Forum Math. **19** (2007), 971–1004
DOI 10.1515/FORUM.2007.038

Forum
Mathematicum

A generalization of Kronecker function rings and Nagata rings

Marco Fontana and Alan Loper

(Communicated by Rüdiger Göbel)

Abstract. Let D be an integral domain with quotient field K. The Nagata ring $D(X)$ and the Kronecker function ring $\mathrm{Kr}(D)$ are both subrings of the field of rational functions $K(X)$ containing as a subring the ring $D[X]$ of polynomials in the variable X. Both of these function rings have been extensively studied and generalized. The principal interest in these two extensions of D lies in the reflection of various algebraic and spectral properties of D and $\mathrm{Spec}(D)$ in algebraic and spectral properties of the function rings. Despite the obvious similarities in definitions and properties, these two kinds of domains of rational functions have been classically treated independently, when D is not a Prüfer domain. The purpose of this note is to study two different unified approaches to the Nagata rings and the Kronecker function rings, which yield these rings and their classical generalizations as special cases.

2000 Mathematics Subject Classification: 13F05, 13A15, 13A18, 13G05, 13H99.

1 Introduction

Let D be a commutative integral domain with quotient field K. Let X be an indeterminate over D and let $f \in D[X]$. We denote by $\boldsymbol{c}(f)$ *the content of the polynomial* f, i.e. $\boldsymbol{c}(f)$ is the ideal of D generated by the coefficients of f. Moreover, if $\mathscr{V}(D)$ is the set of all the valuation overrings of D, for each ideal I of D, we set $I^b := \bigcap\{IV \mid V \in \mathscr{V}(D)\}$ (cf. [14, page 398] and [30, Appendix 4]).

Two classical rings related to D which have both been well studied are the Nagata ring $D(X)$ and the Kronecker function ring $\mathrm{Kr}(D)$ defined as follows.

$$\mathrm{Na}(D) := D(X) := \left\{ \frac{f}{g} \,\middle|\, f, g \in D[X], \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g), \boldsymbol{c}(g) \text{ is invertible} \right\}$$

is *the Nagata ring of* D. (Note that this is not the most common definition: $D(X)$ is usually defined by designating that $f, g \in D[X]$ and that $\boldsymbol{c}(g) = D$, [14, Section 33]. The definition above is equivalent to this one and fits our program better.)

On the other hand, if D is integrally closed, then:

$$\mathrm{Kr}(D) := \left\{ \frac{f}{g} \,\middle|\, f, g \in D[X], g \neq 0, \boldsymbol{c}(f)^b \subseteq \boldsymbol{c}(g)^b \right\}$$

is *the Kronecker function ring of D*, [14, Section 32].

These two rings of rational functions are the same if and only if D is a Prüfer domain [14, Theorem 33.4]. In fact, both rings arose as generalizations of Kronecker's original function rings which specified that D should be a ring of algebraic numbers or, more generally, a Dedekind domain (and, hence, a Prüfer domain), (cf. [21], [29] and [6]). When D is any arbitrary integrally closed domain it is easy to see that $\mathrm{Na}(D) \subseteq \mathrm{Kr}(D)$, [14, Theorem 33.3].

There are obvious similarities in the two definitions in spite of the fact that they generally yield different rings. Next we give equivalent definitions (or, characterizations) for each type of ring in which there are also obvious similarities, i.e. both these rings can be constructed by intersection of families of Nagata rings of quasilocal overrings. (Note: We do assume for both of these results that we know how to construct the Nagata ring $R(X)$ for a *quasilocal* domain R; in this situation, the condition "$\boldsymbol{c}(g)$ is invertible" becomes "$\boldsymbol{c}(g)$ is principal", [14, Proposition 7.4].)

Theorem/Definition 1.1 [14, Theorem 33.3, Theorem 32.10 and the proof of Corollary 32.14]. *Let D be an integral domain, let* $\mathrm{Max}(D)$ *[respectively,* $\mathrm{Spec}(D)$*] represent the set of all maximal [respectively, prime] ideals of D and let $\mathscr{V}(D)$ [respectively, $\mathscr{V}_{min}(D)$] denote the set of all the valuation [respectively, minimal valuation] overrings of D.*

(1) $\mathrm{Na}(D) = \bigcap\{D_M(X) \mid M \in \mathrm{Max}(D)\} = \bigcap\{D_P(X) \mid P \in \mathrm{Spec}(D)\}$.

(2) *If D is integrally closed,* $\mathrm{Kr}(D) = \bigcap\{V(X) \mid V \in \mathscr{V}(D)\} = \bigcap\{W(X) \mid W \in \mathscr{V}_{min}(D)\}$.

Both the Kronecker function ring and the Nagata ring have been generalized and intensively studied (cf. for instance [22], [3], [4], [1], [19], [27], [9], [11] and [17]). However, in spite of their common origin, they have been studied separately. There are generalized Nagata rings and generalized Kronecker function rings which are distinct objects of study. A major goal of this paper is to define and study a single construction for a class of function rings which includes the Kronecker function ring, the Nagata ring and their generalizations as special cases.

Following the double characterization of the Kronecker function rings and Nagata rings above, we approach our generalized function rings from two separate directions, as rings of individually chosen rational functions and as intersections of Nagata rings of quasilocal overrings.

In generalizing the rational function approach, we note that the standard generalization of Kronecker function rings, introduced by Krull [22], involves replacing the b-operation with a more general "star-operation" (for short, $\star$-operation) belonging to a special class of star operations known as the "e.a.b. star operations" (the explicit definitions are recalled in Section 2). The point is that e.a.b. operations have some nice properties in common with the b-operation which make possible the proof that the Kronecker function ring, as defined by Krull (cf. the definition after Remark 3.4), is actually a ring for any arbitrary integrally closed domain.

Note that in the definition of the Kronecker function ring, any nonzero polynomial is eligible to be the denominator of a rational function. Note further that we do not allow arbitrary nonzero polynomials in the denominators for the Nagata ring. Rather, we only allow a very restricted class of polynomials, those with invertible content. Finally note that the definition of the Nagata ring given is formally comparable to the Kronecker definition except that the b-operation is replaced by "the trivial star-operation", called the *identity operation* (for short, *d-operation*), acting as the identity map, i.e.

$$\mathrm{Na}(D) = \left\{ \frac{f}{g} \,\middle|\, f, g \in D[X], \boldsymbol{c}(f)^d \subseteq \boldsymbol{c}(g)^d, \boldsymbol{c}(g) \text{ is invertible} \right\}.$$

A way to combine the ideas of the previous two paragraphs is to view the e.a.b. property not as a property of a $\star$-operation, as has been done classically, but to view it as a property of a certain class of ideals. The e.a.b. $\star$-operations should be those for which every nonzero finitely generated ideal is an "e.a.b.-ideal" (Definition 3.6). On the other hand, the invertible ideals should be the only e.a.b.-ideals associated with the identity operation. So given any $\star$-operation, we can combine the two definitions by specifying that the content ideals of denominators must be e.a.b.-ideals associated to the given $\star$-operation.

Note that, from the beginning of the present paper, we move from just "star-operations" to the more general setting of "semistar-operations" introduced by Okabe-Matsuda in 1994 [26] (the definition is recalled in Section 2). In fact, our generalizations work directly, and more naturally, in this setting.

We also want to define our generalized function rings using the method of intersecting Nagata rings of quasilocal overrings as is done above. Suppose then that we are given a domain D and a semistar operation on D. For the b-operation we chose the class of all valuation overrings of D to define the Kronecker function ring and we chose the class of all localizations of D to define the Nagata ring of D. When we consider our discussion of e.a.b.-ideals above, we note that all finitely generated ideals of D extend to principal ideals in any valuation overring. On the other hand, invertible ideals are the only finitely generated ideals which extend to principal ideals in every localization of D. So the way to proceed seems to be to combine the overring characterizations of the Kronecker and Nagata rings by choosing the overrings in which the e.a.b.-ideals extend to principal ideals.

Let D be a domain and $\star$ a semistar operation on D. Given either a collection of overrings of D or a ring of rational functions (overring of $D[X]$) it is easy to define a new semistar operation on D by either extending to all of the overrings of the collection and intersecting, or by extending to the ring of rational functions and contracting back to the quotient field of D. We explore both of these mechanisms for defining new semistar operations associated with the given $\star$ and compare the properties we obtain with those proven in the classical Kronecker and Nagata settings. In particular, we deepen the study of the ring of rational functions called *the $\star$-Nagata* *ring* $\mathrm{Na}(D,\star)$ (Definition 3.5) and *the* $\star$*-Kronecker function ring* $\mathrm{Kr}(D,\star)$ (Definition 3.2) and we give a complete positive answer to the following question: Is it possible to find a "new" integral domain of rational functions denoted by $\mathrm{KN}(D,\star)$ ("$\star$-*Kronecker-Nagata ring*", obtained as an intersection of Nagata domains of quasilocal domains associated to a given arbitrary semistar operation $\star$) such that:

- $\mathrm{Na}(D,\star) \subseteq \mathrm{KN}(D,\star) \subseteq \mathrm{Kr}(D,\star)$;

- $\mathrm{KN}(D,\star)$ "generalizes" at the same time $\mathrm{Na}(D,\star)$ and $\mathrm{Kr}(D,\star)$ and coincides with $\mathrm{Na}(D,\star)$ or $\mathrm{Kr}(D,\star)$, when the semistar operation $\star$ assumes the "extreme values" of an interval $\star' \leq \star \leq \star''$ (i.e. $\mathrm{KN}(D,\star') = \mathrm{Na}(D,\star') = \mathrm{Na}(D,\star)$ and $\mathrm{KN}(D,\star'') = \mathrm{Kr}(D,\star'') = \mathrm{Kr}(D,\star)$)?

2 Background

In this section we give some definitions and some basic results, some new and some not.

We begin by designating the following terms.

- $\boldsymbol{f}(D)$ is the set of all nonzero finitely generated fractional ideals of D.
- $\boldsymbol{F}(D)$ is the set of all nonzero fractional ideals of D.
- $\overline{\boldsymbol{F}}(D)$ is the set of all nonzero D submodules of K.

In 1994 A. Okabe and R. Matsuda [26] introduced the notion of a semistar operation. A semistar operation is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$ which obeys the following axioms, for all $z \in K$, $z \neq 0$ and for all $E, F \in \overline{\boldsymbol{F}}(D)$.

$(\star_1)$ $(zE)^\star = zE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

The classical notion of a star operation [14] involves a map from $\boldsymbol{F}(D)$ to $\boldsymbol{F}(D)$ which requires, in addition to the semistar axioms, that $(\alpha D)^\star = \alpha D$, for each nonzero principal ideal αD of D.

The key difference here is that if $\star$ is a star operation then $D^\star = D$, whereas $D^\star$ may be properly larger than D (possibly, $D^\star \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$) if $\star$ is a semistar operation. Note that if $\star$ is a semistar operation on a domain D then we obtain a classical star operation on $D^\star$ when we restrict $\star$ to $\boldsymbol{F}(D^\star)$.

Now we give some basic information concerning definitions/terminology, general properties of semistar operations, and concerning the construction of specific semistar operations on integral domains.

- As in the classical star operation setting, we associate to a semistar operation $\star$ of D a new semistar operation $\star_f$ as follows. Let $\star$ be a semistar operation of a domain D. If $E \in \overline{\boldsymbol{F}}(D)$ we set:

$$E^{\star_f} := \bigcup\{F^\star \mid F \subseteq E, F \in \boldsymbol{f}(D)\}.$$

We call $\star_f$ *the semistar operation of finite type of* D associated to $\star$. If $\star = \star_f$, we say that $\star$ is *a semistar operation of finite type of* D. Note that $\star_f \leq \star$ and $(\star_f)_f = \star_f$, so $\star_f$ is a semistar operation of finite type of D.

• *A (semi)star operation* $\star$ *on* D is a semistar operation on D such that $D^\star = D$; i.e. a semistar operation such that:

$$\star|_{\boldsymbol{F}(D)} : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$$

is a "classical" star operation [14, Section 32].

• d_D denotes the identity (semi)star operation on D.

• If $\iota : D \hookrightarrow T$ is the canonical embedding of D in the overring T of D and if $\star$ is a semistar operation on D, then $\star_\iota$ is the semistar operation on T defined, for each $E \in \overline{\boldsymbol{F}}(T)$ $(\subseteq \overline{\boldsymbol{F}}(D))$, by

$$E^{\star_\iota} := E^\star.$$

• If T is an overring of D, we denote by $\star_{\{T\}}$ the semistar operation on D defined as follows: for each $E \in \overline{\boldsymbol{F}}(D)$:

$$E^{\star_{\{T\}}} := ET.$$

Obviously, $(\star_{\{T\}})_\iota = d_T$ (where d_T denotes the identity semistar operation on T).

• If $\{\star_\lambda \mid \lambda \in \Lambda\}$ is a family of semistar operations on D then $\bigwedge\{\star_\lambda \mid \lambda \in \Lambda\}$ is the semistar operation on D defined as follows: for each $E \in \overline{\boldsymbol{F}}(D)$:

$$E^{\wedge\{\star_\lambda \mid \lambda \in \Lambda\}} := \bigcap\{E^{\star_\lambda} \mid \lambda \in \Lambda\}.$$

In particular, if $\mathscr{T} := \{T_\lambda \mid T_\lambda \in \Lambda\}$ is a given family of overrings of D, then $\bigwedge_{\mathscr{T}}$ denotes the semistar operation $\bigwedge\{\star_{\{T_\lambda\}} \mid \lambda \in \Lambda\}$.

• b_D is the b-semistar operation on D, i.e.

$$b_D := \bigwedge\{\star_{\{V\}} \mid V \text{ is a valuation overring of } D\}.$$

It seems natural in the context of the "$\bigwedge$-construction" above, i.e. $\bigwedge_{\mathscr{T}}$, to view semistar operations "extensions to the overrings", i.e. $\star_{\{T_\lambda\}}$, as canonical components of the semistar operation on D. We have defined the b-operation as a $\bigwedge$-construction using the valuation overrings. In that setting we can think of the b-operation as being decomposed into component semistar operations, each defined by extension to a valuation overring.

A question that seems not to have been dealt with in the literature (in either the star or semistar setting) is the extent to which a given star (or semistar) operation on

D can be approximated by one built from component parts of the type "extensions to the overrings" of D in the above manner. One use for our generalized Kronecker-Nagata theory is to associate in a natural way a semistar operation defined by a $\bigwedge$-construction to a given semistar operation.

In the setting of star operations, the class of $\star$-ideals (i.e. those ideals I such that $I^\star = I$) assume a role of great importance. When $\star$ is a semistar operation there frequently are no integral ideals of D which are $\star$-ideals. Instead we use the following more general concept.

Definition 2.1. Let $I \subseteq D$ be a nonzero ideal of D and let $\star$ be a semistar operation on D. We say that I is *a quasi-$\star$-ideal of D* if $I^\star \cap D = I$. Similarly, we designate by *quasi-$\star$-prime* [respectively, *$\star$-prime*] of D a quasi-$\star$-ideal [respectively, an integral $\star$-ideal] of D which is also a prime ideal. We designate by *quasi-$\star$-maximal* [respectively, *$\star$-maximal*] of D a maximal element in the set of all proper quasi-$\star$-ideals [respectively, integral $\star$-ideals] of D.

Note that if $I \subseteq D$ is a $\star$-ideal, it is also a quasi-$\star$-ideal and, when $D = D^\star$ the notions of quasi-$\star$-ideal and integral $\star$-ideal coincide.

We then give the following designations related to quasi-star ideals.

- $\mathrm{QSpec}^\star(D)$ is the set of all the quasi-$\star$-prime ideals of D.
- $\mathcal{M}(\star)$ is the set of all the maximal quasi-$\star$-ideals of D.

It is well known that *if $\star$ is a semistar operation of finite type then $\mathcal{M}(\star)$ is nonempty* [11, Lemma 2.3 (1)].

A particular important semistar operation (of finite type) on D is the following:

- $\tilde{\star} := \bigwedge\{\star_{\{D_Q\}} \mid Q \in \mathcal{M}(\star_f)\}$ $(= \bigwedge_{\mathcal{S}}$, where $\mathcal{S} := \{D_Q \mid Q \in \mathcal{M}(\star_f)\})$.

For the motivations, examples and the basic properties of this type of semistar operation cf. for instance [11, Corollary 2.7 and Remark 2.8].

3 e.a.b.-ideals

In this section we give the promised modification of the e.a.b. condition motivated by the definitions of the Nagata rings and Kronecker function rings.

We begin by giving more general definitions of the Kronecker function ring and the Nagata ring.

Recall that, for general semistar operations, we can consider the notion of semistar invertible ideals.

Definition 3.1. A fractional ideal $I \in \boldsymbol{F}(D)$ is called *$\star$-invertible* if $(II^{-1})^\star = D^\star$.

For the motivations, examples and the basic properties of this type of invertibility see [13].

Definition 3.2. Let D be a domain with quotient field K and let $\star$ be a semistar operation on D. Set

$$\mathrm{Na}(D,\star) := \left\{ \frac{f}{g} \,\middle|\, f,g \in D[X], \boldsymbol{c}(f)^{\star} \subseteq \boldsymbol{c}(g)^{\star}, \boldsymbol{c}(g) \text{ is a } \star_f\text{-invertible ideal of } D \right\}.$$

This is called *the* $\star$*-Nagata ring of* D. If the semistar operation is the identity operation, i.e. $\star = d_D$, then $\mathrm{Na}(D, d_D)$ coincides with the "classical" Nagata ring $D(X)$ of D.

It is known that, *for each* $E \in \bar{\boldsymbol{F}}(D)$,

$$E^{\tilde{\star}} = E\,\mathrm{Na}(D,\star) \cap K, \quad [11, \text{Proposition 3.4 (3)}].$$

The definition of an e.a.b. semistar operation is as follows.

Definition 3.3. Let D be a domain and let $\star$ be a semistar operation on D. Then we say that $\star$ *is an e.a.b. semistar operation* provided $(IJ)^{\star} \subseteq (IH)^{\star}$ implies $J^{\star} \subseteq H^{\star}$ whenever $I, J, H \in \boldsymbol{f}(D)$. (We say that $\star$ is *an a.b. semistar operation* if we weaken the hypotheses to only require that J, H lie in $\bar{\boldsymbol{F}}(D)$.)

Remark 3.4. (1) The paper [12, in preparation] is devoted to a deeper study on the relations between the e.a.b. and the a.b. semistar operations.

(2) Recall that *the e.a.b. semistar operation on* D *of finite type* $\star_a$ *associated to a semistar operation* $\star$ is defined on D by setting for each $G \in \boldsymbol{f}(D)$:

$$G^{\star_a} := \bigcup\{((GH)^{\star} : H^{\star}) \mid H \in \boldsymbol{f}(D)\}$$

[18], [15] and [16]. It is known that a semistar operation of finite type is e.a.b. if and only if $\star = \star_a$ (cf., for instance, [9, Proposition 4.5]).

Let D be an integrally closed domain with quotient field K and let $\star$ be an e.a.b. semistar operation on D. Set

$$\mathrm{Kr}(D,\star) := \left\{ \frac{f}{g} \,\middle|\, f,g \in D[X], g \neq 0, \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)^{\star} \right\}.$$

In the case where $\star$ is an e.a.b. *star* operation on (an integrally closed domain) D this definition yields the classical Krull's extension of the Kronecker function ring of D associated with $\star$ [14, Section 32].

This is not the most general definition of the Kronecker function ring, but it is the one most suited to our program. Recall that the reason customarily given for the assumption that $\star$ be e.a.b. in Krull's definition of the Kronecker function ring is that the classical proof that $\mathrm{Kr}(D)$ is a ring does not work unless $\star$ is assumed to be e.a.b. The use that the e.a.b. property is put to in this context is to show that an equation of the form $(c(g)J)^\star = (c(g)H)^\star$ implies that $J^\star = H^\star$ where $J, H \in \boldsymbol{f}(D)$ and g is a nonzero polynomial in $D[X]$ and the denominator of a rational function $\frac{f}{g}$ in $\mathrm{Kr}(D, \star)$.

A general definition for the Kronecker function ring, without restrictions on D and $\star$, is recalled next.

Definition 3.5. Let D be any integral domain (not necessarily integrally closed) with quotient field K and let $\star$ be any semistar operation on D (not necessarily e.a.b.). Set

$$\mathrm{Kr}(D, \star) := \left\{ \frac{f}{g} \,\middle|\, \begin{array}{rl} f, g \in D[X],\ g \neq 0, & \text{such that there exists } h \in D[X], \\ h \neq 0, & \text{with } \boldsymbol{c}(fh) \subseteq \boldsymbol{c}(gh)^\star \end{array} \right\}.$$

This is called *the $\star$-Kronecker function ring of D*, [9, Theorem 5.1]. Obviously if $\star$ is an e.a.b. semistar operation on D, then this definition coincides with the previous one.

It is known that, *for each $E \in \overline{\boldsymbol{F}}(D)$,*

$$E^{\star_a} = E\,\mathrm{Kr}(D, \star) \cap K, \quad [11, \text{Proposition 4.1 (5)}].$$

Since invertible ideals can be cancelled in any conceivable context, it is clear that the "modified" e.a.b. property, that we want to introduce for generalizing semistar Nagata rings and Kronecker function rings, should be a "cancellation-type" property.

The definition is actually quite straightforward.

Definition 3.6. Let D be an integral domain and let $\star$ be a semistar operation on D. If F is in $\boldsymbol{f}(D)$, we say that *F is a $\star$-e.a.b.-ideal* if $(FG)^\star \subseteq (FH)^\star$, with $G, H \in \boldsymbol{f}(D)$, implies that $G^\star \subseteq H^\star$. As with the semistar operations, we say that *F is an a.b.-ideal* if the conclusion holds with the requirement weakened to say that $G, H \in \overline{\boldsymbol{F}}(D)$.

It is clear that *a semistar operation $\star$ on a domain D is e.a.b. if and only if every finitely generated ideal of D is a $\star$-e.a.b. ideal.*

Remark 3.7. It is clear that *invertible ideals are $\star$-e.a.b. for any semistar operation $\star$.* In fact, if the semistar operation in question is the identity operation d, the d-e.a.b. ideals of a domain D correspond to what D. D. Anderson and D. F. Anderson called quasi-cancellation ideals. They proved that, in the finitely generated setting, the d-e.a.b. ideals are exactly the invertible ideals [2, Lemma 1 and Theorem 1].

It is easy to see that, in general, *a finitely generated $\star$-invertible ideal is also $\star$-e.a.b.* Unlike the case for the identity operation though (Remark 3.7), it is not true in general that all (possibly, finitely generated) $\star$-e.a.b. ideals are $\star$-invertible. For instance, let D be a Noetherian domain of dimension greater than one and let M be a maximal ideal of D of height greater than one. Since the b-operation is an e.a.b. star operation on D, then M (which is finitely generated) is a b-e.a.b. ideal, by the observation preceding Remark 3.7. However, since M has height greater than one, it is not an invertible ideal of D. Hence $MM^{-1} = M$. Then $(MM^{-1})^b = M^b = M$. Hence, M is not b-invertible.

We close this section with a collection of basic results concerning $\star$-e.a.b. ideals, invertible ideals, and $\star$-invertible ideals.

It is known that *if $\star$ is an e.a.b. star operation on an integral domain D, then there exists an a.b. star operation $*$ on D such that* $\star|_{\boldsymbol{f}(D)} = *|_{\boldsymbol{f}(D)}$ [14, Corollary 32.13]. This motivates our next statement proven in [12].

Lemma 3.8. *Let D be an integral domain and let $\star$ be a semistar operation on D.*

(1) *If $\star = \star_f$, then: $\star$ is an e.a.b. semistar operation if and only if $\star$ is an a.b. semistar operation.*

(2) *Let $F \in \boldsymbol{f}(D)$, then: F is $\star$-e.a.b. if and only if F is $\star_f$-a.b.* □

The following result is known [13, Theorem 2.23].

Lemma 3.9. *Let $\star$ be a semistar operation on an integral domain D. Let $F \in \boldsymbol{f}(D)$, then the following are equivalent:*

(i) *F is $\star_f$-invertible;*
(ii) *FD_Q is invertible as a fractional ideal of D_Q, for each $Q \in \mathcal{M}(\star_f)$;*
(iii) *$F\,\mathrm{Na}(D, \star)$ is invertible as a fractional ideal of* $\mathrm{Na}(D, \star)$. □

Remark 3.10. (1) Let $F \in \boldsymbol{f}(D)$. As a consequence of Lemma 3.9, note that, since $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$ [11, Corollary 3.5] (2), then: *F is $\star_f$-invertible if and only if F is $\tilde{\star}$-invertible*, cf. [13, Proposition 2.18].

(2) Let $F \in \boldsymbol{f}(D)$. From [12] recall that *F is $\star$-e.a.b. [respectively, $\star$-a.b.] if and only if $((FH)^\star : F^\star) = H^\star$, for each $H \in \boldsymbol{f}(D)$ [respectively, for each $H \in \bar{\boldsymbol{F}}(D)$].* (Note that $((FH)^\star : F^\star) = ((FH)^\star : F)$, and so the previous equivalences can be stated in a formally slightly different way.)

4 Some distinguished classes of overrings

We begin by considering a class of overrings of a domain D associated with a semistar operation $\star$ which have already been well studied.

Definition 4.1. Let $\star$ be a semistar operation on an integral domain D. We say that an overring T of D is a *$\star$-overring* of D provided for each $F \in \boldsymbol{f}(D)$ we have $F^\star \subseteq FT$ (or equivalently $F^\star T = FT$).

The following lemma gives some basic results concerning $\star$-overrings.

Lemma 4.2. *Let D be an integral domain with quotient field K and let $\star$ be a semistar operation on D.*

(1) *The following are equivalent:*
 (i) *T is a $\star$-overring of D;*
 (ii) *T is a $\star_f$-overring of D;*
 (iii) *$\star_f \leq \star_{\{T\}}$, (i.e. $E^{\star_f} \subseteq ET$, $\forall E \in \bar{\boldsymbol{F}}(D)$);*
 (iv) *$(\star_f)_\iota = d_T$.*

In particular, if T is a $\star$-overring of D, then $D^\star \subseteq T^{\star_f} = T$.

(2) *Every overring of a $\star$-overring is a $\star$-overring.*

(3) *K is a $\star$-overring of D, for each semistar operation $\star$ on D.*

(4) *$D^\star$ is a $\star$-overring of D if and only if $\star_f = \star_{\{D^\star\}}$. More generally, T is a $\star$-overring of D and $T = D^\star$ if and only if $\star_f = \star_{\{T\}}$.*

(5) *If $\star_1 \leq \star_2$ are two semistar operations on D, then: T is a $\star_2$-overring of D implies T is a $\star_1$-overring of D.*

(6) *A Bézout overring B of D is a $\star$-overring of D if and only if $B = B^{\star_f}$. In particular, a valuation overring V of D is a $\star$-overring of D if and only if $V = V^{\star_f}$ (in this situation, V is called a* $\star$-valuation overring of D*).*

(7) *The valuation overrings of a $\star$-overring T of an integral domain D coincide with the $\star$-valuation overrings of D containing T.*

(8) *Let T be a $\star$-overring of D and let $\iota : D \hookrightarrow T$ be the canonical inclusion. Then:* $\mathrm{Kr}(D, \star_{\{T\}}) = \mathrm{Kr}(T, d_T) = \mathrm{Kr}(T, (\star_f)_\iota) = \mathrm{Kr}(T, b_T)$.

(9) *If N is a prime ideal of a $\star$-overring T of D and if $N \cap D \neq 0$, then $N \cap D$ is a quasi-$\star_f$-prime of D.*

(10) *Let $\mathscr{T} := \{T_\lambda \mid \lambda \in \Lambda\}$ be a family of overrings of D. The semistar operation $\bigwedge_{\mathscr{T}}$ (defined in Section 2) is such that: $ET_\lambda = E^{\wedge_{\mathscr{T}}} T_\lambda = (ET_\lambda)^{\wedge_{\mathscr{T}}}$, for each $E \in \bar{\boldsymbol{F}}(D)$. In particular, each T_λ is a $\bigwedge_{\mathscr{T}}$-overring of D.*

Proof. (1) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) are obvious consequences of the definition.

(iii) $\Rightarrow$ (iv): $d_T \leq (\star_f)_\iota \leq (\star_{\{T\}})_\iota = d_T$.

(iv) $\Rightarrow$ (iii): For each $E \in \bar{\boldsymbol{F}}(D)$, $ET = (ET)^{(\star_f)_\iota} = (ET)^{\star_f} \supseteq E^{\star_f}$.

(2) is an easy consequence of (1).

(3) and (5) are obvious.

(4) follows from (1) and from the fact that, in general, $\star_{\{D^\star\}} \leq \star_f$.

(6) The "only if" part is obvious from (1). For the "if" part recall that, for each $F \in \boldsymbol{f}(D)$, there exists a nonzero element $x \in K$ such that $FB = xB$, thus $F^\star \subseteq (FB)^{\star_f} = (xB)^{\star_f} = xB^{\star_f} = xB = FB$.

(7) follows from (2) and (6).

(8) Since T is a $\star$-overring of D, then, by (1), $(\star_f)_\iota = d_T$. Therefore, by (7), V is a (d_T-)valuation overring of T if and only if V is a $\star$-valuation overring of D

containing T if and only if V is a $\star_{\{T\}}$-valuation overring of D. The conclusion is a straightforward consequence of the fact that, if $*$ is a semistar operation on an integral domain R, then $\mathrm{Kr}(R, *) = \bigcap\{W(X) \mid W$ is a $*$-valuation overring of $R\}$ [10, Theorem 3.5].

(9) $(N \cap D)^{\star_f} \subseteq (N \cap D)T \subseteq N$, hence $N \cap D \subseteq (N \cap D)^{\star_f} \cap D \subseteq N \cap D$.

(10) Note that $(ET_\lambda)^{\wedge_{\mathscr{T}}} = \bigcap\{ET_\lambda T_\mu \mid \mu \in \Lambda\} = ET_\lambda \cap (\bigcap\{ET_\lambda T_\mu \mid \mu \in \Lambda, \mu \neq \lambda\}) = ET_\lambda$. $\square$

Corollary 4.3. *Let $\star$ be a semistar operation on an integral domain D, let $F \in \boldsymbol{f}(D)$ be $\star_f$-invertible and let (L, N) be a local $\star$-overring of D. Then FL is a principal fractional ideal of L.*

Proof. Recall that an invertible ideal in a local domain is principal [20, Theorem 59]. By Lemma 3.9, we know that FD_Q is principal in D_Q, for each $Q \in \mathscr{M}(\star_f)$. Hence, $FD_{N\cap D}$ is principal in $D_{N\cap D}$, because $N \cap D$ is a quasi-$\star_f$-prime ideal of D (Lemma 4.2 (9)), hence $N \cap D \subseteq Q$, for some $Q \in \mathscr{M}(\star_f)$. The conclusion follows immediately, since (L, N) dominates $(D_{N\cap D}, N \cap D)$. $\square$

Remark 4.4. Let D be an integral domain and let $\star$ be a semistar operation on D.

(1) An overring T of D such that $T = T^{\star_f}$ is not necessarily a $\star$-overring of D. For instance, let $\mathscr{F}$ be a localizing system of D, and let $\star := \star_{\mathscr{F}}$ be the semistar operation on D, defined by $E^{\star_{\mathscr{F}}} := E_{\mathscr{F}} := \bigcup\{(E : I) \mid I \in \mathscr{F}\}$, for each $E \in \bar{\boldsymbol{F}}(D)$ (cf. [7, Proposition 2.4]). Set $T := D^{\star_{\mathscr{F}}} = D_{\mathscr{F}}$, then in general, $ED_{\mathscr{F}} \subsetneq E_{\mathscr{F}}$. More precisely, $\star_{\{D^{\star_{\mathscr{F}}}\}} = \star_{\mathscr{F}}$ if and only if $D^{\star_{\mathscr{F}}}$ is D-flat and $\mathscr{F} = \{I$ ideal of $D \mid ID^{\star_{\mathscr{F}}} = D^{\star_{\mathscr{F}}}\}$ (cf. [7, Proposition 2.6]).

(2) We have mentioned in the proof of Lemma 4.2 (8) that:

$$\mathrm{Kr}(D, \star) = \bigcap\{V(X) \mid V \text{ is a } \star\text{-valuation overring of } D\} \quad [10, \text{Theorem } 3.5].$$

From this property and from the observation following Definition 3.5 it is possible to prove that:

$$\star_a = \wedge_{\mathscr{V}(D,\star)}, \quad \text{where } \mathscr{V}(D, \star) := \{V \mid V \text{ is a } \star\text{-valuation overring of } D\}.$$

In the introduction we alluded to classes of quasilocal overrings of a domain D that are associated with a given semistar operation on D. The two classes of quasilocal overrings that arise in the Kronecker and Nagata settings are localizations and valuation overrings. Note first that a finitely generated ideal of a domain D is invertible if and only if it is locally principal. Also note that every finitely generated ideal of D extends to a principal ideal in any valuation overring of D. Since the collection of ideals we have been concerned with are the $\star$-e.a.b. ideals it seems reasonable that what we need are quasilocal overrings of D in which each $\star$-e.a.b. ideal extends to a principal ideal. It turns out that assuming just this property is not quite sufficient. We give two refinements, each of which we will use to create separate generalized function rings.

Definition 4.5. Let $\star$ be a semistar operation on an integral domain D and let T be a quasilocal overring of D.

We say that T is *a $\star$-monolocality of D* provided $T^{\star_f} = T$ and every $\star$-e.a.b. ideal of D extends to a principal ideal in T.

We say that T is *a strong $\star$-monolocality of D* provided T is a $\star$-overring of D and every $\star$-e.a.b. ideal of D extends to a principal ideal in T.

Let $\star$ be a semistar operation on a domain D and let T be a $\star$-overring of D. Then $T^{\star_f} = T$. It follows then that *a strong $\star$-monolocality is a $\star$-monolocality.*

Also note that if $\star$ is the identity semistar operation on a domain D then every quasilocal overring of D is a strong $\star$-monolocality (and hence they are all $\star$-monolocalities). At the opposite extreme, if $\star$ is the b-operation on a domain D, then the collection of all strong $\star$-monolocalities and the collection of all $\star$-monolocalities are both equal to the collection of all valuation overrings of D.

Our goal for this concept then is to identify semistar operations such that the collection of $\star$-monolocalities (strong or not) is not all quasilocal overrings and does not consist entirely of valuation domains. Or, conversely, to identify collections of overrings and associate semistar operations, using the $\bigwedge$-constructions, which will give the collection of overrings back as (strong) $\star$-monolocalities.

We adopt the following notation. Set:

$$\mathscr{L} := \mathscr{L}(\star) := \mathscr{L}(D, \star) := \{L \mid L \text{ is a } \star\text{-monolocality of } D\},$$

$$\mathscr{L}' := \mathscr{L}'(\star) := \mathscr{L}'(D, \star) := \{L' \mid L' \text{ is a strong-}\star\text{-monolocality of } D\}.$$

Note that the set: $\mathscr{V} := \mathscr{V}(\star) := \mathscr{V}(D, \star)$ $(= \{V \mid V \text{ is a } \star\text{-valuation overring of } D\})$ is obviously a subset of $\mathscr{L}'$.

Lemma 4.6. *Let $\star$ be a semistar operation on an integral domain D.*

(1) $\mathscr{L}(\star) = \mathscr{L}(\star_f)$ *and* $\mathscr{L}'(\star) = \mathscr{L}'(\star_f)$.

(2) *A quasilocal overring T of a $\star$-monolocality of D is also a $\star$-monolocality of D if and only if $T = T^{\star_f}$.*

(3) *A quasilocal overring S of a strong-$\star$-monolocality of D is always a strong-$\star$-monolocality of D.*

(4) *Let $F := (a_1, a_2, \dots, a_n)D \in \boldsymbol{f}(D)$ be $\star$-e.a.b. and let L be a $\star$-monolocality of D. Then $FL = F^\star L = (FL)^{\star_f} = a_i L$, for some i, with $1 \leq i \leq n$.*

Proof. (1) is obvious and (2) and (3) follow from Lemma 4.2 (1) and (2).

(4) We start by recalling the following well known fact:

Claim. *Let $F = (a_1, a_2, \dots, a_n)D \in \boldsymbol{f}(D)$ and let L be a quasilocal overring of D. If FL is principal in L, then, for some i, with $1 \le i \le n$, $FL = a_i L$.*

If $FL = (a_1, a_2, \dots, a_n)L = zL$, for some $z \in L$ then, for each i, with $1 \le i \le n$, we can find a nonzero $x_i \in L$, such that $x_i z = a_i$. Therefore, $zL = (a_1, a_2, \dots, a_n)L = (x_1, x_2, \dots, x_n)L \cdot zL$, hence $(x_1, x_2, \dots, x_n)L = L$. Since L is quasilocal then, for some i, with $1 \le i \le n$, we have that x_i is a unit in L, thus $zL = x_i zL = a_i L$.

Now we conclude the proof of (4). Since F is $\star$-e.a.b. and $L \in \mathscr{L}$, then FL is principal, and thus, for some i, with $1 \le i \le n$, $(FL)^{\star_f} = (a_i L)^{\star_f} = a_i L^{\star_f} = a_i L$ (since $L = L^{\star_f}$). Therefore $a_i L \subseteq FL \subseteq F^\star L \subseteq (FL)^{\star_f} = (a_i L)^{\star_f} = a_i L$. □

5 Generalized Kronecker-Nagata rings

Now we turn to the construction of the generalized Kronecker and Nagata rings. We define two classes of rings which we refer to as *Kronecker-Nagata ring* (for short, KN) and *Strong Kronecker-Nagata ring* (for short, KN$'$) according to whether we use monolocalities or strong monolocalities.

Proposition 5.1. *Let $\star$ be a semistar operation on an integral domain D. Set:*

$$\mathrm{KN}(D, \star) := \bigcap \{ L(X) \mid L \in \mathscr{L}(D, \star) \},$$

$$\mathrm{KN}'(D, \star) := \bigcap \{ L'(X) \mid L' \in \mathscr{L}'(D, \star) \},$$

then:

$$\mathrm{Na}(D, \star) \subseteq \mathrm{KN}'(D, \star) \subseteq \mathrm{Kr}(D, \star),$$

$$\mathrm{KN}(D, \star) \subseteq \mathrm{KN}'(D, \star).$$

Proof. By Lemma 4.2 (9) and [11, Proposition 3.1 (4)], we know that $\mathrm{Na}(D, \star) = \bigcap \{ D_Q(X) \mid Q \in \mathrm{QSpec}^{\star_f}(D) \} \subseteq \bigcap \{ D_{N' \cap D}(X) \mid (L', N') \in \mathscr{L}' \} \subseteq \mathrm{KN}'(D, \star)$. The inclusions $\mathrm{KN}(D, \star) \subseteq \mathrm{KN}'(D, \star) \subseteq \mathrm{Kr}(D, \star)$ follow from the fact that $\mathscr{V}(D, \star) \subseteq \mathscr{L}'(D, \star) \subseteq \mathscr{L}(D, \star)$. □

We have shown that the Strong $\star$-Kronecker-Nagata ring $\mathrm{KN}'(D, \star)$ lies properly in between the $\star$-Nagata ring and the $\star$-Kronecker function ring. We have also shown that the $\star$-Kronecker-Nagata ring $\mathrm{KN}(D, \star)$ lies inside the $\star$-Kronecker function ring. We will show later (Theorem 5.11 (7)) that, in general, $\mathrm{Na}(D, \star) \subseteq \mathrm{KN}(D, \star)$.

Proposition 5.1 gives a positive result concerning containment relations for KN and KN$'$. The containment/inequality relations between these concepts is not always as clean as we would like however. For example, it seems reasonable that if $\star_1 \le \star_2$ are semistar operations on a domain D then we would have $\mathrm{KN}(D, \star_1) \subseteq \mathrm{KN}(D, \star_2)$ and $\mathrm{KN}'(D, \star_1) \subseteq \mathrm{KN}'(D, \star_2)$. In Example 7.7 we give an example of a star operation $\star$

on a two-dimensional Noetherian local integrally closed domain D such that $b < \star$ and yet $\mathrm{KN}(D, \star) = \mathrm{Na}(D, \star) = D(Z) \subsetneq \mathrm{KN}(D, b) = \mathrm{Kr}(D, b)$. Hence, in general, we do not get the containment we wish for KN. We do not know whether KN′ behaves well with regard to containment and inequality or not. We can give a positive result in this direction when $\star_2$ is a stable semistar operation.

Recall that *a semistar operation* $\star$ is *stable on D* provided

$$(E \cap F)^\star = E^\star \cap F^\star, \quad \text{for all } E, F \in \bar{\boldsymbol{F}}(D).$$

Corollary 5.2. *Let $\star_1 \leq \star_2$ be two semistar operations on an integral domain D. For $i = 1, 2$, set:*

$$\mathscr{L}_i := \mathscr{L}(D, \star_i), \quad \mathscr{L}_i' := \mathscr{L}'(D, \star_i).$$

Assume that $\star_2$ is stable.

(1) *Let $F \in \boldsymbol{f}(D)$. If F is $\star_1$-e.a.b. then F is also $\star_2$-e.a.b.*

(2) $\mathscr{L}_1 \supseteq \mathscr{L}_2$ *and* $\mathscr{L}_1' \supseteq \mathscr{L}_2'$.

(3) $\mathrm{KN}(D, \star_1) \subseteq \mathrm{KN}(D, \star_2)$ *and* $\mathrm{KN}'(D, \star_1) \subseteq \mathrm{KN}'(D, \star_2)$.

Proof. (1) is a consequence of Remark 3.10 (2), since:

$$((FH)^{\star_1} : F) = H^{\star_1} \Rightarrow ((FH)^{\star_1} : F)^{\star_2} = (H^{\star_1})^{\star_2},$$

therefore, by the stability of $\star_2$ [7, Theorem 2.10 (B)]:

$$H^{\star_2} \subseteq ((FH)^{\star_2} : F) = (((FH)^{\star_1})^{\star_2} : F) = (((FH)^{\star_1} : F))^{\star_2} = H^{\star_2},$$

for each $H \in \boldsymbol{f}(D)$.

(2) follows from (1), from Lemma 4.2 (5) and from the fact that, if T is a quasilocal overring of D such that $T = T^{(\star_2)_f}$, then necessarily $T = T^{(\star_1)_f}$.

(3) is a trivial consequence of (2). □

We have noted that the KN and KN′ constructions are not always well behaved in terms of preserving relationships between distinct semistar operations. Nonetheless, it seems worthwhile to pursue this idea with regards to the operations $\star_a$ and $\tilde{\star}$ associated to the semistar operations $\star$ on a domain D. The semistar operations $\star_a$ and $\tilde{\star}$ are generally well behaved and the results work out as we would hope.

We need a preparatory lemma first.

Proposition 5.3. *Let $\star$ be a semistar operation on an integral domain D. Then:*

(1) *For each $Q \in \mathscr{M}(\star_f)$, D_Q is strong-$\tilde{\star}$-monolocality of D.*

(2) *If $F \in \boldsymbol{f}(D)$, then F is $\tilde{\star}$-e.a.b. if and only if F is $\tilde{\star}$-invertible.*

Proof. (1) It is clear from the definition of the semistar operation $\tilde{\star}$ that, for each $Q \in \mathcal{M}(\star_f)$ $(= \mathcal{M}(\tilde{\star})$, [11, Corollary 3.5]), D_Q is a quasilocal $\tilde{\star}$-overring of D since, for each $F \in \boldsymbol{f}(D)$, $F^{\tilde{\star}} \subseteq FD_Q$.

Note, more generally, that for each $Q \in \mathcal{M}(\star_f)$ and for each $E \in \overline{\boldsymbol{F}}(D)$:

$$ED_Q = E^{\tilde{\star}}D_Q = (ED_Q)^{\tilde{\star}},$$

(Lemma 4.2 (10), since $\tilde{\star} = \bigwedge_{\mathcal{S}}$, with $\mathcal{S} := \{D_Q \mid Q \in \mathcal{M}(\star_f)\}$).

(2) Let $F \in \boldsymbol{f}(D)$ be $\tilde{\star}$-e.a.b., thus $((FH)^{\tilde{\star}} : F) = H^{\tilde{\star}}$ and so:

$$HD_Q = H^{\tilde{\star}}D_Q = ((FH)^{\tilde{\star}} : F)D_Q = ((FH)^{\tilde{\star}}D_Q : FD_Q) = (FHD_Q : FD_Q),$$

for each $H \in \boldsymbol{f}(D)$, i.e. FD_Q is a quasi-cancellation ideal of D_Q or, equivalently, it is a principal fractional ideal of D_Q, for each $Q \in \mathcal{M}(\star_f)$ (Remark 3.7).

From Lemma 3.9 and Remarks 3.10 (1) and 3.7, we deduce that if $F \in \boldsymbol{f}(D)$, then F is $\tilde{\star}$-e.a.b. if and only if F is $\tilde{\star}$-invertible. □

Proposition 5.4. *Let $\star$ be a semistar operation on an integral domain D, then:*

(1) $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \tilde{\star}) = \mathrm{KN}'(D, \tilde{\star})$.

(2) $\mathrm{KN}'(D, \star_a) = \mathrm{Kr}(D, \star_a) = \mathrm{Kr}(D, \star)$.

Proof. (1) By Proposition 5.3 we know that $\mathcal{L}'(D, \tilde{\star}) \supseteq \{D_Q \mid Q \in \mathcal{M}(\star_f)\}$ and if $(L', N') \in \mathcal{L}'(D, \tilde{\star})$ then $L' \supseteq D_{N' \cap D} \supseteq D_Q$, where Q is any prime ideal in $\mathcal{M}(\star_f)$ $(= \mathcal{M}(\tilde{\star}))$ which contains the quasi-$\tilde{\star}$-prime ideal $N' \cap D$.

(2) If $\star = \star_f$ is e.a.b., then each $F \in \boldsymbol{f}(D)$ is $\star$-e.a.b., thus every quasilocal $\star$-overring (in particular, a strong-$\star$-monolocality) is necessarily a valuation domain, hence $\mathcal{L}'(D, \star) = \mathcal{V}(D, \star)$. Therefore, in this situation, $\mathrm{KN}'(D, \star) = \mathrm{Kr}(D, \star)$. Using the previous argument (and [11, Proposition 4.1 (2)]), for each semistar operation $\star$, passing to the e.a.b. semistar operation of finite type $\star_a$, we have:

$$\mathrm{KN}'(D, \star_a) = \mathrm{Kr}(D, \star_a) = \mathrm{Kr}(D, \star). \qquad \square$$

Corollary 5.5. *Let $\star$ be a semistar operation on an integral domain D. Then D is a P$\star$MD if and only if* $\mathrm{KN}'(D, \tilde{\star}) = \mathrm{KN}'(D, \star_a)$.

Proof. This statement is a straightforward consequence of Proposition 5.4 and [8, Theorem 3.1 and Remark 3.1]. □

We have defined and done some analysis on the generalized Kronecker-Nagata rings using the (strong) monolocalities. This was motivated by characterizations of the classical Kronecker and Nagata rings. Both the Kronecker and Nagata rings have definitions involving rational functions, content ideals, and semistar operations. We turn toward generalizing along these lines now.

Proposition 5.6. *Let $\star$ be a semistar operation on an integral domain D with quotient field K and let $\mathscr{L}' = \mathscr{L}'(D,\star)$ be the set of all the strong-$\star$-monolocalities of D. Set:*

$$\mathrm{KN}'_{\#}(D,\star) := \{z \in K(X) \mid \forall L' \in \mathscr{L}', \exists g_{L'} \in D[X], g_{L'} \neq 0, \textit{ with } zg_{L'} \in D[X],\ \boldsymbol{c}(zg_{L'}) \subseteq \boldsymbol{c}(g_{L'})L' \textit{ and } \boldsymbol{c}(g_{L'})L' \textit{ is principal in } L'\}.$$

Then $\mathrm{KN}'(D,\star) = \mathrm{KN}'_{\#}(D,\star)$.

Proof. Let $z \in \mathrm{KN}'_{\#}(D,\star)$, let $L' \in \mathscr{L}'$ and let $g_{L'} \in D[X]$ be such that $\boldsymbol{c}(g_{L'})L'$ is a nonzero principal ideal of L' and $c(zg_{L'}) \subseteq c(g_{L'})L'$. Set $f_{L'} := zg_{L'}$. Write $g_{L'} := a_0 + a_1X + \cdots + a_nX^n \in D[X]$. Since L' is a (strong-)$\star$-monolocality of D and $\boldsymbol{c}(g_{L'})L'$ is principal then, by the Claim in the proof of Lemma 4.6 (4), we have $\boldsymbol{c}(g_{L'})L' = a_iL'$ for some a_i. Hence, $\frac{f_{L'}}{a_i} \in L'[X]$ and $\frac{g_{L'}}{a_i} \in L'[X]$. Moreover, $\frac{g_{L'}}{a_i}$ is a primitive polynomial of $L'[X]$ (since one of its coefficients is a unit in L'), hence:

$$z = \frac{f_{L'}}{g_{L'}} = \frac{\frac{f_{L'}}{a_i}}{\frac{g_{L'}}{a_i}} \in L'(X).$$

Therefore, we have proven that $\mathrm{KN}'_{\#}(D,\star) \subseteq \mathrm{KN}'(D,\star)$.

In order to complete the proof, we need to show that $\mathrm{KN}'(D,\star) \subseteq \mathrm{KN}'_{\#}(D,\star)$. If $z \in \mathrm{KN}'(D,\star)$, then, for each strong-$\star$-monolocality L' of D, there exist $\varphi_{L'}, \psi_{L'} \in L'[X]$, with $\psi_{L'} \neq 0$, such that $z = \frac{\varphi_{L'}}{\psi_{L'}}$ and $\boldsymbol{c}(\psi_{L'}) = L'$. Therefore, we can find $f_{L'}, g_{L'} \in D[X]$ and two nonzero elements $\alpha_{L'}, \beta_{L'} \in D$ such that $f_{L'} = \alpha_{L'}\varphi_{L'}$, $g_{L'} = \beta_{L'}\psi_{L'}$ and thus:

$$z = \frac{\varphi_{L'}}{\psi_{L'}} = \frac{\frac{f_{L'}}{\alpha_{L'}}}{\frac{g_{L'}}{\beta_{L'}}} = \frac{\beta_{L'}f_{L'}}{\alpha_{L'}g_{L'}}$$

with $\boldsymbol{c}(f_{L'})L' \subseteq \alpha_{L'}L'$ and $\boldsymbol{c}(g_{L'})L' = \beta_{L'}L'$. Therefore, $z \in \mathrm{KN}'_{\#}(D,\star)$, since $\boldsymbol{c}(\beta_{L'}f_{L'})L' = \beta_{L'}\boldsymbol{c}(f_{L'})L' \subseteq \beta_{L'}\alpha_{L'}L' = \alpha_{L'}\boldsymbol{c}(g_{L'})L' = \boldsymbol{c}(\alpha_{L'}g_{L'})L'$ and $\boldsymbol{c}(\alpha_{L'}g_{L'})L'$ is principal in L', for each $L' \in \mathscr{L}'$. $\square$

Note that it follows immediately from the definition that $D[X] \subseteq \mathrm{KN}'_{\#}(D,\star) \subseteq K(X)$. Hence the quotient field of $\mathrm{KN}'_{\#}(D,\star) = \mathrm{KN}'(D,\star)$ is $K(X)$.

Our next result gives a basic property of $\mathrm{KN}'(D,\star)$ reminiscent of Kronecker function ring and Nagata ring properties.

Proposition 5.7. *Let $\star$ be a semistar operation on an integral domain D. For each $J := (a_0, a_1, \ldots, a_n)D \in \boldsymbol{f}(D)$, with $J \subseteq D$ and J $\star$-e.a.b., let $g := a_0 + a_1X + \cdots + a_nX^n \in D[X]$, then:*

$$J\,\mathrm{KN}'(D,\star) = J^\star\,\mathrm{KN}'(D,\star) = g\,\mathrm{KN}'(D,\star).$$

Proof. First note that, by definition, $J = \boldsymbol{c}(g)D$. Moreover, for each k, with $0 \le k \le n$, we have $a_k/g \in \mathrm{KN}'_{\#}(D,\star)$ $(= \mathrm{KN}'(D,\star))$, since $\boldsymbol{c}(g)L'$ is principal in L', for each $L' \in \mathscr{L}'$. Hence $J\,\mathrm{KN}'(D,\star) \subseteq g\,\mathrm{KN}'(D,\star)$. Clearly, $g \in J\,\mathrm{KN}'(D,\star) = \boldsymbol{c}(g)\,\mathrm{KN}'(D,\star)$. It follows that $J\,\mathrm{KN}'(D,\star) = g\,\mathrm{KN}'(D,\star)$.

On the other hand, let $\alpha = d/d' \in J^\star$, with $d, d' \in D$ and $d' \neq 0$. Then, by Lemma 4.6 (4), for each (strong-)$\star$-monolocality L' of D we have $dD = d'\alpha D \subseteq d'\alpha L' \subseteq d'J^\star L' = d'JL' = d'\boldsymbol{c}(g)L'$, thus $\alpha/g = d'\alpha/d'g \in \mathrm{KN}'_{\#}(D,\star)$ $(= \mathrm{KN}'(D,\star))$, since $\boldsymbol{c}(g)L' = JL'$ is principal in L', for each $L' \in \mathscr{L}'$. Hence, we have that $J^\star \subseteq g\,\mathrm{KN}'(D,\star) = J\,\mathrm{KN}'(D,\star)$, thus $J^\star\,\mathrm{KN}'(D,\star) = J\,\mathrm{KN}'(D,\star)$. $\square$

The rational function definition of the strong Kronecker-Nagata ring is somewhat cumbersome. We introduce now the notion of an "almost e.a.b.-ideal" in an effort to make the definition cleaner.

Definition 5.8. Let $\star$ be a semistar operation on an integral domain D and let $F \in \boldsymbol{f}(D)$, we say that F is *an almost-$\star$-e.a.b.-ideal* if, for each strong-$\star$-monolocality L' of D, FL' is a principal fractional ideal of L'.

We collect in the following statement some of the basic properties of the almost-$\star$-e.a.b. ideals.

Proposition 5.9. *Let $\star$ be a semistar operation on an integral domain D and let $F \in \boldsymbol{f}(D)$*

(1) *If F is $\star$-e.a.b. then F is almost-$\star$-e.a.b.*

(2) *F is almost-$\star$-e.a.b. if and only if F is almost-$\star_f$-e.a.b.*

(3) *If F is almost-$\star$-e.a.b. then FL' is $\star_\iota$-e.a.b., for each strong-$\star$-monolocality L' of D, with $\iota\ (= \iota_{L'}) : D \hookrightarrow L'$ being the canonical embedding.*

(4) *Let $F := (a_1, a_2, \ldots, a_n)D$ be an almost-$\star$-e.a.b.-ideal, then, for each strong-$\star$-monolocality L' of D, $FL' = F^\star L' = (FL')^{\star_f} = a_i L'$, for some i, with $1 \le i \le n$.*

(1) and (2) are obvious since if F is $\star$-e.a.b., then FL' is principal, for each strong-$\star$-monolocality L' of D, and since the strong-$\star$-monolocalities coincide with the strong-$\star_f$-monolocalities.

(3) If $G, H \in \boldsymbol{f}(L')$ and $F \in \boldsymbol{f}(D)$ is an almost-$\star$-e.a.b., then $FL' = zL'$, for some nonzero element z, and thus $(FG)^{\star_\iota} = (FL'G)^\star \subseteq (FL'H)^\star = (FH)^{\star_\iota}$, then $(zL'G)^\star \subseteq (zL'H)^\star$. Hence, $G^{\star_\iota} = G^\star \subseteq H^\star = H^{\star_\iota}$.

(4) This statement is a consequence of the Claim in the proof of Lemma 4.6 (4). $\square$

This allows us to state a new definition.

Definition 5.10. Let $\star$ be a semistar operation on an integral domain D with quotient field K. Then we define

$$\mathrm{KN}'_{\boldsymbol{c}}(D,\star) := \{z \in K(X) \mid \exists g \in D[X], g \neq 0, \text{ with } zg \in D[X], \boldsymbol{c}(zg) \subseteq \boldsymbol{c}(g)^{\star} \text{ and } \boldsymbol{c}(g) \text{ is an almost-}\star\text{-e.a.b. ideal of } D\},$$

$$\mathrm{KN}_{\boldsymbol{c}}(D,\star) := \{z \in K(X) \mid \exists g \in D[X], g \neq 0, \text{ such that } zg \in D[X], \text{ and } \boldsymbol{c}(zg) \subseteq \boldsymbol{c}(g)^{\star}, \boldsymbol{c}(g) \text{ is a } \star\text{-e.a.b. ideal}\}.$$

It is clear that, in general, we have $\mathrm{KN}'_{\boldsymbol{c}}(D,\star) \subseteq \mathrm{KN}'(D,\star)$ $(= \mathrm{KN}'_{\#}(D,\star))$.

In fact, note that, for each $g \in D[X]$, $g \neq 0$, and for each $L' \in \mathscr{L}'$, $\boldsymbol{c}(g)^{\star}L' = \boldsymbol{c}(g)L'$; moreover, $\boldsymbol{c}(g)$ is an almost-$\star$-e.a.b. ideal of D if and only if (by definition) it is a principal ideal of L', for each $L' \in \mathscr{L}'$.

We suspect that in fact $\mathrm{KN}'_{\boldsymbol{c}}(D,\star) = \mathrm{KN}'(D,\star)$, but we do not have a proof. We do demonstrate below that $\mathrm{KN}_{\boldsymbol{c}}(D,\star) = \mathrm{KN}(D,\star)$.

We turn now to investigating the properties of $\mathrm{KN}(D,\star)$. With this ring we will have more luck demonstrating properties that reflect the classical properties of the Kronecker function rings and Nagata rings. In particular, when we localize a Kronecker function ring $\mathrm{Kr}(D,\star)$ at a maximal ideal we obtain $V(X)$ for some ($\star$-)valuation overring V of D. Similarly, when we localize a Nagata ring $\mathrm{Na}(D,\star)$ at a maximal ideal we obtain $D_Q(X)$ for some (quasi-$\star$-)prime ideal Q of D. We obtain similar results with $\mathrm{KN}(D,\star)$.

Theorem 5.11. *Let $\star$ be a semistar operation on an integral domain D with quotient field K.*

(1) $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$ *is an integral domain with quotient field $K(X)$.*

(2) $\mathrm{Na}(D,\star) \subseteq \mathrm{KN}_{\boldsymbol{c}}(D,\star) \subseteq \mathrm{KN}'_{\boldsymbol{c}}(D,\star)$ $(\subseteq \mathrm{KN}'_{\#}(D,\star) = \mathrm{KN}'(D,\star))$.

(3) *For each $J := (a_0, a_1, \dots, a_n)D \in \boldsymbol{f}(D)$, with $J \subseteq D$ and J $\star$-e.a.b., let $g := a_0 + a_1X + \cdots + a_nX^n \in D[X]$, then:*

$$J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star) = J^{\star}\,\mathrm{KN}_{\boldsymbol{c}}(D,\star) = g\,\mathrm{KN}_{\boldsymbol{c}}(D,\star).$$

(4) *For each prime ideal $\mathfrak{p}$ of $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$ and for each $J := (a_0, a_1, \dots, a_n)D \in \boldsymbol{f}(D)$, with $J \subseteq D$ and $J\star$-e.a.b., let $g := a_0 + a_1X + \cdots + a_nX^n \in D[X]$, then there exists an index i, with $0 \leq i \leq n$, such that:*

$$J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} = J^{\star}\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} = g\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} = a_i\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}}.$$

For each prime ideal $\mathfrak{p}$ of $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$, set $L(\mathfrak{p}) := \mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} \cap K$.

(5) *For each prime ideal $\mathfrak{p}$ of $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$, $L(\mathfrak{p})$ is a $\star$-monolocality of D (with maximal ideal $\mathscr{P} := \mathfrak{p}\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} \cap L(\mathfrak{p})$).*

(6) *For each prime ideal* $\mathfrak{p}$ *of* $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$, *the localization* $\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}}$ *coincides with the Nagata ring* $L(\mathfrak{p})(X)$ *(with maximal ideal* $\mathscr{P}(X) := \mathscr{P}L(\mathfrak{p})(X)$*) and* $\mathfrak{p}$ *coincides with* $\mathscr{P}(X) \cap \mathrm{KN}_{\boldsymbol{c}}(D,\star)$.

(7) *Every* $\star$*-monolocality of an integral domain D contains a minimal* $\star$*-monolocality of D. If we denote by* $\mathscr{L}(D,\star)_{min}$, *or simply by* $\mathscr{L}_{min}$, *the set of all the minimal* $\star$*-monolocalities of D, then* $\mathscr{L}(D,\star)_{min} = \{L(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}_{\boldsymbol{c}}(D,\star))\}$ *and*

$$\mathrm{KN}_{\boldsymbol{c}}(D,\star) = \mathrm{KN}(D,\star) = \bigcap\{L(X) \mid L \in \mathscr{L}_{min}\}.$$

In particular, $\mathrm{Na}(D,\star) \subseteq \mathrm{KN}(D,\star)$ $(\subseteq \mathrm{KN}'(D,\star) \subseteq \mathrm{Kr}(D,\star))$.

Proof. (1) Note that:

Claim 1. *If g and h are two nonzero polynomials of* $D[X]$ *and* $\boldsymbol{c}(g)$ *is a* $\star$*-e.a.b. ideal of D then* $(\boldsymbol{c}(g)\boldsymbol{c}(h))^\star = \boldsymbol{c}(gh)^\star$. *Furthermore, if* $\boldsymbol{c}(h)$ *is also a* $\star$*-e.a.b. ideal of D, then* $\boldsymbol{c}(gh)$ *is a* $\star$*-e.a.b. ideal of D.*

The previous claim is a straightforward consequence of the Dedekind-Mertens Lemma [14, Theorem 28.1] and of the definition of $\star$-e.a.b. ideal.

Let $z := f/g$, $z' := f'/g' \in \mathrm{KN}_{\boldsymbol{c}}(D,\star)$, with $\boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)^\star$, $\boldsymbol{c}(f') \subseteq \boldsymbol{c}(g')^\star$ and $\boldsymbol{c}(g)$ and $\boldsymbol{c}(g')$ $\star$-e.a.b. ideals of D. From Claim 1, we deduce immediately that $zz' = ff'/gg' \in \mathrm{KN}_{\boldsymbol{c}}(D,\star)$.

In order to see that $z - z'$ belongs to $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$, it is sufficient to observe that $z - z' = (fg' - f'g)/gg'$ and

$$\boldsymbol{c}(fg' - f'g) \subseteq \boldsymbol{c}(fg' - f'g)^\star \subseteq (\boldsymbol{c}(fg')^\star + \boldsymbol{c}(f'g)^\star)^\star \subseteq \boldsymbol{c}(gg')^\star.$$

Clearly, $D[X] \subseteq \mathrm{KN}_{\boldsymbol{c}}(D,\star) \subseteq K(X)$, hence the quotient field of $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$ is $K(X)$.

(2) To prove that $\mathrm{Na}(D,\star) \subseteq \mathrm{KN}_{\boldsymbol{c}}(D,\star)$, note that the definition of $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$ generalizes the definition of $\mathrm{Na}(D,\star)$ (Definition 3.2) by replacing $\star_f$-invertible ideals with the larger class of $\star$-e.a.b. ideals. The result is then clear. The second inclusion is an easy consequence of the fact that, if $\boldsymbol{c}(zg) \subseteq \boldsymbol{c}(g)^\star$ and $\boldsymbol{c}(g)$ is $\star$-e.a.b. then, for each $L \in \mathscr{L}$, $\boldsymbol{c}(g)^\star L = \boldsymbol{c}(g)L$ is a principal ideal of L (Lemma 4.6 (4)). Therefore we have $\mathrm{KN}_{\boldsymbol{c}}(D,\star) \subseteq \mathrm{KN}'_{\boldsymbol{c}}(D,\star)$ (Definition 5.8).

(3) *Mutatis mutandis* the proof ot the equality $J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star) = \boldsymbol{c}(g)\,\mathrm{KN}_{\boldsymbol{c}}(D,\star) = g\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)$ is analogous to the proof of Proposition 5.7.

More precisely, first, note that by definition $J = \boldsymbol{c}(g)D$. Moreover, for each k, with $0 \leq k \leq n$, we have $a_k/g \in \mathrm{KN}_{\boldsymbol{c}}(D,\star)$. Hence $J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star) \subseteq g\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)$. Clearly, $g \in \boldsymbol{c}(g)\,\mathrm{KN}_{\boldsymbol{c}}(D,\star) = J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)$, and so $J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star) = g\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)$.

On the other hand, let $\alpha := d/d' \in J^\star$, with $d, d' \in D$, $d' \neq 0$. Then $\alpha/g = d'\alpha/d'g = d/d'g \in \mathrm{KN}_{\boldsymbol{c}}(D,\star)$, since $dD \subseteq d'J^\star = d'\boldsymbol{c}(g)^\star$. Hence, $J^\star \subseteq g\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)$ $= J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)$ and so $J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star) = J^\star\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)$.

(4) All the equalities follow trivially from (3) except the last one, involving a_i. This equality holds because $\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}}$ is quasilocal and $g\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} = J\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} =$

$(a_0, a_1, \ldots, a_n)\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}}$ is principal. By a standard technique (Claim in the proof of Lemma 4.6 (4)), an invertible ideal of a quasilocal domain which is generated by a finite list of elements is actually generated by one of those elements.

(5) It is clear that $L(\mathfrak{p})$ is a quasilocal overring of D, with maximal ideal $\mathscr{P}$. In order to show that $L(\mathfrak{p})$ is a $\star$-monolocality of D, take $J := (a_0, a_1, \ldots, a_n)D$ which is a nonzero $\star$-e.a.b. ideal of D. It is clear that $a_k L(\mathfrak{p}) \subseteq J^\star L(\mathfrak{p})$, for each $0 \le k \le n$. Let $\alpha \in J^\star$. Since, by (4), $J^\star\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} = a_i\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}}$, for some i, then $\alpha/a_i \in \mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} \cap K = L(\mathfrak{p})$. Therefore, $J^\star \subseteq a_i L(\mathfrak{p}) \subseteq JL(\mathfrak{p}) \subseteq J^\star L(\mathfrak{p})$ and so $a_i L(\mathfrak{p}) = J^\star L(\mathfrak{p})$.

(6) We start by proving the following:

Claim 2. *Let L be a $\star$-monolocality of D, then $\mathrm{KN}_{\boldsymbol{c}}(D,\star) \subseteq L(X)$. In particular, $\mathrm{KN}_{\boldsymbol{c}}(D,\star) \subseteq \mathrm{KN}(D,\star)$.*

Let $f/g \in \mathrm{KN}_{\boldsymbol{c}}(D,\star)$ with $\boldsymbol{c}(g)$ a $\star$-e.a.b. ideal of D and $\boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)^\star$. Write $g := a_0 + a_1 X + \cdots + a_n X^n \in D[X]$. Since L is a $\star$-monolocality of D then, by Lemma 4.6 (4), we have:

$$\boldsymbol{c}(g)L = \boldsymbol{c}(g)^\star L = (\boldsymbol{c}(g)L)^{\star_f} = a_i L$$

for some a_i. Hence, $\frac{f}{a_i} \in L[X]$ and $\frac{g}{a_i} \in L[X]$. Moreover, $\frac{g}{a_i}$ is a primitive polynomial in $L[X]$ (since one of its coefficients is a unit in L). Hence

$$\frac{f}{g} = \frac{\frac{f}{a_i}}{\frac{g}{a_i}} \in L(X),$$

and so Claim 2 is proven.

Note that, by (5) and Claim 2, we have $L(\mathfrak{p})(X) \supseteq \mathrm{KN}_{\boldsymbol{c}}(D,\star)$. Note also that, since $L(\mathfrak{p}) \subseteq \mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}}$ and $X \in \mathrm{KN}_{\boldsymbol{c}}(D,\star)$, then $L(\mathfrak{p})[X] \subseteq \mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}}$ and hence $\mathscr{P}[X] = (\mathfrak{p}\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} \cap L(\mathfrak{p}))[X] = \mathfrak{p}\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} \cap (L(\mathfrak{p})[X])$, recalling that $\mathscr{P} = \mathfrak{p}\,\mathrm{KN}_{\boldsymbol{c}}(D,\star)_{\mathfrak{p}} \cap L(\mathfrak{p})$. Clearly, $\mathscr{P}(X) \cap \mathrm{KN}_{\boldsymbol{c}}(D,\star)$ is a proper prime ideal of $\mathrm{KN}_{\boldsymbol{c}}(D,\star)$.

Claim 3. *With the notation introduced above, $\mathscr{P}(X) \cap \mathrm{KN}_{\boldsymbol{c}}(D,\star) \subseteq \mathfrak{p}$.*

Let $\varphi \in \mathscr{P}(X) \cap \mathrm{KN}_{\boldsymbol{c}}(D,\star)$. Then, we can write $\varphi = h/k$ where $h, k \in L(\mathfrak{p})[X]$ and $h \in \mathscr{P}[X]$ and k is primitive in $L(\mathfrak{p})[X]$. We can also write $\varphi = f/g$ where $f, g \in D[X]$, $g \neq 0$, $\boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)^\star$ and $\boldsymbol{c}(g)$ is a $\star$-e.a.b. ideal of D. Since $L(\mathfrak{p})$ is a $\star$-monolocality of D (by (5)), it then follows from Lemma 4.6 (4) that g has a coefficient a_i such that $\boldsymbol{c}(g)L(\mathfrak{p}) = a_i L(\mathfrak{p})$ and, hence, $\boldsymbol{c}(f) \subseteq a_i L(\mathfrak{p})$. Then:

$$\frac{f}{g} = \frac{\frac{f}{a_i}}{\frac{g}{a_i}} = \frac{h}{k}$$

with $\frac{f}{a_i}, \frac{g}{a_i} \in L(\mathfrak{p})[X]$. Therefore

$$k\frac{f}{a_i} = h\frac{g}{a_i}.$$

Since k and $\frac{g}{a_i}$ are primitive in $L(\mathfrak{p})[X]$ and $h \in \mathscr{P}[X]$, then we must have $\frac{f}{a_i} \in \mathscr{P}[X] \subseteq \mathfrak{p}\,\mathrm{KN}_c(D,\star)_{\mathfrak{p}}$. Since $\frac{g}{a_i}$ is a unit in $\mathrm{KN}_c(D,\star)_{\mathfrak{p}}$ this implies that f/g $(= h/k = \varphi)$ belongs to $\mathfrak{p}\,\mathrm{KN}_c(D,\star)_{\mathfrak{p}} \cap \mathrm{KN}_c(D,\star) = \mathfrak{p}$.

We conclude the proof of (6). By Claim 3, the prime ideal $\mathfrak{p}' := \mathscr{P}(X) \cap \mathrm{KN}_c(D,\star)$ is contained in $\mathfrak{p}$, thus it is clear that:

$$L(\mathfrak{p})[X] \subseteq \mathrm{KN}_c(D,\star)_{\mathfrak{p}} \subseteq \mathrm{KN}_c(D,\star)_{\mathfrak{p}'} \subseteq L(\mathfrak{p})(X) = L(\mathfrak{p})[X]_{\mathscr{P}[X]}.$$

Moreover, $\mathfrak{p}\,\mathrm{KN}_c(D,\star)_{\mathfrak{p}} \cap (L(\mathfrak{p})[X]) = \mathscr{P}[X]$. Therefore,

$$L(\mathfrak{p})[X]_{\mathscr{P}[X]} = \mathrm{KN}_c(D,\star)_{\mathfrak{p}} = \mathrm{KN}_c(D,\star)_{\mathfrak{p}'} = L(\mathfrak{p})(X),$$

hence we deduce that $\mathfrak{p}' = \mathfrak{p}$.

(7) is an easy consequence of (5) and (6). In fact, let $L \in \mathscr{L}(D,\star)$ and let N be the maximal ideal of L. We know by Claim 2 that $L(X) \supseteq \mathrm{KN}_c(D,\star)$. Set $\mathfrak{n} := N(X) \cap \mathrm{KN}_c(D,\star)$. Let $\mathfrak{m}$ be a maximal ideal of $\mathrm{KN}_c(D,\star)$ containing the prime ideal $\mathfrak{n}$. Then, by (5) and (6), we know that:

$$L(\mathfrak{m})(X) = \mathrm{KN}_c(D,\star)_{\mathfrak{m}} \subseteq \mathrm{KN}_c(D,\star)_{\mathfrak{n}} \subseteq L(X).$$

This fact implies that $\mathscr{L}(D,\star)_{min} = \{L(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}_c(D,\star))\}$, and so:

$$\begin{aligned}
\mathrm{KN}_c(D,\star) &= \bigcap\{\mathrm{KN}_c(D,\star)_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}_c(D,\star))\} \\
&= \bigcap\{L(\mathfrak{m})(X) \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}_c(D,\star))\} \\
&= \bigcap\{L(X) \mid L \in \mathscr{L}(D,\star)_{min}\} \\
&= \bigcap\{L(X) \mid L \in \mathscr{L}(D,\star)\} = \mathrm{KN}(D,\star).
\end{aligned}$$

The last statement of (7) follows from (2). □

6 New semistar operations

Given a semistar operation $\star$ on a domain D we have associated two collections of overrings $\mathscr{L}$ $(= \mathscr{L}(D,\star))$ and $\mathscr{L}'$ $(= \mathscr{L}'(D,\star))$ and using these collections we have constructed two rings of rational functions $\mathrm{KN}(D,\star)$ and $\mathrm{KN}'(D,\star)$. We can use these two collections of overrings and two rings of rational functions to construct four new semistar operations associated to $\star$.

Definition 6.1. Let D be a domain with quotient field K and $\star$ a semistar operation on D. We define new semistar operations on D as follows. For each $E \in \bar{\boldsymbol{F}}(D)$,

(a) $\bigwedge_{\mathscr{L}'}$ *defined by* $E^{\wedge_{\mathscr{L}'}} =: \bigcap\{EL' \mid L' \in \mathscr{L}'\}$;
(b) $\bigwedge_{\mathscr{L}}$ *defined by* $E^{\wedge_{\mathscr{L}}} =: \bigcap\{EL \mid L \in \mathscr{L}\}$;
(c) $\star_\ell$ *defined by* $E^{\star_\ell} := E\,\mathrm{KN}(D,\star) \cap K$;
(d) $\star_{\ell'}$ *defined by* $E^{\star_{\ell'}} := E\,\mathrm{KN}'(D,\star) \cap K$.

Next we give some simple relations between these operations.

Proposition 6.2. *Let D be a domain and $\star$ a semistar operation on D. Then $\star_\ell$ and $\star_{\ell'}$ are semistar operations of finite type of D and*

$$\bigwedge_{\mathscr{L}} \le \bigwedge_{\mathscr{L}'}, \quad \star_\ell \le \bigwedge_{\mathscr{L}}, \quad \star_{\ell'} \le \bigwedge_{\mathscr{L}'}, \quad \star_\ell \le \star_{\ell'}, \quad \star_f \le \bigwedge_{\mathscr{L}'}.$$

Proof. It is easy to verify that, for each integral domain R with quotient field $K(X)$, such that $D \subseteq R \cap K$, the operation $\circlearrowleft_R$ defined by $E^{\circlearrowleft_R} := ER \cap K$, for each $E \in \bar{\boldsymbol{F}}(D)$ is a semistar operation of finite type on D. As a matter of fact, note that, for each nonzero element $x \in K$ and for each $E \in \bar{\boldsymbol{F}}(D)$, we have $(xER) \cap K = x(ER \cap K)$, $ER = \bigcup\{FR \mid F \subseteq E, F \in \boldsymbol{f}(D)\}$ and $(\bigcup\{FR \mid F \subseteq E, F \in \boldsymbol{f}(D)\}) \cap K = \bigcup\{FR \cap K \mid F \subseteq E, F \in \boldsymbol{f}(D)\}$. Therefore, in particular, $\star_\ell$ $(= \circlearrowleft_{\mathrm{KN}(D,\star)})$ and $\star_{\ell'}$ $(= \circlearrowleft_{\mathrm{KN}'(D,\star)})$ are semistar operations of finite type on D. For each strong-$\star$-monolocality L' of D, we have that $F^\star \subseteq FL'$, thus $F^\star L' = FL'$, hence in particular $F^{\wedge_{\mathscr{L}'}} = (F^\star)^{\wedge_{\mathscr{L}'}}$. Therefore, $\star_f \le \bigwedge_{\mathscr{L}'}$. Moreover,

$$\begin{aligned} F\,\mathrm{KN}'(D,\star) \cap K &= (F(\textstyle\bigcap\{L'(X) \mid L' \in \mathscr{L}'\})) \cap K \\ &\subseteq (\textstyle\bigcap\{FL'(X) \mid L' \in \mathscr{L}'\}) \cap K \\ &= \textstyle\bigcap\{FL'(X) \cap K \mid L' \in \mathscr{L}'\} = \bigcap\{FL' \mid L' \in \mathscr{L}'\}. \end{aligned}$$

We conclude that $\star_{\ell'} \le \bigwedge_{\mathscr{L}'}$. Similarly, it can be shown that $\star_\ell \le \bigwedge_{\mathscr{L}}$. Finally, since $\mathrm{KN}(D,\star) \subseteq \mathrm{KN}'(D,\star)$ (Proposition 5.1), then $\star_\ell \le \star_{\ell'}$. $\square$

If we restrict to just $\star_\ell$ and $\bigwedge_{\mathscr{L}}$ we can prove more.

Proposition 6.3. *Let D be a domain and $\star$ a semistar operation on D. Then $\star_\ell$ of D satisfies*

$$\tilde{\star} \le \star_\ell = \bigwedge_{\mathscr{L}} \le \star_f.$$

Proof. For each $E \in \bar{\boldsymbol{F}}(D)$:

$$
\begin{aligned}
E^{\star_\ell} &= E\,\mathrm{KN}(D,\star) \cap K = \left(\bigcap\{E\,\mathrm{KN}(D,\star)_{\mathfrak{m}} \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}(D,\star)\}\right) \cap K \\
&= \bigcap\{E\,\mathrm{KN}(D,\star)_{\mathfrak{m}} \cap K \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}(D,\star)\} \\
&= \bigcap\{EL(\mathfrak{m})(X) \cap K \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}(D,\star)\} \\
&= \bigcap\{EL(\mathfrak{m}) \mid \mathfrak{m} \in \mathrm{Max}(\mathrm{KN}(D,\star)\} = E^{\wedge_{\mathscr{L}_{min}}} = E^{\wedge_{\mathscr{L}}}.
\end{aligned}
$$

Let $J := (a_1, a_2, \dots, a_n)D \in \boldsymbol{f}(D)$. Suppose that $\alpha \in J^{\star_\ell}$. Since $\mathrm{KN}_{\boldsymbol{c}}(D,\star) = \mathrm{KN}(D,\star)$, then in $\mathrm{KN}(D,\star)$ we can write:

$$
\alpha = a_1\left(\frac{f_1}{g_1}\right) + a_2\left(\frac{f_2}{g_2}\right) + \cdots + a_n\left(\frac{f_n}{g_n}\right)
$$

with $\boldsymbol{c}(f_i) \subseteq \boldsymbol{c}(g_i)^\star$ and $\boldsymbol{c}(g_i)$ a $\star$-e.a.b. ideal of D, for each i, $1 \le i \le n$.

Let $g := g_1 g_2 \cdots g_n$ and, for each i, let $\breve{g}_i := g_1 g_2 \cdots g_{i-1} g_{i+1} \cdots g_n = g/g_i$. Then we can write:

$$
g\alpha = a_1 f_1 \breve{g}_1 + a_2 f_2 \breve{g}_2 + \cdots + a_n f_n \breve{g}_n.
$$

Therefore

$$
\begin{aligned}
\boldsymbol{c}(g\alpha)^\star &= \boldsymbol{c}(a_1 f_1 \breve{g}_1 + a_2 f_2 \breve{g}_2 + \cdots + a_n f_n \breve{g}_n)^\star \\
&\subseteq \boldsymbol{c}(a_1 f_1 \breve{g}_1)^\star + \boldsymbol{c}(a_2 f_2 \breve{g}_2)^\star + \cdots + \boldsymbol{c}(a_n f_n \breve{g}_n)^\star \\
&\subseteq \boldsymbol{c}(a_1 g_1 g_2 \cdots g_n)^\star + \boldsymbol{c}(a_2 g_1 g_2 \cdots g_n)^\star + \cdots + \boldsymbol{c}(a_n g_1 g_2 \cdots g_n)^\star \\
&\subseteq (a_1, a_2, \dots, a_n)\boldsymbol{c}(g_1 g_2 \cdots g_n)^\star = (a_1, a_2, \dots, a_n)\boldsymbol{c}(g)^\star \\
&\subseteq ((a_1, a_2, \dots, a_n)\boldsymbol{c}(g))^\star.
\end{aligned}
$$

Since each $\boldsymbol{c}(g_i)$ is a $\star$-e.a.b. ideal of D, then we know that $(\boldsymbol{c}(g_1)\boldsymbol{c}(g_2)\cdots\boldsymbol{c}(g_n))^\star = \boldsymbol{c}(g_1 g_2 \cdots g_n)^\star = \boldsymbol{c}(g)^\star$ and that $\boldsymbol{c}(g)$ is a $\star$-e.a.b. ideal of D (Claim 1 in the proof of Theorem 5.11). It follows that:

$$
(\boldsymbol{c}(g)\alpha)^\star \subseteq ((a_1, a_2, \dots, a_n)\boldsymbol{c}(g))^\star \Rightarrow \alpha \in ((a_1, a_2, \dots, a_n)D)^\star = J^\star.
$$

Therefore $J^{\star_\ell} \subseteq J^\star$.

Finally, since $\mathrm{Na}(D,\star) \subseteq \mathrm{KN}(D,\star)$ (Theorem 5.11 (7)) then, for each $E \in \overline{\boldsymbol{F}}(D)$, $E^{\tilde{\star}} = E\,\mathrm{Na}(D,\star) \cap K \subseteq E\,\mathrm{KN}(D,\star) \cap K = E^{\star_\ell}$. □

The statements (1), (2) and (3) of our next result are essentially the "KN-analogues" of Proposition 5.4 and Corollary 5.5; statement (4) is an "$\mathscr{L}$-analogue" of Lemma 4.2 (9).

Corollary 6.4. *Let $\star$ be a semistar operation on an integral domain D and let $\mathscr{L} = \mathscr{L}(D, \star)$ be the set of all $\star$-monolocalities of D. Then:*

(1) $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \tilde{\star}) = \mathrm{KN}(D, \tilde{\star})$, *(in particular, $\tilde{\star} = (\tilde{\star})_\ell$).*

(2) $\mathrm{KN}(D, \star_a) = \mathrm{Kr}(D, \star_a) = \mathrm{Kr}(D, \star)$, *(in particular, $\star_a = (\star_a)_\ell$).*

(3) *D is a P$\star$MD if and only if* $\mathrm{KN}(D, \tilde{\star}) = \mathrm{KN}(D, \star_a)$.

(4) *For each $(L, N) \in \mathscr{L}$, $N \cap D$ is a quasi-$\tilde{\star}$-prime of D.*

Proof. (1) is an easy consequence of Theorem 5.11 (7) and Proposition 5.4 (1).

(2) follows from the fact that, for the e.a.b. semistar operation $\star_a$, each $F \in \boldsymbol{f}(D)$ is $\star_a$-e.a.b., hence $\mathscr{L}(D, \star_a)$ is the set of all the $\star$-valuation overrings of D (since $T \in \mathscr{L}(D, \star_a)$ is necessarily a valuation overring of D and $T = T^{\star_a}$, i.e. T is a $\star_a$-valuation overring (Lemma 4.2 (6)), or equivalently a $\star$-valuation overring, of D [10, Proposition 3.3]).

(3) follows from (1), (2) and [8, Theorem 3.1 and Remark 3.1].

(4) Using Theorem 5.11 (6) and (7), we have $(N \cap D)^{\tilde{\star}} = (N \cap D)\,\mathrm{Na}(D, \star) \cap K \subseteq (N \cap D)\,\mathrm{KN}(D, \star) \cap K \subseteq (N(X) \cap \mathrm{KN}(D, \star)) \cap K$. Therefore $(N \cap D)^{\tilde{\star}} \cap D \subseteq ((N(X) \cap \mathrm{KN}(D, \star)) \cap K) \cap D = ((N(X) \cap \mathrm{KN}(D, \star)) \cap L) \cap D = N \cap D$. $\square$

Remark 6.5. So far we have given no indication that $\mathrm{KN}(D, \star)$ and $\mathrm{KN}'(D, \star)$ are ever different. In Example 7.7 we exhibit a (semi)star operation (of finite type) $\star$ on a Noetherian integrally closed integral domain D such that $\mathrm{KN}(D, \star) \subsetneq \mathrm{KN}'(D, \star)$ and thus, in particular, $\mathscr{L}(D, \star) \supsetneq \mathscr{L}'(D, \star)$ (and so $\bigwedge_{\mathscr{L}} = \star_\ell \lneq \star_{\ell'} \leq \bigwedge_{\mathscr{L}'}$). Moreover in this example we will see that:

$$(\star =)\ \star_f \lneq \star_{\ell'}\ (= \star_a = t_D),$$

$$(d_D =)\ \tilde{\star} = \star_\ell \lneq \star_f.$$

Moreover, it is not difficult to give an example of a semistar operation $\star$ such that $\tilde{\star} \lneq \star_\ell$ (cf. the following Example 7.8).

Now that we have made note that $\mathscr{L}$ and $\mathscr{L}'$ are not always the same we investigate the implications of assuming that they or the related rational function rings or the related semistar operations are the same.

Corollary 6.6. *Let $\star$ be a semistar operation on an integral domain D. Then the following are equivalent.*

(i) $\mathscr{L} = \mathscr{L}'$;
(ii) $\bigwedge_{\mathscr{L}} = \bigwedge_{\mathscr{L}'}$;
(iii) $\star_f \leq \star_\ell$;
(iv) $\star_\ell = \star_f$;
(v) $\star_\ell = \bigwedge_{\mathscr{L}} = \bigwedge_{\mathscr{L}'} = \star_{\ell'} = \star_f$.

Proof. (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) and (i) $\Rightarrow$ (ii) are trivial.

(iii) $\Rightarrow$ (i) because (iii) implies that each $L \in \mathscr{L}(D, \star)$ is a $\star$-overring of D, since $\star_\ell = \bigwedge_{\mathscr{L}}$.

(ii) $\Rightarrow$ (iv) because we know that, in general, $\bigwedge_{\mathscr{L}} \le \star_f \le \bigwedge_{\mathscr{L}'}$ (Propositions 6.2 and 6.3).

(i) $\Rightarrow$ (v) is obvious, using the fact that we already know that (i) $\Leftrightarrow$ (iv). $\square$

Corollary 6.7. *Let $\star$ be a semistar operation on an integral domain D. If $\star_f$ is stable (i.e. $\tilde{\star} = \star_f$ [7, Corollary 3.9]) then* $\mathrm{KN}(D, \star) = \mathrm{KN}'(D, \star)$ *and* $\star_\ell = \bigwedge_{\mathscr{L}} = \bigwedge_{\mathscr{L}'} = \star_{\ell'}$ $(= \star_f = \tilde{\star})$.

Proof. The result follows from Corollary 6.6 and the fact that, in general, $\tilde{\star} \le \star_\ell \le \star_f$ (Proposition 6.3). $\square$

7 Constructions and Examples

Thus far, we have focussed on the situation where we begin with a semistar operation and investigate related overrings determined by the e.a.b.-ideals associated to the semistar operation. Now we reverse that and begin with a collection of overrings and use them to define a semistar operation. The major questions will concern how the (strong) monolocalities relate to the defining collection of overrings.

Proposition 7.1. *Let $\mathscr{T} := \{T_\lambda \mid \lambda \in \Lambda\}$ be a family of quasilocal overrings of an integral domain D and let $F \in \boldsymbol{f}(D)$. Then F is $\bigwedge_{\mathscr{T}}$-e.a.b. if and only if FT_λ is principal as a fractional ideal of T_λ, for each $\lambda \in \Lambda$.*

Proof. The "only if" part.

Claim. *Let $\lambda \in \Lambda$ and let $F, G, H \in \boldsymbol{f}(D)$. Assume that F is $\bigwedge_{\mathscr{T}}$-e.a.b. and $FGT_\lambda \subseteq FHT_\lambda$, then $GT_\lambda \subseteq HT_\lambda$.*

Note that $FGT_\lambda \subseteq FHT_\lambda$ implies that $(FG)^{\wedge_{\mathscr{T}}} \subseteq (FGT_\lambda)^{\wedge_{\mathscr{T}}} \subseteq (FHT_\lambda)^{\wedge_{\mathscr{T}}}$. Since F is $\bigwedge_{\mathscr{T}}$-e.a.b., thus also $(\bigwedge_{\mathscr{T}})_f$-a.b. (Lemma 3.8 (2)), then $G^{(\wedge_{\mathscr{T}})_f} = G^{\wedge_{\mathscr{T}}} \subseteq (HT_\lambda)^{(\wedge_{\mathscr{T}})_f} \subseteq (HT_\lambda)^{\wedge_{\mathscr{T}}}$. Therefore $GT_\lambda = (G^{\wedge_{\mathscr{T}}})T_\lambda \subseteq ((HT_\lambda)^{\wedge_{\mathscr{T}}})T_\lambda = HT_\lambda$ (Lemma 4.2 (10)).

The conclusion of the "only if" part follows from the previous Claim and Remark 3.7, since T_λ is quasilocal and each finitely generated T_λ-submodule of K, $G_\lambda \in \boldsymbol{f}(T_\lambda)$, is of the type GT_λ, for some $G \in \boldsymbol{f}(D)$.

For the "if" part, assume that $F, G, H \in \boldsymbol{f}(D)$, $(FG)^{\wedge_{\mathscr{T}}} \subseteq (FH)^{\wedge_{\mathscr{T}}}$ and FT_λ is principal as a fractional ideal of T_λ, for each $\lambda \in \Lambda$. Then, clearly, $FGT_\lambda = (FG)^{\wedge_{\mathscr{T}}}T_\lambda \subseteq (FH)^{\wedge_{\mathscr{T}}}T_\lambda = FHT_\lambda$. Since FT_λ is principal, then $GT_\lambda \subseteq HT_\lambda$, for each $\lambda \in \Lambda$, and thus $G^{\wedge_{\mathscr{T}}} \subseteq H^{\wedge_{\mathscr{T}}}$. $\square$

We digress momentarily to give a corollary to the last result which illustrates some nice closure properties of the $(-)_\ell$ operation.

Corollary 7.2. *Let $\star$ be a semistar operation on an integral domain D.*

(1) *Let $F \in \boldsymbol{f}(D)$. If F is $\star$-e.a.b. then F is $\star_\ell$-e.a.b.*

(2) $\mathscr{L}(D,\star) \subseteq \mathscr{L}(D,\star_\ell)$ *(more precisely,* $\mathscr{L}(D,\star) = \{L \in \mathscr{L}(D,\star_\ell) \mid L = L^{\star_f}\}$*).*

(3) $\mathrm{KN}(D,\star) = \mathrm{KN}(D,\star_\ell)$.

(4) $(\star_\ell)_\ell = \star_\ell$.

Proof. Recall that $\star_\ell = \bigwedge_{\mathscr{L}}$, where $\mathscr{L} = \mathscr{L}(D,\star)$ (Proposition 6.3).

(1) Assume that F is $\star$-e.a.b. then, by definition of $\star$-monolocality, FL is principal for each $L \in \mathscr{L}(D,\star)$. Therefore, by Proposition 7.1, F is $\bigwedge_{\mathscr{L}}$-e.a.b. ($= \star_\ell$-e.a.b.).

(2) Let $L \in \mathscr{L}(D,\star)$ and let F be $\star_\ell$-e.a.b. ($= \bigwedge_{\mathscr{L}}$-e.a.b.). As above, by Proposition 7.1, we know that FL is principal, thus L belongs also to $\mathscr{L}(D,\star_\ell)$, since $L = L^{\star_f}$ and $\star_\ell \leq \star_f$ (Proposition 6.3).

For the parenthetical statement, let $L \in \mathscr{L}(D,\star_\ell)$ and let F be $\star$-e.a.b. By (1) F is $\star_\ell$-e.a.b. ($= \bigwedge_{\mathscr{L}}$-e.a.b.), thus FL is principal, for each $L \in \mathscr{L}(D,\star)$ (Proposition 7.1). We conclude that $L \in \mathscr{L}(D,\star_\ell)$ belongs to $\mathscr{L}(D,\star)$ if and only if $L = L^{\star_f}$.

(3) From (2) we deduce that $\mathrm{KN}(D,\star) \supseteq \mathrm{KN}(D,\star_\ell)$. The opposite inclusion follows from Theorem 5.11 (1) and (7), since if $g \in D[X]$ is such that $\boldsymbol{c}(g)$ is $\star$-e.a.b. then $\boldsymbol{c}(g)$ is also $\star_\ell$-e.a.b. by (1).

(4) is a straightforward consequence of (3). □

We now turn back to the special case where we begin with a collection of overrings of a domain D and use them to define a semistar operation.

Proposition 7.3. *Let $\mathscr{T} := \{T_\lambda \mid \lambda \in \Lambda\}$ be a family of quasilocal overrings of an integral domain D and set $* := \bigwedge_{\mathscr{T}}$. Then:*

(1) $\mathscr{T} \subseteq \mathscr{L}'(D,*) \subseteq \mathscr{L}(D,*)$.

(2) $\bigwedge_{\mathscr{L}} \leq \bigwedge_{\mathscr{L}'} \leq *$ *and* $(\bigwedge_{\mathscr{L}'})_f = *_f$.

(3) $\mathrm{KN}(D,\bigwedge_{\mathscr{L}}) = \mathrm{KN}(D,\bigwedge_{\mathscr{L}'}) = \mathrm{KN}(D,*)$ *and* $\mathrm{KN}(D,*) = \mathrm{KN}'(D,*)$.

(4) $*_\ell = *_{\ell'} = \bigwedge_{\mathscr{L}} \leq (\bigwedge_{\mathscr{L}'})_f = *_f$ *and* $*_\ell = (*_{\ell'})_\ell = (\bigwedge_{\mathscr{L}})_\ell = (\bigwedge_{\mathscr{L}'})_\ell$.

Proof. (1) Each T_λ is obviously a $*$-overring of D, since if $F \in \boldsymbol{f}(D)$ then $FT_\lambda = F^*T_\lambda = (FT_\lambda)^*$ (Lemma 4.2 (10)). Furthermore, note that if $F \in \boldsymbol{f}(D)$ is $*$-e.a.b. then FT_λ is principal in T_λ, for each $\lambda \in \Lambda$. As a matter of fact, if $G, H \in \boldsymbol{f}(D)$ are such that $FGT_\lambda \subseteq FHT_\lambda$, then $(FGT_\lambda)^* = FGT_\lambda \subseteq FHT_\lambda = (FHT_\lambda)^*$, thus $(GT_\lambda)^* = GT_\lambda \subseteq HT_\lambda = (HT_\lambda)^*$, because F is $*$-e.a.b. Therefore FT_λ is quasi-cancellative and so it is principal in T_λ (Remark 3.7).

(2) From (1), we deduce immediately that:

$$\bigwedge\nolimits_{\mathscr{L}} \leq \bigwedge\nolimits_{\mathscr{L}'} \leq \bigwedge\nolimits_{\mathscr{T}} = *.$$

Moreover, $*_f = (\bigwedge_{\mathscr{L}'})_f$, since for each $F \in \boldsymbol{f}(D)$, we have:

$$
\begin{aligned}
F^{\wedge_{\mathscr{L}'}} &= \left(\bigcap\{FT_\lambda \mid \lambda \in \Lambda\}\right) \cap \left(\bigcap\{FL' \mid L' \in \mathscr{L}'\backslash\mathscr{T}\}\right) \\
&= F^* \cap \left(\bigcap\{FL' \mid L' \in \mathscr{L}'\backslash\mathscr{T}\}\right) \\
&= F^* \cap \left(\bigcap\{F^*L' \mid L' \in \mathscr{L}'\backslash\mathscr{T}\}\right) = F^*.
\end{aligned}
$$

(3) Since $\mathrm{KN}(D,*) = \mathrm{KN}(D,*_\ell)$ (Corollary 7.2 (3)), $*_\ell = \bigwedge_{\mathscr{L}}$ (Proposition 6.3), and $*_f = (\bigwedge_{\mathscr{L}'})_f$ by (b), then we easily deduce that $\mathrm{KN}(D,\bigwedge_{\mathscr{L}}) = \mathrm{KN}(D,*_\ell) = \mathrm{KN}(D,*) = \mathrm{KN}(D,\bigwedge_{\mathscr{L}'})$.

From Lemma 4.2 (10) we deduce immediately:

Claim 1. *Let $g \in D[X]$, $g \neq 0$, then $\boldsymbol{c}(g)^{\wedge_{\mathscr{T}}} = D^{\wedge_{\mathscr{T}}}$ if and only if $\boldsymbol{c}(g)T_\lambda = T_\lambda$, for each $\lambda \in \Lambda$.*

Claim 1 implies:

Claim 2. $\mathrm{Na}(D,\bigwedge_{\mathscr{T}}) = \bigcap\{\mathrm{Na}(D,\star_{\{T_\lambda\}}) \mid \lambda \in \Lambda\} = \bigcap\{T_\lambda(X) \mid \lambda \in \Lambda\}$.

From Claim 2 and from the fact that $\mathscr{T} \subseteq \mathscr{L}'(D,*)$ we deduce that $\mathrm{KN}'(D,*) = \bigcap\{L'(X) \mid L' \in \mathscr{L}'(D,*)\} \subseteq \bigcap\{T_\lambda(X) \mid \lambda \in \Lambda\} = \mathrm{Na}(D,\bigwedge_{\mathscr{T}}) \subseteq \mathrm{KN}(D,\bigwedge_{\mathscr{T}}) = \mathrm{KN}(D,*)$. Since in general, $\mathrm{KN}(D,*) \subseteq \mathrm{KN}'(D,*)$, we conclude that $\mathrm{KN}(D,*) = \mathrm{KN}'(D,*)$.

(4) From (3) and from Proposition 6.3, we have that $*_\ell = \bigwedge_{\mathscr{L}} = *_{\ell'}$ and so, by Corollary 7.2 (4), $*_\ell = (\bigwedge_{\mathscr{L}})_\ell = (*_{\ell'})_\ell$. From (b) we know that $(\bigwedge_{\mathscr{L}'})_f = *_f$. Since $\bigwedge_{\mathscr{L}}(= *_\ell)$ is a semistar operation of finite type then, clearly, $\bigwedge_{\mathscr{L}} \leq (\bigwedge_{\mathscr{L}'})_f$. Finally, note that $((\bigwedge_{\mathscr{L}'})_f)_\ell = (\bigwedge_{\mathscr{L}'})_\ell$ and $(*_f)_\ell = *_\ell$. □

If we assume in addition to the hypotheses of Proposition 7.3 that each T_λ is integrally closed we can prove a little more.

Corollary 7.4. *Suppose in addition to the hypotheses of Proposition 7.3 that each T_λ is integrally closed. Then* $\mathrm{KN}(D,*) = \mathrm{KN}'(D,*) = \bigcap\{T_\lambda(X) \mid \lambda \in \Lambda\}$.

Proof. We already know that $\mathrm{KN}(D,*) = \mathrm{KN}'(D,*)$. Since each T_λ is a strong $*$-monolocality (Proposition 7.3 (1)), it follows immediately from the definitions that $\mathrm{KN}'(D,\star) \subseteq \bigcap\{T_\lambda(X) \mid \lambda \in \Lambda\}$. We will finish the proof by showing that $\bigcap\{T_\lambda(X) \mid \lambda \in \Lambda\} \subseteq \mathrm{KN}(D,*)$. Let $f/g \in \bigcap\{T_\lambda(X) \mid \lambda \in \Lambda\}$ and suppose that $f, g \in D[X]$, with $g \neq 0$, and that they have no common factors over $K[X]$. Choose a particular T_λ. We will consider content ideals of polynomials as ideals of T_λ. By definition, we know that $f/g = h/k$ where $h, k \in T_\lambda[X]$ and $\boldsymbol{c}_{T_\lambda}(k) = T_\lambda$. Since f, g have no common factors over $K[X]$, by Euclid's Lemma we know that g is a factor of k in $K[X]$. If we rewrite h/k as dh/dk for an appropriate nonzero constant $d \in T_\lambda$, we have g being a factor over $T_\lambda[X]$ of dk and $\boldsymbol{c}_{T_\lambda}(dk) = dT_\lambda$. Then since T_λ is integrally closed we know that g has invertible (hence principal) content [25, Theorem 1.5].

Finally, since $\boldsymbol{c}_{T_\lambda}(h) \subseteq \boldsymbol{c}_{T_\lambda}(k) = T_\lambda$ and $f/g = h/k$ it follows easily that $\boldsymbol{c}_{T_\lambda}(f) \subseteq \boldsymbol{c}_{T_\lambda}(g)$.

Since we have been working with an arbitrary T_λ we have proven:

- $\boldsymbol{c}_D(f)^* \subseteq \boldsymbol{c}_D(g)^*$ (with content ideals considered now as ideals of D).
- $\boldsymbol{c}_D(g)$ is a $*$-e.a.b. ideal of D since its extension to each T_λ is principal.

This proves that $f/g \in \mathrm{KN}(D,*)$ which finishes the proof. $\square$

In the setting where we begin with a collection of overrings of a domain D it can seem that differences between strong monolocalities and monolocalities and the associated constructions should disappear. In particular, we could hope that the inequality in part (4) of Proposition 7.3 should be an equality. We next give an example to demonstrate that this inequality can be strict.

Example 7.5. Let k be a field and let $\{X, Y, W, X_1', Y_1', X_2', Y_2', \ldots; Z\}$ be an infinite family of indeterminates over k. Set $R := k[X, Y, W, X_1', Y_1', X_2', Y_2', \ldots]$. Let M be the maximal ideal of R generated by $\{X, Y, W, X_1', Y_1', X_2', Y_2', \ldots\}$, let $D := R_M$ and let K be the quotient field of D. For each positive integer i define

$$T_i := D\left[\frac{W}{XX_i' + YY_i'}\right].$$

Let $\mathscr{T} := \{T_i \mid i > 0\}$ and set $* := \bigwedge_{\mathscr{T}}$ (i.e. $E^* := \bigcap_{i \geq 1} ET_i$, for each $E \in \overline{\boldsymbol{F}}(D)$). Also, let P_W be the prime ideal of D generated by W. Note that $\bar{k} := D_{P_W}/P_W D_{P_W}$ is isomorphic to $k(X, Y, X_1', Y_1', X_2', Y_2', \ldots)$. Let φ be the canonical homomorphism from D_{P_W} to $\bar{k} = D_{P_W}/P_W D_{P_W}$. Let $\overline{V}$ be a minimal valuation overring of $\overline{D} := D/P_W$ (in its quotient field isomorphic to $\bar{k}$) and let $V = \varphi^{-1}(\overline{V})$. Then V is a minimal valuation overring of D which has D_{P_W} as an overring.

(1) *V is a minimal valuation overring of each T_i.*
(2) *$V(Z)$ is a minimal valuation overring of each $T_i(Z)$.*
(3) $\mathrm{KN}(D,*) = \mathrm{KN}'(D,*) = \bigcap_{i \geq 1} T_i(Z)$.
(4) *Let M_V be the contraction of the maximal ideal of $V(Z)$ to* $\mathrm{KN}(D,*)$. *Then M_V is a maximal ideal of* $\mathrm{KN}(D,*)$.
(5) *M_V is the only maximal ideal of* $\mathrm{KN}(D,*)$.
(6) *$D^* = D$.*
(7) $(\mathrm{KN}(D,*) = \mathrm{KN}'(D,*) =)$ $\bigcap_{i \geq 1} T_i(Z) = D(Z)$.
(8) *D is a $*$-monolocality but not a strong $*$-monolocality.*
(9) *The (semi)star operation $*$ is such that the inequality in Proposition 7.3 (4) is strict (i.e., $\bigwedge_{\mathscr{L}(*)} \lneq (\bigwedge_{\mathscr{L}'(*)})_f$).*

Proof. (1) is a consequence of the fact that if $\frac{W}{XX_i'+YY_i'} \notin V$ then $\frac{XX_i'+YY_i'}{W} \in V \subseteq D_{P_W}$, which is a contradiction.

(2) follows from (1) and (3) is a consequence of Corollary 7.4, since each T_i is integrally closed. (The claim that each T_i is integrally closed follows easily from [28, Theorem 2].)

(4) Proposition 7.3 implies that each T_i is a $*$-monolocality. It follows then from Theorem 5.11 that $T_i(Z)$ is an overring of $\mathrm{KN}(D,*)$ for each i. Hence $V(Z)$ is an overring of $\mathrm{KN}(D,*)$. We assumed V to be a minimal valuation overring of D. It follows that $V(Z)$ is a minimal valuation overing of $D(Z)$. It is clear then that $V(Z)$ is also a minimal valuation overring of any ring properly in between $D(Z)$ and $V(Z)$. In particular, $V(Z)$ is a minimal valuation overring of $\mathrm{KN}(D,*)$. The result follows immediately.

(5) Let $d \in M_V$. Then $d \in \mathrm{KN}(D,*)$. As we noted above, $T_i(Z)$ is an overring of $\mathrm{KN}(D,*)$. Also recall that $V(Z)$ is a minimal valuation overring of each $T_i(Z)$. In particular, d is a nonunit in each $T_i(Z)$. Hence, $1+d$ is a unit in each $T_i(Z)$ and so, by (3), it is a unit in $\mathrm{KN}(D,*)$.

(6) It is easy to see that $\bigcap_{i\geq 1} T_i = D$, hence $D^* = \bigcap_{i\geq 1} DT_i = \bigcap_{i\geq 1} T_i = D$.

(7) Combine Theorem 5.11 (6) and (7) with the fact that we already know that $\mathrm{KN}(D,*)$ is quasilocal and that $\mathrm{KN}(D,*) \cap K = \bigcap_{i\geq 1} T_i(Z) \cap K = \bigcap_{i\geq 1} T_i = D$.

(8) D being a $*$-monolocality follows from the fact that $\mathrm{KN}(D,*) = D(Z)$ (Theorem 5.11 (5)). D is not a strong $*$-monolocality because it is not a $*$-overring: let $J = (X,Y)D$, then $W \in J^*$ but $W \notin JD = J$.

(9) Since D itself is a $*$-monolocality (by (8)), then $\bigwedge_{\mathscr{L}}$ is the identity function. However, the proof of (8) above demonstrates that $*_f$ $(= (\bigwedge_{\mathscr{L}'})_f$ by Proposition 7.3 (4)) is not the identity.

So we have an example of a semistar operation $\star$ on a domain D of the form $\bigwedge_{\mathscr{T}}$, derived from a family $\mathscr{T}$ of overrings of D, such that $\mathrm{KN}(D,\star) = \mathrm{KN}'(D,\star)$ but there exists a $\star$-monolocality which is not strong (Example 7.5).

We next give an example of a semistar operation, also defined by a $\bigwedge$-construction, in which things work exactly as one might hope.

Example 7.6. Let k be a field and let $R := k[X,Y]$ be the ring of polynomials over k in the two variables X and Y. Let $P := (X,Y)$ be the maximal ideal of R generated by the variables and let $D := R_P$. Let M denote the maximal ideal of the local ring D. Set

- $D_1 := D[X/Y]$, $D_2 := D[Y/X]$;
- $\mathscr{T} := \{T_\lambda \mid T_\lambda \in \Lambda\}$ is the collection of all localizations of D_1 and D_2 at their maximal ideals.

Set $* := \bigwedge_{\mathscr{T}}$. *Then* $*$ *is an example of a semistar operation on a integrally closed Noetherian local domain D such that:*

(1) $*_\ell = \bigwedge_{\mathscr{L}} = \bigwedge_{\mathscr{L}'} = *_{\ell'} = *_f = *$ *(i.e.* $\mathscr{L} = \mathscr{L}'$*, by Corollary 6.6).*

(2) $\tilde{*} \lneq * \lneq *_a$.

We know by Proposition 7.3 (1) that $\mathscr{T} \subseteq \mathscr{L}'(D,*) \subseteq \mathscr{L}(D,*)$. Suppose then that T is a $*$-monolocality of D. It is clear from Proposition 7.1 that the ideal $I = (X,Y)$ is a $*$-e.a.b. ideal of D. Hence the ideal I must extend to a principal ideal in T. Since T is quasilocal it follows that IT is generated by either X or Y. Hence either Y/X

or X/Y lies in T. Hence $T_\lambda \subseteq T$ for some $T_\lambda \in \mathscr{T}$. It follows that $\mathscr{T}$ consists exactly of the minimal $*$-monolocalities of D. It follows from this then that the $*$-monolocalities and the strong $*$-monolocalities coincide. This is sufficient to prove that

$$*_\ell = \bigwedge\nolimits_{\mathscr{L}} = \bigwedge\nolimits_{\mathscr{L}'} = *_{\ell'} = *_f = *.$$

Now observe that the T_λ's are neither localizations of D nor valuation overrings of D. This proves that $\tilde{*} \lneq * \lneq *_a$. Suppose that, in fact, $\tilde{*} = *$. Then Corollary 6.4 indicates that $\mathrm{KN}(D,*) = \mathrm{Na}(D,*)$. Recall that the localizations of $\mathrm{Na}(D,*)$ at maximal ideals have the form $D_P(X)$ where P is a prime ideal of D. Similarly, if $* = *_a$ then Corollary 6.4 indicates that $\mathrm{KN}(D,*) = \mathrm{Kr}(D,*)$. Recall that the localizations of $\mathrm{Kr}(D,*)$ at maximal ideals have the form $V(X)$ where V is a valuation overring of D. In the present setting the localizations of $\mathrm{KN}(D,\star)$ are exactly the rings $T_i(X)$. The result follows immediately.

Example 7.6 is significant because we indicated that an important objective of this article was to demonstrate that the Nagata ring construction and the Kronecker function ring construction were at opposite ends of a spectrum. For the generalization to have any real power we need to demonstrate that we can find something which is properly in between these two extremes. We also indicated that we wanted to give a method for approximating a given semistar operation by a semistar operation which was constructed by means of extension to a collection of overrings. We have given such a mechanism, but again we need to show that this is meaningful by demonstrating that the semistar operation obtained can turn out to be associated with a collection of overrings which consists neither of localizations nor of valuation overrings of the domain D. In this example we have a star operation $*$ such that $*$ is equal to all four of the approximations developed in this work (Definition 6.1). And yet we also have

$$\tilde{*} \lneq * \lneq *_a.$$

This indicates that $*$ and all of its KN and KN$'$ derivatives lie properly in between "the localization constructions" associated to $\tilde{*}$ and "the valuation domain constructions" associated to $*_a$.

Example 7.7. *Example of a (semi)star operation $\star$ on a Noetherian integrally closed domain D such that $\star_\ell \lneq \star\ (= \star_f) \lneq \star_{\ell'}$ and so $\mathrm{KN}'(D,\star) \subsetneq \mathrm{KN}(D,\star)$, in particular $\mathscr{L}(D,\star) \subsetneq \mathscr{L}'(D,\star)$.*

Let k be a field, $D := k[X,Y]_{(X,Y)}$ $M := (X,Y)D$ and let $K := k(X,Y)$. Using the techniques of [14, (32.4)], we construct a new (semi)star operation $\star$ on D as follows:

1. If dD is any nonzero principal ideal of D, then $(dD)^\star := dD$.

2. If $J \subseteq D$ is a nonzero ideal of D which is not contained in any proper principal ideal of D, then $J^\star := M$.

3. If $J \subseteq D$ is a nonzero ideal of D which is not principal, but is contained in a principal ideal, then we factor J as $J = \alpha I$, where α is a GCD of a set of generators of J and $I := (J :_D \alpha D)$ is not contained in any proper principal ideal of D by the choice of α. Then $J^\star := \alpha M$.

4. If J is a nonzero fractional ideal of D which is not contained in D, choose a nonzero element $\beta \in D$ such that $\beta J \subseteq D$. Then define $J^\star := (1/\beta)(\beta J)^\star$.

5. If $J \in \bar{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$ we define $J^\star := K$.

Since D is Noetherian, then $\star$ is a (semi)star operation of finite type on D. Henceforth, it is clear that $\mathscr{M}(\star_f) = \{M\}$. Thus, $\tilde{\star}$ coincides with the identity semistar operation d_D, i.e.:

$$d_D = \tilde{\star} \lneq \star_f = \star.$$

Moreover, we have already proved in [11, Example 5.3] that $\widetilde{(\star_a)} = \star_a = t_D$. Therefore, if we denote by Z a (new) indeterminate over the field of quotients K of D, by $\mathscr{W}$ the set of all rank one discrete valuation overrings of D, then $t_D = \bigwedge_{\mathscr{W}}$, thus:

$$\mathrm{Na}(D, \star) = \mathrm{Na}(D, \tilde{\star}) = \mathrm{Na}(D, d_D) = D(Z),$$

$$\mathrm{Kr}(D, \star) = \mathrm{Kr}(D, \star_a) = \mathrm{Kr}(D, t_D) = \mathrm{Kr}(D, \textstyle\bigwedge_{\mathscr{W}}) = \bigcap\{W(Z) \mid W \in \mathscr{W}\}.$$

Claim 1. *Let $J \in \boldsymbol{f}(D)$. If J is $\star$-e.a.b. then J is principal.*

Without loss of generality we can assume that $J \subseteq D$. If J is not principal then either J is not contained in any principal ideal of D or J is contained in a principal ideal of D in both cases $J = \delta I$, for some nonzero ideal I of D such that $I^\star = M$ and for some nonzero element $\delta \in D$ (eventually $\delta = 1$). Therefore, $J^\star = (\delta I)^\star = \delta I^\star = \delta M$. On the other hand, since by definition of $\star$ we have that $(M^2)^\star = M^\star = M$, then:

$$(JI)^\star = (\delta I^2)^\star = \delta (I^\star I^\star)^\star = \delta (M^2)^\star = \delta M = J^\star = (JD)^\star.$$

If J is $\star$-e.a.b. (since I is finitely generated) then $I^\star = D^\star = D$, which is a contradiction.

Claim 2. *Let $\mathscr{V}(D, \star)$ be the set of all $\star$-valuation overrings of D. If L' is a strong-$\star$-monolocality of D then L' is a localization of D at a height-one prime ideal of D, thus $\mathscr{L}'(D, \star) = \mathscr{W} = \mathscr{V}(D, \star)$. In particular,*

$$\mathrm{KN}'(D, \star) = \bigcap\{W(Z) \mid W \in \mathscr{W}\} \ (= \mathrm{Kr}(D, \star)).$$

As a matter of fact $M = (M^2)^\star \subseteq M^2 L'$ and, since M is finitely generated, by Nakayama's Lemma we have $ML' = L'$. Therefore, for some element $f \in M$, $fL' = L'$ and thus $D_f \subseteq L'$. Since D_f is a one-dimensional Krull domain and L' is a quasilocal overring of D_f, then necessarily L' is a localization of D at a height-one prime ideal of D, hence $L' \in \mathscr{W}$, i.e. $\mathscr{L}' \subseteq \mathscr{W}$. Conversely, if Q is an height-one prime ideal of D, then from the definition of $\star$ it follows immediately that the discrete valuation overring $W := D_Q$ is a strong-$\star$-monolocality of D. Note that in general $\mathscr{L}'(D,\star) \supseteq \mathscr{V}(D,\star)$ and, in this case, each $W \in \mathscr{W}$ is a $\star$-valuation overring of D (by [10, Theorem 3.5], since $W(Z) \supseteq \mathrm{Kr}(D,\star)$), that is $\mathscr{V}(D,\star) \supseteq \mathscr{W}$.

Claim 3. *We have that* $b_D \lneq \star_a$ $(= \widetilde{(\star_a)} = t_D)$ *and* $\mathscr{L}'(D,b_D) \supsetneq \mathscr{L}'(D,\star_a)$ *thus, in particular,*

$$(\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_a) =)\ \mathrm{KN}'(D,\star_a) \supsetneq \mathrm{KN}'(D,b_D)\ (= \mathrm{Kr}(D,b_D) = \mathrm{Kr}(D,d_D)).$$

(cf. also Proposition 5.4 (2)).

We have observed above that $\star_a = \widetilde{(\star_a)} = t_D = \bigwedge_{\mathscr{W}}$ and thus it is easy to see that $\mathscr{L}'(D,\star_a) = \mathscr{L}(D,\star_a) = \mathscr{W} = \{D_Q \mid Q \text{ is an height one prime ideal of } D\}$. On the other hand, b_D coincides by definition with $\bigwedge_{\mathscr{V}}$, where (in the present Example) the set $\mathscr{V} := \mathscr{V}(D,d_D)$ $(= \mathscr{V}(D,b_D))$ is the set of all the valuation overrings of D and thus it is easily seen that $\mathscr{L}'(D,\bigwedge_{\mathscr{V}}) = \mathscr{L}(D,\bigwedge_{\mathscr{V}}) = \mathscr{V}$. Since there are plenty of two dimensional valuation overrings of D, then clearly $\mathscr{L}'(D,b_D) \supsetneq \mathscr{L}'(D,\star_a)$. Finally, by [11, Proposition 4.1 (5)], it is clear that $b_D \lneq \star_a$ if and only if $\mathrm{Kr}(D,b_D) \subsetneq \mathrm{Kr}(D,\star_a)$.

Claim 4. *We have that* $\star \lneq \star_a$ *(more precisely, every nonprincipal ideal of D is* $\star_a$*-e.a.b., but not* $\star$*-e.a.b.) and* $\mathscr{L}'(D,\star) = \mathscr{L}'(D,\star_a)$*. In particular,* $\mathrm{KN}'(D,\star) = \mathrm{KN}'(D,\star_a)$.

Note that every nonzero ideal of D is clearly $\star_a$-e.a.b. but from Claim 1 we know that if an ideal of D is $\star$-e.a.b. then it is a principal ideal. We have observed in the Claims 2 and 3 above that $\mathscr{L}'(D,\star) = \mathscr{W} = \mathscr{L}'(D,\star_a)$.

Claim 5. $\mathscr{L}(D,\star) \supsetneq \mathscr{L}'(D,\star)$ *and* $\mathrm{KN}(D,\star) \subsetneq \mathrm{KN}'(D,\star)$*. More precisely:*

$$\begin{aligned}\mathrm{KN}(D,\star) &= D(Z)\ (= \mathrm{Na}(D,\star))\\ &\subsetneq \mathrm{KN}'(D,\star) = \bigcap\{D_Q(Z) \mid Q \in \mathrm{Spec}(D) \textit{ and } \mathrm{ht}(Q) = 1\}\\ &\quad (= \mathrm{Kr}(D,\star)).\end{aligned}$$

By the previous considerations it is sufficient to show that $\mathrm{KN}(D,\star) = D(Z)$. This is an easy consequence of Claim 1, since each $J \in \boldsymbol{f}(D)$ which is a $\star$-e.a.b. is a principal fractional ideal of D and, by definition D is quasilocal and $D = D^\star$, thus D is a $\star$-monolocality of D. Therefore $\mathscr{L}(D,\star) = \{L \mid L \text{ is a quasilocal overring of } D \text{ such that } L = L^{\star_f}\}$ and $D \in \mathscr{L}(D,\star)\backslash\mathscr{L}'(D,\star)$.

Claim 6. $\mathscr{L}(D, \star_a) = \mathscr{L}'(D, \star_a)$ $(= \mathscr{L}'(D, \star) = \mathscr{W})$ *thus* $(\mathrm{KN}(D, t_D) = \mathrm{KN}(D, \star_a) = \mathrm{KN}'(D, \star_a)$ $(= \mathrm{KN}'(D, \star))$ *and so* $\mathrm{KN}(D, \star) \subsetneq \mathrm{KN}(D, \star_a)$.

This is a consequence of Corollary 6.7 since in this case we know that $\star_a = \widetilde{(\star_a)}$ is a stable (semi)star operation on D.

In conluding, in this example, we have:

$$d_D = \tilde{\star} = \star_\ell = \bigwedge\nolimits_{\mathscr{L}} \lneqq \star_f \lneqq \star_{\ell'} = \bigwedge\nolimits_{\mathscr{L}'} = \star_a = t_D.$$

We end with an easy example announced in Remark 6.5.

Example 7.8. *An example of an Noetherian integrally closed domain D with a (semi)star operation $\star$ such that $\tilde{\star} \lneqq \star_\ell$.*

Let D be as in Example 7.7. Note that, in this case, $t_D = \bigwedge_{\mathscr{W}}$ and $b_D = \bigwedge_{\mathscr{V}}$, where $\mathscr{W}$ is the set of all the rank 1 valuation overrings of the Krull domain D [14, Proposition 44.13] and $\mathscr{V}$ is the set of all the valuation overrings of D. Therefore t_D and b_D are both (e.)a.b. (semi)star operations on D.

Let $\star := b_D$. Since it is easy to see that $\mathscr{M}(b_D) = \mathrm{Max}(D)$ [11, Theorem 4.3 (3)], then $\widetilde{b_D} = d_D$. Moreover, we have already seen (in the proof of Claim 3, Example 7.7) that $\mathscr{L}(D, b_D) = \mathscr{L}'(D, b_D) = \mathscr{V}$, thus $(b_D)_\ell = (b_D)_{\ell'} = b_D$.

References

[1] Anderson D. D.: Some remarks on the ring $R(X)$. Comm. Math. Univ. St. Pauli **26** (1977), 137–140

[2] Anderson D. D. and Anderson D. F.: Some remarks on cancellation ideals. Math. Japonica **6** (1984), 879–886

[3] Arnold J.: On the ideal theory of the Kronecker function ring and the domain $D(X)$. Canad. J. Math. **21** (1969), 558–563

[4] Arnold J. and Brewer J.: Kronecker function rings and flat $D[X]$-modules. Proc. Amer. Math. Soc. **27** (1971), 483–485

[5] Bourbaki N.: Algèbre Commutative, Ch. II. Hermann, 1961

[6] Edwards H. M.: Divisor Theory. Birkhäuser, 1990

[7] Fontana M. and Huckaba J.: Localizing systems and semistar operations. In: Non Noetherian Commutative Ring Theory (S. Chapman and S. Glaz, Eds.). Kluwer Academic Publishers, Dordrecht 2000, Chapter 8, 169–187

[8] Fontana M., Jara P., and Santos E.: Prüfer $\star$-multiplication domains and semistar operations. J. Algebra Appl. **2** (2003), 21–50

[9] Fontana M. and Loper K. A.: Kronecker function rings: a general approach. In: Ideal Theoretic Methods in Commutative Algebra (D. D. Anderson and I. J. Papick, Eds.). M. Dekker Lecture Notes Pure Appl. Math. 220 (2001), 189–205

[10] Fontana M. and Loper K. A.: A Krull-type theorem for the semistar integral closure of an integral domain. ASJE Theme Issue "Commutative Algebra" 26 (2001), 89–95

[11] Fontana M. and Loper K. A.: Nagata rings, Kronecker function rings and related semistar operations. Comm. Algebra **31** (2003), 4775–4805

[12] Fontana M. and Loper K. A.: An example of an e.a.b. star operation which is not a.b. (work in progress)
[13] Fontana M. and Picozza G.: Semistar invertibility on integral domains. Algebra Colloquium 12 (2005), 645–664
[14] Gilmer R.: Multiplicative Ideal Theory. M. Dekker, New York 1972
[15] Halter-Koch F.: Generalized integral closures. In: Factorization in Integral Domains (D. D. Anderson, Ed.). M. Dekker Lecture Notes Pure Appl. Math. 187 (1997), 349–358
[16] Halter-Koch F.: Ideal Systems: An Introduction to Multiplicative Ideal Theory. M. Dekker, New York 1998
[17] Halter-Koch F.: Kronecker function rings and generalized integral closures. Comm. Algebra **31** (2003), 45–59
[18] Jaffard P.: Les Systèmes d'Idéaux. Dunod, Paris 1960
[19] Kang B. G.: Prüfer v-multiplication domains and the ring $R[X]_{N_v}$. J. Algebra **123** (1989), 151–170
[20] Kaplansky I.: Commutative Ring Theory. Allyn and Bacon Inc., Boston 1970
[21] Kronecker L.: Grundzüge einer arithmetischen Theorie der algebraischen Grössen. J. Reine Angew. Math. **92** (1882), 1–122; Werke 2, 237–387
[22] Krull W.: Beiträge zur Arithmetik kommutativer Integritätsbereiche, I–II. Math. Z. **41** (1936), 545–577; 665–679
[23] Matsuda R. and Sato I.: Note on star operations and semistar operations. Bull. Fac. Sci. Ibaraki Univ. Ser. A **28** (1996), 5–22
[24] Matsuda R. and Sugatani T.: Semistar operations on integral domains, II. Math. J. Toyama Univ. **18** (1995), 155–161
[25] Mott J., Nashier B., and Zafrullah M.: Contents of polynomials and invertibility. Comm. Algebra **18** (1990), 1569–1583
[26] Okabe A. and Matsuda R.: Semistar operations on integral domains. Math. J. Toyama Univ. **17** (1994), 1–21
[27] Okabe A. and Matsuda R.: Kronecker function rings of semistar operations. Tsukuba J. Math. **21** (1997), 529–540
[28] Sally J.: A note on integral closure. Proc. Amer. Math. Soc. **36** (1972), 93–96
[29] Weyl H.: Algebraic Theory of Numbers. Princeton University Press, 1940
[30] Zariski O. and Samuel P.: Commutative Algebra, Vol. II. Van Nostrand, 1960

Received 9 November 2005; revised 27 January 2006

Dipartimento di Matematica Università degli Studi Roma Tre, Largo San Leonardo Murialdo, 1, 00146 Roma, Italy
fontana@mat.uniroma3.it

Department of Mathematics, Ohio State University-Newark, Newark, Ohio 43055, USA
lopera@math.ohio-state.edu