

IDEMPOTENCE AND DIVISORIALTY IN PRÜFER-LIKE DOMAINS

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ABSTRACT. Let D be a Prüfer \star -multiplication domain, where \star is a semistar operation on D . We show that certain ideal-theoretic properties related to idempotence and divisoriality hold in Prüfer domains, and we use the associated semistar Nagata ring of D to show that the natural counterparts of these properties also hold in D .

1. INTRODUCTION AND PRELIMINARIES

Throughout this work, D will denote an integral domain, and K will denote its quotient field. Recall that Arnold [1] proved that D is a Prüfer domain if and only if its associated Nagata ring $D[X]_N$, where N is the set of polynomials in $D[X]$ whose coefficients generate the unit ideal, is a Prüfer domain. This was generalized to Prüfer v -multiplication domains (PvMDs) by Zafrullah [16] and Kang [14] and to Prüfer \star -multiplication domains (P \star MDs) by Fontana, Jara, and Santos [8].

Our goal in this paper is to show that certain ideal-theoretic properties that hold in Prüfer domains transfer in a natural way to P \star MDs. For example, we show that an ideal I of a Prüfer domain is idempotent if and only if it is a radical ideal each of whose minimal primes is idempotent (Theorem 2.9), and we use a Nagata ring transfer “machine” to transfer a natural counterpart of this characterization to P \star MDs. For another example, in Theorem 3.5 we show that an ideal in a Prüfer domain of finite character is idempotent if and only if it is a product of idempotent prime ideals and, perhaps more interestingly, we characterize ideals that are simultaneously idempotent and divisorial as (unique) products of incomparable divisorial idempotent primes; and we then extend this to P \star MDs.

Let us review terminology and notation. Denote by $\overline{\mathbf{F}}(D)$ the set of all nonzero D -submodules of K , and by $\mathbf{F}(D)$ the set of all nonzero fractional ideals of D , i.e., $E \in \mathbf{F}(D)$ if $E \in \overline{\mathbf{F}}(D)$ and there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\mathbf{f}(D)$ be the set of all nonzero finitely generated D -submodules of K . Then, obviously, $\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \overline{\mathbf{F}}(D)$.

Date: November 26, 2018.

The first-named author was partially supported by *GNSAGA of Istituto Nazionale di Alta Matematica*.

The second-named author was supported by a grant from the Simons Foundation (#354565).

The third-named author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government(MSIP) (No. 2015R1C1A2A01055124).

Following Okabe-Matsuda [15], a *semistar operation* on D is a map $\star : \overline{\mathbf{F}}(D) \rightarrow \overline{\mathbf{F}}(D)$, $E \mapsto E^\star$, such that, for all $x \in K$, $x \neq 0$, and for all $E, F \in \overline{\mathbf{F}}(D)$, the following properties hold:

- (\star_1) $(xE)^\star = xE^\star$;
- (\star_2) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
- (\star_3) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Of course, semistar operations are natural generalizations of star operations—see the discussion following Corollary 2.5 below.

The semistar operation \star is said to have *finite type* if $E^\star = \bigcup\{F^\star \mid F \in \mathbf{f}(D), F \subseteq E\}$ for each $E \in \overline{\mathbf{F}}(D)$. To any semistar operation \star we can associate a finite-type semistar operation \star_f given by

$$E^{\star_f} = \bigcup\{F^\star \mid F \in \mathbf{f}(D), F \subseteq E\}.$$

We say that a nonzero ideal I of D is a *quasi- \star -ideal* if $I = I^\star \cap D$, a *quasi- \star -prime ideal* if it is a prime quasi- \star -ideal, and a *quasi- \star -maximal ideal* if it is maximal in the set of all proper quasi- \star -ideals. A quasi- \star -maximal ideal is a prime ideal. We will denote by $\text{QMax}^\star(D)$ ($\text{QSpec}^\star(D)$) the set of all quasi- \star -maximal ideals (quasi- \star -prime ideals) of D . While quasi- \star -maximal ideals may not exist, quasi- \star_f -maximal ideals are plentiful in the sense that each proper quasi- \star_f -ideal is contained in a quasi- \star_f -maximal ideal. (See [9] for details.) Now we can associate to \star yet another semistar operation: for $E \in \overline{\mathbf{F}}(D)$, set

$$E^{\tilde{\star}} = \bigcap\{ED_Q \mid Q \in \text{QMax}^{\star_f}(D)\}.$$

Then $\tilde{\star}$ is also a finite-type semistar operation, and we have $I^{\tilde{\star}} \subseteq I^{\star_f} \subseteq I^\star$ for all $I \in \overline{\mathbf{F}}(D)$.

Let \star be a semistar operation on D . Set $N(\star) = \{g \in D[X] \mid c(g)^\star = D^\star\}$, where $c(g)$ is the *content* of the polynomial g , i.e., the ideal of D generated by the coefficients of g . Then $N(\star)$ is a saturated multiplicatively closed subset of $D[X]$, and we call the ring $\text{Na}(D, \star) := D[X]_{N(\star)}$ the *semistar Nagata ring of D with respect to \star* . The domain D is called a *Prüfer \star -multiplication domain* ($\text{P}\star\text{MD}$) if $(FF^{-1})^{\star_f} = D^{\star_f}$ ($= D^\star$) for each $F \in \mathbf{f}(D)$ (i.e., each such F is \star_f -invertible). (Recall that $F^{-1} = (D : F) = \{u \in K \mid uF \subseteq D\}$.)

In the following two lemmas, we assemble the facts we need about Nagata rings and $\text{P}\star\text{MD}$ s. Most of the proofs can be found in [6], [9], or [5].

Lemma 1.1. *Let \star be a semistar operation on D . Then:*

- (1) $D^\star = D^{\star_f}$.
- (2) $\text{QMax}^{\star_f}(D) = \text{QMax}^{\tilde{\star}}(D)$.
- (3) *The map $\text{QMax}^{\star_f}(D) \rightarrow \text{Max}(\text{Na}(D, \star))$, $P \mapsto P\text{Na}(D, \star)$, is a bijection with inverse map $M \mapsto M \cap D$.*
- (4) $P \mapsto P\text{Na}(D, \star)$ defines an injective map $\text{QSpec}^{\tilde{\star}}(D) \rightarrow \text{Spec}(\text{Na}(D, \star))$.
- (5) $N(\star) = N(\star_f) = N(\tilde{\star})$ and (hence) $\text{Na}(D, \star) = \text{Na}(D, \star_f) = \text{Na}(D, \tilde{\star})$.
- (6) *For each $E \in \overline{\mathbf{F}}(D)$, $E^{\tilde{\star}} = E\text{Na}(D, \star) \cap K$, and $E^{\tilde{\star}}\text{Na}(D, \star) = E\text{Na}(D, \star)$.*
- (7) *A nonzero ideal I of D is a quasi- $\tilde{\star}$ -ideal if and only if $I = I\text{Na}(D, \star) \cap D$.*

Lemma 1.2. *Let \star be a semistar operation on D .*

- (1) *The following statements are equivalent.*
- (a) *D is a $P\star MD$.*
 - (b) *$\text{Na}(D, \star)$ is a Prüfer domain.*
 - (c) *The ideals of $\text{Na}(D, \star)$ are extended from ideals of D .*
 - (d) *D_P is a valuation domain for each $P \in \text{QMax}^{\star f}(D)$.*
- (2) *Assume that D is a $P\star MD$. Then:*
- (a) *$\tilde{\star} = \star_f$ and (hence) $D^\star = D^{\tilde{\star}}$.*
 - (b) *The map $\text{QSpec}^{\star f}(D) \rightarrow \text{Spec}(\text{Na}(D, \star))$, $P \mapsto P\text{Na}(D, \star)$, is a bijection with inverse map $Q \mapsto Q \cap D$.*
 - (c) *Finitely generated ideals of $\text{Na}(D, \star)$ are extended from finitely generated ideals of D .*

2. IDEMPOTENCE

We begin with our basic definition.

Definition 2.1. *Let \star be a semistar operation on D . An element $E \in \overline{\mathbf{F}}(D)$ is said to be \star -idempotent if $E^\star = (E^2)^\star$.*

Our primary interest will be in (nonzero) \star -idempotent *ideals* of D . Let \star be a semistar operation on D , and let I be a nonzero ideal of D . It is well known that $I^\star \cap D$ is a quasi- \star -ideal of D . (This is easy to see: we have

$$(I^\star \cap D)^\star \subseteq I^{\star\star} = I^\star = (I \cap D)^\star \subseteq (I^\star \cap D)^\star,$$

and hence $I^\star = (I^\star \cap D)^\star$; it follows that $I^\star \cap D = (I^\star \cap D)^\star \cap D$.) It therefore seems natural to call $I^\star \cap D$ the *quasi- \star -closure* of I . If we also call I *\star -proper* when $I^\star \subsetneq D^\star$, then it is easy to see that I is \star -proper if and only if its quasi- \star -closure is a proper quasi- \star -ideal. Now suppose that I is \star -idempotent. Then

$$(I^\star \cap D)^\star = I^\star = (I^2)^\star = ((I^\star)^2)^\star = (((I^\star \cap D)^\star)^2)^\star = ((I^\star \cap D)^2)^\star,$$

whence $I^\star \cap D$ is a \star -idempotent quasi- \star -ideal of D . A similar argument gives the converse. Thus a (\star -proper) nonzero ideal is \star -idempotent if and only if its quasi- \star -closure is a (proper) \star -idempotent quasi- \star -ideal.

Our study of idempotence in Prüfer domains and $P\star MD$ s involves the notions of sharpness and branchedness. We recall some notation and terminology.

Given an integral domain D and a prime ideal $P \in \text{Spec}(D)$, set

$$\begin{aligned} \nabla(P) &:= \{M \in \text{Max}(D) \mid M \not\supseteq P\} \text{ and} \\ \Theta(P) &:= \bigcap \{D_M \mid M \in \nabla(P)\}. \end{aligned}$$

We say that P is *sharp* if $\Theta(P) \not\subseteq D_P$ (see [11, Lemma 1] and [3, Section 1 and Proposition 2.2]). The domain D itself is *sharp (doublesharp)* if every maximal (prime) ideal of D is sharp. (Note that a Prüfer domain D is doublesharp if and only if each overring of D is sharp [7, Theorem 4.1.7].) Now let \star be a semistar operation on D . Given a prime ideal $P \in \text{QSpec}^{\star f}(D)$, set

$$\begin{aligned} \nabla^{\star f}(P) &:= \{M \in \text{QMax}^{\star f}(D) \mid M \not\supseteq P\} \text{ and} \\ \Theta^{\star f}(P) &:= \bigcap \{D_M \mid M \in \nabla^{\star f}(P)\}. \end{aligned}$$

Call P \star_f -sharp if $\Theta^{\star_f}(P) \not\subseteq D_P$. For example, if $\star = d$ is the identity, then the \star_f -sharp property coincides with the sharp property. We then say that D is \star_f -(double)sharp if each quasi- \star_f -maximal (quasi- \star_f -prime) ideal of D is \star_f -sharp. (For more on sharpness, see [10], [11], [13], [7, page 62], [3], [4, Chapter 2, Section 3], and [5].)

Recall that a prime ideal P of a ring is said to be *branched* if there is a P -primary ideal distinct from P . Also, recall that the domain D has *finite character* if each nonzero ideal of D is contained in only finitely many maximal ideals of D .

We now prove a lemma that discusses the transfer of ideal-theoretic properties between D (on which a semistar operation \star has been defined) and its associated Nagata ring.

Lemma 2.2. *Let \star be a semistar operation on D .*

- (1) *Let $E \in \overline{\mathbf{F}}(D)$. Then E is $\tilde{\star}$ -idempotent if and only if $E\text{Na}(D, \star)$ is idempotent. In particular, if D is a $P\star\text{MD}$, then E is \star_f -idempotent if and only if $E\text{Na}(D, \star)$ is idempotent.*
- (2) *Let P be a quasi- $\tilde{\star}$ -prime of D and I a nonzero ideal of D . Then:*
 - (a) *I is P -primary in D if and only if I is a quasi- $\tilde{\star}$ -ideal of D and $I\text{Na}(D, \star)$ is $P\text{Na}(D, \star)$ -primary in $\text{Na}(D, \star)$.*
 - (b) *P is branched in D if and only if $P\text{Na}(D, \star)$ is branched in $\text{Na}(D, \star)$.*
- (3) *D has \star_f -finite character (i.e., each nonzero element of D belongs to only finitely many (possibly zero) $M \in \text{QMax}^{\star_f}(D)$) if and only if $\text{Na}(D, \star)$ has finite character.*
- (4) *Let I be a quasi- $\tilde{\star}$ -ideal of D . Then I is a radical ideal if and only if $I\text{Na}(D, \star)$ is a radical ideal of $\text{Na}(D, \star)$.*
- (5) *Assume that D is a $P\star\text{MD}$. Then:*
 - (a) *If $P \in \text{QSpec}^{\star_f}(D)$, then P is \star_f -sharp if and only if $P\text{Na}(D, \star)$ is sharp in $\text{Na}(D, \star)$.*
 - (b) *D is \star_f -(double)sharp if and only if $\text{Na}(D, \star)$ is (double)sharp.*

Proof. (1) We use Lemma 1.1(6). If $E\text{Na}(D, \star)$ is idempotent, then $E^{\tilde{\star}} = E\text{Na}(D, \star) \cap K = E^2\text{Na}(D, \star) \cap K = (E^2)^{\tilde{\star}}$. Conversely, if E is $\tilde{\star}$ -idempotent, then $(E\text{Na}(D, \star))^2 = E^2\text{Na}(D, \star) = (E^2)^{\tilde{\star}}\text{Na}(D, \star) = E^{\tilde{\star}}\text{Na}(D, \star) = E\text{Na}(D, \star)$. The ‘‘in particular’’ statement follows because $\star_f = \tilde{\star}$ in a $P\star\text{MD}$ (Lemma 1.2(2a)).

(2) (a) Suppose that I is P -primary. Then $ID[X]$ is $PD[X]$ -primary. Since P is a quasi- $\tilde{\star}$ -prime of D , $P\text{Na}(D, \star)$ is a prime ideal of $\text{Na}(D, \star)$ (Lemma 1.1(4)), and then, since $\text{Na}(D, \star)$ is a quotient ring of $D[X]$, $I\text{Na}(D, \star)$ is $P\text{Na}(D, \star)$ -primary in $\text{Na}(D, \star)$. Also, again using the fact that $ID[X]$ is $PD[X]$ -primary (along with Lemma 1.1(6)), we have

$$I^{\tilde{\star}} \cap D = I\text{Na}(D, \star) \cap D \subseteq ID[X]_{PD[X]} \cap D[X] \cap D = ID[X] \cap D = I,$$

whence I is a quasi- $\tilde{\star}$ -ideal of D . Conversely, assume that I is a quasi- $\tilde{\star}$ -ideal of D and that $I\text{Na}(D, \star)$ is $P\text{Na}(D, \star)$ -primary. Then for $a \in P$, there is a positive integer n for which $a^n \in I\text{Na}(D, \star) \cap D = I^{\tilde{\star}} \cap D = I$. Hence $P = \text{rad}(I)$. It now follows easily that I is P -primary.

(b) Suppose that P is branched in D . Then there is a P -primary ideal I of D distinct from P , and $INa(D, \star)$ is $PNa(D, \star)$ -primary by (a). Also by (a), I is a quasi- $\tilde{\star}$ -ideal, from which it follows that $INa(D, \star) \neq PNa(D, \star)$. Now suppose that $PNa(D, \star)$ is branched and that J is a $PNa(D, \star)$ -primary ideal of $Na(D, \star)$ distinct from $PNa(D, \star)$. Then it is straightforward to show that $J \cap D$ is distinct from P and is P -primary.

(3) Let ψ be a nonzero element of $Na(D, \star)$, and let N be a maximal ideal with $\psi \in N$. Then $\psi Na(D, \star) = fNa(D, \star)$ for some $f \in D[X]$, and writing $N = MNa(D, \star)$ for some $M \in QMax^{\star_f}(D)$ (Lemma 1.1(3)), we must have $f \in MD[X]$ and hence $c(f) \subseteq M$. Therefore, if D has finite \star_f -character, then $Na(D, \star)$ has finite character. The converse is even more straightforward.

(4) Suppose that I is a radical ideal of D , and let $\psi^n \in INa(D, \star)$ for some $\psi \in Na(D, \star)$ and positive integer n . Then there is an element $g \in N(\star)$ with $(g\psi^n$ and hence) $(g\psi)^n \in ID[X]$. Since $ID[X]$ is a radical ideal of $D[X]$, $g\psi \in ID[X]$ and we must have $\psi \in INa(D, \star)$. Therefore, $INa(D, \star)$ is a radical ideal of $Na(D, \star)$. The converse follows easily from the fact that $INa(D, \star) \cap D = I\tilde{\star} \cap D = I$ (Lemma 1.1(7)).

(5) (a) This is part of [5, Proposition 3.5], but we give here a proof more in the spirit of this paper. Let $P \in QSpec^{\star_f}(D)$. If P is \star_f -sharp, then by [5, Proposition 3.1] P contains a finitely generated ideal I with $I \not\subseteq M$ for all $M \in \nabla^{\star_f}(P)$, and, using the description of $Max(Na(D, \star))$ given in Lemma 1.1(3), $INa(D, \star)$ is a finitely generated ideal of $Na(D, \star)$ contained in $PNa(D, \star)$ but in no element of $\nabla(PNa(D, \star))$. Therefore, $PNa(D, \star)$ is sharp in the Prüfer domain $Na(D, \star)$. For the converse, assume that $PNa(D, \star)$ is sharp in $Na(D, \star)$. Then $PNa(D, \star)$ contains a finitely generated ideal J with $J \subseteq PNa(D, \star)$ but $J \not\subseteq N$ for $N \in \nabla(PNa(D, \star))$ [13, Corollary 2]. Then $J = INa(D, \star)$ for some finitely generated ideal I of D (necessarily) contained in P by Lemma 1.2(2c), and it is easy to see that $I \not\subseteq M$ for $M \in \nabla^{\star_f}(D)$. Then by [5, Proposition 3.1], P is \star_f -sharp. Statement (b) follows easily from (a) (using Lemma 1.2). \square

Let D be an almost Dedekind domain with a non-finitely generated maximal ideal M . Then $M^{-1} = D$, but M is not idempotent (since MD_M is not idempotent in the Noetherian valuation domain D_M). Our next result shows that this cannot happen in a sharp Prüfer domain.

Theorem 2.3. *Let D be a Prüfer domain. If D is (d -)sharp and I is a nonzero ideal of D with $I^{-1} = D$, then I is idempotent.*

Proof. Assume that D is sharp. Proceeding contrapositively, suppose that I is a nonzero, non-idempotent ideal of D . Then, for some maximal ideal M of D , ID_M is not idempotent in D_M . Since D is a sharp domain, we may choose a finitely generated ideal J of D with $J \subseteq M$ but $J \not\subseteq N$ for all maximal ideals $N \neq M$. Since ID_M is a non-idempotent ideal in the valuation domain D_M , there is an element $a \in I$ for which $I^2D_M \subsetneq aD_M$. Let $B := J + Da$. Then $I^2D_M \subseteq BD_M$ and, for $N \in Max(D) \setminus \{M\}$, $I^2D_N \subseteq D_N = BD_N$. Hence $I^2 \subseteq B$. Since B is a proper finitely generated ideal, we then have $(I^2)^{-1} \supseteq B^{-1} \supsetneq D$. Hence $(I^2)^{-1} \neq D$, from which it follows that $I^{-1} \neq D$, as desired. \square

Kang [14, Proposition 2.2] proves that, for a nonzero ideal I of D , we always have $I^{-1}\text{Na}(D, v) = (\text{Na}(D, v) : I)$. This cannot be extended to general semistar Nagata rings; for example, if D is an almost Dedekind domain with non-invertible maximal ideal M and we define a semistar operation \star by $E^\star = ED_M$ for $E \in \overline{\mathbf{F}}(D)$, then $(D : M) = D$ and hence $(D : M)\text{Na}(D, \star) = \text{Na}(D, \star) = D[X]_{M[X]} = D_M(X) \subsetneq (D_M : MD_M)D_M(X) = (\text{Na}(D, \star) : M\text{Na}(D, \star))$ (where the proper inclusion holds because MD_M is principal in D_M). At any rate, what we really require is the equality $(D^\star : E)\text{Na}(D, \star) = (\text{Na}(D, \star) : E)$ for $E \in \overline{\mathbf{F}}(D)$. In the next lemma, we show that this holds in a $P\star\text{MD}$ but not in general. The proof of part (1) of the next lemma is a relatively straightforward translation of the proof of [14, Proposition 2.2] to the semistar setting. In carrying this out, however, we discovered a minor flaw in the proof of [14, Proposition 2.2]. The flaw involves a reference to [12, Proposition 34.8], but this result requires that the domain D be integrally closed. To ensure complete transparency, we give the proof in full detail.

Lemma 2.4. *Let \star be a semistar operation on D . Then:*

- (1) $(D^\star : E)\text{Na}(D, \star) \supseteq (\text{Na}(D, \star) : E)$ for each $E \in \overline{\mathbf{F}}(D)$.
- (2) *The following statements are equivalent:*
 - (a) $(D^\star : E)\text{Na}(D, \star) = (\text{Na}(D, \star) : E)$ for each $E \in \overline{\mathbf{F}}(D)$.
 - (b) $D^\star = D^{\tilde{\star}}$.
 - (c) $D^\star \subseteq \text{Na}(D, \star)$.
- (3) $(D^{\tilde{\star}} : E)\text{Na}(D, \star) = (\text{Na}(D, \star) : E)$ for each $E \in \overline{\mathbf{F}}(D)$.
- (4) *If D is a $P\star\text{MD}$, then the equivalent conditions in (2) hold.*

Proof. (1) Let $E \in \overline{\mathbf{F}}(D)$, and let $\psi \in (\text{Na}(D, \star) : E)$. For $a \in E$, $a \neq 0$, we may find $g \in N(\star)$ such that $\psi ag \in D[X]$. This yields $\psi g \in a^{-1}D[X] \subseteq K[X]$, and hence $\psi = f/g$ for some $f \in K[X]$. We claim that $c(f) \subseteq (D^\star : E)$. Granting this, we have $f \in (D^\star : E)D[X]$, from which it follows that $\psi = f/g \in (D^\star : E)\text{Na}(D, \star)$, as desired. To prove the claim, take $b \in E$, and note that $fb \in \text{Na}(D, \star)$. Hence $fbh \in D[X]$ for some $h \in N(\star)$, and so $c(fh)b \subseteq D$. By the content formula [12, Theorem 28.1], there is an integer m for which $c(f)c(h)^{m+1} = c(fh)c(h)^m$. Using the fact that $c(h)^\star = D^\star$, we obtain $c(f)^\star = c(fh)^\star$ and hence that $c(f)b \subseteq c(fh)^\star b \subseteq D^\star$. Therefore, $c(f) \subseteq (D^\star : E)$, as claimed.

(2) Under the assumption in (c), $D^\star \subseteq \text{Na}(D, \star) \cap K = D^{\tilde{\star}}$ (Lemma 1.1(6)). Hence (c) \Rightarrow (b). Now assume that $D^\star = D^{\tilde{\star}}$. Then for $E \in \overline{\mathbf{F}}(D)$, we have $(D^\star : E)E \subseteq D^\star = D^{\tilde{\star}} \subseteq \text{Na}(D, \star)$; using (1), the implication (b) \Rightarrow (a) follows. That (a) \Rightarrow (c) follows upon taking $E = D$ in (a).

(3) This follows easily from (2), because $\text{Na}(D, \star) = \text{Na}(D, \tilde{\star})$ by Lemma 1.1(5).

(4) This follows from (2), since if D is a $P\star\text{MD}$, then $D^\star = D^{\tilde{\star}}$ by Lemma 1.2(2a). \square

The conditions in Lemma 2.4(2) need not hold: Let $F \subsetneq k$ be fields, $V = k[[x]]$ the power series ring over V in one variable, and $D = F + M$, where $M = xk[[x]]$. Define a (finite-type) semistar operation \star on D by $A^\star = AV$ for $A \in \overline{\mathbf{F}}(D)$. Then $D^\star = V \supsetneq D = D_M = D^{\tilde{\star}}$.

We can now extend Theorem 2.3 to $P\star\text{MDs}$.

Corollary 2.5. *Let \star be a semistar operation on D such that D is a \star_f -sharp $P\star MD$, and let I be a nonzero ideal of D with $(D^\star : I) = D^\star$. Then I is \star_f -idempotent.*

Proof. By Lemma 2.4(3), we have

$$(\text{Na}(D, \star) : I\text{Na}(D, \star)) = (D^\star : I)\text{Na}(D, \star) = D^\star\text{Na}(D, \star) = \text{Na}(D, \star).$$

Hence $I\text{Na}(D, \star)$ is idempotent in the Prüfer domain $\text{Na}(D, \star)$ by Theorem 2.3. Lemma 2.2(1) then yields that I is \star_f -idempotent. \square

Many semistar counterparts of ideal-theoretic properties in domains result in equations that are “external” to D , since for a semistar operation \star on D and a nonzero ideal I of D , it is possible that $I^\star \not\subseteq D$. Of course, \star -idempotence is one such property. Often, one can obtain a “cleaner” counterpart by specializing from $P\star MD$ s to “ordinary” $PvMD$ s. We recall some terminology. Semistar operations are generalizations of *star* operations, first considered by Krull and repopularized by Gilmer [12, Sections 32, 34]. Roughly, a star operation is a semistar operation restricted to the set $\mathbf{F}(D)$ of nonzero fractional ideals of D with the added requirement that one has $D^\star = D$. The most important star operation (aside from the d -, or trivial, star operation) is the v -operation: For $E \in \mathbf{F}(D)$, put $E^{-1} = \{x \in K \mid xE \subseteq D\}$ and $E^v = (E^{-1})^{-1}$. Then v_f (restricted to $\mathbf{F}(D)$) is the t -operation and \tilde{v} is the w -operation. Thus a $PvMD$ is a domain in which each nonzero finitely generated ideal is t -invertible. Corollary 2.5 then has the following restricted interpretation (which has the advantage of being *internal* to D).

Corollary 2.6. *If D is a t -sharp $PvMD$ and I is a nonzero ideal of D for which $I^{-1} = D$, then I is t -idempotent.*

Our next result is a partial converse to Theorem 2.3.

Proposition 2.7. *Let D be a Prüfer domain such that I is idempotent whenever I is a nonzero ideal of D with $I^{-1} = D$. Then, every branched maximal ideal of D is sharp.*

Proof. Let M be a branched maximal ideal of D . Then $MD_M = \text{rad}(aD_M)$ for some nonzero element $a \in M$ [12, Theorem 17.3]. Let $I := aD_M \cap D$. Then I is M -primary, and since $ID_M = aD_M$, (ID_M and hence) I is not idempotent. By hypothesis, we may choose $u \in I^{-1} \setminus D$. Since $Iu \subseteq D$ and $ID_N = D_N$ for $N \in \text{Max}(D) \setminus \{M\}$, then $u \in \bigcap \{D_N \mid N \in \text{Max}(D), N \neq M\}$. On the other hand, since $u \notin D$, $u \notin D_M$. It follows that M is sharp. \square

Now we extend Proposition 2.7 to $P\star MD$ s.

Corollary 2.8. *Let \star be a semistar operation on D , and assume that D is a $P\star MD$ such that I is \star_f -idempotent whenever I is a nonzero ideal of D with $(D^\star : I) = D^\star$. Then, each branched quasi- \star_f -maximal ideal of D is \star_f -sharp. (In particular if D is a $PvMD$ in which I is t -idempotent whenever I is a nonzero ideal of D with $I^{-1} = D$, then each branched maximal t -ideal of D is t -sharp.)*

Proof. Let J be a nonzero ideal of the Prüfer domain $\text{Na}(D, \star)$ with $(\text{Na}(D, \star) : J) = \text{Na}(D, \star)$. By Lemma 1.2(1c), $J = I\text{Na}(D, \star)$ for some ideal I of D . Applying Lemma 2.4(3) and Lemma 1.1(6), we obtain $(D^\star : I) = D^\star$. Hence, by hypothesis, I is \star_f -idempotent, and this yields that $J = I\text{Na}(D, \star)$ is idempotent in the Prüfer domain $\text{Na}(D, \star)$ (Lemma 2.2(1)). Now, let M be a branched quasi- \star_f -maximal ideal of D . Then, by Lemma 2.2(2), $M\text{Na}(D, \star)$ is a branched maximal ideal of $\text{Na}(D, \star)$. We may now apply Proposition 2.7 to conclude that $M\text{Na}(D, \star)$ is sharp. Therefore, M is \star_f -sharp in D by Lemma 2.2(5). \square

If P is a prime ideal of a Prüfer domain D , then powers of P are P -primary by [12, Theorem 23.3(b)]; it follows that P is idempotent if and only if PD_P is idempotent. We use this fact in the next result.

It is well known that a proper idempotent ideal of a valuation domain must be prime [12, Theorem 17.1(3)]. In fact, according to [12, Exercise 3, p. 284], a proper idempotent ideal in a Prüfer domain must be a radical ideal. We (re-)prove and extend this fact and add a converse:

Theorem 2.9. *Let D be a Prüfer domain, and let I be an ideal of D . Then I is idempotent if and only if I is a radical ideal each of whose minimal primes is idempotent.*

Proof. The result is trivial for $I = (0)$ and vacuously true for $I = D$. Suppose that I is a proper nonzero idempotent ideal of D , and let P be a prime minimal over I . Then ID_P is idempotent, and we must have $ID_P = PD_P$ [12, Theorem 17.1(3)]. Hence PD_P is idempotent, and therefore, by the comment above, so is P . Now let M be a maximal ideal containing I . Then ID_M is idempotent, hence prime (hence radical). It follows (checking locally) that I is a radical ideal.

Conversely, let I be a radical ideal each of whose minimal primes is idempotent. If M is a maximal ideal containing I and P is a minimal prime of I contained in M , then $ID_M = PD_M$. Since P is idempotent, this yields $ID_M = I^2D_M$. It follows that I is idempotent. \square

We next extend Theorem 2.9 to $P\star$ MDs.

Corollary 2.10. *Let D be a $P\star$ MD, where \star is a semistar operation on D , and let I be a quasi- \star_f -ideal of D . Then I is \star_f -idempotent if and only if I is a radical ideal each of whose minimal primes is \star_f -idempotent. (In particular, if D is a PvMD and I is a t -ideal of D , then I is t -idempotent if and only if I is a radical ideal each of whose minimal primes is t -idempotent.)*

Proof. Suppose that I is \star_f -idempotent. Then $I\text{Na}(D, \star)$ is an idempotent ideal in $\text{Na}(D, \star)$ by Lemma 2.2(1). By Theorem 2.9, $I\text{Na}(D, \star)$ is a radical ideal of $\text{Na}(D, \star)$, and hence, by Lemma 2.2(4), I is a radical ideal of D . Now let P be a minimal prime of I in D . Then P is a quasi- \star_f -prime of D . By Lemma 1.2(2b) $P\text{Na}(D, \star)$ is minimal over $I\text{Na}(D, \star)$, whence $P\text{Na}(D, \star)$ is idempotent, again by Theorem 2.9. The \star_f -idempotence of P now follows from Lemma 2.2(1).

The converse follows by similar applications of Theorem 2.9 and Lemma 2.2. \square

Recall that a Prüfer domain is said to be *strongly discrete* (*discrete*) if it has no nonzero (branched) idempotent prime ideals. Since unbranched primes in a Prüfer domain must be idempotent [12, Theorem 23.3(b)], a Prüfer domain is strongly discrete if and only if it is discrete and has no unbranched prime ideals. We have the following straightforward application of Theorem 2.9.

Corollary 2.11. *Let D be a Prüfer domain.*

- (1) *If D is discrete, then an ideal I of D is idempotent if and only if I is a radical ideal each of whose minimal primes is unbranched.*
- (2) *If D is strongly discrete, then D has no proper nonzero idempotent ideals.*

Let us call a $P\star MD$ \star_f -strongly discrete (\star_f -discrete) if it has no (branched) \star_f -idempotent quasi- \star_f -prime ideals. From Lemma 2.2(1,2), we have the usual connection between a property of a $P\star MD$ and the corresponding property of its \star -Nagata ring:

Proposition 2.12. *Let \star be a semistar operation on D . Then D is \star_f -(strongly) discrete $P\star MD$ if and only if $\text{Na}(D, \star)$ is a (strongly) discrete Prüfer domain.*

Applying Corollary 2.10 and Lemma 2.2(1,2), we have the following extension of Corollary 2.11.

Corollary 2.13. *Let D be a domain.*

- (1) *Assume that D is a $P\star MD$ for some semistar operation \star on D .*
 - (a) *If D is \star_f -discrete, then a nonzero quasi- \star_f -ideal I of D is \star_f -idempotent if and only if I is a radical ideal each of whose minimal primes is unbranched.*
 - (b) *If D is \star_f -strongly discrete, then D has no \star_f -proper \star_f -idempotent ideals.*
- (2) *Assume that D is a PvMD.*
 - (a) *If D is t -discrete, then a t -ideal I of D is t -idempotent if and only if I is a radical ideal each of whose minimal primes is unbranched.*
 - (b) *If D is t -strongly discrete, then D has no t -proper t -idempotent ideals.*

3. DIVISORIALITY

According to [7, Corollary 4.1.14], if D is a doublesharp Prüfer domain and P is a nonzero, nonmaximal ideal of D , then P is divisorial. The natural question arises: If D is a \star_f -doublesharp $P\star MD$ and $P \in \text{QSpec}^{\star_f}(D) \setminus \text{QMax}^{\star_f}(D)$, is P necessarily divisorial? Since \star is an arbitrary semistar operation and divisoriality specifically involves the v -operation, one might expect the answer to be negative. Indeed, we give a counterexample in Example 3.4 below. However, in Theorem 3.2 we prove a general result, a corollary of which does yield divisoriality in the “ordinary” PvMD case. First, we need a lemma, the first part of which may be regarded as an extension of [14, Proposition 2.2(2)].

Lemma 3.1. *Let \star be a semistar operation on D . Then*

- (1) $(D^{\tilde{\star}} : (D^{\tilde{\star}} : E))\text{Na}(D, \star) = (\text{Na}(D, \star) : (\text{Na}(D, \star) : E))$ for each $E \in \overline{\mathbf{F}}(D)$,
and

(2) if I is a nonzero ideal of D , then $I^{\tilde{\star}}$ is a divisorial ideal of $D^{\tilde{\star}}$ if and only if $INa(D, \star)$ is a divisorial ideal of $Na(D, \star)$.

In particular, if D is a $P\star MD$, then $(D^{\star} : (D^{\star} : E))Na(D, \star) = (Na(D, \star) : (Na(D, \star) : E))$ for each $E \in \overline{F}(D)$; and, for a nonzero ideal I of D , I^{\star_f} is divisorial in D^{\star} if and only if $INa(D, \star)$ is divisorial in $Na(D, \star)$.

Proof. Set $\mathcal{N} = Na(D, \star)$. For (1), applying Lemma 2.4, we have

$$(D^{\tilde{\star}} : (D^{\tilde{\star}} : E))\mathcal{N} = (\mathcal{N} : (D^{\tilde{\star}} : E)) = (\mathcal{N} : (\mathcal{N} : E)).$$

(2) Assume that I is a nonzero ideal of D . If $I^{\tilde{\star}}$ is divisorial in $D^{\tilde{\star}}$, then (using (1))

$$(\mathcal{N} : (\mathcal{N} : IN)) = (D^{\tilde{\star}} : (D^{\tilde{\star}} : I^{\tilde{\star}}))\mathcal{N} = I^{\tilde{\star}}\mathcal{N} = IN.$$

Now suppose that IN is divisorial. Then

$$(D^{\tilde{\star}} : (D^{\tilde{\star}} : I^{\tilde{\star}}))\mathcal{N} = (\mathcal{N} : (\mathcal{N} : I)) = IN,$$

whence

$$(D^{\tilde{\star}} : (D^{\tilde{\star}} : I^{\tilde{\star}})) \subseteq IN \cap K = I^{\tilde{\star}}.$$

The “in particular” statement follows from standard considerations. \square

Theorem 3.2. *Let \star be a semistar operation on D such that D is a \star_f -doublesharp $P\star MD$, and let $P \in \text{QSpec}^{\star_f}(D) \setminus \text{QMax}^{\star_f}(D)$. Then P^{\star_f} is a divisorial ideal of D^{\star} .*

Proof. Since $Na(D, \star)$ is a doublesharp Prüfer domain (Lemma 2.2(5)), $PNa(D, \star)$ is divisorial by [7, Corollary 4.1.14]. Hence P^{\star_f} is divisorial in D^{\star} by Lemma 3.1. \square

Corollary 3.3. *If D is a t -doublesharp $PvMD$, and P is a non- t -maximal t -prime of D , then P is divisorial.*

Proof. Take $\star = v$ in Theorem 3.2. (More precisely, take \star to be any extension of the star operation v on D to a semistar operation on D , so that \star_f (restricted to D) is the t -operation on D .) Then $P = P^t = P^{\star_f}$ is divisorial by Theorem 3.2. \square

Example 3.4. *Let p be a prime integer and let $D := \text{Int}(\mathbb{Z}_{(p)})$. Then D is a 2-dimensional Prüfer domain by [2, Lemma VI.1.4 and Proposition V.1.8]. Choose a height 2 maximal ideal M of D , and let P be a height 1 prime ideal of D contained in M . Then $P = q\mathbb{Q}[X] \cap D$ for some irreducible polynomial $q \in \mathbb{Q}[X]$ [2, Proposition V.2.3]. By [2, Theorems VIII.5.3 and VIII.5.15], P is not a divisorial ideal of D . Set $E^{\star} = ED_M$ for $E \in \overline{F}(D)$. Then, \star is a finite-type semistar operation on D . Clearly, M is the only quasi- \star -maximal ideal of D , and, since D_M is a valuation domain, D is a $P\star MD$ by Lemma 1.2. Moreover, $Na(D, \star) = D_M(X)$ is also a valuation domain and hence a doublesharp Prüfer domain, which yields that D is a \star_f -doublesharp $P\star MD$ (Lemma 2.2). Finally, since $P = PD_M \cap D = P^{\star} \cap D$, P is a non- \star_f -maximal quasi- \star_f -prime of D . \square*

In the remainder of the paper, we impose on Prüfer domains ($P\star MD$ s) the finite character (finite \star_f -character) condition. As we shall see, this allows improved versions of Theorem 2.9 and Corollary 2.10. It also allows a type of unique factorization for (quasi- \star_f -)ideals that are simultaneously $(\star_f$ -)idempotent and $(\star_f$ -)divisorial.

Theorem 3.5. *Let D be a Prüfer domain with finite character, and let I be a nonzero ideal of D . Then:*

- (1) *I is idempotent if and only if I is a product of idempotent prime ideals.*
- (2) *The following statements are equivalent.*
 - (a) *I is idempotent and divisorial.*
 - (b) *I is a product of non-maximal idempotent prime ideals.*
 - (c) *I is a product of divisorial idempotent prime ideals.*
 - (d) *I has a unique representation as the product of incomparable divisorial idempotent primes.*

Proof. (1) Suppose that I is idempotent. By Theorem 2.9, I is the intersection of its minimal primes, each of which is idempotent. Since D has finite character, I is contained in only finitely many maximal ideals, and, since no two distinct minimal primes of I can be contained in a single maximal ideal, I has only finitely many minimal primes and they are comaximal. Hence I is the product of its minimal primes (and each is idempotent). The converse is trivial.

(2) (a) \Rightarrow (b): Assume that I is idempotent and divisorial. By (1) and its proof, $I = P_1 \cdots P_n = P_1 \cap \cdots \cap P_n$, where the P_i are the minimal primes of I . We claim that each P_i is divisorial. To see this, observe that

$$(P_1)^v P_2 \cdots P_n \subseteq (P_1 \cdots P_n)^v = I^v = I \subseteq P_1.$$

Since the P_i are incomparable, this gives $(P_1)^v \subseteq P_1$, that is, P_1 is divisorial. By symmetry each P_i is divisorial. It is well known that in a Prüfer domain, a maximal ideal cannot be both idempotent and divisorial. Hence the P_i are non-maximal.

(b) \Rightarrow (c): Since D has finite character, it is a (d) -doublesharpe Prüfer domain [13, Theorem 5], whence nonmaximal primes are automatically divisorial by [7, Corollary 4.1.14].

(c) \Rightarrow (a): Write $I = Q_1 \cdots Q_m$, where each Q_j is a divisorial idempotent prime. Since I is idempotent (by (1)), we may also write $I = P_1 \cdots P_n$, where the P_i are the minimal primes of I . For each i , we have $Q_1 \cdots Q_m = I \subseteq P_i$, from which it follows that $Q_j \subseteq P_i$ for some j . By minimality, we must then have $Q_j = P_i$. Thus each P_i is divisorial, whence $I = P_1 \cap \cdots \cap P_n$ is divisorial.

Finally, we show that (d) follows from the other statements. We use the notation in the proof of (c) \Rightarrow (a). In the expression $I = P_1 \cdots P_n$, the P_i are (divisorial, idempotent, and) incomparable, and it is clear that no P_i can be omitted. To see that this is the only such expression, consider a representation $I = Q_1 \cdots Q_m$, where the Q_i are divisorial, idempotent, and incomparable. Fix a Q_k . Then $P_1 \cdots P_n = I \subseteq Q_k$, and we have $P_i \subseteq Q_k$ for some i . However, as above, $Q_j \subseteq P_i$ for some j , whence, by incomparability, $Q_k = P_i$. The conclusion now follows easily. \square

We note that incomparability is necessary for uniqueness above—for example, if D is a valuation domain and $P \subsetneq Q$ are non-maximal (necessarily divisorial) primes, then $P = PQ$.

We close by extending Theorem 3.5 to $P\star$ MDs and then to “ordinary” Pv MDs. We omit the (by now) straightforward proofs.

Corollary 3.6. *Let \star be a semistar operation on D such that D is a $P\star MD$ with finite \star_f -character, and let I be a quasi- \star_f -ideal of D . Then:*

- (1) *I is \star_f -idempotent if and only if I^{\star_f} is a \star_f -product of \star_f -idempotent quasi- \star_f -prime ideals in D , that is, $I^{\star_f} = (P_1 \cdots P_n)^{\star_f}$, where the P_i are \star_f -idempotent quasi- \star_f -primes of D .*
- (2) *The following statements are equivalent.*
 - (a) *I is \star_f -idempotent and \star_f -divisorial (I^{\star_f} is divisorial in D^\star).*
 - (b) *I is a \star_f -product of non-quasi- \star_f -maximal idempotent quasi- \star_f -prime ideals.*
 - (c) *I is a \star_f -product of \star_f -divisorial \star_f -idempotent prime ideals.*
 - (d) *I has a unique representation as a \star_f -product of incomparable \star_f -divisorial \star_f -idempotent primes.*

Corollary 3.7. *Let D be a PvMD with finite t -character, and let I be a nonzero t -ideal of D . Then:*

- (1) *I is t -idempotent if and only if I is a t -product of t -idempotent t -prime ideals in D ,*
- (2) *The following statements are equivalent.*
 - (a) *I is t -idempotent and divisorial.*
 - (b) *I is a t -product of non- t -maximal t -idempotent t -primes.*
 - (c) *I is a t -product of divisorial t -idempotent t -primes.*
 - (d) *I has a unique representation as a t -product of incomparable divisorial t -idempotent t -primes.*

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