt{s}_\nabla}$.

3. Sharpness in Prüfer $\star$-multiplication domains

The goal of this section is to investigate the notion of sharpness in the more general setting of Prüfer $\star$-multiplication domains.

Given a semistar operation $\star$ on an integral domain D and a prime ideal $P \in \mathrm{QSpec}^{\star_f}(D)$, set

$$\begin{aligned}\nabla^{\star_f}(P) &:= \{M \in \mathrm{QMax}^{\star_f}(D) \mid M \not\supseteq P\}\,,\\ \Delta^{\star_f}(P) &:= \nabla^{\star_f}(P) \cup \{P\}\,,\\ \Theta^{\star_f}(P) &:= \bigcap\{D_M \mid M \in \nabla^{\star_f}(P)\}\,.\end{aligned}$$

As in the previous section, we use the simpler notation $\nabla^{\star_f}$ (respectively, $\Delta^{\star_f}$; $\Theta^{\star_f}$) when no possible confusion can arise from the omission of the prime ideal P.

We say that P is $\star_f$*-sharp* (or, $\star_f$*-#*) if $\Theta^{\star_f}(P) \not\subseteq D_P$. For example, if $\star = d$ is the identity, the sharp property coincides -by definition- with the d-sharp property (see, also, [15], [16], [18], [8, page 62], [6], [7, Chapter 2, Section 3]).

Clearly, for each semistar operation $\star$ and for each $P \in \mathrm{QSpec}^{\star_f}(D)$, we have:

$$\mathtt{s}_{\Delta^{\star_f}} := \mathtt{s}_{\Delta^{\star_f}(P)} \leq \mathtt{s}_{\nabla^{\star_f}(P)} =: \mathtt{s}_{\nabla^{\star_f}}\,, \text{ and}$$

$$P \text{ is } \star_f\text{-sharp} \;\Rightarrow\; (\mathtt{s}_{\Delta^{\star_f}(P)})_f \lneq (\mathtt{s}_{\nabla^{\star_f}(P)})_f \;\Rightarrow\; \mathtt{s}_{\Delta^{\star_f}(P)} \lneq \mathtt{s}_{\nabla^{\star_f}(P)}\,.$$

As we have already observed in Example 2.3, (even) in the Prüfer domain case (i.e., when $\star_f = d$), the condition $\mathtt{s}_{\Delta^{\star_f}(P)} \lneq \mathtt{s}_{\nabla^{\star_f}(P)}$ does not imply that P is a $\star_f$-sharp prime ideal.

The next goal is to show that, if D is a Prüfer $\star$-multiplication domain, then the condition $(\mathtt{s}_{\Delta^{\star_f}(P)})_f \lneq (\mathtt{s}_{\nabla^{\star_f}(P)})_f$ coincides with the property that P is $\star_f$-sharp.

We start by recalling that, if D is a Prüfer domain and $P \in \mathrm{Spec}(D)$, Gilmer and Heinzer [18, Corollary 2] proved that P is sharp if and only if there exists a finitely generated ideal I of D such that $I \subseteq P$ and $I \not\subseteq M$ for each $M \in \nabla(P)$. This was generalized to PvMDs in [14, Theorem 1.6].

Proposition 3.1. *Let D be a P$\star$MD and $P \in \mathrm{QSpec}^{\star_f}(D)$. Then P is $\star_f$-sharp if and only if P contains a finitely generated ideal I of D such that $I \not\subseteq M$ for each $M \in \nabla^{\star_f}(P)$.*

Proof. Suppose that P contains a finitely generated ideal I such that $I \not\subseteq M$ for $M \in \nabla^{\star_f}(P)$. If $I^{-1} \subseteq D_P$, then $II^{-1} \subseteq ID_P \cap D \subseteq PD_P \cap D = P$, and hence $D = (II^{-1})^{\star_f} \cap D \subseteq P^{\star_f} \cap D = P$, a contradiction. Therefore, $I^{-1} \not\subseteq D_P$. Choose $u \in I^{-1} \setminus D_P$, and let $M \in \nabla^{\star_f}(P)$. Then, since $I \subseteq (D :_D u)$, we must have $(D :_D u) \not\subseteq M$, that is, $u \in D_M$. Hence, $\bigcap\{D_M \mid M \in \nabla^{\star_f}(P)\} = \Theta^{\star_f}(P) \not\subseteq D_P$, i.e., P is $\star_f$-sharp.

For the converse, suppose that there is an element $v \in \Theta^{\star_f}(P) \setminus D_P$, so that $(1, v)^{-1} = (D :_D v) \subseteq P$. Since $(1, v)$ is $\star_f$-invertible, $(1, v)^{-1}$ is $\star_f$-finite, i.e., there is a finitely generated ideal $J \subseteq (1, v)^{-1}$ for which $J^\star = ((1, v)^{-1})^{\star_f}$ [13, Proposition 2.6]. We have $J \subseteq P$. If $J \subseteq M$ for some $M \in \nabla^{\star_f}(P)$, then $(1, v)^{-1} \subseteq J^\star \cap D \subseteq M^{\star_f} \cap D = M$; however, this contradicts the fact that $v \in D_M$. □

Corollary 3.2. *Let D be a P$\star$MD and $P \in \mathrm{QSpec}^{\star_f}(D)$, then the following statements are equivalent.*

(i) *P is $\star_f$-sharp.*
(ii) $\widetilde{\mathtt{s}_{\Delta^{\star_f}(P)}} \lneq \widetilde{\mathtt{s}_{\nabla^{\star_f}(P)}}$.
(iii) $(\mathtt{s}_{\Delta^{\star_f}(P)})_f \lneq (\mathtt{s}_{\nabla^{\star_f}(P)})_f$.
(iv) $(\mathtt{s}_{\Delta^{\star_f}(P)})_a \lneq (\mathtt{s}_{\nabla^{\star_f}(P)})_a$.

Proof. Recall that $\widetilde{\star} = \mathtt{s}_{\mathrm{QMax}^{\star_f}(D)} = \mathtt{s}_{\mathrm{QSpec}^{\star_f}(D)}$. Since $\nabla^{\star_f} \subseteq \Delta^{\star_f} \subseteq \mathrm{QSpec}^{\star_f}(D)$, $\mathtt{s}_{\mathrm{QSpec}^{\star_f}(D)} \leq \mathtt{s}_{\Delta^{\star_f}} \leq \mathtt{s}_{\nabla^{\star_f}}$. Since D is a P$\star$MD, i.e., a P$\widetilde{\star}$MD, D is also a P$\mathtt{s}_{\Delta^{\star_f}}$MD and a P$\mathtt{s}_{\nabla^{\star_f}}$MD. Therefore, by Lemma 1.4(1), we have that $\widetilde{\mathtt{s}_{\Delta^{\star_f}}} = (\mathtt{s}_{\Delta^{\star_f}})_f = (\mathtt{s}_{\Delta^{\star_f}})_a$ and that $\widetilde{\mathtt{s}_{\nabla^{\star_f}}} = (\mathtt{s}_{\nabla^{\star_f}})_f = (\mathtt{s}_{\nabla^{\star_f}})_a$. Thus, we have the implications (i)⇒(ii)⇔(iii)⇔(iv).

For the implication (ii)⇒(i), recall the fact that $\widetilde{\mathtt{s}_{\Delta^{\star_f}}} \neq \widetilde{\mathtt{s}_{\nabla^{\star_f}}}$ if and only if $\mathtt{Cl}^{inv}(\Delta^{\star_f}) \neq \mathtt{Cl}^{inv}(\nabla^{\star_f})$, i.e., $P \notin \mathtt{Cl}^{inv}(\nabla^{\star_f})$, which is equivalent to the existence of a finitely generated ideal J of D such that $J \subseteq P$ but $J \not\subseteq M$ for all $M \in \nabla^{\star_f}$. Then the conclusion follows from Proposition 3.1. □

An integral domain D is called a *$\star_f$-sharp domain* (or, *$\star_f$-#-domain*) if each $M \in \mathrm{QMax}^{\star_f}(D)$ is $\star_f$-sharp, and it is called a *$\star_f$-doublesharp domain* (or *$\star_f$-##-domain*) if each $P \in \mathrm{QSpec}^{\star_f}(D)$ is $\star_f$-sharp. A d-(double)sharp domain is simply called a *(double)sharp domain*.

We connect the notion of semistar sharpness (and semistar doublesharpness) to that of star sharpness (and star doublesharpness) of "special" overrings.

Proposition 3.3. *Let D be an integral domain, let $\star$ be a semistar operation on D, and let $R := D^{\widetilde{\star}}$. Assume that D is a P$\star$MD. Then,*

(1) *D is a $\star_f$-sharp domain if and only if R is a t_R-sharp domain.*
(2) *D is a $\star_f$-doublesharp domain if and only if R is a t_R-doublesharp domain.*

Proof. This is a straightforward consequence of Lemma 1.8. □

In 1967 Gilmer and Heinzer proved that a Prüfer domain D is a doublesharp domain if and only if every overring of D is a sharp domain [18, Theorem 3]. In order to extend this to the P$\star$MD setting, we need to recall a general version of the notion of linked overring.

Let D be an integral domain, T an overring of D, and let $\star, \star'$ be semistar operations on D, T, respectively. Following [3, Section 3], we say that T is $(\star, \star')$-*linked to D* if $F^\star = D^\star$ implies $(FT)^{\star'} = T^{\star'}$ for each nonzero finitely generated integral ideal F of D. Clearly, T is $(\star, \star')$-linked to D if and only if T is $(\star_f, \star'_f)$-linked to D. In particular, when $\star = t_D$ and $\star' = t_T$, we say that T is a *t-linked overring of D*. Note that $D^\star$ is automatically $(\star, \star^{|_{D^\star}})$-linked to D, where $\star^{|_{D^\star}}$ is the (semi)star operation on $D^\star$ formed by restricting $\star$ to $\overline{\boldsymbol{F}}(D^\star)$.

There is a bijection between the t-linked overrings of a Prüfer v-multiplication domain and the overrings of its t-Nagata ring. We can generalize this to the case of Prüfer $\star$-multiplication domains.

Theorem 3.4. *Let D be an integral domain with quotient field K and let $\star$ be a semistar operation on D. If D is a P$\star$MD, then the map $T \mapsto \mathrm{Na}(T, t_T)$ from the set of $(\star, t_T)$-linked overrings T of D to the set of overrings of $\mathrm{Na}(D, \star)$ is a* *bijection with inverse map $\mathcal{T} \mapsto \mathcal{T} \cap K$. In particular, if R is a PvMD with quotient field K, then the map $T \mapsto \mathrm{Na}(T, t_T)$ from the set of t-linked overrings of R to the set of overrings of $\mathrm{Na}(R, t_R)$ is a bijection with inverse map $\mathcal{T} \mapsto \mathcal{T} \cap K$.*

Proof. We begin by proving the "in particular" statement. Let T be a t-linked overring of R. Then by [24, Theorem 3.8 and Corollary 3.9], T is a Pv_TMD, and it is clear that $\mathrm{Na}(T, t_T)$ is an overring of $\mathrm{Na}(R, t_R)$. Moreover, $\mathrm{Na}(T, t_T) \cap K = T$ by Lemma 1.2(5). Now let $\mathcal{T}$ be an overring of $\mathrm{Na}(R, t_R)$. Then $\mathcal{T}$ is a Prüfer domain, and hence there is a subset Λ of $\mathrm{Spec}^{t_R}(R)$ for which $\mathcal{T} = \bigcap_{P \in \Lambda} R_P(X)$ [17, Theorem 26.1]. It follows that $T := \mathcal{T} \cap K = \bigcap_{P \in \Lambda} R_P$. In particular, T is a Pv_TMD [24, Corollary 3.9] and is t-linked over R [24, Theorem 3.8]. Moreover, since $\mathrm{Na}(T, t_T) \subseteq R_P(X)$ for $P \in \Lambda$, we have $\mathrm{Na}(T, t_T) \subseteq \mathcal{T}$. It remains to show that this inclusion is an equality. To this end, let $\psi \in \mathcal{T}$, and write $\psi = g/f$ with $g, f \in T[X]$. We have $g\mathrm{Na}(T, t_T) = \boldsymbol{c}(g)\mathrm{Na}(T, t_T)$ and $f\mathrm{Na}(T, t_T) = \boldsymbol{c}(f)\mathrm{Na}(T, t_T)$ [24, Lemma 2.11]. Moreover, for any nonzero finitely generated ideal I of T, we have $I^{-1}\mathrm{Na}(T, t_T) = (\mathrm{Na}(T, t_T) : I\mathrm{Na}(T, t_T))$ [24, Proposition 2.2]. Since $\boldsymbol{c}(f)\boldsymbol{c}(f)^{-1}\mathrm{Na}(T, t_T) = \mathrm{Na}(T, t_T)$, we also have $f^{-1}\mathrm{Na}(T, t_T) = \boldsymbol{c}(f)^{-1}\mathrm{Na}(T, t_T)$. Therefore,

$$\boldsymbol{c}(g)\boldsymbol{c}(f)^{-1} \subseteq gf^{-1}\mathrm{Na}(T, t_T) \subseteq gf^{-1}\mathcal{T} \subseteq \mathcal{T},$$

whence $\boldsymbol{c}(g)\boldsymbol{c}(f)^{-1} \subseteq \mathcal{T} \cap K = T$. It follows that $\psi = g/f \in \mathrm{Na}(T, t_T)$, as desired.

For the general statement, let $\star$ be a semistar operation on D, and assume that D is a P$\star$MD. Let T be $(\star, t_T)$-linked to D. Then $\mathrm{Na}(T, t_T) \supseteq \mathrm{Na}(D, \star)$ by [3, Theorem 3.8]. Now let $\mathcal{T}$ be an overring of $\mathrm{Na}(D, \star) = \mathrm{Na}(R, t_R)$, where $R := D^\star$. By what has already been proved, $T := \mathcal{T} \cap K$ is t-linked to R, and since R is $(\star, t_R)$-linked to D, T is $(\star, t_T)$-linked to D by transitivity [3, Lemma 3.1(b)]. We also have $\mathrm{Na}(T, t_T) = \mathcal{T}$. This completes the proof. □

We can now characterize $\star_f$-sharpness (or $\star_f$-doublesharpness) on D with sharpness (or doublesharpness) on $\mathrm{Na}(D, \star)$, in the case of Prüfer $\star$-multiplication domains. Statement (2) generalizes [14, Theorem 3.6].

Proposition 3.5. *Let D be a P$\star$MD.*

1. *Let $P \in \mathrm{QSpec}^{\star_f}(D)$. Then, P is $\star_f$-sharp in D if and only if $P\mathrm{Na}(D, \star)$ is sharp in the Prüfer domain $\mathrm{Na}(D, \star)$.*
2. *D is $\star_f$-sharp if and only if $\mathrm{Na}(D, \star)$ is sharp.*
3. *D is $\star_f$-doublesharp if and only if $\mathrm{Na}(D, \star)$ is doublesharp.*

Proof. (1) Set $T = \Theta^{\star_f}(P) = \bigcap\{D_M \mid M \in \nabla^{\star_f}(P)\}$. Then T is a $(\star, t_T)$-linked overring of D and D_P is a $(\star, t_{D_P})$-linked overring of D [3, Lemma 3.1]. By Theorem 3.4, $T \not\subseteq D_P$ if and only if $\mathrm{Na}(T, t_T) \not\subseteq \mathrm{Na}(D_P, t_{D_P})$. Note that $\mathrm{Na}(D_P, t_{D_P}) = D_P(X)$. We can also show that $\mathrm{Na}(T, t_T) = \bigcap\{D_M(X) \mid M \in \nabla^{\star_f}(P)\}$ by the same argument as in the proof of Theorem 3.4. Thus, we have the equivalence that $\bigcap\{D_M \mid M \in \nabla^{\star_f}(P)\} \not\subseteq D_P$ if and only if $\bigcap\{D_M(X) \mid M \in \nabla^{\star_f}(P)\} \not\subseteq D_P(X)$, that is, P is $\star_f$-sharp in D if and only if $P\mathrm{Na}(D, \star)$ is sharp in $\mathrm{Na}(D, \star)$.

(2) and (3) are direct consequences of (1) and Lemma 1.4(2). □

One might hope for a correspondence similar to that in Theorem 3.4 for overrings T of a P$\star$MD D that are $(\star, \star')$-linked to D (for some semistar operation $\star'$ on T). However, there are too many such T, as the following result shows.

Proposition 3.6. *Let D be a P$\star$MD, where $\star$ is a semistar operation on D, let S be a t-linked overring of $D^\star$, and let T be a ring with $D \subseteq T \subseteq S$. Then, there is a semistar operation $\star'$ on T such that T is $(\star, \star')$-linked to D.*

Proof. Define $\star'$ on T by $E^{\star'} := (ES)^{t_S}$, for each $E \in \overline{\boldsymbol{F}}(T)$ [10, Proposition 2.9]. Let I be a finitely generated ideal of D with $I^\star = D^\star$. Then, since S is (t-linked over $D^\star$ and therefore) $(\star, t_S)$-linked to D, $(IT)^{\star'} = (IS)^{t_S} = S = T^{\star'}$; that is, T is $(\star, \star')$-linked to D. □

Finally, Gimer-Heinzer's characterization of doublesharp Prüfer domains b(and its generalization to PvMDs [14, Proposition 2.8]) can be extended to the case of P$\star$MDs as follows.

Corollary 3.7. *Let D be a P$\star$MD, where $\star$ is a semistar operation on D. The following are equivalent.*

(i) *D is a $\star_f$-doublesharp domain.*
(ii) *If T is an overring of D and $\star'$ is a semistar operation on T such that T is $(\star, \star')$-linked to D, then T is a $\star'_f$-sharp domain.*
(iii) *If T is a $(\star, t_T)$-linked overring of D, then T is a t_T-sharp domain.*

Proof. Assume that D is a $\star_f$-doublesharp domain. Then $\text{Na}(D, \star)$ is a doublesharp Prüfer domain by Proposition 3.5. Let T be a $(\star, \star')$-linked overring of D. By [3, Theorem 3.8], $\text{Na}(T, \star') \supseteq \text{Na}(D, \star)$, whence $\text{Na}(T, \star')$ is a sharp domain. Since T is a P$\star'$MD [3, Corollary 5.4], T is a $\star'_f$-sharp domain by Proposition 3.5. Thus (i) $\Rightarrow$ (ii).

The implication (ii) $\Rightarrow$ (iii) is trivial.

Assume (iii), and let $\mathcal{T}$ be an overring of the Prüfer domain $\text{Na}(D, \star)$. By Theorem 3.4, $T := \mathcal{T} \cap K$ is $(\star, t_T)$-linked to D and hence t_T-sharp by assumption. Then, since $\mathcal{T} = \text{Na}(T, t_T)$ by Theorem 3.4, $\mathcal{T}$ is sharp by Proposition 3.5. It follows that $\text{Na}(D, \star)$ is doublesharp, and hence, by Proposition 3.5, that D is $\star_f$-doublesharp. Therefore, (iii) $\Rightarrow$ (i). □

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