Semistar linkedness and flatness, Prüfer semistar multiplication domains

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Abstract

In 1994, Matsuda and Okabe introduced the notion of semistar operation, extending the "classical" concept of star operation. In this paper, we introduce and study the notions of semistar linkedness and semistar flatness which are natural generalizations, to the semistar setting, of their corresponding "classical" concepts. As an application, among other results, we obtain a semistar version of Davis' and Richman's overring-theoretical theorems of characterization of Prüfer domains for Prüfer semistar multiplication domains.

1 Introduction

Star operations have a central place in multiplicative ideal theory, this concept arises from the classical theory of ideal systems, based on the work by W. Krull, E. Noether, H. Prüfer, and P. Lorenzen (cf. [15], [22], [19]). Recently, new interest on these theories has been originated by the work by R. Matsuda and A. Okabe [30], where the notion of semistar operation was introduced, as a generalization of the notion of star operation. This concept has been proven, regarding its flexibility, extremely useful in studying the structure of different classes of integral domains (cf. for instance [28], [8], [10], [11], [12], and [20]).

Recall that a domain D, on which a semistar operation $\star$ is defined, is called a *Prüfer semistar multiplication domain* (or $P{\star}MD$), if each nonzero finitely generated

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ideal F of D is $\star_f$-invertible (i.e., $(FF^{-1})^{\star_f} = D^\star$), where $\star_f$ is the semistar operation of finite type associated to $\star$ (cf. Section 2 for details). These domains generalize Prüfer v–multiplication domains [15, page 427] (and, in particular, Prüfer and Krull domains) to the semistar multiplication setting.
Among the various overring-theoretical characterization of Prüfer domains, the following two have relevant consequences:

- Davis' characterization [4, Theorem 1]: *a domain D is a Prüfer domain if and only if each overring of D is integrally closed;*
- Richman's characterization [33, Theorem 4]: *a domain D is a Prüfer domain if and only if each overring of D is D–flat.*

The previous theorems have been extended to the case of Prüfer v–multiplication domains (for short, PvMDs) in [6] and [26], respectively, by means of the v (or the t)–operation.

The purpose of the present work is to deepen the study of a general multiplicative theory for the semistar context, with special emphasis to the linkedness and the flatness, and to pursue the study of Prüfer semistar multiplication domains (cf. [21], [9]).
In Section 2 we recall the main definitions and we collect some background results on semistar operations. In Section 3, we define and study the notion of semistar linked overring, which generalizes the notion of t–linked overring defined in [6]. Several characterizations of this concept have been obtained. Section 4 is devoted to semistar flat overrings, a concept which extends the classical notion of flat overring and gives a very "flexible" general tool, preserving for the "semistar prime ideals" involved, a similar behaviour as in the classical context. As an application, in Section 5, we achieve the proofs for analogues of Davis' and Richman's theorems in the general case of Prüfer semistar multiplication domains.

2 Background and preliminary results on semistar operations

Let D be an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D-submodules of K and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, i.e. all $E \in \overline{\boldsymbol{F}}(D)$ such that there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of K. Then, obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

We recall that a mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $\quad E \mapsto E^\star$ is called a *semistar operation on D* if, for $x \in K, x \neq 0$, and $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

$(\star_1)$ $(xE)^\star = xE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^\star = (E^\star)^\star =: E^{\star\star}$,

cf. for instance [31], [30], [28], [27], [8] and [10].
When $D^\star = D$, the semistar operation $\star$, restricted to $\boldsymbol{F}(D)$, is "the classical" star operation (cf. [15, Sections 32 and 34]). In this case, we will write that $\star$ is *a (semi)star operation on D*.

Example 2.1 **(1)** The constant map $E \mapsto E^e := K$, $E \in \overline{\boldsymbol{F}}(D)$, defines a trivial semistar operation e (or, e_D) on D, called *the e-operation*.

(2) The map $E \mapsto E^d := E$, $E \in \overline{\boldsymbol{F}}(D)$, defines a (semi)star operation d (or, d_D) on D, called *the d–operation* or *the identity semistar operation*.

(3) For each $E \in \overline{\boldsymbol{F}}(D)$, set $E^{-1} := (D :_K E) := \{x \in K, xE \subseteq D\}$. The map $E \mapsto E^v := (E^{-1})^{-1}$ defines a (semi)star operation on D, called *the v–operation* on D (or *the v_D–operation*). This operation, when restricted to $\boldsymbol{F}(D)$, is the classical v–operation on D.

(4) Let $\{T_\lambda \mid \lambda \in \Lambda\}$ be a family of overrings of D, and let $*_\lambda$ be a semistar operation on T_λ, for each $\lambda \in \Lambda$. Then $E \mapsto E^{*_\Lambda} := \cap\{(ET_\lambda)^{*_\lambda} \mid \lambda \in \Lambda\}$, is a semistar operation on D. Moreover, $(E^{*_\Lambda}T_\lambda)^{*_\lambda} = (ET_\lambda)^{*_\lambda}$, for each $\lambda \in \Lambda$. This semistar operation is called *the semistar operation induced by the family* $\{(T_\lambda, *_\lambda) \mid \lambda \in \Lambda\}$ (for the star case, cf. [1, Theorem 2] and, for the semistar case, cf. [8, Example 1.3 (d)], [9, Example 2.1 (g)]). Note that, in general, D is a proper subset of $D^{*_\Lambda} = \cap\{(T_\lambda)^{*_\lambda} \mid \lambda \in \Lambda\}$. In particular, if T is an overring of D, we denote by $\star_{\{T\}}$ the semistar operation induced by $\{(T, d_T)\}$. For example, we have that $e_D = \star_{\{K\}}$ and $d_D = \star_{\{D\}}$.

(5) Spectral semistar operations constitute perhaps the most important class of semistar operations of the type introduced in (4). Given a set Δ of prime ideals of an integral domain D, *the spectral semistar operation $\star_\Delta$ on D associated to* Δ is the semistar operation on D induced by the family $\{(D_P, d_{D_P}) \mid P \in \Delta\}$ (cf. the previous Example (4)); when $\Delta = \emptyset$, then we set $\star_\emptyset := e_D$. A *spectral semistar operation on* D is a semistar operation $\star$ on D such that there exists a set of prime ideals Δ of D with $\star = \star_\Delta$. A spectral semistar operation $\star$ is *a stable semistar operation*, i.e., $(E \cap F)^\star = (E^\star \cap F^\star)$, for all $E, F \in \overline{\boldsymbol{F}}(D)$ (or, equivalently, $(E : F)^\star = (E^\star : F^\star)$, for each $E \in \overline{\boldsymbol{F}}(D)$ and $F \in \boldsymbol{f}(D)$). For more details, see [8, Section 4].

(6) Let D be an integral domain and T an overring of D. Let $\star$ be a semistar operation on D, the map $\dot{\star}^T\colon \overline{\boldsymbol{F}}(T) \to \overline{\boldsymbol{F}}(T)$, $E^{\dot{\star}^T} := E^\star$, for $E \in \overline{\boldsymbol{F}}(T)$ ($\subseteq \overline{\boldsymbol{F}}(D)$), is a semistar operation on T. When $T := D^\star$, then we set simply $\dot{\star}$, instead of $\dot{\star}^{D^\star}$, and we note that $\dot{\star}$ is a (semi)star operation on $D^\star$.
Conversely, let $*$ be a semistar operation on an overring T of D and define $*_D\colon \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, by setting $E^{*_D} := (ET)^*$, for each $E \in \overline{\boldsymbol{F}}(D)$. For each semistar operation $*$ on T, if we set $\star := *_D$, then we have that $\dot{\star}^T = *$ [10, Corollary 2.10].

(7) Given a semistar operation $\star$ on D, we can define a new semistar operation on D, by setting $E \mapsto E^{\star_f} := \cup\{F^\star \mid F \in \boldsymbol{f}(D), F \subseteq E\}$, for each $E \in \overline{\boldsymbol{F}}(D)$. The semistar operation $\star_f$ is called *the semistar operation of finite type* associated to $\star$. Note that if $E \in \boldsymbol{f}(D)$, then $E^\star = E^{\star_f}$. A semistar operation $\star$ is called *a semistar operation of finite type* if $\star = \star_f$. Note that $(\star_f)_f = \star_f$.
An important example of semistar operation of finite type is the (semi)star operation of finite type associated to the v-(semi)star operation, i.e. $t := v_f$, called *the t–(semi)star operation on D* (or *the t_D–operation*). Note that the e–operation and the identity operation d on D are of finite type. A spectral semistar operation $\star$ on D is of finite type if and only if $\star = \star_\Delta$, for some quasi–compact set of prime ideals of D [8, Corollary 4.6 (2)].

If $\star_1$ and $\star_2$ are two semistar operations on an integral domain D, we say that $\star_1 \leq \star_2$ if , for each $E \in \overline{\boldsymbol{F}}(D)$, $E^{\star_1} \subseteq E^{\star_2}$; in this case $(E^{\star_1})^{\star_2} = E^{\star_2}$.
Note that, for each semistar operation $\star$, we have that $\star_f \leq \star$. Moreover, for each (semi)star operation $\star$ on D, we have always that $\star \leq v$ and, hence, $\star_f \leq t$ (easy

consequence of [15, Theorem 34.1 (4)]).

Let $I \subseteq D$ be a nonzero ideal of D and let $\star$ be a semistar operation on D. We say that I is *a quasi–$\star$–ideal* (respectively, *$\star$–ideal*) *of* D if $I^\star \cap D = I$ (respectively, $I^\star = I$). Similarly, we call *a quasi–$\star$–prime* (respectively, *a $\star$–prime*) of D a quasi–$\star$–ideal (respectively, $\star$–ideal) of D which is also a prime ideal. We call *a quasi–$\star$–maximal* (respectively, *a $\star$–maximal*) of D a maximal element in the set of all proper quasi–$\star$–ideals (respectively, $\star$–ideals) of D.

Note that if $I \subseteq D$ is a $\star$–ideal, it is also a quasi–$\star$–ideal and, when $D = D^\star$, the notions of quasi–$\star$–ideal and $\star$–ideal coincide.

When $D \subsetneq D^\star \subsetneq K$ we can "restrict" the semistar operation $\star$ on D to the (semi)star operation $\dot{\star}$ on $D^\star$ (Example 2.1 (6)) and we have a strict relation between the quasi–$\star$–ideals of D and the $\dot{\star}$–ideals of $D^\star$, as shown in the following result:

Lemma 2.2 [12, Lemma 2.2]. *Let D be an integral domain and $\star$ a semistar operation on D and let $\dot{\star}$ be the (semi)star operation on $D^\star$ associated to $\star$. Then:*

- **(a)** *I is a quasi–$\star$–ideal of $D \Leftrightarrow I = L \cap D$, where $L \subseteq D^\star$ is a $\dot{\star}$–ideal of $D^\star$.*
- **(b)** *If $L \subseteq D^\star$ is a $\dot{\star}$–prime ideal of $D^\star$, then $L \cap D$ is a quasi–$\star$–prime ideal of D.* □

Note that, in general, the restriction to D of a $\dot{\star}$–maximal ideal of $D^\star$ is a quasi–$\star$–prime ideal of D, but not necessarily a quasi–$\star$–maximal ideal of D, and if L is an ideal of $D^\star$ and $L \cap D$ is a quasi–$\star$–prime ideal of D, then L is not necessarily a $\dot{\star}$–prime ideal of $D^\star$, [12, Remark 3.6].

Lemma 2.3 *Let $\star$ be a semistar operation of an integral domain D. Assume that $\star$ is not trivial and that $\star = \star_f$. Then:*

- **(a)** *Each proper quasi–$\star$–ideal is contained in a quasi-$\star$-maximal.*
- **(b)** *Each quasi–$\star$–maximal is a quasi–$\star$-prime.*
- **(c)** *If Q is a quasi–$\star$–maximal ideal of D then $Q = M \cap D$, for some $\dot{\star}$–maximal ideal M of $D^\star$.*
- **(d)** *Each minimal prime over a quasi–$\star$–ideal is a quasi–$\star$–prime.*
- **(e)** *Set*
$$\Pi^\star := \{P \in \mathrm{Spec}(D) \mid P \neq 0 \text{ and } P^\star \cap D \neq D\},$$
then each quasi–$\star$-prime of D belongs to $\Pi^\star$ and, moreover, the set of maximal elements of $\Pi^\star$ is nonempty and coincides with the set of all the quasi–$\star$-maximals of D.

Proof. We give a proof of **(d)**, for the other statements see [12, Lemma 2.3].

Let I be a quasi–$\star$–ideal of D and let P a minimal prime ideal of D over I, hence $\mathrm{rad}(ID_P) = PD_P$. Then, for each finitely generated ideal J of D, with $J \subseteq P$, there exists an integer $n \geq 1$ such that $J^n D_P \subseteq ID_P$, i.e. $sJ^n \subseteq I$, for some $s \in D \setminus P$. Therefore:

$$\begin{array}{l} s(J^\star)^n \subseteq s\left((J^\star)^n\right)^\star = s\,(J^n)^\star \subseteq I^\star \Rightarrow \\ s((J^\star)^n \cap D) \subseteq s(J^\star)^n \cap D \subset I^\star \cap D = I \subseteq P \Rightarrow \\ (J^\star \cap D)^n \subseteq (J^\star)^n \cap D \subseteq P \Rightarrow \\ J^\star \cap D \subseteq P. \end{array}$$

Since $\star = \star_f$, then $P^\star = \cup\{J^\star \mid J \in \boldsymbol{f}(D),\ J \subseteq P\}$ and so $P^\star \cap D = \cup\{J^\star \cap D \mid J \in \boldsymbol{f}(D),\ J \subseteq P\} \subseteq P$; thus $P^\star \cap D = P$. $\square$

We denote by $\mathcal{M}(\star_f)$ *the set of all the quasi–$\star_f$–maximals of* D, which is nonempty if and only if $\star_f \neq e$, and we associate to the semistar operation $\star$ on D a new semistar operation $\tilde{\star}$ on D, which is of finite type and spectral, defined as follows $\tilde{\star} := \star_{\mathcal{M}(\star_f)}$ (explicitly, $E^{\tilde{\star}} := \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$, for each $E \in \overline{\boldsymbol{F}}(D)$). Note that $\tilde{\star} \leq \star_f$ [12, Corollary 2.7].

We conclude this section by recalling the definition and the main properties of the semistar Nagata rings.

Let D be an integral domain with field of quotients K and $\star$ a semistar operation on D. Let X be an indeterminate over K, for each $f \in D[X]$, we denote by $\boldsymbol{c}(f)$ the content of f. Let $N_D(\star) := \{h \in D[X] \mid h \neq 0 \text{ and } \boldsymbol{c}(h)^\star = D^\star\}$. Then $N_D(\star) = D[X] \setminus \cup\{Q[X] \mid Q \in \mathcal{M}(\star_f)\}$ is a saturated multiplicative system of $D[X]$. The ring of fractions:

$$\mathrm{Na}(D,\star) := D[X]_{N_D(\star)}$$

is called *the Nagata ring of* D *with respect to the semistar operation* $\star$ (cf. [12]). Obviously, $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star_f)$ and if $\star = d$, where d is the identity (semi)star operation of D, then $\mathrm{Na}(D,d)$ coincides with the "classical" Nagata ring $D(X) := \{f/g \mid f,g \in D[X],\ \boldsymbol{c}(g) = D\}$ of D.

Lemma 2.4 [12, Corollary 2.7, Proposition 3.1 and 3.4, Corollary 3.5]. *Let D be an integral domain with quotient field K and let $\star$ be a semistar operation on D. Then, for each $E \in \overline{\boldsymbol{F}}(D)$, we have:*

- **(a)** $E^{\star_f} = \cap\{E^{\star_f} D_Q \mid Q \in \mathcal{M}(\star_f)\}$.
- **(b)** $E^{\tilde{\star}} = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$.
- **(c)** $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.
- **(d)** $\mathrm{Na}(D,\star) = \cap\{D[X]_{Q[X]} \mid Q \in \mathcal{M}(\star_f)\} = \cap\{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.
- **(e)** $\mathrm{Max}(\mathrm{Na}(D,\star)) = \{Q[X]_{N_D(\star)} \mid Q \in \mathcal{M}(\star_f)\} = \{QD_Q(X) \cap \mathrm{Na}(D,\star) \mid Q \in \mathcal{M}(\star_f)\}$.
- **(f)** $E\mathrm{Na}(D,\star) = \cap\{ED_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.
- **(g)** $E\mathrm{Na}(D,\star) \cap K = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$.
- **(h)** $E^{\tilde{\star}} = E\mathrm{Na}(D,\star) \cap K$.
- **(i)** $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D^{\tilde{\star}},\dot{\tilde{\star}})$. $\square$

An easy consequence of the previous result (in particular, Lemma 2.4 (e)) is the following:

Corollary 2.5 *Let D be an integral domain and let $\star$ be a semistar operation on D. For each prime ideal P of D such that $P^\star \neq D^\star$, $\mathrm{Na}(D,\star)_{P\,\mathrm{Na}(D,\star)} = D_P(X)$.*

$\square$

3 Semistar linkedness

Let D be an integral domain and T be an overring of D. Let $\star$ (respectively, $\star'$) be a semistar operation on D (respectively, on T).
We say that T is $(\star, \star')$*–linked to* D if:

$$F^\star = D^\star \Rightarrow (FT)^{\star'} = T^{\star'},$$

for each nonzero finitely generated integral ideal F of D.
It is straightforward that T is $(\star, \star')$–linked to D if and only if T is $(\star_f, \star'_f)$–linked to D.
Obviously, T is $(d_D, \star')$–linked to D, for each semistar operation $\star'$ on T and T is $(\star, e_T)$–linked to D, for each semistar operation $\star$ on D; in particular, when T coincides with the field of quotients K of D, then there exists a unique (trivial) semistar operation $e_T = d_T$ on T, hence T is $(\star, \star')$–linked to D, for each semistar operation $\star$ on D and for each semistar operation $\star'$ on T.

We say that T is t*–linked to* $(D, \star)$ if T is $(\star, t_T)$–linked. In particular, the classical notion "*T is t–linked to D*" [6] coincides with the notion "T is t–linked to (D, t_D)" (i.e. T is (t_D, t_T)–linked to D).

In the following result we collect some of the basic properties of the semistar linkedness.

Lemma 3.1 *Let S, T be two overrings of an integral domain D, with $D \subseteq T \subseteq S$.*

(a) *Let $D = T$ and $\star', \star''$ be two semistar operations on T. If $\star'_f \leq \star''_f$, then T is $(\star', \star'')$–linked to T.*

(b) *Let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, T, S). Assume that S is $(\star', \star'')$–linked to T and that T is $(\star, \star')$–linked to D, then S is $(\star, \star'')$–linked to D.*

(c) *Let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, two semistar operations on T). Assume that $\star'_f \leq \star''_f$. Then T is $(\star, \star')$–linked to D implies that T is $(\star, \star'')$–linked to D.*

(d) *If $\star'$ is a (semi)star operation on T (i.e. if $T^{\star'} = T$) and if T is $(\star, \star')$–linked to D then T is t–linked to $(D, \star)$.*

(e) *Let $\star$ be a semistar operation on D then T is $(\star, \dot{\star}^T)$–linked to D. In particular, $D^\star$ is $(\star, \dot{\star})$–linked to D.*

(f) *If $\star'$ is a semistar operation on T such that $\dot{\star}^T \leq \star'$, then T is $(\star, \star')$–linked to D. In particular, we deduce that:*

$$(\dot{t_D})^T \leq t_T \Rightarrow T \text{ is } t\text{–linked to } D\,; \quad \text{and more generally,}$$
$$\left(\dot{\star}^T\right)_f \leq t_T \Rightarrow T \text{ is } t\text{–linked to } (D, \star)\,.$$

(g) *Let $\star'$ be a semistar operation on T, then T is $(\star'_{\dot{}D}, \star')$–linked to D.*

(h) *Let $\star_1$ and $\star_2$ be two semistar operations on D and let $\star'$ be a semistar operation on T. If $(\star_1)_f \leq (\star_2)_f$ and if T is $(\star_2, \star')$–linked to D, then T is $(\star_1, \star')$–linked to D.*

(i) *Let $\star$ (respectively, $\star'$) be a semistar operations on D (respectively, T). If $\star \leq \star'_{\dot{}D}$, then T is $(\star, \star')$–linked to D.*

Also, we have:

$$t_D \leq (\dot{t_T})_D \Rightarrow T \text{ is } t\text{–linked to } D\,; \quad \text{and, more generally,}$$
$$\star_f \leq (\dot{t_T})_D \Rightarrow T \text{ is } t\text{–linked to } (D,\star)\,.$$

(j) *Let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, T, S). Assume that S is $(\star, \star'')$–linked to D and that each quasi–$\star'_f$–maximal ideal of T is the contraction of a quasi–$\star''_f$–maximal ideal of S, then T is $(\star, \star')$–linked to D.*
In particular (Lemma 2.3 (c)), if we take $S := T^{\star'}$ and $\star'' := \dot{\star'}$ (note that $\dot{\star'}$ is a (semi)star operation on $T^{\star'}$), then T is $(\star, \star')$–linked to D if and only if $T^{\star'}$ is $(\star, \dot{\star'})$–linked to D.

(k) *Let $\{T_\lambda \mid \lambda \in \Lambda\}$ be a family of overrings of D and let $*_\lambda$ be a semistar operation defined on T_λ, for $\lambda \in \Lambda$. Set $T := \cap\{T_\lambda \mid \lambda \in \Lambda\}$ and let $*_\Lambda$ be the semistar operation on T induced by the family $\{T_\lambda \mid \lambda \in \Lambda\}$ (i.e. for each $E \in \overline{\boldsymbol{F}}(T)$, $E^{*_\Lambda} := \cap\{(ET_\lambda)^{*_\lambda} \mid \lambda \in \Lambda\}$). If T_λ is $(\star, *_\lambda)$–linked to D, for each $\lambda \in \Lambda$, then T is $(\star, *_\Lambda)$–linked to D.*

Proof. Straightforward. □

Let T, S be two overrings of an integral domain D, with $D \subseteq T \subseteq S$ and let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, T, S). Assume that S is $(\star, \star'')$–linked to D. When is S $(\star', \star'')$–linked to T ? A partial answer to this question will be given in Remark 3.13.

Proposition 3.2 *Let D be an integral domain and T be an overring of D. Let $\star$ (respectively, $\star'$) be a semistar operation on D (respectively, on T). The following are equivalent:*

- **(i)** *T is $(\star, \star')$–linked to D;*
- **(ii)** *for each nonzero ideal I of D, $I^{\star_f} = D^\star \Rightarrow (IT)^{\star'_f} = T^{\star'}$;*
- **(iii)** *for each quasi–$\star'_f$–ideal J of T, with $J \neq T$, $(J \cap D)^{\star_f} \neq D^\star$;*
- **(iv)** *for each quasi–$\star'_f$–prime ideal Q of T, $(Q \cap D)^{\star_f} \neq D^\star$;*
- **(v)** *for each quasi–$\star'_f$–maximal ideal N of T, $(N \cap D)^{\star_f} \neq D^\star$.*

Proof. (i) $\Rightarrow$ (ii). Since $D^\star = I^{\star_f} = \cup\{F^\star \mid F \subseteq I,\ F \in \boldsymbol{f}(D)\}$, then $D^\star = F^\star$, for some $F \subseteq I$, $F \in \boldsymbol{f}(D)$. Therefore, we conclude $T^{\star'} = (FT)^{\star'} \subseteq (IT)^{\star'_f} \subseteq T^{\star'}$.

(ii) $\Rightarrow$ (iii). Assume that, for some proper quasi–$\star'_f$–ideal J of T, the ideal $I := J \cap D$ is such that $I^{\star_f} = D^\star$. By assumption, we have $T^{\star'} = (IT)^{\star'_f} = ((J \cap D)T)^{\star'_f} \subseteq J^{\star'_f} \subseteq T^{\star'}$, i.e. $J^{\star'_f} = T^{\star'}$. This fact contradicts the hypothesis that J is a quasi–$\star'_f$–ideal of T, with $J \neq T$.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious.

(v) $\Rightarrow$ (i). Assume that, for some $F \in \boldsymbol{f}(D)$, with $F \subseteq D$, we have $F^\star = D^\star$ and $(FT)^{\star'_f} \subsetneq T^{\star'}$. Let N be a quasi–$\star'_f$–maximal ideal of T containing $(FT)^{\star'_f} \cap T$. By hypothesis, we have $(N \cap D)^{\star_f} \neq D^\star$. On the other hand, $F^\star \subseteq ((FT)^{\star'_f} \cap D)^{\star_f} \subseteq (N \cap D)^{\star_f}$ and this contradicts the choice of F. □

Remark 3.3 **(a)** It follows from Lemma 3.1 (b), (e) and (j) that, if $T^{\star'}$ is a $(\dot{\star}, \dot{\star'})$–linked *overring* of $D^\star$, then T is $(\star, \star')$–linked to D. What about the converse? More precisely, since it is not true in general that $T^{\star'}$ is an overring of $D^\star$, for "the converse" we mean the following statement: Assume $T^{\star'}$ is an overring of $D^\star$ and that T is $(\star, \star')$–linked to D. Is it true that $T^{\star'}$ is $(\dot{\star}, \dot{\star'})$–linked to $D^\star$? The answer to this question is negative, as the following example shows.

Let K be a field and X, Y be two indeterminates over K. Let $R := K[X, Y]$ and $M := (X, Y)$. Set $D := K[X, XY]$ and $T := R_M$. Let $\star := (t_R)_{\dot{D}}$ and $\star' := d_T$. Then:

(1) T is $(\star, \star')$–linked to D.

(2) $D^\star \subseteq T^{\star'}$, but $T^{\star'}$ is not $(\dot{\star}, \dot{\star'})$–linked to $D^\star$.

Clearly $D^\star = R \subseteq T^{\star'} = T$.
(1) Set $M' := MR_M$, then M' is the unique $(\star'$–$)$maximal ideal of T. We have $M' \cap D = M \cap D \subseteq XR$. Therefore, $(M' \cap D)^\star \subseteq (XR)^\star = XR \subsetneq R = D^\star$.
(2) Note that $\dot{\star} = t_R$ (Example 2.1 (6)) and $\dot{\star'} = \star' = d_T$. Moreover, for the maximal ideal M' of $T^{\star'} = T$, we have $(M' \cap D^\star)^{\dot{\star}} = (M' \cap R)^{t_R} = M^{t_R} = R = D^\star$. Therefore, $T^{\star'}$ is not $(\dot{\star}, \dot{\star'})$–linked to $D^\star$ (Proposition 3.2 (v)).

A related question to the previous one will be examined in Theorem 3.8.

(b) If T is $(\star, \star')$–linked to D, then, for each quasi–$\star'_f$–prime ideal Q of T, there exists a quasi–$\star_f$–prime ideal P such that $D_P \subseteq T_{D \setminus P} \subseteq T_Q$. (Since $(Q \cap D)^{\star_f} \neq D^\star$, take a quasi–$\star_f$–prime ideal P of D such that $Q \cap D \subseteq P$, and so $(D \setminus P) \subseteq (T \setminus Q)$.) Therefore, if T is $(\star, \star')$–linked to D, then $D^{\tilde{\star}} \subseteq T^{\tilde{\star'}}$.

Example 3.4 **(1)** Let D be an integral domain and T be an overring of D. Let $\star$ be a semistar operation on D and let P be a quasi–$\star_f$–prime ideal of D. Then, $T_{D \setminus P}$ is $(\star, *)$–linked to D, for *each* semistar operation $*$ on $T_{D \setminus P}$ (equivalently, $T_{D \setminus P}$ is $(\star, d_{T,P})$–linked to D, where $d_{T,P}$ is the identity (semi)star operation on $T_{D \setminus P}$).
As a matter of fact, for each prime ideal N, in particular, for each quasi–$*_f$–prime ideal, of $T_{D \setminus P}$, $N \cap T$ is a prime ideal of T such that $N \cap D \subseteq P = P^{\star_f} \cap D$. Hence $(N \cap D)^{\star_f} \neq D^\star$.

(2) Given a semistar operation $\star$ on an integral domain D, recall that on D we can introduce a new semistar operation of finite type, denoted by $[\star]$, called *the semistar integral closure of* $\star$, by setting:

$$F^{[\star]} := \cup\{((H^\star : H^\star)F)^{\star_f} \mid H \in \boldsymbol{f}(D)\}\,, \ \text{ for each } \ F \in \boldsymbol{f}(D)\,,$$

(and thus in general:

$$E^{[\star]} := \cup\{F^{[\star]} \mid F \in \boldsymbol{f}(D),\ F \subseteq E\}\,, \ \text{ for each } \ E \in \overline{\boldsymbol{F}}(D)\,).$$

It is known that $\star_f \leq [\star]$, hence $D^\star \subseteq D^{[\star]}$, and that $D^{[\star]}$ is integrally closed. Therefore, it is obvious that if $D^\star = D^{[\star]}$ then $D^\star$ is integrally closed. The converse is false, even when $\star$ is a (semi)star operation on D. However, it is known that if $\star_f$ is stable, then $D^\star$ is integrally closed if and only if $D^\star = D^{[\star]}$, (cf. [30, Proposition 34], [10, Proposition 4.3 and Proposition 4.5], [9, Example 2.1 (c)], [18]).
From Lemma 3.1 (e), (a) and (b), we have that $D^{[\star]}$ is $(\star, [\dot{\star}])$–linked to D.

Assume that $T := \cup\{T_\lambda \mid \lambda \in \Lambda\}$ is the direct union of a given direct family of overrings $\{T_\lambda \mid \lambda \in \Lambda\}$ of an integral domain D with field of quotients K (where Λ is

a directly ordered set by setting $\lambda' \leq \lambda''$ if $T_{\lambda'} \subseteq T_{\lambda''}$). Let $*_\lambda$ be a semistar operation defined on the overring T_λ of D, for each $\lambda \in \Lambda$. We say that the family $\{*_\lambda \mid \lambda \in \Lambda\}$ is *a direct family of semistar operations* (or, simply, that $\{(T_\lambda, *_\lambda) \mid \lambda \in \Lambda\}$ is a *direct family*), if λ_2 follows λ_1 inside Λ and if $H \in \boldsymbol{f}(T_{\lambda_1})$, then $H^{*_{\lambda_1}} \subseteq (HT_{\lambda_2})^{*_{\lambda_2}}$.
For each $\lambda \in \Lambda$, let E_λ be a T_λ-submodule of K. We say that $E = \cup\{E_\lambda \mid \lambda \in \Lambda\}$ is *a direct union*, if for each pair $\alpha, \beta \in \Lambda$, and for each $\gamma \in \Lambda$ such that $T_\alpha \subseteq T_\gamma$ and $T_\beta \subseteq T_\gamma$ then $E_\alpha T_\gamma \subseteq E_\gamma$ and $E_\beta T_\gamma \subseteq E_\gamma$.

The following result generalizes [6, Proposition 2.2 (a)].

Lemma 3.5 *Let $\star$ be a semistar operation on an integral domain D. Given a direct family $\{(T_\lambda, *_\lambda) \mid \lambda \in \Lambda\}$, as above. For each $E \in \overline{\boldsymbol{F}}(T)$, set:*

$$E^{*^\Lambda} := \cup\{E^{(*_\lambda)_f} \mid \lambda \in \Lambda\}.$$

(1) *$*^\Lambda$ is a semistar operation of finite type on T.*

(2) *If T_λ is $(\star, *_\lambda)$–linked to D, for each $\lambda \in \Lambda$, then T is $(\star, *^\Lambda)$–linked to D.*

(3) *If T_λ is $(\star, t_{T_\lambda})$–linked to D, for each $\lambda \in \Lambda$, then T is $(\star, t_T)$–linked to D.*

Proof. **(1)** The properties $(\star_1)$ and $(\star_2)$ are straightforward. Before proving $(\star_3)$, we show the following:

Claim. *If $E = \cup\{E_\lambda \mid \lambda \in \Lambda\} \in \overline{\boldsymbol{F}}(T)$ is a direct union, where E_λ is a T_λ-submodule of K, then:*

$$E^{*^\Lambda} = \cup\{E_\lambda^{(*_\lambda)_f} \mid \lambda \in \Lambda\}.$$

Given $\alpha \in \Lambda$, we have $E = \cup\{E_\beta T_\alpha \mid \beta \in \Lambda\}$ is a direct union of T_α-submodules. Since $(*_\alpha)_f$ is of finite type and $E \in \overline{\boldsymbol{F}}(T_\alpha)$ $(\supseteq \overline{\boldsymbol{F}}(T))$, then $E^{(*_\alpha)_f} = \cup\{(E_\beta T_\alpha)^{(*_\alpha)_f} \mid \beta \in \Lambda\}$. Let $\beta \in \Lambda$, then there exists $\gamma \in \Lambda$ such that $T_\alpha \subseteq T_\gamma$ and $T_\beta \subseteq T_\gamma$ and, $E_\alpha \subseteq E_\gamma$ and $E_\beta \subseteq E_\gamma$. Hence $(E_\beta T_\alpha)^{(*_\alpha)_f} \subseteq {E_\gamma}^{(*_\alpha)_f} \subseteq {E_\gamma}^{(*_\gamma)_f}$ (the second inclusion follows from the fact that $\{(T_\lambda, *_\lambda) \mid \lambda \in \Lambda\}$ is direct). So $E^{(*_\alpha)_f} \subseteq \cup\{E_\lambda^{(*_\lambda)_f} \mid \lambda \in \Lambda\}$, and hence $E^{*^\Lambda} \subseteq \cup\{E_\lambda^{(*_\lambda)_f} \mid \lambda \in \Lambda\}$. The other inclusion is trivial.
Now we prove $(\star_3)$. Clearly, for each $E \in \overline{\boldsymbol{F}}(T)$, $E \subseteq E^{*^\Lambda}$. On the other hand, we have $E^{*^\Lambda} = \cup\{E^{(*_\lambda)_f} \mid \lambda \in \Lambda\}$ is a direct union of $E^{(*_\lambda)_f} \in \overline{\boldsymbol{F}}(T_\lambda)$ and so, by the Claim, $(E^{*^\Lambda})^{*^\Lambda} = \cup\{(E^{(*_\lambda)_f})^{(*_\lambda)_f} \mid \lambda \in \Lambda\} = \cup\{E^{(*_\lambda)_f} \mid \lambda \in \Lambda\} = E^{*^\Lambda}$.
Finally, the fact that $*^\Lambda$ is of finite type is an immediate consequence of the definition.
(2) Let I be a nonzero finitely generated ideal of D such that $I^\star = D^\star$. Then, by the Claim, $(IT)^{*^\Lambda} = \cup\{(IT_\lambda)^{*_\lambda} \mid \lambda \in \Lambda\}$. Since T_λ is $(\star, *_\lambda)$–linked to D, then $(IT_\lambda)^{*_\lambda} = T_\lambda^{*_\lambda}$, for each $\lambda \in \Lambda$. Hence, again by the Claim, $(IT)^{*^\Lambda} = \cup\{T_\lambda^{*_\lambda} \mid \lambda \in \Lambda\} = T^{*^\Lambda}$.
(3) Let I be a nonzero finitely generated ideal of D such that $I^\star = D^\star$, then for each λ, $(IT_\lambda)^t = T_\lambda$, i.e. $(IT_\lambda)^{-1} = T_\lambda$. Let $I := (x_1, x_2, \ldots, x_n)D$ and $z \in (IT)^{-1}$. Then, for each i, $zx_i \in T_{\lambda_i}$, for some $\lambda_i \in \Lambda$ and so, for some $\lambda_I \in \Lambda$, $zI \subseteq T_{\lambda_I}$. Hence, $z \in (IT_{\lambda_I})^{-1} = T_{\lambda_I} \subseteq T$. Therefore, $(IT)^{-1} \subseteq T$ and so $(IT)^{-1} = T$. □

The following corollary generalizes [6, Corollary 2.3].

Corollary 3.6 *Let $\star$ be a semistar operation on an integral domain D. Then $D^{[\star]}$ is t–linked to $(D, \star)$. If, moreover $(D : D^\star) \neq (0)$, then the complete integral closure $\widetilde{D}$ of D is t–linked to $(D, \star)$; in particular, the complete integral closure $\widetilde{D}$ of D is always t–linked to D.*

Proof. The statement can be seen as an easy consequence of Example 3.4 (2) and of the fact that $[\dot{\star}] \leq t_{D^{[\star]}}$ (Lemma 3.1 (c)). We give here another proof based on the previous Lemma 3.5, which also shows that the semistar operation $[\star]$ is issued from a semistar operation associated to a directed family of overrings and semistar operations.

For each $E \in \overline{\boldsymbol{F}}(D)$, set $T_E := (E^\star : E^\star)$. Let $*_E$ denote the semistar operation $\dot{\star}^{T_E}$ on T_E. Then T_E is an overring of D, which is $(\star, *_E)$–linked to D (Lemma 3.1 (e)). Note that $*_E$ is a (semi)star operation on T_E (since $(T_E)^{*_E} = T_E$).
We claim that $\{(T_F, *_F) \mid F \in \boldsymbol{f}(D)\}$ and $\{(T_E, *_E) \mid E \in \boldsymbol{F}(D)\}$ are direct families (as in Lemma 3.5). To see this, note that:

$$(H_1^\star : H_1^\star) \subseteq ((H_1H_2)^\star : (H_1H_2)^\star) \supseteq (H_2^\star : H_2^\star),$$

for all $H_1, H_2 \in \overline{\boldsymbol{F}}(D)$.
Therefore, as in Lemma 3.5 (1), $\{(T_F, *_F) \mid F \in \boldsymbol{f}(D)\}$ (respectively, $\{(T_E, *_E) \mid E \in \boldsymbol{F}(D)\}$) defines a (semi)star operation of finite type $*^{\boldsymbol{f}(D)}$ (respectively, $*^{\boldsymbol{F}(D)}$) on $D^{[\star]} = \cup\{(F^\star : F^\star) \mid F \in \boldsymbol{f}(D)\}$ (respectively, on $D^{\langle\star\rangle} := \cup\{(E^\star : E^\star) \mid E \in \boldsymbol{F}(D)\}$). Note that $D^{[\star]}$ is $(\star, *^{\boldsymbol{f}(D)})$–linked to D (Lemma 3.5 (2)) and that $*^{\boldsymbol{f}(D)} \leq t_{D^{[\star]}}$ (since $*^{\boldsymbol{f}(D)}$ is a (semi)star operation of finite type on $D^{[\star]}$). We conclude, by Lemma 3.1 (c), that $D^{[\star]}$ is t–linked to $(D, \star)$.

For the last statement, note that $\widetilde{D} = \cup\{(E : E) \mid E \in \boldsymbol{F}(D)\} \subseteq \cup\{(E^\star : E) \mid E \in \boldsymbol{F}(D)\} = \cup\{(E^\star : E^\star) \mid E \in \boldsymbol{F}(D)\} = D^{\langle\star\rangle} \subseteq \cup\{(H : H) \mid H \in \boldsymbol{F}(D^\star)\} = \widetilde{D^\star}$. If $(D : D^\star) \neq (0)$ then $\widetilde{D} = \widetilde{D^\star} = D^{\langle\star\rangle} = \cup\{T_E \mid E \in \boldsymbol{F}(D)\}$. Arguing as above, we have that $\widetilde{D}$ is $(\star, *^{\boldsymbol{F}(D)})$–linked to D, and $*^{\boldsymbol{F}(D)} \leq t_{\widetilde{D}}$ (since $*^{\boldsymbol{F}(D)}$ is a (semi)star operation of finite type on $D^{\langle\star\rangle} = \widetilde{D}$). Again from Lemma 3.1 (c), we conclude that $\widetilde{D}$ is t–linked to $(D, \star)$. □

Remark 3.7 Let $\star$ be a semistar operation on an integral domain D.
(a) Let $*^{\boldsymbol{f}(D)}$ be the (semi)star operation of finite type over $D^{[\star_f]}$ $(= D^{[\star]})$, associated to the semistar operation of finite type $\star_f$ and defined, in general for any semistar operation, in the proof of the previous corollary. Then:

$$[\star] = \left(*^{\boldsymbol{f}(D)}\right)_{\dot{}\,D}.$$

As a matter of fact, first, note that in this case $T_H = (H^{\star_f} : H^{\star_f}) = (H^\star : H^\star)$, for each $H \in \boldsymbol{f}(D)$ and let now $*_H$ denote the semistar operation of finite type $\dot{\star}_f^{T_H}$ on T_H. For each $E \in \overline{\boldsymbol{F}}(D)$, we have:

$$ED^{[\star]} = E\left(\cup\{(H^\star : H^\star) \mid H \in \boldsymbol{f}(D)\}\right) = \cup\{E(H^\star : H^\star) \mid H \in \boldsymbol{f}(D)\},$$

thus, using the Claim of the proof of Lemma 3.5, we have:

$$\begin{aligned}(ED^{[\star]})^{*^{\boldsymbol{f}(D)}} &= \left(\cup\{E(H^\star : H^\star) \mid H \in \boldsymbol{f}(D)\}\right)^{*^{\boldsymbol{f}(D)}} = \\ &= \cup\{(E\,(H^\star : H^\star))^{\star_f} \mid H \in \boldsymbol{f}(D)\}.\end{aligned}$$

In particular, $F^{[\star]} = (FD^{[\star]})^{*^{\boldsymbol{f}(D)}}$, for each $F \in \boldsymbol{f}(D)$.
As a consequence we have that, for each $E \in \overline{\boldsymbol{F}}(D)$:

$$E^{[\star]} = \cup\{(E\,(H^\star : H^\star))^{\star_f} \mid H \in \boldsymbol{f}(D)\}.$$

(b) If we set:

$$\langle\star\rangle := \left(*^{\boldsymbol{F}(D)}_{\dot{}}\right)_D ,$$

then $\langle\star\rangle$ is a semistar operation of finite type on D, with $D^{\langle\star\rangle} = \cup\{(E^\star : E^\star) \mid E \in \boldsymbol{F}(D)\}$. Moreover,

$$\star_f \leq [\star] \leq \langle\star\rangle \quad \text{and} \quad D^\star \subseteq D^{[\star]} \subseteq D^{\langle\star\rangle} \subseteq \widetilde{D^\star}\,.$$

Theorem 3.8 *Let D be an integral domain with quotient field K and let T be an overring of D. Let $\star$ (respectively, $\star'$) be a semistar operation on D (respectively, on T). The following are equivalent:*

(i) *T is $(\star, \star')$–linked to D;*

(ii) $\mathrm{Na}(D,\star) \subseteq \mathrm{Na}(T,\star')$*;*

(iii) $\tilde{\star} \leq (\widetilde{\star'})_{\dot{D}}$*;*

(iv) *T is an $(\tilde{\star}, \widetilde{\star'})$–linked overring of D;*

(v) *$T^{\widetilde{\star'}}$ is an $(\dot{\tilde{\star}}, \dot{\widetilde{\star'}})$–linked overring of $D^{\tilde{\star}}$.*

Proof. (i) $\Rightarrow$ (ii). Let $g \in D[X]$ such that $(\boldsymbol{c}_D(g))^\star = D^\star$. Then, by the assumption, $(\boldsymbol{c}_T(g))^{\star'} = (\boldsymbol{c}_D(g)T)^{\star'} = T^{\star'}$. Hence $\mathrm{Na}(D,\star) \subseteq \mathrm{Na}(T,\star')$.

(ii)$\Rightarrow$(iii). Let $E \in \overline{\boldsymbol{F}}(D)$. Then $E\mathrm{Na}(D,\star) \subseteq E\mathrm{Na}(T,\star')$. Hence (Lemma 2.4 (h)) $E^{\tilde{\star}} = E\mathrm{Na}(D,\star) \cap K \subseteq E\mathrm{Na}(T,\star') \cap K = (ET)^{\widetilde{\star'}}$ and so we conclude that $\tilde{\star} \leq (\widetilde{\star'})_{\dot{D}}$.

(iii) $\Rightarrow$ (iv). It follows from Lemma 3.1 (i).

(iv) $\Rightarrow$ (ii) follows from (i) $\Rightarrow$ (ii) and from Lemma 2.4 (i).

(ii) $\Rightarrow$ (i). Let G be a nonzero finitely generated integral ideal of D such that $G^\star = D^\star$ and let $g \in D[X]$ be such that $\boldsymbol{c}_D(g) = G$. From the fact that $(\boldsymbol{c}_D(g))^\star = D^\star$, we have that g is a unit in $\mathrm{Na}(D,\star)$ and so, by assumption, g is also a unit in $\mathrm{Na}(T,\star')$. This implies that $(\boldsymbol{c}_T(g))^{\star'} = (\boldsymbol{c}_D(g)T)^{\star'} = T^{\star'}$, i.e. $(GT)^{\star'} = T^{\star'}$.

(ii) $\Leftrightarrow$ (v) is an easy consequence of (ii) $\Leftrightarrow$ (i) and of Lemma 2.4 (i). □

The next result characterizes domains such that each overring is semistar linked and generalizes [6, Theorem 2.6].

Theorem 3.9 *Let D be an integral domain and $\star$ a semistar operation on D. The following statements are equivalent:*

(i) *for each overring T of D and for each semistar operation $\star'$ on T, T is $(\star, \star')$–linked to D;*

(ii) *each overring T of D is $(\star, d_T)$–linked to D;*

(iii) *each overring T of D is t–linked to $(D, \star)$;*

(iv) *for each valuation overring V of D there exists a (semi)star operation $*_V$ on V, such that V is $(\star, *_V)$–linked to D;*

(v) *each maximal ideal of D is a quasi–$\star_f$–maximal ideal;*

(vi) *for each proper ideal I of D, $I^{\star_f} \subsetneq D^\star$;*

(vii) *for each proper finitely generated ideal I of D, $I^\star \subsetneq D^\star$;*

(viii) *for each proper $\star_f$–invertible ideal I of D (i.e. $(II^{-1})^{\star_f} = D^\star$), $I^{\star_f} \subsetneq D^\star$ (hence, each proper $\star_f$–invertible ideal I of D is contained in the proper quasi–$\star_f$–ideal $I^\star \cap D$ of D).*

Proof. (i) $\Rightarrow$ (ii) is obvious. (ii) $\Rightarrow$ (iii) is a consequence of the fact that $d_T \leq t_T$ and Lemma 3.1 (c). (iii) $\Rightarrow$ (iv) is obvious, taking $*_V = t_V$.

(iv) $\Rightarrow$ (v). If M is a maximal ideal of D such that $M \subsetneq M^{\star_f} = D^\star$ then, for some nonzero finitely generated ideal $I \subseteq M$, we have $I^\star = D^\star$. Let (V, N) be a valuation overring of D such that $N \cap D = M$. Then $(IV)^{(*_V)_f} = V^{*_V} = V$. Since a nonzero finitely generated ideal of a valuation domain is principal and $*_V$ is a (semi)star operation on V, then $V = V^{*_V} = (IV)^{(*_V)_f} = IV$. This is a contradiction, because $IV \subseteq N \subsetneq V$.

(v) $\Rightarrow$ (vi) $\Rightarrow$(vii) are obvious.

(vii) $\Rightarrow$ (viii). If $(II^{-1})^{\star_f} = D^\star$ and $II^{-1} \subsetneq D$ then, for some nonzero finitely generated ideal $F \subseteq II^{-1} \subsetneq D$, we have $F^\star = D^\star$ and this contradicts the assumption. Since I is invertible then, in particular, I is a finitely generated proper ideal of D and so, by assumption and (vii), $I^\star \cap D$ is a proper quasi–$\star_f$–ideal of D containing I.

(viii) $\Rightarrow$ (v). Assume that, for some maximal ideal M of D, $M \subsetneq M^{\star_f} = D^\star$. Then $(MM^{-1})^{\star_f} = (MM^{-1})^\star = D^\star$, because $D^{\star_f} \supseteq (MM^{-1})^{\star_f} = (M^{\star_f}(M^{-1})^{\star_f})^{\star_f} = (D^{\star_f}(M^{-1})^{\star_f})^{\star_f} = (M^{-1})^{\star_f} \supseteq D^{\star_f} = D^\star$. Hence, by assumption, $M^{\star_f} \subsetneq D^\star$, but this contradicts the choice of M.

(v) $\Rightarrow$ (i). Assume that, for some overring T of D, for some semistar operation $\star'$ on T and for some quasi–$\star'_f$–maximal ideal N of T, we have $(N \cap D)^{\star_f} = D^\star$ (Proposition 3.2 ((i) $\Leftrightarrow$ (v))). Note that, from the assumption, $N \cap D \subseteq M = M^{\star_f} \cap D$, for some (quasi–$\star_f$–)maximal ideal M of D, and so we reach immediately a contradiction. □

Remark 3.10 Note that the proof of (vii) $\Rightarrow$ (viii) (Theorem 3.9) shows that, in an integral domain verifying the conditions of Theorem 3.9, each $\star_f$–invertible ideal is invertible.

Example 3.11 Let $\star$ be a semistar operation on an integral domain D. Assume that $D^\star$ is faithfully flat on D (for instance, assume that $\star$ is a (semi)star operation on D). In this situation, every principal ideal of D is a quasi–$\star$–ideal of D. If $\mathrm{Spec}(D)$ is a tree (e.g., $\dim(D) = 1$ or D is a GD-domain, in particular, D is a Prüfer domain), then every overring T of D is t–linked to $(D, \star)$.

In order to apply Theorem 3.9 ((v) $\Rightarrow$ (iii)), we show that each maximal ideal M of D is a quasi–$\star_f$–ideal of D. For each nonzero $x \in M$, xD is a quasi–$\star_f$–ideal of D, hence a minimal prime ideal P of xD is a quasi–$\star_f$–prime ideal of D (cf. Lemma 2.3 (d)). Since $\mathrm{Spec}(D)$ is a tree, M is a direct union of a family $\{P_\lambda\}$ of quasi–$\star_f$–prime ideals of D. If $M^{\star_f} = D^\star$, then $1 \in M^{\star_f} = (\cup_\lambda \{P_\lambda\})^{\star_f} = (\cup_\lambda \{(P_\lambda)^{\star_f}\})^{\star_f}$ thus, from the finiteness of $\star_f$, we deduce that $1 \in (P_\lambda)^{\star_f} \cap D = P_\lambda$, for some λ, and this is a contradiction.

Our next goal is the study of a new semistar operation strictly related to semistar linkedness.

Let D be an integral domain, $\star$ a semistar operation on D, and T an overring of D. We define *the semistar operation* $\ell_{\star,T}$ (or, simply, ℓ) *on* T, in the following way:

$$E^{\ell_{\star,T}} := E^{\ell} := \cap\{ET_{D\setminus P} \mid P \text{ is a quasi-}\star_f\text{-prime ideal of } D\},$$

for each $E \in \overline{\boldsymbol{F}}(T)$.
Note that if $T = D$, then $\ell_{\star,D} = \tilde{\star}$ (Lemma 2.4 (b)). Moreover, note that $\ell_{\star,T}$ is the semistar operation on T induced, in the sense described in Example 2.1 (4), by the family of overrings $\{T_{D\setminus P} \mid P \text{ is a quasi-}\star_f\text{-prime ideal of } D\}$ of D (where $T_{D\setminus P}$ is endowed with the identity $d_{T,P}$ (semi)star operation) .

The following proposition collects some interesting properties of the semistar operation $\ell_{\star,T}$.

Proposition 3.12 *Let D be an integral domain, $\star$ a semistar operation on D, T an overring of D and $\star'$ a semistar operation on T.*

(1) *$\ell_{\star,T}$ is a stable semistar operation of T.*

(2) *Assume that T is $(\star, \star')$–linked to D. Then $\ell_{\star,T} \leq \widetilde{\star'}$ $(\leq \star'_f)$; in particular T is $(\ell_{\star,T}, \star')$–linked to T.*

(3) *T is $(\star, \ell_{\star,T})$–linked to D, for each semistar operation $\star$ on D; in particular, D is $(\star, \tilde{\star})$–linked to D, for each semistar operation $\star$ on D.*

(4) *$\ell_{\star,T}$ is a semistar operation of finite type on T and $\widetilde{\ell_{\star,T}} = \ell_{\star,T}$.*

(5) *$\ell_{\star,T}$ is the unique minimal element in set of semistar operations $\star'_f$, where $\star'$ is a semistar operation on T such that T is $(\star, \star')$–linked to D.*

(6) *T is $(\star, \star')$–linked to D if and only if T is $(\ell_{\star,T}, \star')$–linked to T (and T is $(\star, \ell_{\star,T})$–linked to D).*

(7) *T is $(\star, \star')$–linked to D if and only if $\ell_{\star,T} \leq \star'_f$.*

Proof. **(1)** This is a straightforward consequence of the fact that $T_{D\setminus P}$ is flat over T, for each prime ideal P of D.

(2) For each quasi–$\star'_f$–prime ideal Q of T, there exists a quasi–$\star_f$–prime ideal P of D, such that $T_{D\setminus P} \subseteq T_Q$ (Remark 3.3 (b)), and so also $ET_{D\setminus P} \subseteq ET_Q$, for each $E \in \overline{\boldsymbol{F}}(T)$; from this we deduce that $\ell_{\star,T} \leq \widetilde{\star'}$. The last statement follows from Lemma 3.1 (a).

(3) If $I^{\star_f} = D^{\star}$, then $I \not\subseteq P$, i.e. $ID_P = D_P$, and this implies that $IT_{D\setminus P} = T_{D\setminus P}$, for each quasi–$\star_f$–prime ideal P of D. Therefore $(IT)^{\ell_{\star,T}} = T^{\ell_{\star,T}}$.

(4) From (3), we have that T is $(\star, \ell_{\star,T})$–linked to D. From (2) (for $\star' = \ell_{\star,T}$), we deduce that $(\ell_{\star,T})_f \leq \ell_{\star,T} \leq \widetilde{\ell_{\star,T}} \leq (\ell_{\star,T})_f$.

(5) follows from (2) and (3).

(6) It is a direct consequence of (2), (3) and Lemma 3.1 (b).

(7) is equivalent to (6), by (2) and Lemma 3.1 (a). □

Remark 3.13 Let T, S be two overrings of an integral domain D, with $D \subseteq T \subseteq S$ and let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, T, S).

Assume that S is $(\star, \star'')$–linked to D. If T is $(\star', \ell_{\star,T})$–linked to T (e.g. if $\star'_f \leq \ell_{\star,T}$), then S is $(\star', \star'')$–linked to T. As a matter of fact, let Q be a quasi-$\star''_f$-prime ideal of S, then $(Q \cap D)^{\star_f} \neq D^\star$, and hence, by definition of $T^{\ell_{\star,T}}$, $(Q \cap T)^{\ell_{\star,T}} \neq T^{\ell_{\star,T}}$. So $(Q \cap T)^{\star'_f} \neq T^{\star'}$.

In general, for *any* nontrivial semistar operation $\star''$ on S, we can construct a nontrivial semistar operation $\star'$ on T such that S is not $(\star', \star'')$-linked to T: Let Q be a quasi–$\star''_f$–prime ideal of S, and let $0 \neq q \in Q \cap T$. Let T_q be the ring of fractions of T with respect to its multiplicative set $\{q^n \mid n \geq 0\}$ and let $\star' := \star_{\{T_q\}}$. Then S is not $(\star', \star'')$-linked to T, since $(Q \cap T)^{\star'_f} = (Q \cap T)T_q = T_q = T^{\star'}$.

In [6], the authors showed that the equality $T^{\ell_{t_D},T} = T$ characterizes t–linkedness of T to D. The next goal is to investigate the analogous question in semistar setting.

Lemma 3.14 *Let D be an integral domain, T an overring of D, $\star$ a semistar operation on D and $\star'$ a (semi)star operation on T. If T is $(\star, \star')$–linked to D, then $T^{\ell_{\star,T}} = T$.*

Proof. Since $\star'$ is a (semi)star operation on T, then $T = T^{\widetilde{\star'}} = T^{\star'}$. Therefore, by Proposition 3.12 (2), we have $T \subseteq T^{\ell_{\star,T}} \subseteq T^{\widetilde{\star'}} = T$, and so $T^{\ell_{\star,T}} = T$. □

However, "a general converse" of the previous lemma fails to be true as the following example shows.

Example 3.15 Let K be a field and X, Y two indeterminates over K. Let $D := K[X, Y]$ and $M := (X, Y)$. Set $T := D_M$. Then $D \subset T$ is t–linked (since $D \subset T$ is flat [6, Proposition 2.2 (c)]). Hence $T^{\ell_{t_D},T} = T$, by [6, Proposition 2.13 (a)]. On the other hand, we have $MT \neq T$ and $M^{t_D} = D$. Hence T is not (t_D, d_T)–linked to D.

A generalization of [6, Proposition 2.13 (a)] is given next, by showing that the converse of Lemma 3.14 holds when $\star' = t_T$.

Proposition 3.16 *Let $\star$ be a semistar operation on the integral domain D and T an overring of D. Then T is t–linked to $(D, \star)$ if and only if $T^{\ell_{\star,T}} = T$.*

Proof. Assume that $T^{\ell_{\star,T}} = T$, that is $\ell_{\star,T}$ is a (semi)star operation of finite type on T (Proposition 3.12 (4)). In this situation, we have $\ell_{\star,T} \leq t_T$ and thus T is $(\ell_{\star,T}, t_T)$–linked to T. By Proposition 3.12 (3), T is $(\star, \ell_{\star,T})$–linked to D. By transitivity (Lemma 3.1 (b)), we conclude that T is t–linked to $(D, \star)$. □

4 Semistar flatness

Let D be an integral domain and T be an overring of D and let $\star$ (respectively, $\star'$) be a semistar operation on D (respectively, on T). We say that T is $(\star, \star')$*–flat over* D if, for each quasi–$\star'_f$–prime ideal Q of T, $(Q \cap D)^{\star_f} \neq D^\star$ (i.e. T is $(\star, \star')$–linked to D) and, moreover, $D_{Q \cap D} = T_Q$.
We say that T is *t–flat over* D, if T is (t_D, t_T)–flat over D. Note that, from [26, Remark 2.3], this definition of t–flatness coincides with that introduced in [26]. More generally, we say that T is *t-flat over* $(D, \star)$ if T is $(\star, t_T)$–flat over D.

Remark 4.1 **(a)** If $\star := d_D$ (respectively, $\star' := d_T$) the identity (semi)star operation on D (respectively, T), then T is (d_D, d_T)–flat over D if and only if T is flat over D.

(b) Note that T is t-flat over $(D, \star)$ implies T is t–flat over D (for a converse see the following Lemma 4.2 (e)). As a matter of fact, for each $Q \in \mathcal{M}(t_T)$, $D_{Q\cap D} = T_Q$ and thus, by [26], T is a t–flat overring of D.

(c) Recall that an example given by Fossum [13, page 32] shows that, even for a Krull domain (hence, in particular, for a PvMD), t–flatness does not imply flatness (cf. also [26, Remark 2.12]).

The proof of the following lemma, in which we collect some preliminary properties of semistar flatness, is straightforward.

Lemma 4.2 *Let T, S be two overrings of an integral domain D, with $D \subseteq T \subseteq S$.*

- **(a)** *Let $D = T$ and $\star', \star''$ be two semistar operations on T. Then T is $(\star', \star'')$–flat over T if and only if T is $(\star', \star'')$–linked to T. This happens when $\star'_f \leq \star''_f$.*
- **(b)** *Let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, T, S). Assume that S is $(\star', \star'')$–flat over T and that T is $(\star, \star')$–flat over D, then S is $(\star, \star'')$–flat over D.*
- **(c)** *Let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, two semistars operations on T). Assume that $\star'_f \leq \star''_f$. If T is $(\star, \star')$–flat over D, then T is also $(\star, \star'')$–flat over D.*
- **(d)** *Let $\star$ be a semistar operation on D and let $\star'$ be a (semi)star operation on T (hence, $\star'_f \leq t_T$). If T is $(\star, \star')$–flat over D then T is t–flat over $(D, \star)$.*
- **(e)** *Let $\star_1$ and $\star_2$ be two semistar operations on D and let $\star'$ be a semistar operation on T. Assume that $(\star_1)_f \leq (\star_2)_f$. If T is $(\star_2, \star')$–flat over D, then T is $(\star_1, \star')$–flat over D. In particular (cf. also Remark 4.1 (b)), if $\star$ is a (semi)star operation on D (hence $\star_f \leq t_D$), then T is t–flat over $(D, \star)$ if and only if T is t–flat over D.*
- **(f)** *Let $\star$ (respectively, $\star'$) be a semistar operation on D (respectively, T). The overring T is $(\star, \star')$–flat over D if and only if, for each quasi–$\star'_f$–maximal ideal N of T, $(N \cap D)^{\star_f} \neq D^\star$ and $D_{N\cap D} = T_N$.*
- **(g)** *Let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, T, S). Assume that S is $(\star, \star'')$–flat over D and that each quasi–$\star'_f$–maximal ideal of T is the contraction of a quasi–$\star''_f$–(maximal)ideal of S, then T is $(\star, \star')$–flat over D.*
- **(h)** *Let $\star$ (respectively, $\star'$, $\star''$) be a semistar operation on D (respectively, T, S). Assume that S is $(\star, \star'')$–flat over D. Then S is $(\star', \star'')$–flat over T if and only if S is $(\star', \star'')$–linked with T.* □

Remark 4.3 **(a)** When $\star$ is a proper semistar operation on D (that is $D^\star \neq D$), the equivalence of the second part of statement (e) in the previous lemma fails to be true in general. Indeed, if $\star = e_D$ then each t–flat overring T of D is not t–flat over (D, e_D), since T is not (e_D, t_T)–linked with D. An example in case $\star \neq e_D$ is given next.
Let D be a Prüfer domain with two prime ideals $P \not\subseteq Q$. Let $T := D_P$ and consider $\star := \star_{\{D_Q\}}$ as a semistar operation of finite type on D. Then T is t–flat over D (since

T is flat over D), but T is not t–flat over $(D, \star)$. Indeed, we have that $M := PD_P$ is a t–ideal of T and $(M \cap D)^\star = P^\star = PD_Q = D_Q = D^\star$.

(b) Note that, for each semistar operation $\star$ on D, $D^\star$ is $(\star, \dot{\star})$–linked to D (Lemma 3.1 (e)), but in general $D^\star$ is not $(\star, \dot{\star})$–flat over D. For instance, if T is a proper non-flat overring of D and if $\star := \star_{\{T\}}$, then $D^\star = T$, $\dot{\star} = d_T$ and T is not $(\star_{\{T\}}, d_T)$–flat over D.

(c) Let $\{T_\lambda \mid \lambda \in \Lambda\}$ be a family of overrings of D and let $*_\lambda$ be a semistar operation defined on T_λ, for $\lambda \in \Lambda$. Set $T := \cap\{T_\lambda \mid \lambda \in \Lambda\}$ and denote by $*_\Lambda$ the semistar operation on T associated to the family $\{(T_\lambda, *_\lambda) \mid \lambda \in \Lambda\}$ (Example 2.1 (4)). If T_λ is $(\star, *_\lambda)$–flat over D, for each $\lambda \in \Lambda$, is T $(\star, *_\Lambda)$–flat over D ?
The answer is negative, in general. For instance, let $V := \mathbb{C} + M$ be a valuation domain with unbranched maximal ideal M and let $D := \mathbb{R} + M \subseteq V$. By [15, Exercise 5 (a), p. 340], the domain D has the QQR–property, but it is not a Prüfer domain. By [26, Proposition 2.8], there exists an overring T of D which is not t–flat (note that, necessarily, $T = \cap\{D_P \mid P \in \Lambda\}$ for some subset Λ of the prime spectrum of D). Let $\star := t_D$ and let $*_P := t_{D_P}$, for each $P \in \Lambda$. Then, obviously, D_P is $((\star, *_P)$–)flat over D, for each $P \in \Lambda$, but T is not $(\star, *_\Lambda)$–flat over D. Indeed, we have $(*_\Lambda)_f \leq t_T$, so if T was $(\star, *_\Lambda)$–flat over D, then T would be t–flat over D (Lemma 4.2 (d)).

Let $\star$ be a semistar operation on an integral domain D with field of quotients K, if Σ is a multiplicative system of ideals of D, then we set $\Sigma^\star := \{I^\star \mid I \in \Sigma\}$. It is easy to verify that $\Sigma^\star$ is *a $\dot{\star}$-multiplicative system of $\dot{\star}$-ideals of $D^\star$* (i.e., if $I^\star, J^\star \in \Sigma^\star$ then $(I^\star \cdot J^\star)^{\dot{\star}} = (I \cdot J)^\star \in \Sigma^\star$).
If Σ is a multiplicative system of ideals of D, then:

$$D^{\dot{\star}}_{\Sigma^\star} := \{z \in K \mid zI^\star \subseteq D^\star\,, \text{ for some } I \in \Sigma\}$$

is an overring of $D^\star$ (and of $D_\Sigma := \{z \in K \mid zI \subseteq D\,, \text{ for some } I \in \Sigma\}$), called *the generalized ring of fractions of $D^\star$ with respect to the $\dot{\star}$-multiplicative system $\Sigma^\star$.*

Proposition 4.4 *Let D be an integral domain and T be an overring of D. Let $\star$ (respectively, $\star'$) be a semistar operation on D (respectively, on T). The following statements are equivalent:*

- **(i)** *T is $(\star, \star')$–flat over D;*
- **(ii)** *T is $(\star, \star')$–linked with D and, for each prime ideal P of D, either $(PT)^{\star'_f} = T^{\star'}$ or $T \subseteq D_P$;*
- **(iii)** *T is $(\star, \star')$–linked with D and, for each $x \in T$, $x \neq 0$, $((D :_D xD)T)^{\star'_f} = T^{\star'}$;*
- **(iv)** *T is $(\star, \star')$–linked with D and $T^{\widetilde{\star'}} = \cap\{D_{Q \cap D} \mid Q \in \mathcal{M}(\star'_f)\}$;*
- **(v)** *T is $(\star, \star')$–linked with D and, there exists a multiplicative system of ideals Σ in D such that $T^{\widetilde{\star'}} = D^{\dot{\tilde{\star}}}_{\Sigma^{\tilde{\star}}}$ and $(IT)^{\star'_f} = T^{\star'}$ for each $I \in \Sigma$.*

Moreover, each of the previous statement is a consequence of the following:

- **(vi)** *T is $(\star, \star')$–linked with D and, for each quasi–$\star_f$–prime ideal P of D, $T_{D \setminus P}$ is flat over D_P.*

Proof. (i) $\Rightarrow$ (ii). Let P be a prime ideal of D. Assume that $(PT)^{\star'_f} \neq T^{\star'}$ then there exists $Q \in \mathcal{M}(\star'_f)$ such that $PT \subseteq Q$, and so $P \subseteq Q \cap D$. Therefore, by the assumption, $D_P \supseteq D_{Q\cap D} = T_Q \supseteq T$.

(ii) $\Rightarrow$ (iii). Let $0 \neq x \in T$. Assume that $((D :_D xD)T)^{\star'_f} \neq T^{\star'}$, then there exists $Q \in \mathcal{M}(\star'_f)$ such that $(D :_D xD)T \subseteq Q$. We have $(D :_D xD) \subseteq Q \cap D =: P$ and $(PT)^{\star'_f} \neq T^{\star'}$. Hence, by assumption, $T \subseteq D_P$. Write $x = \frac{d}{s}$, for some $d \in D$ and $s \in D \setminus P$. Then $s \in (D :_D xD) \subseteq P$, which is impossible.

(iii) $\Rightarrow$ (iv). By the definition of $\widetilde{\star'}$ we have that $T^{\widetilde{\star'}} = \cap\{T_Q \mid Q \in \mathcal{M}(\star'_f)\}$, and hence $\cap\{D_{Q\cap D} \mid Q \in \mathcal{M}(\star'_f)\} \subseteq T^{\widetilde{\star'}}$. For the reverse inclusion, let $x \in T$, $x \neq 0$, then $((D :_D xD)T)^{\star'_f} = T^{\star'}$. Let $Q \in \mathcal{M}(\star'_f)$. Then $(D :_D xD)T \not\subseteq Q$, that is $(D :_D xD) \not\subseteq Q \cap D$. So $x \in D_{Q\cap D}$. Thus $T \subseteq D_{Q\cap D}$, and hence $T_Q = D_{Q\cap D}$. Therefore $T^{\widetilde{\star'}} \subseteq D_{Q\cap D}$ for each $Q \in \mathcal{M}(\star'_f)$ and so we conclude that $T^{\widetilde{\star'}} = \cap\{D_{Q\cap D} \mid Q \in \mathcal{M}(\star'_f)\}$.

(iv) $\Rightarrow$ (i). Let $Q \in \mathcal{M}(\star'_f)$. Then $T \subseteq T^{\widetilde{\star'}} \subseteq D_{Q\cap D}$. Hence $T_Q \subseteq D_{Q\cap D}$. The reverse inclusion is trivial.

(ii) $\Rightarrow$ (v). Let $\Sigma := \{I \text{ nonzero ideal of } D \mid (IT)^{\star'_f} = T^{\star'}\}$. The set Σ is a multiplicative system of ideals of D. Hence $\Sigma^{\tilde{\star}} = \{I^{\tilde{\star}} \mid I \in \Sigma\}$ is a $\dot{\tilde{\star}}$–multiplicative system of $\dot{\tilde{\star}}$–ideals of $D^{\tilde{\star}}$. Let $x \in D^{\tilde{\star}}_{\Sigma^{\tilde{\star}}}$. Then $xI \subseteq xI^{\tilde{\star}} \subseteq D^{\tilde{\star}}$, for some $I \in \Sigma$. Since $D^{\tilde{\star}} \subseteq T^{\widetilde{\star'}}$ (Remark 3.3 (b)), then $xIT \subseteq T^{\widetilde{\star'}}$, and hence $x(IT)^{\widetilde{\star'}} \subseteq T^{\widetilde{\star'}}$. On the other hand, since $(IT)^{\star'_f} = T^{\star'}$, then necessarily $(IT)^{\widetilde{\star'}} = T^{\widetilde{\star'}}$. Hence $xT^{\widetilde{\star'}} \subseteq T^{\widetilde{\star'}}$ and so $x \in T^{\widetilde{\star'}}$. Therefore $D^{\tilde{\star}}_{\Sigma^{\tilde{\star}}} \subseteq T^{\widetilde{\star'}}$.
For the opposite inclusion, let $0 \neq x \in T^{\widetilde{\star'}}$. Set $I := (D :_D xD)$. We claim that $(IT)^{\star'_f} = T^{\star'}$ (i.e. $I \in \Sigma$). Otherwise, as in the proof of (ii) $\Rightarrow$ (iii), there exists $Q \in \mathcal{M}(\star'_f)$ such that $I \subseteq Q \cap D$ and $T \subseteq D_{Q\cap D}$. Hence $T^{\widetilde{\star'}} \subseteq T_Q \subseteq D_{Q\cap D}$. Write $x = \frac{d}{s}$ for some $d \in D$ and $s \in D \setminus (Q \cap D)$. Therefore $s \in (D :_D xD) \subseteq Q \cap D$, which is impossible.
Finally, in general, we have $xI^{\tilde{\star}} = (xI)^{\tilde{\star}} \subseteq (x(D :_K xD))^{\tilde{\star}} = D^{\tilde{\star}}$. So $x \in D^{\tilde{\star}}_{\Sigma^{\tilde{\star}}}$ (i.e. $T^{\widetilde{\star'}} \subseteq D^{\tilde{\star}}_{\Sigma^{\tilde{\star}}}$), hence we conclude that $T^{\widetilde{\star'}} = D^{\tilde{\star}}_{\Sigma^{\tilde{\star}}}$.

(v) $\Rightarrow$ (iv). The inclusion $\cap\{D_{Q\cap D} \mid Q \in \mathcal{M}(\star'_f)\} \subseteq T^{\widetilde{\star'}}$ is clear. Now, let $x \in T^{\widetilde{\star'}} = D^{\tilde{\star}}_{\Sigma^{\tilde{\star}}}$. Then there exists a nonzero ideal $I \in \Sigma$ such that $xI \subseteq D^{\tilde{\star}}$. Let $Q \in \mathcal{M}(\star'_f)$. Since $I \in \Sigma$ then, by assumption, $(IT)^{\star'_f} = T^{\star'}$ and, thus, $I \not\subseteq Q \cap D$. Let $s \in I \setminus (Q \cap D)$, then $sx \in D^{\tilde{\star}}$. On the other hand, since $(Q \cap D)^{\star_f} \neq D^{\star}$ (Proposition 3.2), there exists $M \in \mathcal{M}(\star_f)$ such that $Q \cap D \subseteq M$. Therefore we have that $D^{\tilde{\star}} \subseteq D_M \subseteq D_{Q\cap D}$ and so $sx \in D_{Q\cap D}$, thus $x \in D_{Q\cap D}$. Hence we conclude that $T^{\widetilde{\star'}} \subseteq \cap\{D_{Q\cap D} \mid Q \in \mathcal{M}(\star'_f)\}$.

(vi) $\Rightarrow$(i). Let Q be a quasi–$\star'_f$–prime ideal, and let P be a quasi–$\star_f$–prime of D such that $(Q \cap D)^{\star_f} \subseteq P$ (Proposition 3.2). Since $QT_{D\setminus P}$ is a prime ideal of $T_{D\setminus P}$ such that $QT_{D\setminus P} \cap D_P = (Q \cap D)D_P$ and, by assumption, $T_{D\setminus P}$ is flat over D_P, then we conclude that $D_{Q\cap D} = (D_P)_{(Q\cap D)D_P} = (T_{D\setminus P})_{QT_{D\setminus P}} = T_Q$.

□

Theorem 4.5 *Let D be an integral domain and T be an overring of D. Let $\star$ (respectively, $\star'$) be a semistar operation on D (respectively, on T). The following statements are equivalent:*

(i) *T is $(\star, \star')$–flat over D;*

(ii) $\mathrm{Na}(T, \star')$ *is a flat overring of* $\mathrm{Na}(D, \star)$*;*

(iii) *T is $(\widetilde{\star}, \widetilde{\star'})$–flat over D;*

(iv) *$T^{\widetilde{\star'}}$ is a $(\dot{\widetilde{\star}}, \dot{\widetilde{\star'}})$–flat overring of $D^{\widetilde{\star}}$.*

Proof. Since $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \widetilde{\star}) = \mathrm{Na}(D^{\widetilde{\star}}, \dot{\widetilde{\star}})$ and, similarly, $\mathrm{Na}(T, \star') = \mathrm{Na}(T, \widetilde{\star'})$ $= \mathrm{Na}(T^{\widetilde{\star'}}, \dot{\widetilde{\star'}})$ (Lemma 2.4 (i)), it suffices to show that (i) $\Leftrightarrow$ (ii).

(i) $\Rightarrow$ (ii). Since T is $(\star, \star')$–linked to D, then $\mathrm{Na}(D, \star) \subseteq \mathrm{Na}(T, \star')$, by Theorem 3.8. Now, let N be a maximal ideal of $\mathrm{Na}(T, \star')$. Then $N = Q\,\mathrm{Na}(T, \star') = QT_Q(X) \cap \mathrm{Na}(T, \star')$, for some $Q \in \mathcal{M}(\star'_f)$ (cf. also Lemma 2.4 (e)), and $\mathrm{Na}(T, \star')_N = \mathrm{Na}(T, \star')_{Q\,\mathrm{Na}(T,\star')} = T_Q(X) = D_{Q\cap D}(X)$, because of Corollary 2.5 and, by assumption, $D_{Q\cap D} = T_Q$. On the other hand, by semistar linkedness, $(Q \cap D)^{\star_f} \neq D^\star$ (Proposition 3.2) then we have that $\mathrm{Na}(D, \star)_{(Q\cap D)\,\mathrm{Na}(D,\star)} = D_{Q\cap D}(X)$ (Corollary 2.5). One can easily check that $N \cap \mathrm{Na}(D, \star) = (Q \cap D)\,\mathrm{Na}(D, \star)$. Therefore $\mathrm{Na}(T, \star')_N = \mathrm{Na}(D, \star)_{N\cap \mathrm{Na}(D,\star)}$, as desired.

(ii) $\Rightarrow$ (i). Since $\mathrm{Na}(D, \star) \subseteq \mathrm{Na}(T, \star')$, then T is $(\star, \star')$–linked to D (Theorem 3.8). Let Q be a quasi–$\star'_f$–maximal ideal of T and set $N := Q\,\mathrm{Na}(T, \star')$, then $\mathrm{Na}(T, \star')_N = \mathrm{Na}(T, \star')_{Q\,\mathrm{Na}(T,\star')} = T_Q(X)$ (Corollary 2.5). On the other hand, by flatness, we have $\mathrm{Na}(T, \star')_N = \mathrm{Na}(D, \star)_{N\cap\mathrm{Na}(D,\star)} = \mathrm{Na}(D, \star)_{(Q\cap D)\,\mathrm{Na}(D,\star)}$. Since, by semistar linkedness, $(Q \cap D)^{\star_f} \neq D^\star$ (Proposition 3.2), then we have that $\mathrm{Na}(D, \star)_{(Q\cap D)\,\mathrm{Na}(D,\star)} = D_{Q\cap D}(X)$ (Corollary 2.5). Therefore $T_Q(X) = D_{Q\cap D}(X)$ and so $T_Q = D_{Q\cap D}$. Hence T is $(\star, \star')$–flat over D. □

The following result sheds new light on the statement (vi) of Proposition 4.4.

Proposition 4.6 *Let D be an integral domain and T be an overring of D. Let $\star$ be a semistar operation on D and let $\ell := \ell_{\star,T}$ be the semistar operation on T introduced in Section 3. The following statements are equivalent:*

(i) *T is $(\star, \ell)$–flat over D;*

(ii) *for each prime ideal P of D, either $(PT)^\ell = T^\ell$ or $T \subseteq D_P$;*

(iii) *for each $x \in T$, $x \neq 0$, $((D :_D xD)T)^\ell = T^\ell$;*

(iv) *$T^\ell = \cap\{D_{N\cap D} \mid N$ is a prime ideal of T, maximal with the property $(N \cap D)^{\star_f} \neq D^\star\}$;*

(v) *for each prime ideal Q of T such that $(Q \cap D)^{\star_f} \neq D^\star$, then $D_{Q\cap D} = T_Q$;*

(vi) *for each prime ideal N of T, maximal with respect to the property $(N\cap D)^{\star_f} \neq D^\star$, then $D_{N\cap D} = T_N$;*

(vii) *for each quasi–$\star_f$–prime ideal P of D, $T_{D\setminus P}$ is flat over D_P;*

(viii) *for each nonzero finitely generated fractional ideal F of D, $((D :_K F)T)^\ell = (T^\ell :_K FT)$.*

Proof. Note that the set of quasi–ℓ–prime (respectively, quasi–ℓ–maximal) ideals of T coincides with the set of prime ideals Q of T such that $(Q \cap D)^{\star_f} \neq D^\star$ (respectively, the set of prime ideals N of T, maximal with the property $(N\cap D)^{\star_f} \neq D^\star$). Therefore, the statements (i) – (vi) are equivalent by Proposition 4.4 and Proposition 3.12 (3).

(v) $\Rightarrow$ (vii). Let P be a quasi–$\star_f$–prime ideal of D. Let N be a maximal ideal of $T_{D\setminus P}$. Then $N\cap D\subseteq P$ and, hence, $((N\cap T)\cap D)^{\star_f}\neq D^\star$. So $D_{N\cap D}=T_{N\cap T}$. On the other hand, we have $(T_{D\setminus P})_N=T_{N\cap T}$, and $(D_P)_{N\cap D_P}=(D_P)_{(N\cap D)D_P}=D_{N\cap D}$. Hence $(D_P)_{N\cap D_P}=(T_{D\setminus P})_N$, as desired.

(vii) $\Rightarrow$ (viii). We have $((D:_K F)T)^\ell=\cap\{(D:_K F)T_{D\setminus P}\mid P$ is a quasi–$\star_f$– prime of $D\}$. As $T_{D\setminus P}$ is D_P–flat (hence, $T_{D\setminus P}$ is also D–flat) and F is finitely generated, then $(D:_K F)T_{D\setminus P}=(T_{D\setminus P}:_K FT_{D\setminus P})=(T:_K FT)T_{D\setminus P}$, for each quasi–$\star_f$–prime P of D. Hence $((D:_K F)T)^\ell=\cap\{(T:_K FT)T_{D\setminus P}\mid P$ is a quasi–$\star_f$–prime of $D\}$ $=(T:_K FT)^\ell=(T^\ell:_K FT)$ (since ℓ is stable; Example 2.1 (5) and Proposition 3.12 (1)).

(viii) $\Rightarrow$ (iii). Take $F:=D+xD$. $\square$

It is well-known that a domain with all its overrings flat (or, equivalently, with all its overrings t–flat) coincides with a Prüfer domain (cf. [33, Theorem 4], [26, Proposition 2.8]). The following proposition deals with a similar question in the semistar case.

Theorem 4.7 *Let D be an integral domain and $\star$ a semistar operation on D. The following statements are equivalent:*

- **(i)** *For each overring T of D and for each semistar operation $\star'$ on T, T is $(\star,\star')$–flat over D;*
- **(ii)** *Each overring T of D is $(\star,d_T)$–flat over D;*
- **(iii)** *Each overring T of D is t–flat over $(D,\star)$;*
- **(iv)** *D is a Prüfer domain in which each maximal ideal is a quasi–$\star_f$–maximal ideal.*

Proof. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii) is a consequence of $d_T\leq t_T$ (Lemma 4.2 (c)).

(iii) $\Rightarrow$ (iv). Since semistar flatness implies semistar linkedness, then, by Theorem 3.9, each maximal ideal is a quasi–$\star_f$–maximal ideal. On the other hand, since an overring t–flat over $(D,\star)$ is also t–flat over D (Remark 4.1 (b)), then each overring of D is t–flat over D. Hence, by [26, Proposition 2.8], D is a Prüfer domain.

(iv) $\Rightarrow$ (i). Let T be an overring of D and $\star'$ a semistar operation on T. Let Q be a quasi–$\star'_f$–prime ideal of T. Then $Q\cap D$ is contained in a maximal ideal of D which is, by assumption, a quasi–$\star_f$–maximal ideal of D. Therefore $(Q\cap D)^{\star_f}\neq D^\star$, and so T is $(\star,\star')$–linked with D. The equality $T_Q=D_{Q\cap D}$ is a consequence of the fact that T is an overring of the Prüfer domain D [15, Theorem 26.1]. $\square$

5 Prüfer semistar multiplication domains

As an application of the previous sections, our goal is to give new characterizations of Prüfer semistar multiplication domains, in terms of semistar linked overrings and semistar flatness.

Let D be an integral domain and $\star$ a semistar operation on D. Recall that D is *a P$\star$MD (Prüfer $\star$–multiplication domain)*, if each $F\in\boldsymbol{f}(D)$ is $\star_f$-invertible (i.e., $(FF^{-1})^{\star_f}=D^\star$).

The notion of P$\star$MD is a generalization of the notion of Prüfer v–multiplication domain (cf. [15, page 427], [17], [29]) and so, in particular, of Prüfer domain. When $\star = d$ (where d is the identity (semi)star operation on D) the PdMDs are just the Prüfer domains. If $\star = v$ (where v is the v-(semi)star operation on D), we obtain the notion of PvMD.

Remark 5.1 **(a)** The notions of P$\star$MD and P$\star_f$MD coicide. In particular, a PvMD coincides with a PtMD.

(b) Let $\star_1$ and $\star_2$ be two semistar operations on D such that $\star_1 \leq \star_2$. If D is a P$\star_1$MD, then D is also a P$\star_2$MD. In particular, if $\star$ is a (semi)star operation on D, and hence $\star \leq v$ [15, Theorem 34.1 (4)], then a P$\star$MD is a PvMD. Also, since $d \leq \star$ for any semistar operation $\star$, then a Prüfer domain is a P$\star$MD for any arbitrary semistar operation $\star$ on D.

(c) In the semistar case (i.e. if $\star$ is a proper semistar operation), a P$\star$MD is not necessarily integrally closed [9, Example 3.10].

We recall some of the characterizations of P$\star$MDs proved in [9]:

Theorem 5.2 [9, Theorem 3.1, Remark 3.2] *Let D be an integral domain and $\star$ a semistar operation on D. The following statements are equivalent:*

- **(i)** *D is a P$\star$MD;*
- **(ii)** *D_Q is a valuation domain, for each $Q \in \mathcal{M}(\star_f)$;*
- **(iii)** $\mathrm{Na}(D, \star)$ *is a Prüfer domain;*
- **(iv)** *D is a P$\tilde{\star}$MD.*

Moreover, if D is a P$\star$MD, then $\tilde{\star} = \star_f$. □

The following theorem is "a semistar version" of a characterization of the Prüfer domains proved by E. Davis [4, Theorem 1]. It generalizes properly [6, Theorem 2.10], stated in the case of t-operations (cf. also [29, Theorem 5.1] and [23, Corollary 3.9]). Recall that an integral domain D, with field of quotients K, is *seminormal* if, whenever $x \in K$ satisfies $x^2, x^3 \in D$, then $x \in D$, [16].

Theorem 5.3 *Let D be an integral domain , T an overring of D, $\star$ a semistar operation on D and let $\ell := \ell_{\star,T}$ be the semistar operation on T introduced in Section 3. The following statements are equivalent:*

- **(i)** *For each overring T and for each semistar operation $\star'$ such that T is $(\star, \star')$–linked to D, $T^{\tilde{\star'}}$ is integrally closed.*
- **(ii)** *For each overring T of D, $T^{\ell_{\star,T}}$ is integrally closed.*
- **(iii)** *Each overring T, t–linked to $(D, \star)$, is integrally closed.*
- **(iv)** *Each overring T, $(\star, d_T)$-linked to D, is integrally closed.*
- **(v)** *$D^{\tilde{\star}}$ is integrally closed and, for each overring T and for each semistar operation $\star'$ on T such that T is $(\star, \star')$–linked to D, $T^{\tilde{\star'}}$ is seminormal.*

(vi) *$D^{\tilde{\star}}$ is integrally closed and each overring T, t–linked to $(D,\star)$, is seminormal.*

(vii) *$D^{\tilde{\star}}$ is integrally closed and each overring T, $(\star, d_T)$-linked to D, is seminormal.*

(viii) *D is a P$\star$MD.*

Proof. (i) $\Rightarrow$ (ii). It follows from Proposition 3.12 (3) and (4), by taking $\star' = \ell_{\star,T}$.

(ii) $\Rightarrow$ (iii) follows from Proposition 3.16.

(iii) $\Rightarrow$ (iv). Obvious since $d_T \leq t_T$ (Lemma 3.1 (c)).

(iv) $\Rightarrow$ (v). Let $(T,\star')$ be such that T is $(\star,\star')$–linked to D. Let P be a quasi–$\star_f$–prime ideal of D. By Example 3.4 (1), $T_{D\setminus P}$ is $(\star, d_{T,P})$–linked to D. Hence, by assumption, $T_{D\setminus P}$ is integrally closed. In particular (for $(T,\star') = (D,\star)$), D_P is integrally closed, and hence $D^{\tilde{\star}}$ is integrally closed. On the other hand, if Q is a quasi–$\star'_f$–prime ideal of T, there exists P a quasi–$\star_f$–prime ideal of D such that $Q \cap D \subseteq P$ (Proposition 3.2). Hence $T_{D\setminus P} \subseteq T_Q$ and so T_Q is integrally closed, since $T_{D\setminus P}$ is. Therefore, $T^{\tilde{\star'}}$ is integrally closed; in particular, $T^{\tilde{\star'}}$ is seminormal.

(v) $\Rightarrow$ (vi) is obvious and (vi) $\Rightarrow$ (vii) is a consequence of $d_T \leq t_T$ (Lemma 3.1 (c)).

(vii) $\Rightarrow$ (viii). We want to show that, for each quasi–$\star_f$–maximal ideal P of D, D_P is a valuation domain (Theorem 5.2), i.e., if x is a nonzero element of the quotient field K of D, then either x or x^{-1} is in D_P . Note that, from the assumption, it follows that $D_P = D^{\tilde{\star}}_{PD_P\cap D^{\tilde{\star}}}$ is integrally closed. If we set $T := D[x^2,x^3]$ then (by Example 3.4 (1)) $T_{D\setminus P} = D[x^2,x^3]_{D\setminus P} = D_P[x^2,x^3]$ is $(\star, d_{T,P})$–linked to D thus, by assumption, $D_P[x^2,x^3]$ is seminormal, i.e., $x \in D_P[x^2,x^3]$. Hence x is the root of some polynomial f with coefficients in D_P and with the coefficient of the linear term equal to 1. This implies that either x or x^{-1} is in D_P, by [24, Theorem 67].

(viii) $\Rightarrow$ (i). Let T be an overring of D $(\star,\star')$–linked to D. For each quasi–$\star'_f$–maximal ideal N of T, let P be a quasi–$\star_f$–maximal ideal P of D, such that $N \cap D \subseteq P$ (Proposition 3.2), thus $D_P \subseteq T_{D\setminus P} \subseteq T_N$. Since D is a P$\star$MD, then D_P is a valuation domain, hence T_N is also a valuation domain and so $T^{\tilde{\star'}} = \cap\{T_N \mid N \in \mathcal{M}(\star'_f)\}$ is integrally closed. □

The following result generalizes [29, Theorem 5.1] (cf. also [23, Corollary 3.9]).

Corollary 5.4 *Let D be an integral domain and T be an overring of D. Let $\star$ (respectively, $\star'$) be a semistar operation on D (respectively, on T). Assume that D is a P$\star$MD and that T is $(\star,\star')$–linked to D, then T is a P$\star'$MD.*

Proof. If S is an overring of T and $\star''$ a semistar operation on S such that S is $(\star',\star'')$–linked to T, then S is $(\star,\star'')$–linked to D (Lemma 3.1 (b)). By Theorem 5.3 ((viii) $\Rightarrow$ (i)) $S^{\widetilde{\star''}}$ is integrally closed. The conclusion follows from Theorem 5.3 ((i) $\Rightarrow$ (viii)). □

Corollary 5.5 *Let D be P$\star$MD for some semistar operation $\star$ on D. Then:*

(a) *For each overring T of D, T is a P$\dot{\star}^T$MD.*

(b) *Each t–linked overring to $(D,\star)$ is a PvMD. In particular, $D^{[\star]}$ is a PvMD and if, moreover, $(D : D^{\star}) \neq 0$, then the complete integral closure $\widetilde{D}$ of D is a PvMD.*

Proof. **(a)** follows from Corollary 5.4 and Lemma 3.1 (e). The first statement in **(b)** is a particular case of Corollary 5.4; the remaining part is a consequence of the first part and of Corollary 3.6. □

Note that Corollary 5.5 (b) generalizes the fact that the pseudo–integral closure, $D^{[v]}$, of a PvMD, D, is still a PvMD [3, Proposition 1.3].

Remark 5.6 The integral closure D' of an integral domain D is not in general t-linked over D [5, Example 4.1]. But, each domain D has a smallest integrally closed t-linked overring, namely $D'^{\ell_{t_D,D'}} = \cap\{D'_{D\setminus P} \mid P \text{ is a } t-\text{prime ideal of } D\}$ [6, Proposition 2.13 (b)].
In the semistar case, D' is always $(\star, \ell_{\star,D'})$–linked to D, for any semistar operation $\star$ on D (Proposition 3.12 (3)). Also note that $\ell_{\star,D'}$ is the unique minimal semistar operation in the set of semistar operations $\star'_f$, where $\star'$ is a semistar operation on D' such that D' is $(\star, \star')$–linked to D (Proposition 3.12 (5)). Therefore, D' is t–linked over D if and only if $\ell_{t_D,D'} \leq t_{D'}$ (Lemma 3.1 (c)) or, equivalently, if and only if $\ell_{t_D,D'}$ is a (semi)star operation on D' (i.e. $D' = D'^{\ell_{t_D,D'}}$).

The next theorem of characterization of P$\star$MDs is a "semistar analogue" of Richman's flat-theoretic theorem of characterization of Prüfer domains [33, Theorem 4]. A special case of the following result, concerning the t-operations, was obtained in [26, Proposition 2.10].

Theorem 5.7 *Let D be an integral domain, $\star$ a semistar operation on D, T an overring of D and let $\ell := \ell_{\star,T}$ be the semistar operation on T introduced in Section 3. The following statements are equivalent:*

- **(i)** *D is a P$\star$MD.*
- **(ii)** *For each overring T of D and for each semistar operation $\star'$ such that T is $(\star, \star')$–linked to D, T is $(\star, \star')$–flat over D.*
- **(iii)** *For each overring T of D, T is $(\star, \ell_{\star,T})$-flat over D.*
- **(iv)** *For each overring T of D, t–linked to $(D, \star)$, T is t–flat over $(D, \star)$.*
- **(v)** *For each overring T of D such that T is $(\star, d_T)$-linked to D, T is $(\star, d_T)$–flat over D.*

Proof. (i) $\Rightarrow$ (ii). Let T be an overring and $\star'$ a semistar operation on T such that T is $(\star, \star')$–linked to D. Let Q be a quasi–$\star'_f$–prime of T such that $(Q \cap D)^{\star_f} \neq D^\star$. Then $Q \cap D \subseteq P$ for some quasi–$\star_f$–maximal ideal P of D. Thus $D_P \subseteq D_{Q\cap D} \subseteq T_Q$. Since D is a P$\star$MD, then D_P is a valuation domain (Theorem 5.2), hence T_Q is also a valuation domain and $T_Q = D_{Q\cap D}$. Hence T is $(\star, \star')$–flat over D.

(ii) $\Rightarrow$ (iii) is a trivial consequence of Proposition 3.12 (3).

(iii) $\Rightarrow$ (ii). Let T be an overring and $\star'$ a semistar operation on T such that T is $(\star, \star')$–linked to D. Then $\ell_{\star,T} \leq \star'_f$ (Proposition 3.12 (5)). Hence T is $(\star, \star')$–flat over D (Lemma 4.2 (c)).

(ii) $\Rightarrow$ (iv) is obvious.

(iv) $\Rightarrow$ (v). Let T be an overring $(\star, d_T)$-linked to D, and let Q be a prime ideal of T. We have $(Q \cap D)^{\star_f} \neq D^\star$ (Proposition 3.2). Let P be a quasi–$\star_f$–maximal ideal of D such that $Q \cap D \subseteq P$, thus $D_P \subseteq T_{D \setminus P} \subseteq T_Q$. Let (V, M) be a valuation overring of D such that $M \cap D = P$. Then $D \subseteq V_{D \setminus P} = V$ is t–linked with $(D, \star)$ (Example 3.4 (1)), and hence V is t–flat over $(D, \star)$, by assumption. So $V = D_{M \cap D} = D_P$. Therefore T_Q $(\supseteq D_P)$ is also a valuation domain and $T_Q = D_{Q \cap D}$, thus T is $(\star, d_T)$-flat over D.

(v) $\Rightarrow$ (i). Let P be a quasi–$\star_f$–prime ideal of D. Let T be an overring of D_P (and hence of D). Note that, in this situation, $T = T_{D \setminus P}$. Hence T is $(\star, d_T)$–linked to D (Example 3.4 (1)). So T is $(\star, d_T)$–flat over D, by assumption. Therefore, if N is a maximal ideal of T, then $T_N = D_{N \cap D}$. Hence $T_N = (D_P)_{(N \cap D) D_P} = (D_P)_{N \cap D_P}$ (since $N \cap D \subseteq P$). That is, T is D_P–flat. By a result proved by Richman [33, Lemma 4], we deduce that D_P is a valuation domain. Hence D is a P$\star$MD (Theorem 5.2). □

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