The Arabian Journal for Science and Engineering, Volume 35, Number 1D
May 2010, Pages 45–49

CANCELLATION PROPERTIES IN IDEAL SYSTEMS: AN *e.a.b.* NOT *a.b.* STAR OPERATION

MARCO FONTANA, K. ALAN LOPER AND RYÛKI MATSUDA

ABSTRACT. We show that Krull's a.b. cancellation condition is a properly stronger condition than Gilmer's e.a.b. cancellation condition for star operations.

1. INTRODUCTION

Let D be an integral domain with quotient field K. Let $\boldsymbol{F}(D)$ [respectively, $\boldsymbol{f}(D)$] be the set of all nonzero fractional ideals [respectively, nonzero finitely generated fractional ideals] of D.

A *star operation* $*$ *on* D is a mapping $* : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $E \mapsto E^*$ such that the following properties hold: $(*_1)$ $(zD)^* = zD$ and $(zE)^* = zE^*$, $(*_2)$ $E \subseteq F \Rightarrow E^* \subseteq F^*$, $(*_3)$ $E \subseteq E^*$ and $E^{**} := (E^*)^* = E^*$, for all nonzero $z \in K$, and for all $E, F \in \boldsymbol{F}(D)$.

Examples of star operations include the *v-operation*, defined by $E^v := (D : (D : E))$, for each $E \in \boldsymbol{F}(D)$ [2, page 396]; the *t-operation*, defined by $E^t := \bigcup\{F^v \mid F \in \boldsymbol{f}(D), F \subseteq E\}$, for each $E \in \boldsymbol{F}(D)$ [2, page 406]; the *w-operation* (with the notation proposed by Wang-McCasland) defined by $E^w := \bigcap\{ED_Q \mid Q \in \mathrm{Max}^t(D)\}$ (where $\mathrm{Max}^t(D)$ is the (nonempy) set of all maximal t-ideals of D) for all $E \in \boldsymbol{F}(D)$ [4].

Let $*$ be a star operation on D. If F is in $\boldsymbol{f}(D)$, we say that F is $*$-*eab* [respectively, $*$-*ab*], if the inclusion $(FG)^* \subseteq (FH)^*$ implies that $G^* \subseteq H^*$, with $G, H \in \boldsymbol{f}(D)$, [respectively, with $G, H \in \boldsymbol{F}(D)$].

The operation $*$ is said to be *eab* [respectively, *ab*] if each $F \in \boldsymbol{f}(D)$ is $*$-eab [respectively, $*$-ab]. An ab operation is obviously an eab operation. Recall also that $E \in \boldsymbol{F}(D)$ is called *a* (*fractional*) $*$-*ideal* of D if $E = E^*$.

In the classical (Krull's) setting, the study of Kronecker function rings on an integral domain generally focusses on the collection of "a*rithmetisch* b*rauchbar*" (for short, a.b. or, simply, ab, as above) $*$-operations [3]. Gilmer's presentation of Kronecker function rings [2, Section 32] makes use of the (presumably larger class of) "e*ndlich* a*rithmetisch* b*rauchbar*" (for short, e.a.b. or, simply, eab, as above) $*$-operations. In this paper, we show that the e.a.b. cancellation condition is really strictly weaker than the a.b. cancellation condition. This goal is reached by modifying an example given in the recent paper [1].

2. THE EXAMPLE

In [1, Example 16], the authors consider the following example.

Let k be a field, $X_1, X_2, \ldots, X_n, \ldots$ an infinite set of indeterminates over k and $N := (X_1, X_2, \ldots, X_n, \ldots)k[X_1, X_2, \ldots, X_n, \ldots]$. Clearly, N is a maximal ideal in $k[X_1, X_2, \ldots, X_n, \ldots]$. Set $D := k[X_1, X_2, \ldots, X_n, \ldots]_N$, $M := ND$ be the maximal ideal of the local domain D and $K := k(X_1, X_2, \ldots, X_n, \ldots)$ the quotient field of D. Note that D is a UFD and consider $\mathscr{W}$ the set of all the rank one essential valuation overrings of D. Let $\wedge_{\mathscr{W}}$ be the star ab operation on D defined by $\mathscr{W}$ [2, page 398], *i.e.*, for each $E \in \boldsymbol{F}(D)$,

$$E^{\wedge_{\mathscr{W}}} := \bigcap\{EW \mid W \in \mathscr{W}\}.$$

It is well known that the t-operation on D is an ab star operation, since $F^t = F^{\wedge_{\mathscr{W}}}$ for all $F \in \boldsymbol{f}(D)$ [2, Proposition 44.13] (more precisely, in this case, we have $v = t = w = \wedge_{\mathscr{W}}$).

Received June 15, 2009; Accepted April 13, 2010.
2010 *Mathematics Subject Classification*. 13A15, 13G05, 13F30, 13E99.
Key words and phrases. Cancellation properties for multiplicative ideal systems, star operation, v-operation, t-operation, b-operation.
Corresponding author: Marco Fontana.

Consider the following subset of fractional ideals of D:

$$\mathcal{J} := \{xF^t, yM, zM^2 \mid x, y, z \in K\backslash\{0\}, F \in \boldsymbol{f}(D)\}.$$

Since each nonzero principal fractional ideal of D is in $\mathcal{J}$ and, for each ideal $J \in \mathcal{J}$ and for each nonzero $a \in K$, the ideal aJ belongs to $\mathcal{J}$, then, as above, [2, Proposition 32.4] guarantees that the set $\mathcal{J}$ defines on D a star operation $*$, by setting:

$$E^* := \bigcap\{J \mid J \in \mathcal{J}, J \supseteq E\}, \quad \text{for each } E \in \boldsymbol{F}(D).$$

Since, for each $F \in \boldsymbol{f}(D)$, $F^t \in \mathcal{J}$, it was claimed in [1, Example 16] that $*|_{\boldsymbol{f}(D)} = t|_{\boldsymbol{f}(D)}$. This would imply that $*$ was an eab operation on D, since the operation t—as observed above—is an ab star operation on D.

Unfortunately, it is not true that $F^* = F^t$ for all $F \in \boldsymbol{f}(D)$ and, in particular, this equality does not hold if $F \subset D$ and $F^t = D$. For instance, if $I := (X_1, X_2)$, then clearly, in the Krull domain D, we have $I^v = I^t = D$. On the other hand, $I^* \subseteq M^* = M$, since $M \in \mathcal{J}$. More generally, and with a more careful analysis, we claim that, if $I := I_{ij} := (X_i, X_j)$, with $i \neq j \geq 1$, then $I^* = M$.

Case 1. For every $G \in \boldsymbol{f}(D)$, if $I \subseteq G^t$, then $I \subseteq I^* \subseteq M^* = M \subsetneq D = I^t \subseteq G^t$. Note that the same conclusion holds for every proper ideal A of D such that $A^t = D$, *i.e.*, for every $G \in \boldsymbol{f}(D)$ if $A \subseteq G^t$, then $A \subseteq M^* = M \subsetneq G^t$.

Case 2. If $I \subseteq yM$, for some $0 \neq y \in K$, then in particular $I \subseteq yD$ and so $D = I^t \subseteq yD$, hence, $y^{-1} \in D$. There are two possibilities here: either $y^{-1} \in M$ or $y^{-1} \in D\backslash M$. In the first case, *i.e.*, if $y^{-1} \in M$, then $1 \in yM$ and so $D \subseteq yM$. In the second case, *i.e.*, if $y^{-1} \in D\backslash M$, then y^{-1} is invertible in D, and so $y, y^{-1} \in D$. Thus, $yM = M$.

Note that the same conclusion holds for every proper ideal A of D such that $A^t = D$, *i.e.*, if $A \subseteq yM$, for some $0 \neq y \in K$ and $A^t = D$, then either $D \subseteq yM$ or $M = yM$.

Case 3. If $I \subseteq zM^2 \subseteq zM$, for some $0 \neq z \in K$, then as above $z^{-1} \in D$. Two cases are possible: either $z^{-1} \in M$ or $z^{-1} \in D\backslash M$. If $z^{-1} \in D\backslash M$, then z^{-1} is invertible in D and so $z, z^{-1} \in D$. Thus, $zM^2 = M^2$. However, this is impossible, since $I \nsubseteq M^2$. If $z^{-1} \in M$, then $M \subseteq zM^2$.

Note that a variation of the previous conclusion holds for every proper ideal A of D such that $A^t = D$ and $A \subseteq M^2$ (for instance, for $A = I^3$), *i.e.*, if $A \subseteq zM^2$, for some $0 \neq z \in K$, $A^t = D$ and $A \subseteq M^2$, then either $A \subseteq zM^2 = M^2$ or $A \subseteq M^2 \subset M \subseteq zM^2$.

By the previous analysis, we conclude in particular that $I^* = \bigcap\{J \in \mathcal{J} \mid J \supseteq I\} = M$. Moreover, since $I^* = M$, we obtain $(I^2)^* = (I \cdot I)^* = (I^* \cdot I^*)^* = (M^2)^* = M^2$. Furthermore, by the more general analysis for a proper ideal A of D such that $A^t = D$, in case $A = I^3$ we deduce in particular that $(I^3)^*$ also coincides with M^2. Therefore,

$$(I^3)^* = M^2 = (I^2)^* \quad \text{but} \quad (I^2)^* = M^2 \subsetneq I^* = M,$$

and so $*$ is not an *eab* star operation on D.

Remark 1. Let $\mathcal{J}' := \{xD, yM, zM^2 \mid x, y, z \in K\backslash\{0\}\}$. It is easy to see that [2, Proposition 32.4] guarantees that the set $\mathcal{J}'$ defines on D a star operation that coincides with the star operation $*$ defined above by the set $\mathcal{J}$, since $F^t = F^v = \bigcap\{xD \mid x \in K, F \subseteq xD\}$, for each $F \in \boldsymbol{f}(D)$ [2, Theorem 34.1 (1)].

We provide next a variation of the previous example in order to construct an eab star operation that is not ab.

Example 2. (*Example of an* eab *star operation that is not an* ab *star operation*) Let D, M and K be as above. Consider the following subset of fractional ideals of D:

$$\mathcal{S} := \{xF^b, yM \mid x, y \in K\backslash\{0\}, F \in \boldsymbol{f}(D)\},$$

where b is the standard ab operation on D defined by the set $\mathcal{V}$ of all valuation overrings of D, *i.e.*, for each $E \in \boldsymbol{F}(D)$,

$$E^b := E^{\wedge_{\mathcal{V}}} := \bigcap\{EV \mid V \in \mathcal{V}\}.$$

Since each nonzero principal fractional ideal of D is in $\mathcal{S}$ and, for each (fractional) ideal $J \in \mathcal{S}$ and for each nonzero $a \in K$, the (fractional) ideal aJ belongs to $\mathcal{S}$, as above, [2, Proposition 32.4] guarantees that the set $\mathcal{S}$ defines on D a star operation $*$.

We claim that $*$ is an eab operation. Since the b-operation is an ab operation, it is sufficient to prove that $*|_{f(D)} = b|_{f(D)}$. Suppose then that $F \in \boldsymbol{f}(D)$. Since $F^b \in \mathscr{S}$, it is clear that $F^* \subseteq F^b$. Note also that it is well-known that each prime ideal P of an integrally closed domain D is a b-ideal, since there always exists a valuation overring of D centered on P [2, Theorem 19.6]. It follows that each ideal of the form yM is a b-ideal and, hence, each ideal of $\mathscr{S}$ is a b-ideal. Since F^b is the intersection of all b-ideals which contain F, this implies that $F^b \subseteq F^*$ (the same conclusion follows also from [2, Proposition 32.2 (b)]). It follows that $*|_{f(D)} = b|_{f(D)}$ and, hence, $*$ is an eab operation.

Now, we claim that $*$ is not an ab operation on D.

To show this, we let $I := (X_1, X_2)$ and we prove that $(IM)^* = I^* = I$. This will show that $*$ is not ab, because we clearly cannot cancel I in the previous equation, *i.e.*, $(IM)^* = (ID)^*$ but $M^* = M \neq D = D^*$.

Therefore, we try to determine which (fractional) ideals in $\mathscr{S}$ contain IM. We know that I is in $\mathscr{S}$ (since $I \in \boldsymbol{f}(D)$ and I is a prime ideal of D. Thus, $I = I^b$) and I contains IM. What we really want to prove is that any (fractional) ideal in $\mathscr{S}$ which contains IM also contains I.

(1) First, suppose that $IM \subseteq yM$ for some nonzero element $y \in K$. This causes no problems if it also implies that $D \subseteq yM$, since then, in particular, we have $I \subseteq yM$, which is what we want.

Assume that y is a nonzero element of K and that $D \nsubseteq yM$. There are four possibilities here.

- (1, a) If y is not in D and y^{-1} is not in D, then $yM \cap D \subseteq yD \cap D \neq D$. Hence, $yD \cap D$ is a proper divisorial ideal of D containing IM. This contradicts the fact that $(IM)^v = D$.
- (1, b) If y is not in D and y^{-1} is in D, then y^{-1} is in M (since D is local) and so $D \subseteq yM$, which is a contradiction.
- (1, c) If y is in D and y is invertible in D, then $yM = M$, and so in this case $I \subseteq yM$, which is what we want.
- (1, d) If y is in D and y is not invertible in D, then $IM \subseteq yM \subseteq yD \subseteq M \neq D$. Again, this contradicts $(IM)^v = D$.

(2) Now suppose that $G \in \boldsymbol{f}(D)$ is such that $IM \subseteq G^* = G^b$. We extend everything to the b-Kronecker function ring of D, which is the following subring of the field of rational functions in one indeterminate, denoted by T, over K, *i.e.*:

$$\mathrm{Kr}(D,b) := \{f/g \in K(T) \mid f, g \in D[T], 0 \neq g, \boldsymbol{c}(f) \subseteq \boldsymbol{c}(g)^b\} = \bigcap\{V(T) \mid V \in \mathscr{V}\},$$

where $\boldsymbol{c}(h)$ is the content of a polynomial $h \in D[X]$ and $V(T) := \{f/g \in K(T) \mid f, g \in V[T], 0 \neq g$ and $\boldsymbol{c}(g) = V\}$ is *the trivial valuation extension of V to $K(T)$* [2, definitions at pages 218 and 401, Theorems 32.7 and 32.11, Proposition 33.1]. Then, we should have $I\,\mathrm{Kr}(D,b)M\,\mathrm{Kr}(D,b) \subseteq G^b\,\mathrm{Kr}(D,b) = G\,\mathrm{Kr}(D,b)$. Recall that $\mathrm{Kr}(D,b)$ is a Bézout domain and so both $I\,\mathrm{Kr}(D,b)$ and $G\,\mathrm{Kr}(D,b)$ are principal ideals. This means that we actually have $M\,\mathrm{Kr}(D,b) \subseteq G\,\mathrm{Kr}(D,b)(I\,\mathrm{Kr}(D,b))^{-1}$, the latter (fractional) ideal being principal.

There are two possibilities here.

- (2, a) $\mathrm{Kr}(D,b) \subseteq G\,\mathrm{Kr}(D,b)(I\,\mathrm{Kr}(D,b))^{-1}$. This would imply that $I\,\mathrm{Kr}(D,b) \subseteq G\,\mathrm{Kr}(D,b)$. This would in turn imply that $I = I^b \subseteq G^b = G^*$, which was our goal.
- (2, b) $\mathrm{Kr}(D,b) \nsubseteq G\,\mathrm{Kr}(D,b)(I\,\mathrm{Kr}(D,b))^{-1}$. Rename the principal (fractional) ideal $G\,\mathrm{Kr}(D,b) \cdot (I\,\mathrm{Kr}(D,b))^{-1}$ as $\mathscr{H}$. We know that $M\,\mathrm{Kr}(D,b) \subseteq \mathscr{H}$.

If $\mathscr{H}$ is an integral ideal of $\mathrm{Kr}(D,b)$, then obviously $M\,\mathrm{Kr}(D,b)$ is contained in a proper principal ideal of $\mathrm{Kr}(D,b)$. On the other hand, if $\mathscr{H}$ is not an integral ideal, then $\mathscr{H} \cap \mathrm{Kr}(D,b)$ is a proper integral ideal of $\mathrm{Kr}(D,b)$. Moreover, it is also finitely generated [2, Proposition 25.4 (1)] (hence, principal) in the Bézout domain $\mathrm{Kr}(D,b)$.

Therefore, in either case $M\,\mathrm{Kr}(D,b)$ is contained in a proper principal ideal of $\mathrm{Kr}(D,b)$. This will lead to a contradiction. As a matter of fact, suppose that $\varphi \in \mathrm{Kr}(D,b)$ is a nonzero nonunit rational function and that $M\,\mathrm{Kr}(D,b) \subseteq \varphi\,\mathrm{Kr}(D,b)$. This means that, for any natural number $n \geq 1$, we have $X_n \in \varphi\,\mathrm{Kr}(D,b)$. On the other hand, there are only a finite number of X_n that are part of the reduced representation of φ. Without loss of generality, suppose that these finitely many indices are $1, 2, \ldots, r$, *i.e.*, $\varphi \in k(X_1, X_2, \ldots, X_r; T)$ $(\subset K(T))$. Since φ is a nonunit in $\mathrm{Kr}(D,b)$, there must be a valuation overring V of D such that φ is a nonunit in the valuation overring $V(T)$ of $\mathrm{Kr}(D,b)$. Contract V to the subfield $k(X_1, X_2, \ldots, X_r)$ of K. Call this valuation domain V_r. Then, extend V_r trivially to K. Call this valuation domain W, *i.e.*, $W := V_r(X_{r+1}, X_{r+2}, \ldots)$. Clearly, W is a valuation overring of D. Then we have a contradiction, because φ is still a nonunit in the valuation overring $W(T)$ of $\mathrm{Kr}(D,b)$ and each X_n with $n > r$ is a unit in $W(T)$. This contradicts the fact that each X_n lies in the principal ideal $\varphi\,\mathrm{Kr}(D,b)$.

Therefore, Possibility (2, b) does not occur. Therefore, we have to fall back on Possibility (2, a) which implies that $I \subseteq G^b = G^*$, which was what we needed.

Acknowledgment. The first-named author was partially supported by a MIUR-PRIN grant 2008-2011, No. 2008WYH9NY.

References

[1] M. Fontana and K. A. Loper, *Cancellation properties in ideal systems: A classification of* e.a.b. *semistar operations*, J. Pure Appl. Algebra 213 (2009), 2095–2103.
[2] R. Gilmer, *Multiplicative ideal theory.* M. Dekker, New York, 1972.
[3] W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche*, *I–II*, Math. Z. 41 (1936), 545–577; 665–679.
[4] Wang Fanggui and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra 25 (1997), 1285–1306.

Dipartimento di Matematica, Università degli Studi "Roma Tre", 00146 Rome, Italy.
E-mail address: fontana@mat.uniroma3.it

Department of Mathematics, Ohio State University, Newark, Ohio 43055, USA.
E-mail address: lopera@math.ohio-state.edu

Department of Mathematics, Ibaraki University, Mito, Ibaraki 310-8512, Japan.
E-mail address: rmazda@adagio.ocn.ne.jp

CANCELLATION PROPERTIES IN IDEAL SYSTEMS: AN *e.a.b.* NOT *a.b.* STAR OPERATION

MARCO FONTANA, K. ALAN LOPER, AND RYÛKI MATSUDA

الملخص:

نبين أن شرط اختزال *a.b.* لعمليّات النجمة الذي قدمه Krull أقوى من شرط اختزال *e.a.b.* الذي قدمه Gilmer.