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A Spectral Construction of a Treed Domain that is not Going-Down

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Abstract

It is proved that if $2 \leq d \leq \infty$, then there exist a treed domain R of Krull dimension d and an integral domain T containing R as a subring such that the extension $R \subseteq T$ does not satisfy the going-down property. Rather than proceeding ring-theoretically, we construct a suitable spectral map φ connecting spectral (po)sets, then use a realization theorem of Hochster to infer that φ is essentially $\text{Spec}(f)$ for a suitable ring homomorphism f, and finally replace f with an inclusion map $R \hookrightarrow T$ having the asserted properties.

1 Introduction and Summary

The purpose of this note is to construct a ring-theoretic example by using some order-theoretic machinery and relatively little calculation. In the next paragraph, we review the relevant ring-theoretic background and state the main result. In the following paragraph, we review the relevant order-theoretic machinery and outline our approach. Full details are given in Section 2.

All rings considered below are commutative with identity; all ring extensions and all ring homomorphisms are unital. A ring homomorphism $f : A \to B$ is said to *satisfy going-down* if, whenever $P_2 \subseteq P_1$ are prime ideals of A and Q_1 is a prime ideal of B such that $f^{-1}(Q_1) = P_1$, there exists a prime ideal Q_2 of B such that $Q_2 \subseteq Q_1$ and $f^{-1}(Q_2) = P_2$. A ring extension $A \subseteq B$ is said to *satisfy going-down* if the inclusion map $i : A \hookrightarrow B$ satisfies going-down. Following [2] and [7], we say that an integral domain R is a *going-down domain* in case $R \subseteq T$ satisfies going-down for all integral domains T containing R as a subring (equivalently, for all integral domains T contained between R and its quotient field). The most natural examples of going-down domains are arbitrary Prüfer domains and integral domains of Krull dimension at most 1. The fundamental order-theoretic fact about such rings is [2, Theorem 2.2]: any going-down domain is a treed domain. (For each integral domain A, the set $\text{Spec}(A)$ of all prime ideals of A is a poset via inclusion; A is said to be a *treed domain* in case $\text{Spec}(A)$, as a poset, is a tree, that is, in case no prime ideal of A contains incomparable prime ideals of A.) Remarkably, the converse is false, as [8, Example 4.4] presents a construction, due to W. J. Lewis, of an extension $R \subseteq T$ of two-dimensional domains such that R is a treed domain and $R \subseteq T$ does not satisfy going-down. Like the construction of Lewis, the only other known example of this phenomenon [4, Example 2.3] depends on a specific type of ring-theoretic construction ($k + J(A)$, as in [13, (E2.1), p. 204]) whose analysis involves a considerable amount of calculation. It seems natural to ask if one can use order-theoretic methods to produce a treed domain R that is not a going-down domain without having to appeal to the details of a specific ring-theoretic construction. We do so here for all possible Krull dimensions d of R, namely, $2 \leq d \leq \infty$.

A key concept in our approach is that of an $L-$spectral set. Recall from [11, p. 53] that the underlying set of any T_0-topological space Z can be given the structure of a poset as follows: for $x, y \in Z, x \leq y \Leftrightarrow y \in \overline{\{x\}}$. A T_0-topology $\mathcal{T}$ on a poset $(W, \leq)$ is said to be *compatible with* $\leq$ in case $\leq$ coincides with the partial order induced by $\mathcal{T}$ on W. Recall from [1, Exercice 2, p. 89] that the finest topology on W that is compatible with the given partial order $\leq$ is the *left topology on* W, namely, the topology having an open basis consisting of the sets $w^{\downarrow} := \{v \in W \mid v \leq w\}$ as w runs through the elements of W. Let W^L denote W equipped with the left topology. As in [6], a poset W is called an *$L-$spectral set* if W^L is a *spectral space*, i.e., is homeomorphic to $\text{Spec}(A)$ (with the Zariski topology) for some ring A. (As usual, the Zariski topology on $\text{Spec}(A)$ is defined to be the topology that has an open basis consisting of the sets $\{P \in \text{Spec}(A) \mid a \notin P\}$ as a runs through the elements of A.) In Section 2, we construct $L-$spectral sets Y, X and a spectral map $\varphi : Y^L \to X^L$, in the sense of [11, p. 43], namely, a continuous map of spectral spaces for which the inverse image of any quasi-compact open set is quasi-compact. Y and X are chosen as small as possible for φ to fail to satisfy the order-theoretic analogue of the going-down property. Verification of the above-stated topological properties of X, Y and φ proceeds order-theoretically, by appealing to some results in [6]. Then, since the spectral map φ is surjective, we can apply [11, Theorem 6 (a)]. This result allows us to avoid introducing — or analyzing — a specific ring theoretic construction, for it essentially permits the identifcations $Y = \text{Spec}(B), X = \text{Spec}(A)$ and $\varphi = \text{Spec}(f)$, for a suitable ring homomorphism $f : A \to B$. (As usual, $\text{Spec}(f) : \text{Spec}(B) \to \text{Spec}(A)$ is defined by $Q \mapsto f^{-1}(Q)$.) The proof concludes by using standard ring-theoretic tools to replace $f : A \to B$ with an inclusion map $i : R \hookrightarrow T$ having the desired properties.

2 The construction

We begin by defining the three-element poset $Y := \{y_0, y_1, y_2\}$ by imposing the requirements that $y_0 \leq y_1$ and $y_0 \leq y_2$ (with y_1 and y_2 unrelated). Before analyzing Y order-theoretically with essentially no calculations, we indicate how detailed a ring-theoretical approach to the properties of Y would be. It can be seen ring-theoretically that Y is a spectral set: consider, for instance, the poset structure imposed by the Zariski topology on $\text{Spec}(D)$, where D is the localization of $\mathbb{Z}$ at the multiplicatively closed set $\mathbb{Z} \setminus (2\mathbb{Z} \cup 3\mathbb{Z})$. From this point of view, the Prime Avoidance Lemma (cf. [10, Proposition 4.9]) allows the identifications $y_0 = \{0\}, y_1 = 2D$ and $y_2 = 3D$. Using the definition of the Zariski topology, one can then show after some case analysis that the open sets of $Y = \text{Spec}(D)$ are $\emptyset, Y, \{y_0\}, \{y_0, y_1\}$ and $\{y_0, y_2\}$.

Fortunately, Y can be studied directly by order-theoretic means, without recourse to the above ring D. In the process, one learns even more: Y is an $L-$spectral set. To see this, one need only verify the four order-theoretic conditions $(\alpha) - (\delta)$ in the characterization of $L-$spectral sets in [6, Theorem 2.4]. Since Y is finite, it is evident that the following three conditions

(α) each nonempty linearly ordered subset of Y has a least upper bound,

(γ) Y has only finitely many maximal elements, and

(δ) for each pair of distinct elements $x, y \in Y$, there exist only finitely many elements of Y which are maximal in the set of common lower bounds of x and y

all hold in Y. Moreover, checking (β) amounts to the easy verification that each nonempty lower-directed subset Z of Y has a greatest lower bound z such that $\{y \in Y \mid z \leq y\} = \{y \in Y \mid w \leq y \text{ for some } w \in Z\}$. By using the definition of the left topology on Y, we obtain the same list of open sets as in the above ring-theoretic approach. This is not a coincidence, since an application of either the Main Theorem (whose order-theoretic criteria evidently hold in any finite poset) or Corollary 2.6 of [3] reveals that any finite poset has only one order-compatible topology.

We next introduce the three-element linearly ordered poset $X := \{x_0, x_1, x_2\}$ by imposing the requirements that $x_0 \leq x_1 \leq x_2$. (Since X has a unique maximal element, the eventual treed domain A will be automatically quasilocal, that is, will have a unique maximal ideal.) One could verify ring-theoretically that X is a spectral set (arising from, for instance, a valuation domain of Krull dimension 2) and then, by considering the Zariski topology, identify the open sets of X as $\emptyset, X, \{x_0\}$ and $\{x_0, x_1\}$. We leave these details to the reader, as the above-cited results from [3] ensure that a "left topology" approach would produce the same list of open sets in X. Of course, such an approach is appropriate, for by considering conditions $(\alpha)-(\delta)$ in [6, Theorem 2.4], one shows easily that any finite linearly ordered set is an $L-$spectral set.

The function $\varphi : Y \to X$ is defined by $\varphi(y_i) = x_i$ for $i = 1, 2, 3$. Observe that φ is surjective and order-preserving. Of course, φ is *not* an order-isomorphism). Indeed, we have constructed φ so as to fail to have the order-theoretic analogue of the going-down property, for no y_i satisfies both $y_i \leq y_2$ and $\varphi(y_i) = x_1$.) One could use the above lists of open sets to check that when viewed as a map $Y^L \to X^L$, φ is continuous, since $\varphi^{-1}(\{x_0\}) = \{y_0\}$ and $\varphi^{-1}(\{x_0, x_1\}) = \{y_0, y_1\}$. However, this detail can be avoided by appealing to [6, Lemma 2.6 (a)], which states that any order-preserving map of posets is continuous when these posets are each equipped with the left topology. Being a continuous function between finite spectral spaces, φ is also a spectral map (as the quasi-compact open subsets are the same as the open subsets). In short, $\varphi : Y^L \to X^L$ is spectral and surjective.

The above data are made to order for the realization assertion in [11, Theorem 6 (b)]. This result states that when Spec is viewed as a contravariant functor from the category of commutative rings (and ring homomorphisms) to the category of spectral spaces (and spectral maps), then Spec is invertible on the (nonfull) subcategory of all spectral spaces and surjective spectral maps. In particular, one infers the existence of a ring homomorphism $f : A \to B$ and homeomorphisms $\alpha : \mathrm{Spec}(A) \to X$, $\beta : \mathrm{Spec}(B) \to Y$ (where $\mathrm{Spec}(A)$ and $\mathrm{Spec}(B)$ are each endowed with the Zariski topology) such that $\alpha \circ \mathrm{Spec}(f) = \varphi \circ \beta$. It follows that $\mathrm{Spec}(f)$ is surjective. Moreover, since the homeomorphisms α, β are necessarily order-isomorphisms, it also follows that $\mathrm{Spec}(f)$ has all the order-theoretic properties of φ. In particular, f does not satisfy going-down.

We next reduce to the case of injective f. Indeed, the First Isomorphism

Theorem gives the factorization $f = j \circ \pi$, where $\pi : A \to A/\ker(f)$ is the canonical projection and $j : A/\ker(f) \hookrightarrow B$ is the canonical injection. Note that $\mathrm{Spec}(\pi)$ is a homeomorphism (hence, an order-isomorphism), the key point being that $P \supseteq \ker(f)$ for each prime ideal P of A. (To see this, take a prime ideal Q of B such that $P = \mathrm{Spec}(f)(Q) = f^{-1}(Q)$ and observe that $\ker(f) = f^{-1}(\{0\}) \subseteq f^{-1}(Q)$.) As $\mathrm{Spec}(j) = (\mathrm{Spec}(\pi))^{-1} \circ \mathrm{Spec}(f)$, we see that j does not satisfy going-down. By *abus de langage*, we henceforth replace f with j, viewed as an inclusion (and thus replace A with $A/\ker(f)$). Notice also that (either the "old" or the "new") A is a quasilocal ring of Krull dimension 2, thanks to the order-isomorphism α and the construction of X.

Since f does not satisfy going-down, we see via [5, Lemma 3.2 (a)] that the injection $f_{red} : A_{red} \to B_{red}$ of associated reduced rings also does not satisfy going-down. (Recall that if E is any ring, then $E_{red} := E/\sqrt{E}$, where $\sqrt{E}$ denotes the set of all nilpotent elements of E. It is well known that applying the Spec functor to the canonical projection $E \to E_{red}$ produces a homeomorphism. Of course, f_{red} is defined by $a + \sqrt{A} \mapsto f(a) + \sqrt{B}$.) By more *abus de langage*, we replace f with f_{red} (which is now viewed as an inclusion). Observe that (the "new") A is quasilocal and of Krull dimension 2. Moreover, we have now reduced to the case in which both A and B are reduced rings (that is, rings with no nonzero nilpotents) each having a unique minimal prime ideal, that is, integral domains.

For $d = 2$, putting $(R, T, i) := (A, B, f)$ produces, as asserted, an inclusion map $i : R \hookrightarrow T$ of integral domains such that R is (quasilocal and) of Krull dimension d and i does not satisfy going-down. To produce such an example in which T is contained between R and its quotient field, one need only invoke the characterization of going-down domains in [7, Theorem 1].

Suppose next that $3 \leq d \leq \infty$. Take R and (either) T as above, and let F denote the quotient field of T. Using, for instance, the proof of [10, Corollary 18.5], we can construct a valuation domain of the form $V = F + M$ such that V has Krull dimension $d - 2$ and M is the maximal ideal of V. (As usual, we take $\infty \pm r := \infty$ for each real number r.) Observe that the integral domains $R+M \subseteq T+M$ have the same quotient field, since they share M as a common nonzero ideal. The standard lore of the classical $(D+M)-$construction, as in [10, Exercise 12, p. 202], yields that $\mathrm{Spec}(R+M) = \mathrm{Spec}(V) \cup \{P+M \mid P \in \mathrm{Spec}(R)\}$; of course, one also has a similar description of $\mathrm{Spec}(T + M)$. (The same conclusions are available via [9, Theorem 1.4] since, for instance, $R + M$ is the pullback of the canonical projection $V \to V/M \cong F$ and the inclusion $R \hookrightarrow F$.) As valuation domains are quasilocal treed domains, it follows that $R + M$ inherits from R the property of being a (quasilocal) treed domain. Moreover, with "dim" denoting Krull dimension, we have that $\dim(R+M) = \dim(R) + \dim(V) = 2 + (d-2) = d$. Finally, as in the proof of [7, Corollary], the above description of prime spectra implies that the extension $R+M \subseteq T+M$ inherits from $R \subseteq T$ the failure of the going-down property. Consequently, $R + M \hookrightarrow T + M$ has the asserted properties, to complete the proof.

In closing, we contrast the above role of pullbacks (which we used only in the case $d \geq 3$) with their role in Lewis's two-dimensional example. That example had been only sketched in [8]. A fuller explanation of it, as in [4, Remark 2.1 (a), second paragraph], involves the use of either the "maximal quotient map" machinery of [12] or the fundamental gluing result on the prime spectra of pullbacks [9, Theorem 1.4] to analyze a pullback of the form $k + J(A)$. On the other hand, our approach needed such gluing information only for (the arguably more computationally tractable) pullbacks of classical $D + M$ type. In sum, our approach has used the order-theoretic characterization of spectral spaces when the ambient topology on a poset is the left topology and an order-theoretic verification that the function φ is a spectral map, Hochster's fundamental result on invertibility of the Spec functor for surjective spectral maps, and relatively straightforward ring theory consisting of isomorphism theorems and a description of the prime spectrum of the classical $(D + M)-$ construction.

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