Cancellation properties in ideal systems: A classification of `e.a.b.` semistar operations

Marco Fontana [a,*], K. Alan Loper [b]

[a] Dipartimento di Matematica, Università degli Studi "Roma Tre", 00146 Rome, Italy
[b] Department of Mathematics, Ohio State University, Newark, OH 43055, USA

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ABSTRACT

We give a classification of `e.a.b.` semistar (and star) operations by defining four different (successively smaller) distinguished classes. Then, using a standard notion of equivalence of semistar (and star) operations to partition the collection of all `e.a.b.` semistar (or star) operations, we show that there is exactly one operation of finite type in each equivalence class and that this operation has a range of nice properties. We give examples to demonstrate that the four classes of `e.a.b.` semistar (or star) operations we defined can all be distinct. In particular, we solve the open problem of showing that `a.b.` is really a stronger condition than `e.a.b.`

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1. Introduction

In the classical (Krull's) setting, the study of Kronecker function rings on an integral domain generally focusses on the collection of `a.b.` (= *arithmetisch brauchbar*) star operations on the domain. Gilmer's presentation of star operations [1, Section 32] covers the class of `a.b.` star operations and also the (presumably larger class of) `e.a.b.` (= *endlich arithmetisch brauchbar*) star operations (the definitions are recalled in the following section).

This paper began with an attempt to clarify the relation between the `e.a.b.` and `a.b.` conditions and trying to solve the open problem of showing that `a.b.` is really a stronger condition than `e.a.b.` In Section 2 of the paper we give some general background and prove some elementary results concerning star operations (and the more general concept of semistar operations) and the related issue of cancellation properties of ideals (since the `e.a.b.` condition is essentially a cancellation property). We also expand our goal and define four different (successively smaller) classes of `e.a.b.` semistar (and star) operations. Given two `e.a.b.` semistar operations, we say that they are equivalent if they agree on the class of all finitely generated ideals. Using this notion of equivalence to partition the collection of all `e.a.b.` semistar (or star) operations, we show that there is exactly one operation of finite type in each equivalence class and that this operation has a range of nice properties. These operations of finite type constitute the smallest of our four classes. Then, in Section 3, we give examples to demonstrate that the four classes of semistar (or star) operations we defined can all be distinct, including the motivating example of a star operation that is `e.a.b.` but not `a.b.` Then, in a brief final section, we approach the question of generalizing the results beyond the scope of `e.a.b.` operations. In particular, we note that for general star or semistar operations, an operation of finite type may not have the various nice properties that an `e.a.b.` operation of finite type has. We suggest an alternative construction to the standard finite-type construction which agrees with the finite-type

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* Corresponding author. Tel.: +39 06 5733 8232; fax: +39 06 5733 8080.
 E-mail addresses: fontana@mat.uniroma3.it (M. Fontana), lopera@math.ohio-state.edu (K.A. Loper).

construction in the `e.a.b.` case and does appear to give results similar to our `e.a.b.` results in the general setting. This generalization is based on recent results from [2].

2. Classification of `e.a.b.` semistar operations and cancellation properties

Let D be an integral domain with quotient field K. Let $\boldsymbol{F}(D)$ [respectively, $\boldsymbol{f}(D)$] be the set of all nonzero fractionary ideals [respectively, nonzero finitely generated fractionary ideals] of D. Let $\overline{\boldsymbol{F}}(D)$ represent the set of all nonzero D-submodules of K (thus, $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$).

W. Krull introduced the concept of a star operation in 1936 in his first Beiträge paper [3] (or [4]). In 1994, Okabe and Matsuda [29] introduced the more "flexible" notion of semistar operation $\star$ of an integral domain D, as a natural generalization of the notion of star operation, allowing $D \neq D^{\star}$.

A mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^{\star}$, is called *a semistar operation of D* if, for all $z \in K, z \neq 0$ and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold: $(\star_1)$ $(zE)^{\star} = zE^{\star}$; $(\star_2)$ $E \subseteq F \Rightarrow E^{\star} \subseteq F^{\star}$; $(\star_3)$ $E \subseteq E^{\star}$ and $E^{\star\star} := (E^{\star})^{\star} = E^{\star}$.

When $D^{\star} = D$, the map $\star$, restricted to $\boldsymbol{F}(D)$, defines a star operation [1] of D [1, Section 32]; in this situation, we say that $\star$ is a *(semi)star operation of D*. A *proper semistar operation of D* is a semistar operation $\star$ of D such that $D \subsetneq D^{\star}$.

For several examples we construct, we use results that were proven for star operations rather than semistar. However, if $*$ is a star operation on an integral domain D (hence, defined only on $\boldsymbol{F}(D)$), we can extend it trivially to a semistar (in fact, (semi)star) operation of D, denoted $*_e$, by defining E^{*_e} to be the quotient field of D whenever $E \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$. Hence, our star operation examples can be considered to be semistar examples as well.

As in the classical star-operation setting, we associate to a *semistar* operation $\star$ of D a new semistar operation $\star_f$ of D as follows. If $E \in \overline{\boldsymbol{F}}(D)$ we set:

$$E^{\star_f} := \bigcup \{ F^{\star} \mid F \subseteq E, \ F \in \boldsymbol{f}(D) \}.$$

We call $\star_f$ *the semistar operation of finite type of D* associated to $\star$. If $\star = \star_f$, we say that $\star$ is *a semistar operation of finite type on D*. Given two semistar operations $\star'$ and $\star''$ of D, we say that $\star' \leq \star''$ if $E^{\star'} \subseteq E^{\star''}$ for all $E \in \overline{\boldsymbol{F}}(D)$. Note that $\star_f \leq \star$ and $(\star_f)_f = \star_f$, so $\star_f$ is a semistar operation of finite type of D.

If $\star$ coincides with the semistar v-*operation of D*, defined by $E^v := (D : (D : E))$, for each $E \in \overline{\boldsymbol{F}}(D)$, then v_f is denoted by t. Note that v [respectively, t] restricted to $\boldsymbol{F}(D)$ coincides with the classical *star v-operation* [respectively, *t-operation*] of D.

Let $\star$ be a semistar operation on D. If F is in $\boldsymbol{f}(D)$, we say that F is $\star$-*eab* [respectively, $\star$-*ab*] if $(FG)^{\star} \subseteq (FH)^{\star}$ implies that $G^{\star} \subseteq H^{\star}$, with $G, H \in \boldsymbol{f}(D)$, [respectively, with $G, H \in \overline{\boldsymbol{F}}(D)$].

The operation $\star$ is said to be *eab* [respectively, *ab*] if each $F \in \boldsymbol{f}(D)$ is $\star$-eab [respectively, $\star$-ab]. An ab operation is obviously an eab operation.

Remark 1. W. Krull, in [3], only considered the concept of "arithmetisch brauchbar" (for short, `a.b.` or, simply ab as above) $\star$-operation (more precisely, Krull's original notation was " $'$-Operation", instead of "$\star$-operation"). He did not consider the concept of "endlich arithmetisch brauchbar" $\star$-operation.

The `e.a.b.` (or, more simply, eab as above) concept stems from the original version of Gilmer's book [5]. The results of Section 26 in [5] show that this (presumably) weaker concept is all that one needs to develop a complete theory of Kronecker function rings. Robert Gilmer explained to us that «I believe I was influenced to recognize this because during the 1966 calendar year in our graduate algebra seminar (Bill Heinzer, Jimmy Arnold, and Jim Brewer, among others, were in that seminar) we had covered Bourbaki's Chapters 5 and 7 of *Algèbre Commutative*, and the development in Chapter 7 on the v-operation indicated that `e.a.b.` would be sufficient.»

Remark 2. (1) When $\star$ coincides with the identity star operation d on the integral domain D, the notion of d-eab [respectively, d-ab], *for finitely generated ideals*, coincides with the notion of *quasi-cancellation ideal* [respectively, *cancellation ideal*] studied by Anderson and Anderson [6] (cf. also [7]).

As a matter of fact, a nonzero ideal I (not necessarily finitely generated) of an integral domain D is called a *cancellation* [respectively, *quasi-cancellation*] *ideal of D* if $(IJ : I) = J$, for each nonzero ideal J of D [respectively, if $(IF : I) = F$, for each nonzero finitely generated ideal F of D].

Obviously, a cancellation ideal is a quasi-cancellation ideal, but in general (for nonfinitely generated ideals) the converse does not hold (e.g., a maximal ideal of a nondiscrete rank one valuation domain, [6]).

For a finitely generated ideal, the notion of cancellation ideal coincides with the notion of quasi-cancellation ideal [6, Corollary 1] (thus, in particular, the identity operation d is eab if and only if d is ab and this happens if and only if D is a Prüfer domain (cf. also the following part (4)). More precisely, by [6, Lemma 1 and Theorem 1] we have:

If I is a nonzero finitely generated ideal of an integral domain D, then the following conditions are equivalent:

(i) *I is a quasi-cancellation ideal of D;*
(ii) *$IG \subseteq IH$, with G and H nonzero finitely generated ideals of D, implies that $G \subseteq H$;*

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[1] More explicitly, a *star operation $*$ of an integral domain D* is a mapping $* : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $E \mapsto E^*$ such that the following properties hold: $(*_1)(zD)^* = zD$ and $(zE)^* = zE^*$, $(*_2)E \subseteq F \Rightarrow E^* \subseteq F^*$, $(*_3)E \subseteq E^*$ and $E^{**} := (E^*)^* = E^*$, for all nonzero $z \in K$, and for all $E, F \in \boldsymbol{F}(D)$.

(iii) $IG \subseteq IH$, with G and H nonzero ideals of D, implies that $G \subseteq H$;

(iv) I is a cancellation ideal of D;

 (v) for each prime [maximal] ideal Q of D, ID_Q is an invertible ideal of D_Q;

(vi) I is an invertible ideal of D.

Note that the definitions of quasi-cancellation and cancellation ideal can be extended in a natural way to the case of fractional ideals and, *mutatis mutandis*, the previous equivalent conditions hold for fractional ideals.

(2) The notion of quasi-cancellation ideal was introduced in [6], in relation to the fact that in [1, Exercise 4, page 66] it was erroneously stated that a nonzero ideal I of an integral domain D is a cancellation ideal if and only if $(IF : I) = F$, for each finitely generated ideal F of D (see the counter-example mentioned in part (1)).

(3) Kaplansky, in an unpublished set of notes [1, Exercise 7, page 67], proved a result that, in the integral domain case, affirms that *a nonzero finitely generated ideal I of a local integral domain D is a cancellation ideal if and only if I is principal.* Therefore, the equivalence $((iv) \Leftrightarrow (v))$ in part (1) is a "globalization" of Kaplansky's result. Note also that Kaplansky observed that, if I, G and H are nonzero ideals of an integral domain D with $IG \subseteq IH$ and if I is finitely generated ideal, generated by n elements, then necessarily $G^n \subseteq H$ [8, Theorem 254].

(4) Recall that Jaffard [9] proved that *for each ideal $I \in \boldsymbol{f}(D)$, I is a (quasi-)cancellation ideal if and only if D is a Prüfer domain* (cf. also Jensen [10, Theorem 5]; in that paper Jensen [10, Theorem 6] proved also that *for each ideal $I \in \boldsymbol{F}(D)$, I is a cancellation ideal if and only if D is an almost Dedekind domain).* Recall also that, by [6, Theorem 7], *I is a quasi-cancellation ideal, for each $I \in \boldsymbol{F}(D)$, if and only if D is a completely integrally closed Prüfer domain.*

(5) Note that, *when D is a Prüfer domain*, it is known [6, Theorem 2 and Theorem 5] that:

(5,a) $I \in \boldsymbol{F}(D)$ is a quasi-cancellation ideal $\Leftrightarrow (I : I) = D$.

(5,b) $I \in \boldsymbol{F}(D)$ is a cancellation ideal $\Leftrightarrow ID_M$ is principal for each M maximal ideal of D.

Anderson and Roitman [11, Theorem] extended (5, b) outside of the (Prüfer) domain case and proved that, *given a nonzero ideal [respectively, a regular ideal] I of an integral domain [respectively, a ring] R, then I is a cancellation ideal of R if and only if IR_M is a principal [respectively, principal regular] ideal of R_M, for each maximal ideal M of R.*

Note that the previous statement was "extended" further to submodules of the quotient field of an integral domain D by Goeters and Olberding [12]. Let $E \in \overline{\boldsymbol{F}}(D)$, E is called *a cancellation module for D* if, for $G, H \in \overline{\boldsymbol{F}}(D)$, $EG = EH$ implies that $G = H$. Then, by [12, Theorem 2.3], *E is a cancellation module for D if and only if ED_M is principal, for each $M \in \mathrm{Max}(D)$, or, equivalently, if and only if ED_M is a cancellation module for D_M, for each $M \in \mathrm{Max}(D)$.*

We note that if $\star$ is an eab semistar operation then $\star_f$ is also an eab semistar operation, since they agree on all finitely generated ideals. The following easy result generalizes the fact, already observed in Remark 2(1), that the identity semistar operation d is eab if and only if it is ab.

Lemma 3. *Let $\star$ be a semistar operation of finite type, then $\star$ is an eab semistar operation if and only if $\star$ is an ab semistar operation.*

Proof. Since it is obvious that an ab semistar operation is always eab, we need only to prove the converse. Let $I \in \boldsymbol{f}(D)$ and $J, L \in \overline{\boldsymbol{F}}(D)$. Assume that $(IJ)^\star \subseteq (IL)^\star$. By the assumption, we have $(IJ)^\star = \bigcup\{H^\star \mid H \in \boldsymbol{f}(D), H \subseteq IJ\} = \bigcup\{(IF)^\star \mid F \in \boldsymbol{f}(D), F \subseteq J\}$ and similarly $(IL)^\star = \bigcup\{(IG)^\star \mid G \in \boldsymbol{f}(D), G \subseteq L\}$. Therefore, for each $F \in \boldsymbol{f}(D), F \subseteq J$, we have $IF \subseteq \bigcup\{(IG)^\star \mid G \in \boldsymbol{f}(D), G \subseteq L\}$. Thus we can find $G_1, G_2, \ldots, G_r$ in $\boldsymbol{f}(D)$ with the property that $G_i \subseteq L$, for $1 \leq i \leq r$, such that:

$$(IF)^\star \subseteq (IG_1 \cup IG_2 \cup \cdots \cup IG_r)^\star \subseteq (I(G_1 \cup G_2 \cup \cdots \cup G_r))^\star.$$

Since $\star$ is an eab semistar operation then $F^\star \subseteq (G_1 \cup G_2 \cup \cdots \cup G_r)^\star \subseteq \bigcup\{G^\star \mid G \in \boldsymbol{f}(D), G \subseteq L\} = L^{\star_f} = L^\star$ and so $J^\star = J^{\star_f} = \bigcup\{F^\star \mid F \in \boldsymbol{f}(D), F \subseteq J\} \subseteq L^\star$. $\quad\square$

The next result provides a useful generalization of Lemma 3.

Proposition 4. *Let D be an integral domain, let $\star$ be a semistar operation on D, and let $F \in \boldsymbol{f}(D)$. Then F is $\star$-eab if and only if F is $\star_f$-(e)ab. In particular, the notions of $\star$-eab semistar operation and $\star_f$-(e)ab semistar operation coincide.*

Proof. Since from the definition it follows that the notion of $\star$-eab coincides with the notion of $\star_f$-eab and, by Lemma 3, the notion of $\star_f$-eab coincides with the notion of $\star_f$-ab, it remains to show that if F is $\star$-eab then F is $\star_f$-ab, when F belongs to $\boldsymbol{f}(D)$. Let $G, H \in \overline{\boldsymbol{F}}(D)$ and assume that $(FG)^{\star_f} \subseteq (FH)^{\star_f}$, then arguing as in Lemma 3, for each $G' \in \boldsymbol{f}(D)$, with $G' \subseteq G$, we can find a $H'_{G'} \in \boldsymbol{f}(D)$, with $H'_{G'} \subseteq H$, in such a way that $(FG')^\star \subseteq (FH'_{G'})^\star$. Since F is $\star$-eab, then $(G')^\star \subseteq (H'_{G'})^\star$ and so $G^{\star_f} = \bigcup\{(G')^\star \mid G' \in \boldsymbol{f}(D), G' \subseteq G\} \subseteq \bigcup\{(H'_{G'})^\star \mid G' \in \boldsymbol{f}(D), G' \subseteq G\} \subseteq \bigcup\{(H')^\star \mid H' \in \boldsymbol{f}(D), H' \subseteq H\} = H^{\star_f}. \quad\square$

If $\mathcal{W}$ is a given family of valuation overrings of D, then the mapping $\wedge_{\mathcal{W}}$ defined as follows: for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\wedge_{\mathcal{W}}} := \bigcap\{EW \mid W \in \mathcal{W}\}$$

defines an ab semistar operation of D, since FW is principal in W, for each $F \in \boldsymbol{f}(D)$ and for each $W \in \boldsymbol{\mathcal{W}}$. We call a semistar operation of the previous type a $\boldsymbol{\mathcal{W}}$-*operation of* D. If $\boldsymbol{\mathcal{W}}$ coincides with the set $\boldsymbol{\mathcal{V}}$ of all valuation overrings of D, then we call $\wedge_{\boldsymbol{\mathcal{V}}}$ the b-*operation of* D.

If we assume that, given a family of valuation overrings overrings $\boldsymbol{\mathcal{W}}$ of D, the overring $T := \bigcap \{W \mid W \in \boldsymbol{\mathcal{W}}\}$ of D coincides with D, then the map $\wedge_{\boldsymbol{\mathcal{W}}}$ defines a (semi)star operation of D. In particular, if (and only if) D is integrally closed, the b-operation is a (semi)star operation of D.

Remark 5. Given an integrally closed domain D, note that Gilmer discusses star operations defined as above (on the fractional ideals of D) using collections of valuation overrings $\boldsymbol{\mathcal{W}}$ of D with the property that $D = \bigcap \{W \mid W \in \boldsymbol{\mathcal{W}}\}$ and refers to them as w-*operations*. Since the terminology of w-operation was re-introduced recently by Wang and McCasland (see [13,14]) for denoting a very different kind of star operation, in order to avoid a possible confusion, we have slightly modified Gilmer's original terminology (i.e., star "$\boldsymbol{\mathcal{W}}$-operation" instead of "w-operation"), by emphasizing the set of valuation overrings occurring in the definition.

Gilmer proves that, given any eab star operation $*$ of a domain D, there exists a (star) $\boldsymbol{\mathcal{W}}$-operation of D which agrees with $*$ on all finitely generated ideals [1, Theorem 32.12]. It would seem then that $\boldsymbol{\mathcal{W}}$-operations may be the most refined class of eab operations. We have one more class to define, however.

For a domain D and a semistar operation $\star$ of D, we say that a valuation overring V of D is a $\star$-*valuation overring of* D provided $F^{\star} \subseteq FV$, for each $F \in \boldsymbol{f}(D)$. Set $\boldsymbol{\mathcal{V}}(\star) := \{V \mid V$ is a $\star$-valuation overring of $D\}$ and let $b(\star) := \wedge_{\boldsymbol{\mathcal{V}}(\star)}$ the ab semistar operation on D defined as follows: for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{b(\star)} := \bigcap \{EV \mid V \in \boldsymbol{\mathcal{V}}(\star)\}.$$

Clearly, when $\star$ coincides with d, the identity (semi)star operation, then $b(d) = b$. Note that, this example shows that even if $\star$ is a (semi)star operation, $b(\star)$ may be a *proper* semistar operation (e.g., $b(d) = b$ is a (semi)star operation of D if and only if D is integrally closed).

We call the semistar operation $b(\star)$ defined as above, using the $\star$-valuation overrings of a domain D associated with a given semistar operation $\star$ on D, *the complete $\boldsymbol{\mathcal{W}}$-operation associated with* $\star$. From the definition, it follows that $\star_f \leq b(\star)$. *A complete ab operation* is a semistar operation $\star$ such that $\star = b(\star)$. Clearly, a complete ab operation is a $\boldsymbol{\mathcal{W}}$-operation and so, without loss of generality, we may consider just the complete $\boldsymbol{\mathcal{W}}$-operations. Since $F^{b(\star)}V = F^{\star}V$, for all $F \in \boldsymbol{f}(D)$ and for all $V \in \boldsymbol{\mathcal{V}}(\star)$, then clearly, $b(b(\star)) = b(\star)$ and so $b(\star)$ is a complete ab operation.

Let D be a domain and $\star$ a semistar operation. Note that, by definition, the $\star$-valuation overrings coincide with the $\star_f$-valuation overrings. Hence, the above construction could be done using $\star_f$ in place of $\star$, i.e., $b(\star) = b(\star_f)$.

Remark 6. Note that not all $\boldsymbol{\mathcal{W}}$-operations are complete (see Example 14). In a work in progress on the ultrafilter topology of abstract Riemann surfaces (in the sense of Zariski [15]), we will describe the complete ab semistar operation $b(\wedge_{\boldsymbol{\mathcal{W}}})$ for any family $\boldsymbol{\mathcal{W}}$ of valuation domains sharing the same field of quotients.

The four distinguished classes of semistar operations introduced above are related as follows.

Proposition 7. *Let D be a domain and let $\star$ be a semistar operation on D. Consider the following four propositions.*

(1) $\star$ *is an eab operation.*
(2) $\star$ *is an ab operation.*
(3) $\star$ *is a $\boldsymbol{\mathcal{W}}$-operation.*
(4) $\star$ *is a complete $\boldsymbol{\mathcal{W}}$-operation.*

Then (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1).

Proof. The only implication which is not trivial is (3) $\Rightarrow$ (2). This follows immediately though from the observation that any finitely generated ideal of a domain D extends to a principal ideal in any valuation overring V of D. □

The next goal is to give examples to show that each of the implications is not reversible. In fact, this paper began with the desire to demonstrate that the ab property was properly stronger than the eab property (we were unable to find an example in the literature of an eab operation which was not ab) and expanded to a broader study and finer classification of eab operations. In particular, we pay special attention to the class of complete $\boldsymbol{\mathcal{W}}$-operations and give several characterizations of them.

It is not so simple to demonstrate that the implications in Proposition 7 are not reversible. We will give three examples covering the three pairs of classes, including the desired example of an eab operation which is not an ab operation. First, however, we will give the promised additional characterizations of the class of complete $\boldsymbol{\mathcal{W}}$-operations.

We start by extending Gilmer's notion of equivalent star operation to the semistar setting: let $\star_1$ and $\star_2$ be two semistar operations defined on an integral domain D, we say that $\star_1$ *is equivalent to* $\star_2$ if they agree on $\boldsymbol{f}(D)$, i.e., $F^{\star_1} = F^{\star_2}$ for each $F \in \boldsymbol{f}(D)$. It is very plausible that there can be numerous eab semistar operations that are all equivalent to the same (e)ab semistar operation of finite type. Note that, if $\star_1$ and $\star_2$ are equivalent and $\star_1$ is eab, then $\star_2$ is also eab. Hence, we can

partition the set of all eab semistar operations on a domain D into classes of equivalent operations. Each equivalence class has a single distinguished member, the one of finite type.

A result proven in [16, Proposition 3.4 and Theorem 3.5] ensures that each eab semistar operation $\star$ is equivalent to $b(\star)$. The preceding fact seems to give us two distinguished members (i.e., $\star_f$ and $b(\star)$) in each equivalence class of eab semistar operations. We resolve this apparent conflict by introducing yet another semistar construction.

Suppose that D is a domain with quotient field K, $\star$ is a semistar operation on D, X is an indeterminate over D and $\boldsymbol{c}(h)$ is the content of a polynomial $h \in D[X]$. Then, we define

$$\mathrm{Kr}(D, \star) := \{f/g \mid f, g \in D[X],\ g \neq 0,\ \text{and there exists } h \in D[X] \setminus \{0\} \text{ with } (\boldsymbol{c}(f)\boldsymbol{c}(h))^{\star} \subseteq (\boldsymbol{c}(g)\boldsymbol{c}(h))^{\star}\}.$$

This is a Bézout domain with quotient field $K(X)$, called *the semistar Kronecker function ring associated to semistar operation* $\star$ [16, Theorem 5.1 and Theorem 3.11 (3)]. We can then define an eab semistar operation on D, denoted by $\circlearrowleft_{\mathrm{Kr}}$, as follows:

$$E^{\circlearrowleft_{\mathrm{Kr}}} := E\mathrm{Kr}(D, \star) \cap K, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D),\ [16, \text{Corollary 5.2}].$$

From [17, Proposition 3.4 and its proof] (or, in a more general context, from [2, Proposition 6.3]) it follows that, given an eab semistar operation $\star$, $\circlearrowleft_{\mathrm{Kr}} = b(\star)$.

On the other hand, it is proven in [16, Corollary 5.2] that $\circlearrowleft_{\mathrm{Kr}}$ is a semistar operation of finite type. Hence, the preceding results, all collected, yield the following: for any eab semistar operation $\star$, $\circlearrowleft_{\mathrm{Kr}} = b(\star) = \star_f$.

There is still another construction, with a more classical origin, for associating to a semistar operation an (e)ab semistar operation of finite type. In order to introduce this construction we need first to generalize, in the semistar operation setting, one of the useful characterizations, given in [1, Theorem 6.5] and [6, Lemma 1] for cancellation and quasi-cancellation ideals.

Lemma 8. *Let D be a domain, let $F \in \boldsymbol{f}(D)$ and let $\star$ be a semistar operation on D. Then, F is $\star$-eab [respectively, $\star$-ab] if and only if $((FH)^{\star} : F^{\star}) = H^{\star}$, for each $H \in \boldsymbol{f}(D)$ [respectively, for each $H \in \overline{\boldsymbol{F}}(D)$].*

(Note that $((FH)^{\star} : F^{\star}) = ((FH)^{\star} : F)$, so the previous equivalences can be stated in a formally slightly different way.)

Proof. We consider only the ab case, since $\star$-eab coincides with $\star_f$-ab (Proposition 4). As a matter of fact, $((FH)^{\star} : F^{\star}) = H^{\star}$, coincides with $((FH)^{\star_f} : F^{\star_f}) = H^{\star_f}$, when $F, H \in \boldsymbol{f}(D)$; if $H \in \overline{\boldsymbol{F}}(D)$ and $F \in \boldsymbol{f}(D)$, then $((FH)^{\star_f} : F^{\star_f}) = ((FH)^{\star_f} : F) = \left(\left(\bigcup\{(FL)^{\star} \mid L \subseteq H, L \in \boldsymbol{f}(D)\}\right) : F\right) = \bigcup\{((FL)^{\star} : F) \mid L \subseteq H, L \in \boldsymbol{f}(D)\}$.

The "if" part: it is easy to see that, F is $\star$-ab if and only if $(FG)^{\star} = (FH)^{\star}$, with $G, H \in \overline{\boldsymbol{F}}(D)$, implies that $G^{\star} = H^{\star}$. Then,

$$(FG)^{\star} = (FH)^{\star} \;\Rightarrow\; \left((FG)^{\star} : F^{\star}\right) = \left((FH)^{\star} : F^{\star}\right).$$

The conclusion now is a straightforward consequence of the assumption.

The "only if" part: given $H \in \overline{\boldsymbol{F}}(D)$, clearly $H^{\star} \subseteq ((FH)^{\star} : F^{\star})$. Conversely, note that $F((FH)^{\star} : F^{\star}) \subseteq (FH)^{\star}$, and so we have $(F((FH)^{\star} : F^{\star}))^{\star} \subseteq (FH)^{\star}$. Therefore, by the assumption, $((FH)^{\star} : F^{\star})^{\star} \subseteq H^{\star}$. $\square$

Using the characterization in Lemma 8, we can associate to any semistar operation $\star$ of D an (e)ab semistar operation of finite type $\star_a$ of D, called *the (e)ab semistar operation associated to* $\star$, defined as follows for each $F \in \boldsymbol{f}(D)$ and for each $E \in \overline{\boldsymbol{F}}(D)$:

$$F^{\star_a} := \bigcup\{((FH)^{\star} : H^{\star}) \mid H \in \boldsymbol{f}(D)\},$$
$$E^{\star_a} := \bigcup\{F^{\star_a} \mid F \subseteq E, F \in \boldsymbol{f}(D)\},$$

[16, Definition 4.4 and Proposition 4.5]. The previous construction, in the ideal systems setting, is essentially due to Jaffard [9] and Halter-Koch [18,19].

Obviously $(\star_f)_a = \star_a$. Note also that, when $\star = \star_f$, then $\star$ is (e)ab if and only if $\star = \star_a$ [16, Proposition 4.5(5)].

It follows that if $\star$ is any eab semistar operation then $\star_a$ is the unique (e)ab semistar operation which is of finite type and is equivalent to $\star$. Hence, we can extend our previous characterization.

Proposition 9. *Let D be a domain and let $\star$ be an eab semistar operation. Then $\star_f = \circlearrowleft_{\mathrm{Kr}} = b(\star) = \star_a$.*

Remark 10. Note that, with the notation introduced above, $D^{\star_a}$ is integrally closed and contains the integral closure of D in its field of quotients [16, Proposition 4.3 and Proposition 4.5 (10)]. In particular, when $\star = v$, then D^{v_a} coincides with the pseudo-integral closure of D introduced by Anderson, Houston and Zafrullah [20]. Therefore, $\star_a$ is a semistar operation which might be a proper semistar operation, even if $\star$ is a (semi)star operation.

The next goal is to show that in many cases the properties eab and ab coincide. Probably, the most important (semi)star operation which is not generally of finite type is the v-operation. In this case, from [1, Theorem 34.6] it follows that *the following properties are equivalent*:

 (i) *For each $F \in \boldsymbol{f}(D)$, $(FF^{-1})^v = D$.*
 (ii) *For each $F \in \boldsymbol{f}(D)$, F is v-ab (i.e., v is ab).*
 (iii) *For each $F \in \boldsymbol{f}(D)$, F is v-eab (i.e., v is eab or, equivalently, D is a v-domain [1, page 418]).*

We have already observed that, for a semistar operation $\star$, if $\star = \star_f$, then the notions of $\star$-ab and $\star$-eab coincide (Proposition 4). The following result provides further information, but to state it we need to recall some standard facts on semistar operations and related ideals.

We say that a nonzero ideal I of D is a *quasi-$\star$-ideal* if $I^\star \cap D = I$, a *quasi-$\star$-prime* if it is a prime quasi-$\star$-ideal, and a *quasi-$\star$-maximal* if it is maximal in the set of all proper quasi-$\star$-ideals. A quasi-$\star$-maximal ideal is a prime ideal. It is possible to prove that each proper quasi-$\star_f$-ideal is contained in a quasi-$\star_f$-maximal ideal. More details can be found in [21, page 4781]. We will denote by $\mathrm{QMax}^\star(D)$ the set of the quasi-$\star$-maximal ideals of D. By the previous considerations we have that $\mathrm{QMax}^\star(D)$ is not empty, for all semistar operations $\star$ *of finite type*. Then, for each $E \in \overline{\boldsymbol{F}}(D)$, we can consider

$$E^{\widetilde{\star}} := \bigcap \left\{ ED_P \mid P \in \mathrm{QMax}^{\star_f}(D) \right\}.$$

It is well known that the previous definition gives rise to a semistar operation $\widetilde{\star}$ of D which is stable (i.e., $(E \cap F)^{\widetilde{\star}} = E^{\widetilde{\star}} \cap F^{\widetilde{\star}}$, for each $E, F \in \overline{\boldsymbol{F}}(D)$) and of finite type [22, Corollary 3.9]. Recall that, if K is the quotient field of D and X is an indeterminate over K, we set $\mathrm{Na}(D, \star) := \{f/g \in K(X) \mid f, g \in D[X], 0 \neq g \text{ and } \boldsymbol{c}(g)^\star = D^\star\}$. It is known that $E^{\widetilde{\star}} = E\mathrm{Na}(D, \star) \cap K$ for all $E \in \overline{\boldsymbol{F}}(D)$ [21, Proposition 3.4(3)].

Proposition 11. *Let $\star$ be a semistar operation of an integral domain D and let $F \in \boldsymbol{f}(D)$. The following properties are equivalent:*

(i) $(FF^{-1})^{\star_f} = D^\star$.
(ii) F is $\widetilde{\star}$-ab.
(iii) F is $\widetilde{\star}$-eab.

Proof. Since $\mathrm{QMax}^{\star_f}(D) = \mathrm{QMax}^{\widetilde{\star}}(D)$ [21, Corollary 3.5(2)], it is easy to see that $(FF^{-1})^{\star_f} = D^\star$ if and only if $(FF^{-1})^{\widetilde{\star}} = D^{\widetilde{\star}}$. From this observation, it follows immediately that (i) $\Rightarrow$ (ii). By the definitions, it is clear that (ii) $\Rightarrow$ (iii).

(iii) $\Rightarrow$ (i) By [23, Theorem 2.23], recall that (i) is also equivalent to each of the following statements:

(i') F_Q is a nonzero principal fractional ideal in D_Q, for all $Q \in \mathrm{QMax}^{\star_f}(D)$.
(i'') $F\mathrm{Na}(D, \star)$ is an invertible fractional ideal of $\mathrm{Na}(D, \star)$ (i.e., $F\mathrm{Na}(D, \star)_M$ is nonzero principal, for each $M \in \mathrm{Max}(\mathrm{Na}(D, \star))$).

Let $F \in \boldsymbol{f}(D)$ be a $\widetilde{\star}$-eab ideal. We want to show that F_Q is a nonzero fractional principal ideal of D_Q for all quasi-$\star_f$-maximal ($=$quasi-$\widetilde{\star}$-maximal) ideal Q of D. Note that, by the definition of $\widetilde{\star}$, it is easy to see that $H^{\widetilde{\star}}D_Q = HD_Q$, for all $H \in \boldsymbol{f}(D)$ and for all quasi-$\widetilde{\star}$-maximal ideal Q of D. From this observation and from the fact that each finitely generated ideal of D_Q is extended from a finitely generated ideal of D, it follows that $F\widetilde{\star}$-eab implies that FD_Q is nonzero (quasi-)cancellative in D_Q, for all quasi-$\widetilde{\star}$-maximal ideals Q of D. This is equivalent to saying that FD_Q is nonzero principal in D_Q, for all quasi-$\widetilde{\star}$-maximal ideal Q of D by Remark 2(1). $\square$

From the previous proposition, we reobtain some of the characterizations given in [24, Theorem 3.1] of a *Prüfer $\star$-multiplication domain* (i.e., an integral domain in which every nonzero finitely generated ideal is $\star_f$-invertible).

Corollary 12. *Let $\star$ be a semistar operation of an integral domain D. The following properties are equivalent:*

(i) D *is a Prüfer $\star$-multiplication domain.*
(ii) $\widetilde{\star}$ *is* ab.
(iii) $\widetilde{\star}$ *is* eab.
(iv) $\mathrm{Na}(D, \star)$ *is a Prüfer domain.*

Proof. (i), (ii) and (iii) are the direct globalizations to all $F \in \boldsymbol{f}(D)$ of the corresponding properties of Proposition 11. (iv) is equivalent to the globalization of (i'') to all $F_1 \in \boldsymbol{f}(\mathrm{Na}(D, \star))$. $\square$

Remark 13. (1) Note that, even for a star operation (of finite type) $*$, the notions of $*_f$-ab and $\widetilde{*}$-ab do not coincide. For instance, take $*$ equal to the b-operation on an integrally closed non-Prüfer domain D, then clearly $b_f = b$ and $\widetilde{b} = d$. Moreover, b is an ab-operation for every domain D, but d is not an ab-operation if D is not Prüfer. In particular, the previous example shows that there exist star operations (of finite type) $*$ and nonzero finitely generated ideals that are $\widetilde{*}$-ab but not $*_f$-ab.

(2) From the previous observation and from Corollary 12, we also deduce that the notions of Prüfer b–multiplication domain and Prüfer domain coincide.

(3) Note that if $\star_1$ and $\star_2$ are two semistar operations on an integral domain and if $\star_1 \leq \star_2$, then in general there are no relations between the notions of $\star_1$-eab (respectively, $\star_1$-ab) ideal and $\star_2$-eab [respectively, $\star_2$-ab] ideal.

For instance, let K be a field and X and Y be two indeterminates over K. Set $D := K[X, Y]_{(X,Y)}$ and $N := (X, Y)D$. Consider on D the (semi)star operation $\star$ (of finite type) defined in [21, Example 5.3]. In this case, $(\widetilde{\star})_a = b \leq \star \leq \star_a = t$. (The only fact not already explicitly proved in [21, Example 5.3] is that $b \leq \star$, but this follows from examining each type of ideal occurring in the set of ideals "generating" $\star$ and from observing that $N^b = N$, because there is always a valuation overring of D centered on N.) So, obviously, every ideal of D is b-ab and t-ab, but for instance N is not $\star$-ab, since by the definition $(N^k)^\star = N^\star = N$, for all $k \geq 1$.

On the other hand, in general, we know that, given $\star_1$ and $\star_2$ two semistar operations on an integral domain with $\star_1 \leq \star_2$ and $F \in \boldsymbol{f}(D)$, then F is $\widetilde{\star_1}$-ab implies that F is $\widetilde{\star_2}$-ab [2, Corollary 5.2(1)].

3. Examples

Now we proceed to the promised examples.

Example 14. *An example of a $\mathcal{W}$-operation which is not of finite type (and so it is not a complete $\mathcal{W}$-operation).*

We say that a domain D is an *almost Dedekind domain* if D_M is a DVR for each maximal ideal M of D. Let D be an almost Dedekind domain with the property that each maximal ideal is finitely generated except for one. Let M_∞ be the one maximal ideal of D which is not finitely generated. Explicit examples of such domains can be found for instance in [25, Example 2], [26, Example 30], [27, 6.10] or [28]. Let $\Delta := \mathrm{Max}(D) \setminus \{M_\infty\}$ and let $\mathcal{W} := \{D_M \mid M \in \Delta\}$ and set $\star := \wedge_{\mathcal{W}}$. In this case $D^\star = \bigcap \{D_M \mid M \in \Delta\} = D$ [25, Example 2] and so $\star$ is a (semi)star operation of D.

Since D is a Prüfer domain, each nonzero finitely generated ideal F is invertible. Moreover, any invertible ideal is necessarily a v-ideal and so, in particular, $F^\star = F$ for each $F \in \boldsymbol{f}(D)$. If follows that $\star_f$ is the identity operation of D. However, it is clear from the definition that $(M_\infty)^\star = D$. Hence, $\star$ is not of finite type.

For the next example, we note that [1, Proposition 32.4] provides a way of producing star operations given a collection of ideals. In particular, we begin with a collection $\mathscr{S}$ of fractional ideals of D which contains all of the principal fractional ideals and satisfies the condition that if $J \in \mathscr{S}$ and αD is a principal fractional ideal of D then $\alpha J \in \mathscr{S}$. We then define the star operation $*$ of D (depending on $\mathscr{S}$) by saying that, for each $E \in \boldsymbol{F}(D)$,

$$E^* := \bigcap \{J \mid J \in \mathscr{S} \text{ and } E \subseteq J\}.$$

Example 15. *An example of an $\mathfrak{ab}$ star operation which is not a (star) $\mathcal{W}$-operation.*

As in the previous example, we let D be an almost Dedekind domain with exactly one maximal ideal M_∞ which is not finitely generated. To define the required star operation, we give a generating collection of ideals as in the comments above. In particular, we let $\mathscr{S}$ consist of all fractional invertible ideals and all ideals of the form JM_∞ where J is fractional invertible. As recalled above, [1, Proposition 32.4] guarantees that this collection will generate a star operation $*$ of D and it is well known that any star operation on a Prüfer domain is an $\mathfrak{ab}$ operation. Finally, we note that M_∞^2 cannot be written as an intersection of ideals in $\mathscr{S}$ (in fact, M_∞^2 is only contained in the ideal M_∞ among the ideals belonging to $\mathscr{S}$), thus $(M_\infty^2)^* = M_\infty$. This proves that $*$ cannot be a $\mathcal{W}$-operation. As a matter of fact, clearly, each valuation overring of D is of the form D_N for some maximal ideal N of D. If $*$ were a $\mathcal{W}$-operation for some family of valuation overrings $\mathcal{W}$ of D, then either D_{M_∞} would be included in $\mathcal{W}$ or not. If it were included, then we would have $(M_\infty^2)^* = M_\infty^2$ and, if it were not included, then we would have $(M_\infty)^* = D$, as in the previous example. Both of these possibilities fail in the current example. It follows that $*$ cannot be a $\mathcal{W}$-operation.

Finally, we give the example which motivated the paper.

Example 16. *Example of an $\mathfrak{eab}$ star operation that is not an $\mathfrak{ab}$ star operation.*

Let k be a field, let $X_1, X_2, X_n, \ldots$ be an infinite set of indeterminates over k and let $N := (X_1, X_2, X_n, \ldots)k[X_1, X_2, X_n, \ldots]$. Clearly, N is a maximal ideal in $k[X_1, X_2, X_n, \ldots]$. Set $D := k[X_1, X_2, X_n, \ldots]_N$, let $M := ND$ be the maximal ideal of the local domain D and let K be the quotient field of D.

Note that D is a UFD and consider $\mathcal{W}$ the set of all the rank one valuation overrings of D. Let $\wedge_{\mathcal{W}}$ be the star $\mathfrak{ab}$ operation on D defined by $\mathcal{W}$. It is well known that the t-operation on D is an $\mathfrak{ab}$ star operation, since $t|_{\boldsymbol{f}(D)} = \wedge_{\mathcal{W}}|_{\boldsymbol{f}(D)}$ [1, Proposition 44.13].

We consider the following subset of fractionary ideals of D:

$$\mathscr{J} := \{xF^t, yM, zM^2 \mid x, y, z \in K \setminus \{0\}, F \in \boldsymbol{f}(D)\}.$$

Since each nonzero principal fractional ideal of D is in $\mathscr{J}$ and, for each ideal $J \in \mathscr{J}$ and for each nonzero $a \in K$, the ideal aJ belongs to $\mathscr{J}$, then, as above, [1, Proposition 32.4], guarantees that the set $\mathscr{J}$ defines on D a star operation $*$. Since, for each $F \in \boldsymbol{f}(D)$, $F^t \in \mathscr{J}$, then $*|_{\boldsymbol{f}(D)} = t|_{\boldsymbol{f}(D)}$ and so $*$ is an $\mathfrak{eab}$ operation on D, since t is an $(\mathfrak{e})\mathfrak{ab}$ star operation on D. Note that $(X_1, X_2)M \subset M^2$ and $(M^2)^* = M^2$, because $M^2 \in \mathscr{J}$.

We claim that:

$$((X_1, X_2)M)^* = ((X_1, X_2))^t \cap M^2 = M^2 = ((X_1, X_2)M^2)^*.$$

As a matter of fact, if $(X_1, X_2)M \subseteq G^t$ for some $G \in \boldsymbol{f}(D)$, then we have $((X_1, X_2)D)^t M^t \subseteq G^t$, with $((X_1, X_2)D)^t = M^t = D$, since X_1 and X_2 are coprime in D and so $(X_1, X_2)D$ is not contained in any proper principal ideal of D. Therefore $(X_1, X_2)M$ is not contained in any nontrivial ideal of the type $xF^t (= G^t) \in \mathscr{J}$.

A similar argument shows that $(X_1, X_2)M$ is neither contained in any ideal of the type $yM, zM^2 \in \mathscr{J}$, with y and z nonzero and non-unit in D, and thus the only nontrivial ideals of $\mathscr{J}$ containing $(X_1, X_2)M$ are M^2 and M, hence $((X_1, X_2)M)^* = M^2$. A similar argument shows that $((X_1, X_2)M^2)^* = M^2$.

Since $((X_1, X_2)M)^* = M^2 = ((X_1, X_2)M^2)^*$, if $*$ were an $\mathfrak{ab}$ star operation, then we would deduce that $M^\star (= M)$ is equal to $(M^2)^* (= M^2)$, which is not the case.

4. Generalization: A conjecture

Given an eab semistar operation $\star$ on a domain D, we introduced in Section 2 several natural means to associate a new eab semistar operation to $\star$. In [2] we introduced a ring construction $KN(D, \star)$ which simultaneously generalizes the notions of Kronecker function ring and Nagata ring, for an arbitrarily given semistar operation $\star$ on any domain D. Along with this generalized function ring, we introduced a semistar operation $\star_\ell$ which is a semistar operation on D. What is noteworthy about this is that $\star_\ell$ possesses at least two different interpretations that seem to be natural generalizations of the constructions giving rise to the semistar operations $\circlearrowleft_{Kr}$ and $b(\star)$ (both coinciding with $\star_f$, when $\star$ is eab). On the other hand, $\star_f$ and $\star_\ell$ can be dramatically different for a given semistar operation $\star$ which is not eab. What seems plausible then is that we can unify the theory of $\star_\ell$ and the theory developed in this paper regarding eab operations of finite type if we can give a construction for $\star_\ell$ which agrees with $\star_f$ in the case where $\star$ is eab. We have a candidate for such a construction which seems plausible, but at this time do not have a proof. We start by giving a brief summary of results from [2].

Let D be a domain with quotient field K and let $\star$ be a semistar operation on D. We call an overring T of D a $\star$-*monolocality of* D provided $T^{\star_f} = T$ and FT is a principal fractional ideal of T, for each $\star$-eab $F \in \boldsymbol{f}(D)$. Let $\mathscr{L}(D, \star)$ be the set of all $\star$-monolocalities on D. We can then define the new semistar operation $\star_\ell$ on D by setting, for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\star_\ell} := \bigcap \{ET \mid T \in \mathscr{L}(D, \star)\}.$$

In particular, since a finitely generated ideal extends to a principal ideal in a valuation overring, we have $\mathcal{V}(\star)$ $(=\{V \mid V$ is a $\star$-valuation overring of $D\}) \subseteq \mathscr{L}(D, \star)$. Therefore, $\star_\ell \leq b(\star)$.

Remark 17. (1) Note that, for any semistar operation $\star$, it is known that $\star_\ell \leq \star_f$ [2, Proposition 6.3] and this inequality is stronger than $\star_\ell \leq b(\star)$, since by the definition of $\star$-valuation overring it follows immediately that $\star_f \leq b(\star)$.

(2) Since each $\star$-monolocality contains a minimal $\star$-monolocality [2, Proposition 5.11(7)], if we denote by $\mathscr{L}(D, \star)_{\min}$, or simply by $\mathscr{L}_{\min}$, the set of all minimal $\star$-monolocalities of D, then $E^{\star_\ell} = \bigcap \{ET \mid T \in \mathscr{L}_{\min}\}$, for each $E \in \overline{\boldsymbol{F}}(D)$.

If we define the domain $KN(D, \star)$ to be the subring of the field of rational functions $K(X)$ given by

$$KN(D, \star) := \left\{ \frac{f}{g} \mid f, g \in D[X], \ g \neq 0, \ \boldsymbol{c}(f)^\star \subseteq \boldsymbol{c}(g)^\star, \ \text{and } \boldsymbol{c}(g) \text{ is } \star\text{-eab} \right\},$$

then we know that this ring generalizes both the classical Kronecker function ring construction (the case where $\star = \star_a$, i.e., $KN(D, \star_a) = Kr(D, \star_a) = Kr(D, \star)$ [2, Proposition 5.4(2) and Theorem 5.11(7)]) and the Nagata ring construction (the case where $\star = \widetilde{\star}$, i.e., $KN(D, \widetilde{\star}) = Na(D, \widetilde{\star}) = Na(D, \star)$ [2, Proposition 5.4(1) and Theorem 5.1(7)]).

As we did for $\circlearrowleft_{Kr}$, we can then define a new semistar operation $\circlearrowleft_{KN}$ on D using the previous construction as follows, for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\circlearrowleft_{KN}} := E(KN(D, \star)) \cap K.$$

Since $b(\star) = \circlearrowleft_{Kr}$ and $\star_\ell$ is a generalization of $b(\star)$, the key point of this speculative section is then made clear by the following result.

Theorem 18 ([2, Propositions 5.1, 5.11(7), and 6.3]). *Let D be a domain and let $\star$ be a semistar operation on D. Then,*

(1) $KN(D, \star) = \bigcap \{T(X) \mid T \in \mathscr{L}(D, \star)\}$.
(2) $\star_\ell = \circlearrowleft_{KN}$.

As noted earlier, what is needed now to unify the theory is a construction which begins with a semistar operation $\star$ and yields $\star_\ell$ in the general setting, but obviously yields $\star_f = b(\star)$ in the special case where $\star$ is an eab operation (recall that, if $\star$ is an eab semistar operation, then any finitely generated ideal is an eab ideal and so $KN(D, \star) = Kr(D, \star)$), i.e., $\star_\ell = \star_a$ $(=b(\star) = \star_f)$).

Let $\boldsymbol{f}_\flat(D)$ be the set of all nonzero (finitely generated) $\star$-eab fractional ideals of D. For each $E \in \overline{\boldsymbol{F}}(D)$, we then define

$$E^{\star_\flat} := \bigcup \{F^\star \mid F \subseteq E \text{ and } F \in \boldsymbol{f}_\flat(D)\}.$$

Obviously, $E^{\star_\flat} \subseteq E^{\star_f}$ and if $\star$ is an eab semistar operation, then $\boldsymbol{f}_\flat(D) = \boldsymbol{f}(D)$ and so $\star_\flat = \star_f$ Note also that, $d_\flat = d$.

In general, it is not clear that $E^{\star_\flat}$ is even an ideal. So the proposed new semistar operation would be defined using the ideal generated by the set $E^{\star_\flat}$. The reason that such a definition seems reasonable can be seen if one considers how an ideal of D gets larger when one extends to the ring $KN(D, \star)$ and then contracts to K. Suppose then that J is an ideal of a domain D and that $\star$ is a semistar operation on D. Suppose also that I is a $\star$-eab ideal of D such that $I \subseteq J$. Let $\{a_0, a_1, \ldots, a_n\}$ be a set of generators of I and let $d \in I^\star$. Let $f(X) := a_n X^n + \cdots + a_1 X + a_0$. Then by the definition $\frac{d}{f(X)} \in KN(D, \star)$. It follows that $d \in J^{\star_\ell}$. It remains to be proven that $J^{\star_\ell}$ can be generated by such elements.

We conclude with the following.

Conjecture. *Let D be a domain and let $\star$ be a semistar operation on D. Then, $\star_\flat$, as defined above, is a semistar operation on D and is actually equal to $\star_\ell$.*

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