On some classes of integral domains defined by Krull's *a.b.* operations ☆

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ABSTRACT

Let D be an integral domain with quotient field K. The b-operation that associates to each nonzero D-submodule E of K, $E^b := \bigcap \{EV \mid V$ valuation overring of $D\}$, is a semistar operation that plays an important role in many questions of ring theory (e.g., if I is a nonzero ideal in D, I^b coincides with its integral closure). In a first part of the paper, we study the integral domains that are b-Noetherian (i.e., such that, for each nonzero ideal I of D, $I^b = J^b$ for some a finitely generated ideal J of D). For instance, we prove that a b-Noetherian domain has Noetherian spectrum and, if it is integrally closed, is a Mori domain, but integrally closed Mori domains with Noetherian spectra are not necessarily b-Noetherian. We also characterize several distinguished classes of b-Noetherian domains. In a second part of the paper, we study more generally the e.a.b. semistar operation of finite type $\star_a$ canonically associated to a given semistar operation $\star$ (for instance, the b-operation is the e.a.b. semistar operation of finite type canonically associated to the identity operation). These operations, introduced and studied by Krull, Jaffard, Gilmer and Halter-Koch, play a very important role in the recent generalizations of the Kronecker function ring. In particular, in the present paper, we classify several classes of integral domains having some of the fundamental operations d, t, w and v equal to some of the canonically associated e.a.b. operations b, t_a, w_a and v_a.

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1. Introduction and background results

Let D be an integral domain with quotient field K. We denote by $\overline{\boldsymbol{F}}(D)$ the set of all nonzero D-submodules of K, by $\boldsymbol{F}(D)$ the set of nonzero fractional ideals of D and by $\boldsymbol{f}(D)$ the set of nonzero finitely generated fractional ideals of D. Recall that a *star operation on D* is a map $* : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $I \mapsto I^*$, such that for all $z \in K$, $z \neq 0$ and for all $I, J \in \boldsymbol{F}(D)$, the following properties hold: $(*_1)$ $D^* = D$ and $(zI)^* = zI^\star$; $(*_2)$ $I \subseteq J \Rightarrow I^* \subseteq J^*$; $(*_3)$ $I \subseteq I^*$ and $I^{**} := (I^*)^* = I^*$ [G-1972, Section 32].

In [G-1972], it is shown that if D is an integrally closed domain, the completion of ideals, that is the map which associates to a nonzero fractional ideal I the fractional ideal $I^b := \bigcap IV$, where V varies over all valuations overrings of D, defines a star operation. Note that if D is not integrally closed, the map b still satisfies most of the properties of star operations; the only problem is that D^b (which coincides with the integral closure of D, by Krull Theorem [G-1972, Theorem 19.8]) is a proper overring of D and is not necessarily a fractional ideal of D. This observation leads in a natural way to the notion of semistar operation [OM-1994].

A *semistar operation on D* is a map $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$ such that for all $z \in K$, $z \neq 0$ and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold: $(\star_1)$ $(zE)^\star = zE^\star$; $(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$; $(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

The b-operation is a semistar operation even if D is not integrally closed.

A *(semi)star operation on D* is a semistar operation $\star$ such that $D^\star = D$, i.e., a (semi)star operation is a semistar operation which restricted to $\boldsymbol{F}(D)$ is a star operation. Conversely, if $*$ is a star operation on an integral domain D (hence, defined only on $\boldsymbol{F}(D)$), we can extend it trivially to a semistar (in fact, (semi)star) operation on D, denoted $*_e$, by defining E^{*_e} to be the quotient field of D whenever $E \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$. Therefore, all "classical" star operation examples can be considered semistar examples as well. Note that, in general, a star operation $*$ can be extended in several different ways to a semistar operation and $*_e$ is just one possible (trivial) way to do so.

As in the classical star-operation setting, we associate to a semistar operation $\star$ on D a new semistar operation $\star_f$ of D as follows. If $E \in \overline{\boldsymbol{F}}(D)$, we set:

$$E^{\star_f} := \bigcup \{ F^\star \mid F \subseteq E, \ F \in \boldsymbol{f}(D) \}.$$

We call $\star_f$ *the semistar operation of finite type on D associated to $\star$*. If $\star = \star_f$, we say that $\star$ is *a semistar operation of finite type on D*. Given two semistar operations $\star'$ and $\star''$ of D, we say that $\star' \leqslant \star''$ if $E^{\star'} \subseteq E^{\star''}$ for all $E \in \overline{\boldsymbol{F}}(D)$. Note that $\star_f \leqslant \star$ and $(\star_f)_f = \star_f$, so $\star_f$ is a semistar operation of finite type of D.

If $\star$ coincides with the semistar *v-operation of D*, defined by $E^v := (D : (D : E))$, for each $E \in \overline{\boldsymbol{F}}(D)$, then v_f is denoted by t. Note that v [respectively, t] restricted to $\boldsymbol{F}(D)$ coincides with the classical *star v-operation* [respectively, *t-operation*] of D. Furthermore, the v-semistar operation is the trivial extension of the v-star operation. The *identity semistar operation on D*, i.e., the operation denoted by d and defined by $E^d := E$ for all $E \in \overline{\boldsymbol{F}}(D)$, is the smallest semistar operation on D. Clearly, d, t and v are examples of (semi)star operations on D.

Examples of semistar operations (not necessarily star operations) can be given as follows: Let $\mathcal{T} := \{T_\lambda \mid \lambda \in \Lambda\}$ be a family of overrings of D, then the operation $\wedge_\mathcal{T}$ defined by $E^{\wedge_\mathcal{T}} := \bigcap \{ET_\lambda \mid \lambda \in \Lambda\}$ for all $E \in \overline{\boldsymbol{F}}(D)$ is a semistar operation of D and it is a (semi)star operation if and only if $\bigcap \{T_\lambda \mid \lambda \in \Lambda\} = D$. In particular, if $\mathcal{T} := \{T\}$ is a family consisting of a unique proper overring T of D, then we have that $E^{\wedge_\mathcal{T}} := E^{\wedge_{\{T\}}} = ET$ is a semistar, but not a (semi)star, operation of D. If $\iota : D \hookrightarrow T$ is the canonical embedding of D in one of its overrings and if $\star$ is a semistar operation on T, we denote by $\star^\iota$ the semistar operation defined on D by $E^{\star^\iota} := (ET)^\star$, for all $E \in \overline{\boldsymbol{F}}(D)$. More generally, it is easy to see that, if $\star_\lambda$ is a semistar operation of T_λ and $\iota_\lambda : D \hookrightarrow T_\lambda$ is the canonical embedding for $\lambda \in \Lambda$, then $E^{\wedge\{\star_\lambda \mid \lambda \in \Lambda\}} := E^{\wedge\{(\star_\lambda)^{\iota_\lambda} \mid \lambda \in \Lambda\}} := \bigcap \{(ET_\lambda)^{\star_\lambda} \mid \lambda \in \Lambda\}$, for all $E \in \overline{\boldsymbol{F}}(D)$, defines a semistar operation of D.

We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* if $I^\star \cap D = I$, a *quasi-$\star$-prime* if it is a prime quasi-$\star$-ideal, and a *quasi-$\star$-maximal* if it is maximal in the set of all proper quasi-$\star$-ideals. A quasi-$\star$-maximal ideal is a prime ideal. It is possible to prove that each proper quasi-$\star_f$-ideal is contained

in a quasi-$\star_f$-maximal ideal. More details can be found in [FL-2003, p. 4781]. We will denote by QMax$^\star(D)$ the set of the quasi-$\star$-maximal ideals of D. By the previous considerations we have that QMax$^\star(D)$ is not empty, for all semistar operations $\star$ of finite type. When $\star$ is a (semi)star operation, the condition $I^\star \cap D = I$ becomes $I^\star = I$ and we simply say that I is a $\star$-*ideal* and we will denote by Max$^\star(D)$ the set of the $\star$-maximal ideals of D (i.e., the maximal elements in the set of all proper $\star$-ideals).

A semistar operation $\star$ is called *stable* if $(E \cap F)^\star = E^\star \cap F^\star$, for all $E, F \in \overline{\boldsymbol{F}}(D)$. By using the localizations at the quasi-$\star_f$-maximal ideals, we can associate to $\star$ a semistar operation stable and of finite type, as follows. For each $E \in \overline{\boldsymbol{F}}(D)$, we set

$$E^{\tilde{\star}} := \bigcap \{ ED_Q \mid Q \in \text{QMax}^{\star_f}(D) \}.$$

The previous definition gives rise to a semistar operation $\tilde{\star}$ on D which is stable and of finite type, called *the semistar operation stable of finite type associated to* $\star$ [FH-2000, Corollary 3.9].

Recall that, if K is the quotient field of D and X is an indeterminate over K, the integral domain $\text{Na}(D, \star) := \{ f/g \in K(X) \mid f, g \in D[X], 0 \neq g \text{ and } \boldsymbol{c}(g)^\star = D^\star \}$, overring of $D[X]$, is called *the Nagata ring of D associated to the semistar operation* $\star$. It is known that $E^{\tilde{\star}} = E\text{Na}(D, \star) \cap K$ for all $E \in \overline{\boldsymbol{F}}(D)$ [FL-2003, Proposition 3.4(3)]. It is easy to see that $\tilde{\star} \leqslant \star_f \leqslant \star$. We denote by w the semistar operation stable of finite type associated to v, i.e., $w := \tilde{v}$. Since $D^w = D$ [Gr-1967, Proposition 4], w is also an example of (semi)star operation.

Let $\star$ be a semistar operation on D. If F is in $\boldsymbol{f}(D)$, we say that F is $\star$-*eab* [respectively, $\star$-*ab*] if $(FG)^\star \subseteq (FH)^\star$ implies that $G^\star \subseteq H^\star$, with $G, H \in \boldsymbol{f}(D)$ [respectively, with $G, H \in \overline{\boldsymbol{F}}(D)$].

The operation $\star$ is said to be *eab* [respectively, *ab*] if each $F \in \boldsymbol{f}(D)$ is $\star$-*eab* [respectively, $\star$-*ab*]. An *ab* operation is obviously an *eab* operation.

Note that if $\star$ is an *eab* semistar operation then $\star_f$ is also an *eab* semistar operation, since they agree on all finitely generated ideals. Note also that, if $\star$ is a semistar operation of finite type, then $\star$ is an *eab* semistar operation if and only if $\star$ is an *ab* semistar operation; therefore, in the finite type setting, we use the terminology of (e)ab semistar operation. In general, we have that the notions of $\star$-*eab* ideal and $\star_f$-(e)ab ideal coincide, therefore, $\star$ is an *eab* semistar operation if and only if $\star_f$ is an (e)ab semistar operation [FL-2009, Proposition 4].

Using the fact that, given $F \in \boldsymbol{f}(D)$, F is $\star$-*eab* if and only if $((FH)^\star : F^\star) = H^\star$, for each $H \in \boldsymbol{f}(D)$ [FL-2009, Lemma 8], we can associate to any semistar operation $\star$ on D an (e)ab semistar operation of finite type $\star_a$ on D, called *the (e)ab semistar operation associated to* $\star$, defined as follows for each $F \in \boldsymbol{f}(D)$ and for each $E \in \overline{\boldsymbol{F}}(D)$:

$$F^{\star_a} := \bigcup \{ ((FH)^\star : H^\star) \mid H \in \boldsymbol{f}(D) \},$$

$$E^{\star_a} := \bigcup \{ F^{\star_a} \mid F \subseteq E, \ F \in \boldsymbol{f}(D) \}$$

[FL-2001a, Definition 4.4 and Proposition 4.5]. The previous construction, in the ideal systems setting, is essentially due to P. Jaffard [J-1960] and F. Halter-Koch [HK-1997,HK-1998]. The overring $D^{\star_a} = \bigcup \{ (H^\star : H^\star) \mid H \in \boldsymbol{f}(D) \}$ of D is called *the $\star$-integral closure of* D. Obviously $(\star_f)_a = \star_a$. Moreover, when $\star = \star_f$, then $\star$ is (e)ab if and only if $\star = \star_a$ [FL-2001a, Proposition 4.5(5)].

For a domain D and a semistar operation $\star$ of D, we say that a valuation overring V of D is *a $\star$-valuation overring of D* provided $F^\star \subseteq FV$ (or, equivalently, $F^\star V = FV$) for each $F \in \boldsymbol{f}(D)$. Set $\mathcal{V}(\star) := \{ V \mid V \text{ is a } \star\text{-valuation overring of } D \}$. The semistar operation on D defined as follows: for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{b(\star)} := \bigcap \{ EV \mid V \in \mathcal{V}(\star) \} = E^{\wedge \mathcal{V}(\star)},$$

is an *ab* semistar operation on D [FL-2009, p. 2098]; clearly, $b(\star) = b(\star_f)$ and $b(\star)$ is a (semi)star operation on D if and only if D is $\star$-integrally closed, i.e. $D = D^{\star_a}$.

Let X be an indeterminate over D and $\boldsymbol{c}(h)$ the content of a polynomial $h \in D[X]$. Then, we define

$$\mathrm{Kr}(D, \star) := \big\{ f/g \mid f, g \in D[X], \ g \neq 0, \ \text{and there exists}$$

$$h \in D[X] \setminus \{0\} \ \text{with} \ \boldsymbol{c}(f)\boldsymbol{c}(h) \subseteq \big(\boldsymbol{c}(g)\boldsymbol{c}(h)\big)^{\star}\big\}.$$

This is a Bézout domain with quotient field $K(X)$, called *the semistar Kronecker function ring associated to a semistar operation* $\star$ [FL-2001a, Theorem 5.1 and Theorem 3.11 (3)]. Furthermore, $\mathrm{Kr}(D, \star) = \bigcap\{V(X) \mid V \in \mathcal{V}(\star)\}$ [FL-2001a, Corollary 3.8 and Theorem 5.1]. A key fact is the following ([FL-2001a, Corollary 3.4] and [FL-2003, Proposition 4.1(5)]): for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\star_a} = E^{b(\star)} = E \,\mathrm{Kr}(D, \star) \cap K.$$

Finally, recall that a nonzero fractional ideal I of D is called $\star$-*invertible* if $(II^{-1})^{\star} = D^{\star}$ and a domain D is a *Prüfer $\star$-multiplication domain* (for short, P$\star$MD) if every finitely generated ideal of D is $\star_f$-invertible [HMM-1984, p. 48]. For $\star = d$, a Prüfer d-multiplication domain coincides with a Prüfer domain; for $\star = v$, the PvMD's or Prüfer v-multiplication domains generalize at the same time Prüfer and Krull domains ([Gr-1967] and [MZ-1981]).

After collecting, in Section 2, some properties of the b-operation needed later, in Section 3, we study the integral domains that are b-Noetherian (i.e., such that, for each nonzero ideal I of D, $I^b = J^b$ for some finitely generated ideal J of D). For instance, we prove that a b-Noetherian domain has Noetherian spectrum and, if it is integrally closed, is a Mori domain, but integrally closed Mori domains with Noetherian spectra are not necessarily b-Noetherian. We also characterize several distinguished classes of b-Noetherian domains and we investigate the local-global behavior of the b-Noetherianity.

In Section 4, we study more generally the $\mathtt{eab}$ semistar operation of finite type $\star_a$ canonically associated to a given semistar operation $\star$ (for instance, the b-operation is the $\mathtt{eab}$ semistar operation of finite type canonically associated to the identity operation). These operations, introduced and studied by Krull, Jaffard, Gilmer and Halter-Koch, play a very important role in the recent generalizations of the Kronecker function ring. In particular, in the present section, we classify several classes of integral domains having some of the fundamental operations d, t, w and v equal to some of the canonically associated $\mathtt{eab}$ operations b, t_a, w_a and v_a. For instance, the integral domains such that v coincides with v_a [respectively, t coincides with t_a; w coincides with w_a] (considered as star operations) are exactly the integrally closed domains such that $\mathrm{Na}(D, v)$ is a divisorial domain [respectively, the v-domains; the Prüfer v-multiplication domains].

2. Elementary properties of the *b*-operation

Given $E \in \overline{\boldsymbol{F}}(D)$, an element z of K is said to be *integrally dependent on E* if it satisfies an equation of the form $z^q + a_1 z^{q-1} + \cdots + a_q = 0$, where $q \geqslant 1$ and $a_i \in E^i$ for all $i = 1, 2, \ldots, q$ [ZS-1960, Appendix 4, p. 349]. Equivalently, z is integrally dependent on E if (and only if) there exists a nonzero finitely generated D-submodule H of K such that $zH \subseteq EH$.

It turns out that the set of the elements that are integrally dependent on E is a D-submodule of K which coincides with *the completion of E*, i.e., with the D-submodule (denoted here by) E^b of K [ZS-1960, Appendix 4, Definition 1 and Theorem 1].

In other words, for all $E \in \overline{\boldsymbol{F}}(D)$:

$$\bigcup\big\{(EH : H) \mid H \in \boldsymbol{f}(D)\big\} = E^b = \bigcap\big\{EV \mid V \text{ valuation overring of } D\big\}.$$

Therefore, the b-operation coincides with the $\mathtt{eab}$-semistar operation canonically associated to the identity (semi)star operation, i.e. $b = b(d)$. In particular, the b-operation is an $\mathtt{eab}$ (in fact, $\mathtt{ab}$) semistar operation of finite type. Note that the quasi-b-ideals are exactly the ideals which are integrally closed (in D).

We collect in the following lemma some elementary facts about the b-operation.

Lemma 1. *Let D be an integral domain.*

(1) $b = d_a$.
(2) b is an ab semistar operation of finite type.
(3) $P^b \cap D = P$ for each nonzero prime ideal P of D.
(4) $(\sqrt{I})^b \cap D = \sqrt{I}$ for each nonzero ideal I of D.
(5) $\tilde{b} = d$. In particular, an ideal of D is b-invertible if and only if it is invertible.

Proof. For (1) and (2), see the comments preceding Lemma 1.

(3) If P is a nonzero prime ideal, there exists a valuation overring (V, M) of D centered in P [G-1972, Theorem 19.6], and so $P^b \cap D \subseteq PV \cap D \subseteq M \cap D = P$.

(4) The previous statement ensures that every nonzero prime ideal is a quasi-b-ideal. Radical ideals are also quasi-b-ideals, as intersections of quasi-b-ideals.

(5) The equality $\tilde{b} = d$ follows from (3) (and from the definition of $\tilde{b}$). The fact that b-invertible ideals are invertible is the consequence of the fact that an ideal is $\star_f$-invertible if and only if it is $\tilde{\star}$-invertible (cf. for example [FP-2005, Theorem 2.18]). Explicitly, in the present situation, for all $F \in \boldsymbol{f}(D)$, $(FF^{-1})^b = D^b$ is equivalent to $FF^{-1} \nsubseteq M$ for all $M \in \mathrm{QMax}^b(D) = \mathrm{Max}(D)$, i.e., $FF^{-1} = D$. $\square$

Let $\mathbf{SStar}(D)$ [respectively, $\mathbf{SStar}_f(D)$] be the set of all semistar operations [respectively, all semistar operations of finite type] on D. We can consider the maps $\widetilde{(\ldots)} \colon \mathbf{SStar}(D) \to \mathbf{SStar}_f(D)$, $\star \mapsto \tilde{\star}$, and $(\ldots)_a \colon \mathbf{SStar}(D) \to \mathbf{SStar}_f(D)$, $\star \mapsto \star_a$.

The relations among $\widetilde{(\star_a)}$, $(\tilde{\star})_a$, $\tilde{\star}$, and $\star_a$ were already investigated in [FL-2003]. The next goal is to answer the following natural question: when do the maps $(\ldots)_a$ and $\widetilde{(\ldots)}$ establish a bijection on $\mathbf{SStar}_f(D)$?

Note that $\widetilde{(d_a)} = d$, but $(\tilde{d})_a = b = d_a$, and $(\tilde{b})_a = b$, but $\widetilde{(b_a)} = d = \tilde{b}$. Therefore, using also [G-1972, Theorem 24.7], it is easy to verify the next lemma.

Lemma 2. *Let D be an integral domain. The following statements are equivalent.*

(i) *The maps $(\ldots)_a$ and $\widetilde{(\ldots)}$ establish a bijection on* $\mathbf{SStar}_f(D)$.
(ii) *D is a Prüfer domain.*
(iii) *$d = b$.*
(iv) $\mathbf{SStar}_f(D) = \{d\}$.

3. b-Noetherian domains

Recall that an integral domain D is $\star$-*Noetherian* if the ascending chain condition on the quasi-$\star$-ideals of D holds. If $\star = d$ (where d is the identity (semi)star operation), we have the classical Noetherian domains, if $\star = v$ this definition gives back the Mori domains [Ba-2000, Theorem 2.1], and if $\star = w$ we obtain the class of strong Mori domains [WMc-1997]. It is well known that a Noetherian domain is characterized by the fact that each ideal is finitely generated. The semistar version of this characterization uses the concept of $\star_f$-finiteness: if $E \in \overline{\boldsymbol{F}}(D)$, we say that E is $\star_f$-*finite* if there exists $F \in \boldsymbol{f}(D)$, $F \subseteq E$, such that $F^{\star_f} = E^{\star_f}$ (see for instance [FP-2005, p. 650]).

A $\star$-Noetherian domain is characterized by the fact that each nonzero ideal of D is $\star_f$-finite [EFP-2004, Lemma 3.3].

Note that, from this characterization, it follows that D is $\star$-Noetherian if and only if it is $\star_f$-Noetherian. Finally, we notice that if $\star_1 \leqslant \star_2$ are two semistar operations on D, then D is $\star_1$-Noetherian implies D is $\star_2$-Noetherian. Note that the converse does not hold, since a domain can be $\star_f$-Noetherian, but not $\tilde{\star}$-Noetherian [WMc-1999, p. 159]. When $\star = \tilde{\star}$, i.e. when $\star$ is a stable semistar operation of finite type, Picozza has shown that several classical properties of Noetherian domains can be extended to $\tilde{\star}$-Noetherian domains [Pi-2007]. This is not true for general semistar operations: for instance if D is a (non-integrally closed) v-Noetherian domain, then $D[X]$ is not necessarily v-Noetherian [Q-1980, §3, Théorème 5] and [Ro-1990, Theorem 8.4], but it is true that if D

is w-Noetherian, then $D[X]$ is w-Noetherian [WMc-1999, Theorem 1.13], [Pk-2001, Theorem 4.7] and [C-2005, Theorem 2.2].

Remark 3. When $*$ is a star operation on D, it is clear that, if D is $*$-Noetherian then $*$ is a star operation of finite type (that is, $* = *_f$). Indeed, for each nonzero (fractional) ideal I of D there exists a finitely generated (fractional) ideal F of D, $F \subseteq I$, such that $I^* = F^*$. So, $I^* = F^* = F^{*_f}$, thus $I = I^{*_f}$.

When $\star$ is a semistar operation on D, it is still true that $I^\star = I^{\star_f}$ for each nonzero (fractional) ideal I of D, but this is not enough to say that $\star$ is a semistar operation on D of finite type (even if $\star$ is a (semi)star operation). For instance, let D be a Noetherian domain that is not conducive (that is, there exists a proper overring T of D, $T \neq K$, such that $T \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$, i.e., $(D : T) = (0)$ [DF-1984]). Consider the semistar operation d_e on D defined as follows: $E^{d_e} := E$ if $E \in \boldsymbol{F}(D)$ and $E^{d_e} := K$ otherwise (this semistar operation is the so-called *trivial semistar extension of the identity star operation d on D* [OM-1994, Proposition 17]). It is clear that D is d_e-Noetherian, but it is easy to check that d_e is not a semistar operation of finite type.

In particular, a b-Noetherian domain is a domain in which the ascending chain condition on quasi-b-ideals holds. Equivalently, since b is a semistar operation of finite type, if for every nonzero (fractional) ideal I of D, I is b-finite, that is, there exists a finitely generated (fractional) ideal F (which can be taken inside I by [FP-2005, Lemma 2.3], since b is of finite type), such that $F^b = I^b$.

The next goal is to give an example of a b-Noetherian domain that is not Noetherian.

Lemma 4. *Let D be an integral domain and $\overline{D}$ its integral closure. Set $b := b_D$ and $\bar{b} := b_{\overline{D}}$.*

(1) *If $\overline{D}$ is $\bar{b}$-Noetherian, then D is b-Noetherian.*
(2) *If D is b-Noetherian and $(D : \overline{D}) \neq (0)$, then $\overline{D}$ is $\bar{b}$-Noetherian.*

Proof. Let $\iota : D \hookrightarrow \overline{D}$ be the canonical inclusion. Note that the semistar operation $\bar{b}^\iota$ on D, defined by $E^{\bar{b}^\iota} := (E\overline{D})^{\bar{b}}$ for all $E \in \overline{\boldsymbol{F}}(D)$, coincides with b, since $\overline{D}$ and D have the same valuation overrings. Conversely, for the same reason, the semistar operation b_ι on $\overline{D}$ defined by $E^{b_\iota} := E^b$ for all $E \in \overline{\boldsymbol{F}}(\overline{D})(\subseteq \overline{\boldsymbol{F}}(D))$, coincides with $\bar{b}$.

(1) It is not difficult to see that if $i : D \hookrightarrow T$ is the canonical inclusion of D in its overring T and if $\star'$ is a semistar operation on T, then T $\star'$-Noetherian implies that D is $(\star')^i$-Noetherian [EFP-2004, Lemma 3.1(2)]. For $T = \overline{D}$, $\star' = \bar{b}$, and $i = \iota$, if $\overline{D}$ is $\bar{b}$-Noetherian, then we conclude that D is b-Noetherian.

(2) Suppose that D is b-Noetherian and let J be a nonzero ideal of $\overline{D}$. For any $0 \neq x \in (D : \overline{D})$, $I := xJ$ is a nonzero ideal of D and so for some $F \in \boldsymbol{f}(D)$, with $F \subseteq I$, $F^b = I^b$. Therefore, if $G := x^{-1}F\overline{D}$, then $G \in \boldsymbol{f}(\overline{D})$, $G \subseteq J$ and $G^{\bar{b}} = G^b = x^{-1}(F\overline{D})^b = x^{-1}F^b = x^{-1}I^b = J^b = J^{\bar{b}}$. $\square$

Example 5 (*Example of non-Noetherian b-Noetherian domain with Noetherian integral closure*). Let D be a non-Noetherian domain with Noetherian integral closure $\overline{D}$ (e.g., $D := \mathbb{Q} + X\overline{\mathbb{Q}}[\![X]\!]$, where $\overline{\mathbb{Q}}$ is the field of algebraic numbers, i.e., the algebraic closure of the field of rational numbers $\mathbb{Q}$ in $\mathbb{C}$. Note that D is a 1-dimensional Mori non-Noetherian non-integrally closed local domain with Noetherian spectrum and integral closure $\overline{D} = \overline{\mathbb{Q}}[\![X]\!]$, which is a Noetherian domain [Ba-1983, Theorem 3.2], [F-1980, Corollary 15(5), Propositions 1.8 and 2.1(7)]). Since $\overline{D}$ is Noetherian, it is $b_{\overline{D}}$-Noetherian. If ι is the canonical embedding of D in $\overline{D}$, from Lemma 4(1), we have that D is b_D-Noetherian.

Proposition 6. *Let D be a b-Noetherian domain. Then,*

(1) *D has Noetherian spectrum.*
(2) *Let V be a valuation overring of D. Then, for every (fractional) ideal I of D, IV is a principal (fractional) ideal of V. In particular, if V is an essential valuation overring of D, V is a rank 1 discrete valuation domain.*
(3) *Assume, moreover, that D is integrally closed, then D is a Mori domain.*

Proof. (1) It is well known that $\mathrm{Spec}(D)$ is Noetherian if and only if each prime ideal of D is the radical of a finitely generated ideal (cf., for instance, [FHP-1997, Theorem 3.1.11]). Let P be a nonzero prime ideal of D. Since D is b-Noetherian, there exists a finitely generated ideal F of D, $F \subseteq P$, such that $F^b = P^b$. Since $F \subseteq \sqrt{F} \subseteq P$, we have $P^b = F^b \subseteq \sqrt{F}^b \subseteq P^b$. By Lemma 1(5), $\sqrt{F} = \sqrt{F}^b \cap D = P^b \cap D = P$ and P is the radical of a finitely generated ideal.

(2) Let I be a nonzero (fractional) ideal of D. By b-Noetherianity, there exists a finitely generated (fractional) ideal F of D, $F \subseteq I$, such that $I^b = F^b$. Thus $IV = I^b V = F^b V = FV = aV$ for some $a \in F$. If, moreover, V is an essential valuation overring of D, then $V = D_P$ for some prime ideal P of D. So, for each nonzero prime ideal Q of D, $Q \subseteq P$, $Q D_P$ is a principal ideal of D_P. Therefore, D_P is a Noetherian domain, thus D_P is a rank 1 discrete valuation domain.

(3) Note that, in this case $D = D^b$, since D^b coincides with the integral closure of D, and so $b \leqslant t$ (see also [G-1972, Theorem 34.1(4)]), because we have already observed that b is a semistar operation of finite type (Lemma 1(2)). Therefore, since D is b-Noetherian, D is also t-Noetherian (i.e., Mori). $\quad\square$

Remark 7. (a) By Proposition 6(2), if V is a valuation overring of a b-Noetherian domain D such that each ideal of V is the extension of a fractional ideal of D, then V is a DVR. This is the case, for example, of the following situations:

(1) $(D : V) \neq (0)$ (or, equivalently, D is a *conducive domain* [DF-1984, Theorem 3.2], i.e., for every overring T of D, $(D : T) \neq (0)$);
(2) V is flat over D (or, equivalently, V is an essential valuation overring of D);
(3) V well-centered on D (i.e., if each principal ideal of V is generated by an element of D [HR-2004, p. 435]).

The cases (2) and (3) are considered by Sega [Se-2007, Proposition 3.8].

(b) From (a), we deduce that a b-Noetherian valuation domain (as a valuation overring of a b-Noetherian domain) is a DVR. Note that this property can be observed also as a straightforward consequence of the fact that in a valuation domain $b = d$ (Lemma 2).

Recall that a Noetherian conducive domain (not a field) is one-dimensional and local [DF-1984, Corollary 2.7]. More generally, one can easily deduce from [HR-2004, Proposition 3.21] that the same holds for Mori domains. From Remark 7, we obtain the same result for b-Noetherian conducive domains.

Proposition 8. *Let D be a b-Noetherian conducive domain with quotient field K, $D \neq K$. Then, D is local and one-dimensional.*

Proof. By Remark 7(a), every valuation overring of D is a DVR. So, D has dimension 1 [G-1972, Theorem 30.8]. On the other hand, conducive domains have at most one prime of height 1 [DF-1984, Theorem 2.4], so D is necessarily local. $\quad\square$

From Proposition 6(2), we deduce immediately the following.

By the previous Proposition 6, for finding an example of a b-Noetherian non-Noetherian integrally closed domain, one should look among the examples of (non-Noetherian integrally closed) Mori domains with Noetherian spectrum.

Recall that a Prüfer v-multiplication domain is characterized by the fact that the localizations at its t-maximal ideals are valuation domains [Gr-1967, Theorem 5]. Since each domain is intersection of the localizations at its t-maximal ideals [Gr-1967, Proposition 4], a PvMD is integrally closed.

It is easy to prove the following.

Proposition 9.

(1) *Prüfer b-multiplication domains coincide with Prüfer domains.*

(2) *The following classes of integral domains coincide*:
  (i) *b-Noetherian Prüfer domains*;
  (ii) *b-Dedekind domains* (*i.e., b-Noetherian Prüfer b-multiplication domains* [EFP-2004, Proposition 4.1]);
  (iii) *Dedekind domains.*

Proof. (1) Recall that $\tilde{b} = d$; we have also observed that a nonzero fractional ideal is b-invertible if and only if it is invertible (Lemma 1(3)).

(2) It is enough to recall that an integral domain is Prüfer if and only if $d = b$ (Lemma 2) and that a Noetherian Prüfer domain is a Dedekind domain. $\square$

In Proposition 9(2)(i), if we replace the assumption "Prüfer domain" with the weaker assumption "Prüfer v-multiplication domain", we obtain the following.

Corollary 10. *A b-Noetherian Prüfer v-multiplication domain is a Krull domain.*

Proof. As observed above, a PvMD is integrally closed and an integrally closed b-Noetherian domain is Mori (Proposition 6(3)). The conclusion follows from the fact that a Mori PvMD is Krull [Kg-1989, Theorem 3.2]. $\square$

In Theorem 21 we will further extend the previous corollary.

Remark 11. (a) Note that a Krull domain is not necessarily b-Noetherian (even if it is always a PvMD). For instance, if K is a field and $X_1, X_2, \ldots, X_n, \ldots$ is a countable family of indeterminates over K, then $D := K[X_1, X_2, \ldots, X_n, \ldots; n \geqslant 1]$ is a Krull domain [Bk-1965, Chapitre 7, §1, Exercice 8], but it is not b-Noetherian, since the ascending chain of prime (b-)ideals of D given by $(X_1) \subsetneq (X_1, X_2) \subsetneq (X_1, X_2, X_3) \subsetneq \cdots$ is not stationary. This example (which is, in particular, an integrally closed Mori domain) also shows that the conclusion of statement (3) of Proposition 6 does not imply b-Noetherianity.

(b) The conclusion of statement (2) of Proposition 6 is also not sufficient to have a b-Noetherian domain. Take, for instance, an almost Dedekind domain which is not Dedekind.

(c) A 2-dimensional valuation domain has Noetherian spectrum but it is not b-Noetherian (Remark 7(b)). Therefore, the conclusion of statement (1) of Proposition 6, even in the integrally closed case, does not imply b-Noetherianity

(d) From Proposition 6(2) (or, from Corollary 10) it follows that if D is a b-Noetherian PvMD then it has t-dimension 1, since the localizations D_Q are DVR's for each $Q \in \mathrm{Max}^t(D)$.

Example 12 (*Examples of Mori integrally closed domains with Noetherian spectrum that are not b-Noetherian*). **(a)** Take any DVR (V, M), let $\pi : V \to V/M$ be the canonical projection. Assume that k is a proper subfield of the residue field $\boldsymbol{k}(V) := V/M$ and that k is algebraically closed in $\boldsymbol{k}(V)$. The domain $D := \pi^{-1}(k)$ is a non-Noetherian integrally closed Mori domain [Ba-1983, Theorem 3.2 or Proposition 3.4] (or, [Ba-2000, Theorem 2.2]) and, clearly, $\mathrm{Spec}(D)$ is Noetherian, since $\mathrm{Spec}(D) = \mathrm{Spec}(V)$ [AD-1980, Corollary 3.11]. However, this domain is not b-Noetherian by Proposition 6(2), since D admits valuation overrings with non-principal extended ideals, because in the present situation $\mathrm{tr.deg}_k(\boldsymbol{k}(V)) \geqslant 1$. For instance, let $\mathbb{C}$ be the field of complex numbers and let X and Y be two indeterminates over $\mathbb{C}$. Take $V := \mathbb{C}(X)[\![Y]\!]$, $M := Y\mathbb{C}(X)[\![Y]\!]$, and $k := \mathbb{C}$. Consider $D := \mathbb{C} + Y\mathbb{C}(X)[\![Y]\!]$. Clearly, D is an integrally closed local 1-dimensional Mori domain with Noetherian spectrum (homeomorphic to $\mathrm{Spec}(\mathbb{C}(X)[\![Y]\!])$). Set $W := \mathbb{C}[X]_{(X)} + Y\mathbb{C}(X)[\![Y]\!]$. Then, W is a 2-dimensional discrete valuation overring of D with height 1 prime ideal equal to $M = MW$ (in fact, it is not hard to prove that M remains a prime ideal in all the overrings of D included in V). However, the prime (nonmaximal) ideal M of W is not a principal ideal, since W is a 2-dimensional discrete valuation domain.

(b) A nonlocal example of a Mori integrally closed domain with Noetherian spectrum that is not b-Noetherian can be constructed as follows.

Let K be a field and X, Y two indeterminates over K. Set $D := K + XK[X, Y]$. It is easy to see that $D = K[X, Y] \cap D_1$, where $D_1 := K + XK[X, Y]_{(X)} = K + XK(Y)[X]_{(X)}$. By the same arguments used above, D_1 is an integrally closed local 1-dimensional Mori domain with Noetherian spectrum that is not b-Noetherian. By standard properties of the rings of fractions of pullbacks [F-1980, Proposition 1.9], it is easy to see that D_1 coincides with the localization of D at the maximal ideal $M := XK[X, Y]$. Since D_1 is not b-Noetherian, D also is not b-Noetherian (Proposition 14(1)). Furthermore, since for each maximal ideal N of $K[X, Y]$ such that $N \not\supseteq XK[X, Y]$, $D_{N \cap D}$ is canonically isomorphic to $K[X, Y]_N$ and the canonical continuous map $\mathrm{Spec}(K[X, Y]) \to \mathrm{Spec}(D)$ is surjective [F-1980, Theorem 1.4 and Corollary 1.5], then it is easy to conclude that $\dim(D) = 2$, D has infinitely many maximal ideals of height 2 (different from M, which has height 1), and D is a Mori integrally closed domain with Noetherian spectrum ([F-1980, Corollary 1.5(5)], [BG-1987, Proposition 4.5 and Example 4.6(b)], and also [Lu-2000, Example 2]).

We can look for other possible extensions of Proposition 9(2). We call a *quasi-Prüfer domain* an integral domain with Prüfer integral closure [ACE-1996, Proposition 1.3].

Corollary 13. *Let D be an integral domain.*

(1) *Assume that the integral closure $\overline{D}$ of D is a Dedekind domain. Then D is a (one-dimensional) quasi-Prüfer b-Noetherian domain.*
(2) *Assume that D is a (one-dimensional) quasi-Prüfer b-Noetherian domain and that $(D : \overline{D}) \neq (0)$. Then $\overline{D}$ is a Dedekind domain.*

Proof. The statements are easy consequences of Lemma 4 and Proposition 9. The condition on the dimension is not essential, since it follows from the fact that $\dim(D) = 1$ if and only if $\dim(\overline{D}) = 1$. $\quad\square$

Recall that, if D is an integral domain, P a prime ideal of D, b the b-operation of D and ι_P the canonical embedding of D in D_P, then the semistar operation b_{ι_P} on D_P is defined by $E^{b_{\iota_P}} := E^b$, for each $E \in \overline{\boldsymbol{F}}(D_P)$. Note that b_{ι_P} coincides with b_P, the b-operation of D_P. Indeed, clearly $b_{\iota_P} \leqslant b_P$. Conversely, since b is an ab semistar operation of finite type on D, then b_{ι_P} is an $\mathrm{(e)ab}$ operation of finite type on D_P [Pi-2005, Proposition 3.1((1) and (3))] and obviously $d_P \leqslant b_{\iota_P}$, where d_P is the identity operation on D_P. Therefore, $b_P = (d_P)_a \leqslant (b_{\iota_P})_a = b_{\iota_P}$.

Proposition 14. *Let D be an integral domain. For each $P \in \mathrm{Spec}(D)$, denote by b_P the b-operation on the localization D_P and by $\iota_P : D \hookrightarrow D_P$ the canonical inclusion.*

(1) *If D is b-Noetherian, then D_P is b_P-Noetherian for every $P \in \mathrm{Spec}(D)$.*
(2) *$b = \bigwedge (b_P)^{\iota_P}$, where P varies over $\mathrm{Spec}(D)$*

Proof. (1) Let J be an ideal of D_P. Then, $J = ID_P$ for some ideal $I \subseteq D$. Since D is b-Noetherian, there exists $F \in \boldsymbol{f}(D)$, $F \subseteq I$, such that $F^b = I^b$. Since $b_P = b_{\iota_P}$, then $(FD_P)^{b_P} = (FD_P)^b = (F^b D_P)^b = (I^b D_P)^b = (ID_P)^b = (ID_P)^{b_P} = J^{b_P}$. Therefore, D_P is b_P-Noetherian, since $FD_P \in \boldsymbol{f}(D_P)$.

(2) Recall that $(b_P)^{\iota_P}$ is the semistar operation on D defined by $E^{(b_P)^{\iota_P}} := (ED_P)^{b_P}$ for all $E \in \overline{\boldsymbol{F}}(D)$. The conclusion follows after observing that $\{V \mid V \text{ valuation overring of } D\} = \bigcup \{\mathcal{W}(P) \mid P \in \mathrm{Spec}(D)\}$, where $\mathcal{W}(P) := \{W \mid W \text{ valuation overring of } D_P\}$. $\quad\square$

A variation of Proposition 14(2) can be stated for more general eab semistar operations.

Proposition 15. *Let $\star$ be a semistar operation on an integral domain D. For each $Q \in \mathrm{QMax}^{\star_f}(D)$, let $\iota_Q : D^{\tilde{\star}} \hookrightarrow D_Q$ be the canonical inclusion. Then*

$$(\tilde{\star})_a = \bigwedge \big\{ (b_Q)^{\iota_Q} \mid Q \in \mathrm{QMax}^{\star_f}(D) \big\}.$$

In particular, for $\star = d$, we have

$$b = \bigwedge \big\{ (b_M)^{\iota_M} \mid M \in \mathrm{Max}(D) \big\}.$$

Proof. Note that W is a valuation overring of D_Q for some $Q \in \mathrm{QMax}^{\star_f}(D)$ if and only if W is a $\tilde{\star}$-valuation overring of D [FL-2003, Theorem 3.9].

The second part follows from the first part and from the fact that $d = \tilde{d} = d_f$, $d_a = b$ and $\mathrm{QMax}^d(D) = \mathrm{Max}(D)$. □

The next result generalizes [FPT-2010, Proposition 1.7].

Proposition 16. *Let D be an integral domain and $\mathcal{T} := \{ T_\lambda \mid \lambda \in \Lambda \}$ be a family of overrings of D. Assume that $\mathcal{T}$ has finite character (i.e., each nonzero element of D is non-unit in finitely many T_λ's). For each λ, let $\star_\lambda$ be a given semistar operation of finite type on T_λ and let $\iota_\lambda : D \hookrightarrow T_\lambda$ be the canonical inclusion. Set $\star := \bigwedge \{ (\star_\lambda)^{\iota_\lambda} \mid \lambda \in \Lambda \}$. If I is an ideal of D such that IT_λ is $\star_\lambda$-finite for each λ, then I is $\star$-finite.*

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the finite set of indexes $\lambda \in \Lambda$ such that $IT_\lambda \neq T_\lambda$. Let $G_k \in \boldsymbol{f}(T_{\lambda_k})$, $G_k \subseteq IT_{\lambda_k}$, be such that $(G_k)^{\star_{\lambda_k}} = (IT_{\lambda_k})^{\star_{\lambda_k}}$, for $1 \leqslant k \leqslant r$. Now, every generator $g_i^{(k)}$ of G_k ($\subseteq IT_{\lambda_k}$), for $1 \leqslant i \leqslant n_k$, can be written as a finite linear combination of elements in I and coefficients in T_{λ_k}. Therefore, using all these finite elements of I, varying $g_i^{(k)}$ for all i, we can construct $F_k \in \boldsymbol{f}(D)$, $F_k \subseteq I$, $F_k T_{\lambda_k} = G_k$ for all k. Set $F := F_1 + F_2 + \cdots + F_r \in \boldsymbol{f}(D)$. Then, by a routine argument, it can be shown that $F^\star = I^\star$, with $F \subseteq I$. □

The following corollary is a straightforward consequence of Propositions 14, 15 and 16.

Corollary 17. *Let D be an integral domain with the finite character on maximal ideals. The following are equivalent.*

(i) *D is b-Noetherian.*
(ii) *D_P is b_P-Noetherian for each prime ideal P.*
(iii) *D_M is b_M-Noetherian for each maximal ideal M.*

In relation with Corollary 13, note that if $\mathrm{Kr}(D, b)$ is Noetherian (or, equivalently, Dedekind) then D is b-Noetherian. As a matter of fact, if $\iota : D \hookrightarrow \overline{D}$ is the canonical inclusion, we have already observed in the proof of Lemma 4 that b_ι coincides with $\bar{b} := b_{\overline{D}}$ (the b-operation on $\overline{D}$). Therefore, $\mathrm{Kr}(D, b) = \mathrm{Kr}(\overline{D}, \bar{b})$ [FL-2003, Proposition 4.1(2)]; hence, $\mathrm{Kr}(D, b)$ is Dedekind is equivalent to $\mathrm{Kr}(\overline{D}, \bar{b})$ is Dedekind and this happens if and only if $\overline{D}$ is Dedekind [G-1972, Proposition 38.7]. Note also that, in this case, $b = \wedge_{\{\overline{D}\}}$ since, for every $E \in \overline{\boldsymbol{F}}(D)$, $E^b = \bigcap \{ E \overline{D}_N \mid N \in \mathrm{Max}(\overline{D}) \} = E\overline{D}$. More generally, the fact that $b = \wedge_{\{\overline{D}\}}$ characterizes the quasi-Prüfer domains [Pi-2009, Remark 2.7].

Conversely, if D is (b-)Noetherian, not necessarily $\mathrm{Kr}(D, b)$ is Noetherian. For instance, take a Noetherian 2-dimensional domain D; in this case $2 = \dim(D) = \dim_v(D) = \dim(\mathrm{Kr}(D, b))$ [G-1972, Corollary 30.10 and Proposition 32.16]. Therefore, the Bézout domain $\mathrm{Kr}(D, b)$ is not Noetherian (since it is not a Dedekind domain).

Let $\star_a$ be the eab semistar operation of finite type canonically associated to a given semistar operation $\star$ defined on an integral domain D. Recall that a domain D is said to be $\star$-*integrally closed* if $D = D^{\star_a} = \bigcup \{ (F^\star : F^\star) \mid F \in f(D) \} = \bigcap \{ V \mid V \text{ is a } \star\text{-valuation overring of } D \}$.

In the particular case that $\star = d$, it is clear that the valuation overrings of D coincide with the b-valuation overrings of D, and therefore we reobtain that $D^{d_a} = D^b = \bigcup\{(F : F) \mid F \in f(D)\} = \overline{D} = \bigcap\{V \mid V$ is a valuation overring of $D\}$ [G-1972, Theorem 19.8 and Proposition 34.7]. From the previous observations it follows that D is b-integrally closed if and only if it is integrally closed and a $\star$-integrally closed domain is always $(b\text{-})$integrally closed.

Theorem 18. *Let D be an integral domain. Then, D is a 1-dimensional, integrally closed, b-Noetherian domain if and only if D is a Dedekind domain.*

The proof of the previous result is based on the following fact of independent interest.

Lemma 19. *Let D be a b-Noetherian integrally closed domain and I a nonzero ideal of D. Then there exists $m \geqslant 1$ such that $(\sqrt{I})^m \subseteq I^b \subseteq \sqrt{I}$. In particular, $\sqrt{I^b} = \sqrt{I}$.*

Proof. We have already observed that, in a b-Noetherian domain, radical ideals are quasi-b-ideals (Lemma 1(5)), and so, in b-Noetherian integrally closed domain, radical ideals are b-ideals. Therefore, $\sqrt{I} = (a_1, a_2, \ldots, a_t)^b$, for some $a_k \in \sqrt{I}$. Moreover, for each k, $1 \leqslant k \leqslant t$, there exists n_k such that $a_k^{n_k} \in I$. Now, if we take $m := 1 + \sum_{k=1}^t (n_k - 1)$, then $(\sqrt{I})^m \subseteq ((\sqrt{I})^m)^b = (((a_1, a_2, \ldots, a_t)^b)^m)^b = ((a_1, a_2, \ldots, a_t)^m)^b \subseteq I^b$. Furthermore, if $x \in I^b$ then, for some $n \geqslant 1$, $x^n = \sum_{k=1}^n a_k x^{n-k}$, with $a_k \in I^k$ and $x^{n-k} \in D^b = \overline{D} = D$ and so $x^n \in I\overline{D} = I$.

The last statement follows by observing that $\sqrt{(\sqrt{I})^m} = \sqrt{I}$. $\square$

Proof of Theorem 18.

Claim 1. *Let (D, M) be an integrally closed, one-dimensional, b-Noetherian local domain. Then, M is a principal ideal (i.e., D is a DVR).*

Let $0 \neq t \in M$. Since D is local one-dimensional, $M = \sqrt{(t)}$. By Lemma 19, there exists an integer $m \geqslant 1$ such that $(M^m)^b \subseteq (t)^b \subseteq M^b = M$. Since D is integrally closed, all nonzero principal ideals are integrally closed, in particular $(t^r)^b = (t^r)$ for any $r \geqslant 1$. Thus, $M^m \subseteq (t) \subseteq M$. If $(t) = M$, M is principal and we have done. So, assume that $(t) \subsetneq M$. Since $M^m \subseteq (t)$, there exists $1 \leqslant n \leqslant m$ such that $M^n \subseteq (t)$ but $M^{n-1} \nsubseteq (t)$. Let $a \in M^{n-1} \setminus (t)$, and set $\beta := t/a \in K$. Note that $\beta^{-1} = a/t \notin D$, otherwise $a \in tD$. In particular, β^{-1} is not integral over D. Since D is b-Noetherian, there exists a finitely generated ideal F of D such that $F^b = M$. If $\beta^{-1} M \subseteq M$, we have that $\beta^{-1} \in (F^b : F^b)$ and so it is in the b-integral closure of D. On the other hand, we have already recalled above that the b-integral closure coincides with the integral closure. So, $\beta^{-1} M \subseteq M$ implies that $\beta^{-1} \in \overline{D} = D$, a contradiction. Thus, $\beta^{-1} M \nsubseteq M$. We claim that $\beta^{-1} M \subseteq D$. Indeed, $\beta^{-1} M = (a/t)M \subseteq D$, since $aM \subseteq M^{n-1}M = M^n \subseteq (t)$. Since $\beta^{-1} M \nsubseteq M$ and D is local, we have $\beta^{-1} M = D$ and so $M = \beta D$ is principal. Therefore, D is integrally closed, one-dimensional and its unique nonzero prime ideal is principal, hence, by Cohen's theorem, D is Noetherian [G-1972, Theorem 3.6]. i.e., a DVR.

Claim 2. *Let D be an integrally closed, one-dimensional, b-Noetherian domain. Then, D is an almost Dedekind domain.*

Recall that an *almost Dedekind domain* is an integral domain such that D_M is a DVR for all maximal ideals M of D. Therefore, this claim is a consequence of Proposition 14(1) and Claim 1.

We now conclude that if D is an integrally closed, one-dimensional, b-Noetherian domain, then D is Dedekind by Claim 2 and [G-1972, Theorem 37.2], since a b-Noetherian domain has the `acc` on radical ideals (Proposition 6(1)), and so every nonzero element is contained in a finite number of minimal primes [Ka-1970, Theorem 88], which are also maximal ideals in the present situation. $\square$

Remark 20. Note that, from the properties proved above, *the following are equivalent*:

 (i) *D is integrally closed* 1-*dimensional* b_M-*Noetherian for each* $M \in \mathrm{Max}(D)$.
(ii) *D is an almost Dedekind domain.*

Therefore, a (non-semilocal) integrally closed 1-dimensional domain D which is b_M-Noetherian for each $M \in \mathrm{Max}(D)$ is not necessarily b-Noetherian, because in this situation b-Noetherian coincides with Noetherian (Corollary 17 and Theorem 18).

By the previous results, the localizations of an integrally closed b-Noetherian domain at the primes of height 1 are DVR's. So, if one could prove that an integrally closed b-Noetherian domain D is the intersection of its localizations at the primes of height 1, then D would be a Krull domain. We obtain this property by a simple argument, avoiding the techniques used in the proof of Theorem 3.12 of Fossum's book [Fo-1973]. Note that next theorem generalizes Corollary 10 and Theorem 18.

Theorem 21. *Let D be a b-Noetherian domain. The following are equivalent.*

  (i) *D is integrally closed.*
 (ii) *D is completely integrally closed.*
(iii) *D is a v-domain.*
(iv) *D is a PvMD.*
 (v) *D is a Krull domain.*

Proof. It is well known that (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (i) and (ii) $\Rightarrow$ (i) (see, for example, [FZ-2011, Section 2] and [G-1972, Theorem 13.1 and p. 418]).

In order to show that all the statements are equivalent, it is enough to show that (i) $\Rightarrow$ (ii), since a b-Noetherian integrally closed domain is Mori (Proposition 6(3)) and a Mori completely integrally closed domain is Krull [G-1972, Exercise 15, p. 559].

Recall that D is integrally closed (respectively, completely integrally closed) if and only if $D = \bigcup\{(F : F) \mid F \in \boldsymbol{f}(D)\}$ (respectively, $D = \bigcup\{(I : I) \mid I \in \boldsymbol{F}(D)\}$) [G-1972, Theorem 34.3 and Proposition 34.7].

As mentioned above, if D is integrally closed, it is b-integrally closed and conversely. Therefore, $\bigcup\{(F : F) \mid F \in \boldsymbol{f}(D)\} = D = \bigcup\{(F^b : F^b) \mid F \in \boldsymbol{f}(D)\}$. On the other hand, because of the b-Noetherianity, for every $I \in \boldsymbol{F}(D)$, there exists $F \in \boldsymbol{f}(D)$ such that $F^b = I^b$. Therefore, $\bigcup\{(I : I) \mid I \in \boldsymbol{F}(D)\} \subseteq \bigcup\{(I^b : I^b) \mid I \in \boldsymbol{F}(D)\} \subseteq \bigcup\{(F^b : F^b) \mid F \in \boldsymbol{f}(D)\} = D$. $\square$

Corollary 22. *A conducive domain is b-Noetherian if and only if its integral closure is a DVR. In particular, a conducive integrally closed b-Noetherian domain is a DVR.*

Proof. The "if" part follows directly from Lemma 4(1). Conversely, let D be conducive and b-Noetherian. Then, $\overline{D}$ is b-Noetherian by Lemma 4(2) and integrally closed. Moreover, $\overline{D}$ is one-dimensional and local, since D is one-dimensional and local (Proposition 8). Thus $\overline{D}$ is a local Dedekind domain (i.e., a DVR) by Theorem 18. $\square$

Recall that a *DW-domain* is an integral domain in which $d = w$, that is a domain in which each maximal ideal is a t-ideal (see [Mi-2005, Proposition 2.2] and [PT-2008, Corollary 2.6]); a *treed domain* is an integral domain such that $\mathrm{Spec}(D)$ (as a partially ordered set under $\subseteq$) is a tree. It is well known that treed domains (for example, pseudo-valuation domains or Prüfer domains) are DW-domains [DHLZ-1989, Corollary 2.7] and [PT-2008, p. 1957]. Note that quasi-Prüfer domains, i.e., domains with Prüfer integral closure, are also DW-domains and it is not difficult to give examples of quasi-Prüfer domains that are not treed [Pp-1976, Example 2.28].

It is also clear that a Krull DW-domain is a Dedekind domain, and conversely [Mi-2005, Proposition 2.3]. As a consequence of Theorem 21, the following result relates these classes of domains in the b-Noetherian integrally closed case.

Proposition 23. *The following are equivalent.*

 (i) *b-Noetherian integrally closed treed domain.*
 (ii) *b-Noetherian integrally closed DW-domain.*
(iii) *Dedekind domain.*

Note that the previous result recovers in particular Remark 7(b) and Proposition 9(2). The following result is a straightforward consequence of Lemma 4(2) and Proposition 23.

Corollary 24. *Let D be a b-Noetherian domain such that $(D : \overline{D}) \neq (0)$. If $\overline{D}$ is DW, then $\overline{D}$ is a Dedekind domain.*

Note that the previous result recovers in particular Corollary 13(2). Examples of integral domains whose integral closure is a DW (e.g., finite-dimensional treed domains, domains with finite spectrum, etc.) are mentioned in [PT-2008, Section 3].

4. Classes of domains defined by `eab` semistar operations

As the next lemma shows in the star operation case, the equality of the star operations $*$ and $*_a$ for the classical operations d, w and t characterizes relevant classes of domains. So, it is natural to study more in detail equalities of the previous type, in the general setting of star and semistar operations.

Lemma 25.

(1) $d = d_a$ *is equivalent to Prüfer domain*;
(2) $w = w_a$ *is equivalent to PvMD*;
(3) $t = t_a$ *is equivalent to v-domain*.

Proof. (1) is clear from Corollary 2, since $d_a = b$ (Lemma 1(1)).

(2) is a consequence of a general characterization of P$\star$MD's given by Fontana, Jara and Santos [FJS-2003, Theorem 3.1], from which we have that PvMD is equivalent to an integral domain such that w is an `eab` (semi)star operation.

(3) It is well known that v-domain is equivalent to saying that v is an `eab` (semi)star operation [G-1972, p. 418] and thus, also, $t = v_f$ is an `(e)ab` (semi)star operation, i.e., $t = t_a$ [FL-2001a, Proposition 4.5(5)]. Conversely, if $t = t_a$, then it is easy to see that v is `eab`, since $(FG)^t = (FG)^v \subseteq (FH)^v = (FH)^t$ implies $G^v = G^t \subseteq H^t = H^v$, for $F, G, H \in \boldsymbol{f}(D)$. $\square$

Remark 26. In Lemma 25, we have (implicitly) considered the equality of two operations *as semistar operations*, that is, we have compared them on $\overline{\boldsymbol{F}}(D)$. However, this is not relevant in case of the previous lemma, since the operations considered there are all operations of finite type, so the statements (1), (2) and (3) are respectively equivalent to their analogous "weaker" versions (that is, the equality holds as *star operations*):

($1_{\boldsymbol{F}(D)}$) $d = d_a$ on $\boldsymbol{F}(D)$,
($2_{\boldsymbol{F}(D)}$) $w = w_a$ on $\boldsymbol{F}(D)$,
($3_{\boldsymbol{F}(D)}$) $t = t_a$ on $\boldsymbol{F}(D)$.

Indeed, since a finitely generated D-submodule of K is always a fractional ideal, the semistar operations of finite type are "essentially" defined on $\boldsymbol{f}(D)$ (since, $E^\star = \bigcup\{F^\star \mid F \in \boldsymbol{f}(D), F \subseteq E\}$, for each $E \in \overline{\boldsymbol{F}}(D)$), that is if $\star_1$ and $\star_2$ are semistar operations of finite type, then the following are equivalent:

  (i) $\star_1$ and $\star_2$ coincide on $\boldsymbol{f}(D)$;
 (ii) $\star_1$ and $\star_2$ coincide on $\boldsymbol{F}(D)$;
(iii) $\star_1$ and $\star_2$ coincide (on $\overline{\boldsymbol{F}}(D)$).

As we will see later, when dealing with operations that are not of finite type, the equality as semistar operations is much stronger than the equality as star operations.

The next step is the study of domains for which $v = v_a$. First, we consider the case when $v = v_a$, as star operations.

Proposition 27. *Given an integral domain D, $v = v_a$ on $\boldsymbol{F}(D)$ if and only if D is a PvMD with t-finite character such that each (nonzero) t-prime is contained in only one t-maximal ideal and t-maximal ideals are t-finite (and, therefore, t-invertible).*

Proof. Since v_a is an operation of finite type, then clearly $v = v_a$ (on $\boldsymbol{F}(D)$) is equivalent to $v = t$ (on $\boldsymbol{F}(D)$) and $t = t_a$ (on $\boldsymbol{F}(D)$). Since $t = t_a$ on $\boldsymbol{F}(D)$ is equivalent to v-domain (Lemma 25(3)), $v = v_a$ on $\boldsymbol{F}(D)$ is equivalent to v-domain which is also a TV-domain (i.e., a domain for which $t = v$ on $\boldsymbol{F}(D)$ [HZ-1998] and [E-2009]). The v-domains that also are TV-domains are exactly the PvMD's with t-finite character such that each (nonzero) t-prime is contained in only one t-maximal ideal and t-maximal ideals are t-finite (t-invertible) [HZ-1998, Theorem 3.1]. $\square$

Remark 28. (a) The domains with t-finite character such that each (nonzero) t-prime is contained in only one t-maximal ideal are called *weakly Matlis* in [AZ-1999]. We can say that this is the "t-version" of Matlis' notion of h-*local domain* (i.e., an integral domain such that each nonzero ideal is contained in at most finitely many maximal ideals and each nonzero prime is contained in a unique maximal ideal [Ma-1964]).

    (b) In particular, a domain in which $v = v_a$ (on $\boldsymbol{F}(D)$) is a PvMD with the property that PD_P is a principal ideal in the essential valuation overring D_P, for every (nonzero) t-prime ideal P of D. However, a PvMD (or, even, a Prüfer domain) with this property not necessarily has $v = v_a$, even on $\boldsymbol{F}(D)$. For instance, take an almost Dedekind domain which is non-Dedekind. In this case, $v_a = t_a = w_a = d_a = b = d \lneq v$.

Several characterizations of domains for which $v = v_a$, as star operations, are summarized in the following proposition. Recall that a *domain* is called *divisorial* if every nonzero ideal is divisorial (i.e., if $d = v$ as star operations). Heinzer characterized the integrally closed divisorial domains as the h-local Prüfer domains such that the maximal ideals are finitely generated [H-1968, Theorem 5.1].

Proposition 29. *Let D be an integral domain. The following are equivalent.*

   (i) $v = v_a$ on $\boldsymbol{F}(D)$.
  (ii) D is a PvMD and $v = t$ on $\boldsymbol{F}(D)$.
 (iii) D is a v-domain and $v = t$ on $\boldsymbol{F}(D)$.
 (iv) D is an essential domain and $v = t$ on $\boldsymbol{F}(D)$.
  (v) D is integrally closed and $\mathrm{Na}(D, v)$ is a divisorial domain.
 (vi) D is integrally closed and $v = w$ on $\boldsymbol{F}(D)$.
(vii) $\mathrm{Na}(D, v) = \mathrm{Kr}(D, v)$ is h-local and the maximal ideals are finitely generated.
(viii) $w = v_a$ (on $\boldsymbol{F}(D)$) and $v = t$ on $\boldsymbol{F}(D)$.
 (ix) $v = w_a$ on $\boldsymbol{F}(D)$.
  (x) $w = t = v = w_a = t_a = v_a$ on $\boldsymbol{F}(D)$.

Proof. (i) $\Rightarrow$ (ii) If $v = v_a$, then $v = t$, since v_a is of finite type. That D is a PvMD has been proven in Proposition 27.
    (ii) $\Rightarrow$ (iv) A PvMD is obviously an essential domain.

(iv) $\Rightarrow$ (iii) An essential domain is a v-domain [Kg-1989, Lemma 3.1].

(iii) $\Rightarrow$ (ii) All finitely generated ideals are v-invertible, since D is a v-domain [G-1972, Theorem 34.6]. By assumption $v = t$, so all finitely generated ideals are t-invertible and D is a PvMD.

(ii) $\Rightarrow$ (vi) In a PvMD, $t = w$ and so if $v = t$, we have $v = t = w$. Moreover, a PvMD is integrally closed.

(vi) $\Rightarrow$ (ii) D is a PvMD by [EG-2005, Theorem 3.3]. Moreover, $w = v$ implies $t = v$ (since $w \leqslant t \leqslant v$).

(vi) $\Leftrightarrow$ (v) [GHP-2009, Corollary 3.5 and Proposition 3.2].

(v) $\Rightarrow$ (vii) We have already shown that (v) implies (ii), so D is a PvMD and $\mathrm{Na}(D, v) = \mathrm{Kr}(D, v)$ by [FJS-2003, Remark 3.1]. Furthermore, $\mathrm{Kr}(D, v)$ is always a Prüfer domain (in fact, Bézout) [FL-2001a, Corollary 3.4(2)]. Thus, $\mathrm{Na}(D, v)$ is a divisorial Prüfer domain, hence $\mathrm{Na}(D, v)$ is h-local with the maximal ideals finitely generated by [H-1968, Theorem 5.1].

(vii) $\Rightarrow$ (v) First, D is a PvMD, so integrally closed, by [FJS-2003, Remark 3.1]. Moreover, $\mathrm{Na}(D, v) = \mathrm{Kr}(D, v)$ implies that $\mathrm{Na}(D, v)$ is Prüfer. Finally, a Prüfer h-local domain with the maximal ideals finitely generated is divisorial again by [H-1968, Theorem 5.1].

(ii) $\Rightarrow$ (x) It is an easy consequence of the fact that, in PvMD, $t = w = w_a = t_a \ (= v_a)$.

(x) $\Rightarrow$ (viii) This implication is trivial.

(viii) $\Rightarrow$ (ix) We have $v = t \leqslant t_a = v_a = w \leqslant w_a \leqslant v_a = w \leqslant t$. Thus, $v = w_a$.

(ix) $\Rightarrow$ (i) $v_a = (w_a)_a = w_a = v$. $\quad\square$

Corollary 30. *Let D be an integral domain.*

(1) $v = v_a$ on $\mathbf{F}(D)$ and $\dim(D) = 1$ *if and only if D is a Dedekind domain.*
(2) $v = v_a$ on $\mathbf{F}(D)$ and $\dim_t(D) = 1$ *if and only if D is a Krull domain.*

Proof. (1) If D is 1-dimensional, the maximal ideals of D are t-ideal, so in D we have $w = d$. By Proposition 27, we know that, when $v = v_a$ on $\mathbf{F}(D)$, D is a PvMD. In a PvMD, we know that $w = w_a$ (Lemma 25(2)), thus a PvMD with $d = w$ is a Prüfer domain, since $b = d_a = w_a = w = d$ (Lemma 2). Moreover, again from Proposition 27, we have that PD_P is finitely generated for every (nonzero) (t-)prime ideal P of D. So D_P is a DVR, for each P. Therefore, D is an almost Dedekind domain. On the other hand, we have also that $v = v_a = t = w = b = d$, so D is an almost Dedekind domain in which every nonzero ideal is divisorial, hence a Dedekind domain since the maximal ideals of D are finitely generated by [H-1968, Theorem 5.1]. The converse is obvious.

(2) *Mutatis mutandis*, the proof of this statement follows the lines of the previous proof, using Proposition 27 and recalling that, in a PvMD, $D = \bigcap\{D_P \mid P \in \mathrm{Max}^t(D)\}$. Conversely, in a Krull domain, we have $v = t = t_a$ [G-1972, Corollary 44.3 and Proposition 44.13]. $\quad\square$

Note that a domain in which $\star = \star_a$ is not necessarily a P$\star$MD. For example, in any integral domain $b = b_a$ and, on the other hand, a PbMD is a Prüfer domain. More generally, for a semistar operation $\star$ of finite type which is (e)ab, we have $\star = \star_a$, however a P$\star$MD is an integral domain for which $\tilde{\star} = (\tilde{\star})_a$ [FJS-2003, Theorem 3.1]. Therefore, $\star = \star_a$ does not imply $\tilde{\star} = (\tilde{\star})_a$ and, conversely, $\tilde{\star} = (\tilde{\star})_a$ does not imply $\star = \star_a$, even on $\mathbf{F}(D)$ (for instance, take $\star = v$ in a PvMD which does not verify the other conditions listed in Proposition 27).

Remark 31. Note that, in Proposition 27, we have considered $v = v_a$ as star operations. Suppose now $v = v_a$ as semistar operations, that is, $E^v = E^{v_a}$ for all $E \in \overline{\mathbf{F}}(D)$.

In particular, $v = v_a$ as star operations, so D is a PvMD and $w = t = v$. Assume that $D \neq K$. Let V be a v-valuation overring of D (since $v = w$ and D is a PvMD, one can take as V a localization of D at a t-maximal ideal). If $(D : V) = (0)$, $K = V^v = V^w = V$, a contradiction. So, $(D : V) \neq (0)$ and D is a conducive domain, by [DF-1984, Theorem 3.2]. So, since D is a conducive integrally closed domain, there exists a divided prime ideal P, such that D_P is a valuation domain [BDF-1986, Corollary 4]. In particular, P is a t-ideal, being the contraction to D of the t-ideal PD_P of D_P. So, P is a prime t-ideal and, since it is divided, it is comparable to all other prime ideals of D (see, for instance, [Ak-1967, proof of Theorem 1] and [Gn-1974, Proposition 1.2(ii)], or [D-1976, Proposition 2.1]).

Moreover, since $w = v$, D is weakly Matlis [EG-2005, Theorem 1.5], that is, each nonzero element is contained only in a finite number of t-maximal ideals and each t-prime is contained in a unique t-maximal ideal. In particular, P is contained in only one t-maximal ideal. But, since P is comparable with all primes of D, it follows that D has exactly one t-maximal ideal, say M. Since D is a PvMD, D is the intersection of the localizations of D at its maximal t-ideals, it follows that $D = D_M$ and so it is a valuation domain. Furthermore, since in a valuation domain $t = d$, we have that D is a divisorial domain and so, in particular, its maximal ideal is principal by [H-1968, Theorem 5.1].

Conversely, the fact that if V is a divisorial valuation domain then it is trivial that $v = v_a$ as semistar operations.

So we have proven the following result:

Let D be an integral domain. Then, $v = v_a$ as semistar operations if and only if D is a valuation domain with principal maximal ideal.

We have already observed that the integral domains for which $d = b$ are exactly the Prüfer domains (Lemma 2). The next goal is to understand the domains for which $v = b$. This is a stronger condition than $v = v_a$, since we require not only that v is eab of finite type but, also, precisely that $v_a = b$. First, we consider the case when $v = b$ as star operations.

Proposition 32. *Let D be an integral domain. The semistar operations b and v coincide on $\boldsymbol{F}(D)$ if and only if D is an h-local Prüfer domain such that the maximal ideals are finitely generated or, equivalently, if and only if the (semi)star operations d and v coincide on $\boldsymbol{F}(D)$ and D is integrally closed.*

Proof. Note that if $b = v$ on $\boldsymbol{F}(D)$, then in particular $v = v_a$ on $\boldsymbol{F}(D)$. In this situation, by Proposition 27, D is a PvMD (with further properties). On the other hand, $b = v$ on $\boldsymbol{F}(D)$ also implies that D is a PbMD, i.e., D is Prüfer domain. Furthermore, since in a Prüfer domain $d = b$, D is a divisorial Prüfer domain, hence we conclude by [H-1968, Theorem 5.1], where the second and the third part of the statement are shown to be equivalent.

Conversely it is clear that in a divisorial integrally closed (Prüfer) domain $d = v$ on $\boldsymbol{F}(D)$ and also, at the same time, $d = b$ on $\boldsymbol{F}(D)$, thus $b = v$ on $\boldsymbol{F}(D)$. $\quad\square$

Remark 33. (a) Note that, in an integrally closed b-Noetherian domain, we have $v = t = w = w_a$ on $\boldsymbol{F}(D)$, since it is a Krull domain (Theorem 21). However, for a general integrally closed b-Noetherian (non-Dedekind) domain $b \lneqq w_a$ (Proposition 32).

(b) If we require $b = v$ as semistar operations (i.e., if we require that $b = v$ on $\overline{\boldsymbol{F}}(D)$), we can say something more. In fact, if V is a valuation overring such that $(D : V) = (0)$, we have $V^v = K$ and $V^b = V$. So, if $b = v$ as semistar operations, such a valuation overring of D cannot exist, unless $D = K$. Therefore, if $D \neq K$, D must be a conducive domain [DF-1984, Theorem 3.2] and, also, by Proposition 32, a Prüfer divisorial domain. Therefore, D is a valuation domain [Pi-2005, Lemma 4.6]. In this case, we also have $d = v$ as semistar operations. Moreover, if D is integrally closed and $d = v$ then $b = v_a$ and, in particular, $b = v$ since in this case $b \leqslant t \leqslant v \leqslant v_a$. Therefore, we can conclude, using also Remark 31, that *for an integral domain D, the following are equivalent.*

(i) $b = v$ (*as semistar operations*).
(ii) D *is a divisorial valuation domain* (*i.e., D is a valuation domain with principal maximal ideal*).
(iii) D *is integrally closed and $d = v$* (*as semistar operations*).
(iv) $v = v_a$ (*as semistar operations*).

Note that the condition that D is integrally closed is necessary in (iii). For example, in a pseudo-valuation non-valuation domain D such that the canonically associated valuation domain V is two-generated as a D-module, $d = v$ on $\boldsymbol{F}(D)$ ([HH-1978, Corollary 1.8] and [HZ-1998, Proposition 4.3]). Moreover, in this case, D is conducive, since $(D : V) \neq (0)$, so $d = v$ on $\overline{\boldsymbol{F}}(D)$. Clearly, in this example, D is not integrally closed, since $D^b = V$ and, obviously, $D = D^v \neq D^b$.

We consider next the case $w = w_a = b$.

Proposition 34. *Let D be an integral domain. The following are equivalent.*

 (i) $b = w$.
 (ii) *D is a Prüfer domain.*
(iii) $d = b$.

Proof. The condition $b = w$ implies that b is a stable semistar operation (of finite type) and so $d = \tilde{b} = b = d_a$. Henceforth, D is a Prüfer domain. The converse is clear, since in a Prüfer domain not only $d = b$, but also every nonzero finitely generated ideal is invertible (hence, divisorial) and so $d = t$ and thus, in particular, $d = w$ $\quad\square$

It remains only to study the case when the b-operation coincides with the t-operation. Recall that a domain is called *vacant* if it is integrally closed and it has only one "classical" Kronecker function ring (as in Gilmer's book [G-1972, p. 400]) or, equivalently, if it admits exactly one `eab` star operation of finite type (i.e., the b-operation). For example, any Prüfer domain is a vacant domain.

Proposition 35. *(See [Fa-2011, Remark 2.9].) Let D be an integral domain. The following are equivalent*:

 (i) $b = t$.
 (ii) $b = t_a$.
(iii) *D is a vacant v-domain.*

Proof. (i) $\Rightarrow$ (ii) If $b = t$, then in particular t is `eab`, and so $t = t_a$ [FL-2001a, Proposition 4.5(5)]. Therefore, $b = t_a$.

(ii) $\Rightarrow$ (iii) Now, let $*$ be an `eab` star operation of finite type on D. Clearly, $* = *_a \geqslant d_a = b$, but also $* \leqslant t$, being $*$ a star operation of finite type [G-1972, Theorem 34.1(4)]. Thus, $t_a \geqslant t \geqslant * = *_a \geqslant b$. Therefore, there is a unique star operation which is `eab` and of finite type on D. So D is vacant. Moreover, we have also observed that t is `eab` and so D is a v-domain (Lemma 25(3)).

(iii) $\Rightarrow$ (i) Since D is a v-domain, t is `eab`. Since D is vacant, the b-operation is the only star operation `eab` and of finite type. So, $b = t$. $\quad\square$

Remark 36. (a) Recall that a t-integrally closed ($= v$-integrally closed) domain was also called a *pseudo-integrally closed* domain in [AHZ-1991]. With this terminology, the condition (iii) in Proposition 35 can be equivalently stated (by [FZ-2011, Theorem 2.4 and Remark 2.6]) in the following form:

(iii$'$) *D is a vacant pseudo-integrally closed domain.*

Note also that a vacant domain (which is integrally closed by definition) is not in general pseudo-integrally closed (see [G-1972, Example 12, p. 409] and [AHZ-1991, Proposition 1.8]).

(b) If D verifies $b = t$, then D is a (vacant v-)domain for which d ($= \tilde{b} = \tilde{t}$) $= w$, i.e., a DW-domain. More generally, it can be shown that any vacant domain is a DW-domain [Fa-2011, Proposition 2.6]. Note also that

$$\text{DW-domain} \quad \Leftrightarrow \quad b = w_a.$$

As a matter of fact, in a DW-domain, $b = d_a = w_a$; conversely, if $b = w_a$, then $d = \tilde{b} = \widetilde{w_a} \geqslant \tilde{w} = w \geqslant d$. Finally, note that the condition $b = w_a$ is strictly weaker than the condition "D is a vacant domain" (and so, in particular, also of the condition $b = t_a$) [Fa-2011, Example 6.8].

(c) If, in Proposition 35(iii), we assume PvMD [respectively, Krull domain] instead of v-domain, we have:

$$D \text{ is a vacant } P v MD\text{-domain} \quad \Leftrightarrow \quad D \text{ is a Prüfer domain} \quad \Leftrightarrow \quad b = w \, (= w_a),$$

$$D \text{ is a vacant Krull domain} \quad \Leftrightarrow \quad D \text{ is a Dedekind domain}.$$

As a matter of fact, in the first case, $b = t_a$ and $w = w_a$ easily imply that $b = w_a$ and so $b = w_a = w$. Conversely, from $b = w$, clearly $w = w_a$, and also $d = \tilde{b} = \tilde{w} = w$ and hence $d = b$. For the second case, it is sufficient to recall that a Krull domain is a PvMD and that a Krull Prüfer domain is a Dedekind domain.

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