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Kaplansky Ideal Transform: A Survey

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0 INTRODUCTION

In 1956, M. Nagata [N1] introduced the *ideal transform* $T_R(I) = \cup_n(R : I^n)$ *of an integral domain* R *with respect to an ideal* I of R (cf. (3.3)). This transform proved very useful in his series of papers on the Fourteenth Problem of Hilbert (cf. [N1], [N2], [N3], [N4] and [N5]).

Hilbert's XIV$^{\text{th}}$ Problem
Let k be a field, $x_1, x_2, \ldots, x_n$ algebraically independent elements over k and let L be a subfield of $k(x_1, x_2, \ldots, x_n)$ containing k. Is the ring $k[x_1, x_2, \ldots, x_n] \cap L$ finitely generated over k?

Hilbert's problem was motivated by the following problem of invariant theory:

Hilbert's XIV$^{\text{th}}$ Problem (strict form)
Let k be a field, $x_1, x_2, \ldots, x_n$ algebraically independent elements over k and let G be a subgroup of $GL(n, k)$. Is the ring of invariants, $k[x_1, x_2, \ldots, x_n]^G$, subring of the polynomial ring $k[x_1, x_2, \ldots, x_n]$, finitely generated over k?

Positive answers to the Hilbert's XIV$^{\text{th}}$ problem were given, in particular cases, by D. Hilbert, E. Fischer, E. Noether and H. Weyl (cf. for instance [N5, Chapter 0]). The next significant contributions were made after Zariski generalized, in 1954, the original form of the problem in the following way:

Zariski's Problem [Z]
Let k be a field and A a finitely generated and integrally closed k-algebra with quotient field K. Let L be a subfield of K containing k. Is $A \cap L$ a finitely generated k-algebra?

Zariski answered this question, in the affirmative, when $\text{tr.deg}_k L \leq 2$ and D. Rees in 1957 [Re] gave a counterexample when $\text{tr.deg}_k L = 3$. Finally, in 1959, Nagata [N2] gave a counterexample to Hilbert's XIV$^{\text{th}}$ Problem, when $\text{tr.deg}_k L = 4$.

One of the key steps for a negative solution to this type of problem, made by Nagata [N1], lies in the following result that shows clearly the rôle of the ideal transform:

A finitely generated field extension L of a given field k is called a Zariski field over k, *if for each finitely generated and integrally closed k-algebra A with quotient field K and $K \supseteq L$, then $A \cap L$ is a finitely generated k-algebra. Then, L is a Zariski field if and only if, for each finitely generated and integrally closed k-algebra B, with quotient field L, and for each ideal I of B, the ideal transform $T_B(I)$ is finitely generated over B.*

Ideal transforms have been proved to be very useful in other contexts of commutative algebra.

Nagata [N7] noted that the ideal transform $T_R(I)$ may be used in the study of the Catenary Chain Conditions.

Brewer [Br] introduced the ideal transform in the study of the overrings of an integral domain. After his work, several authors pursued the investigation of overrings by related means (cf., for instance, Brewer-Gilmer [BrG], Arnold-Brewer [AB], Heinzer-Ohm-Pendleton [HOP], Gilmer-Huckaba [GHu], Hedstrom [He1], [He2], Hays [Hy], Anderson-Bouvier [AnB], Fontana-Popescu [FP3], Fontana-Houston [FH]).

The ideal transform was also studied intensively in the not necessarily integral domain setting and the problem of when $T_R(I)$ is finitely generated, flat or integral over R was investigated by Anderson [A], Brodmann [Bro], Játem [J], Katz [Ka], Kiyek [Ki], McAdam-Ratliff [MR], Matijevic [Ma], Nishimura [Ni], Schenzel [S1], [S2], Zöschinger [Zö].

Ideal transform are also closely tied in with local cohomology and with affineness of the open subspaces of the prime spectrum (cf. Hartshorne [Ha], Serre [Se], Arezzo-Ramella [AR], Ohi [O] and the following Section 4).

However, outside of Noetherian setting, the behaviour of the ideal transform, as defined by Nagata, is not entirely satisfactory. For instance, in the study of the overrings, Brewer and Gilmer [BrG] obtained complete results, when considering ideal transforms with respect to finitely generated ideals, but only partial results (and conjectures) in the general case. Another aspect of the non satisfactory behaviour of the ideal transform, as defined by Nagata, will be examined in this paper when, in Section 4, we will look for a general result on the affineness of the open subspace $D(I) := \{P \in \mathrm{Spec}(R) : P \not\supseteq I\}$.

In the case of a non finitely generated ideal I of an integral domain R with quotient field K, a variant of the notion of ideal transform was introduced by Kaplansky [K2]. We call the *Kaplansky (ideal) transform of R with respect to an ideal I of R* the following overring of R:

$$\Omega_R(I) := \{z \in K : \mathrm{rad}(R :_R zR) \supseteq I\} .$$

Note that $\Omega_R(I)$ is an overring of the Nagata (ideal) transform of R with respect to I, since

$$T_R(I) = \{z \in K : (R :_R zR) \supseteq I^n \text{ for some } n \geq 1\}$$

and, if I is finitely generated, $\Omega_R(I) = T_R(I)$.

A natural and general approach to ideal transforms is to use multiplicative systems of ideals and generalized ring of fractions. This is the point of view that will be used in the present paper.

In Section 1, we review and complete some properties, concerning localizing systems of ideals and flatness, partially contained in [FHP, Chapter 5], in [HOP] and in a paper by Gabelli [Ga] published in this volume. In particular, we focus our attention on spectral localizing systems and to related overring properties (e.g. QQR, GQR, $\mathcal{F}QR$).

The second section is devoted to the study of flatness and finiteness of the overrings of an integral domain, by means of localizing systems. In particular, in this section we introduce and study a new class of saturated multiplicative systems of ideals, defined as follows: given a family of ideals of a domain R, $\mathcal{I} := \{I_\alpha : \alpha \in A\}$, consider $\mathcal{K}(\mathcal{I}) := \{J : \mathrm{rad}(J) \supseteq \prod_{k=1}^{n} I_{\alpha_k},$ where $n \geq 1$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq A\}$. The multiplicative system of ideals $\mathcal{K}(\mathcal{I})$, in several relevant cases, is a localizing system. Furthermore, in the Noetherian case, every localizing system is of the type $\mathcal{K}(\mathcal{I})$, for some family of ideals $\mathcal{I}$ (Lemma 2.9). Among other results, following the lines of a paper by E.L. Popescu [Po], we recover, in the integral domain case, a result proved by Schenzel [S3] on when a flat algebra is of finite type (Theorem 2.11).

In Section 3, we introduce the Kaplansky transform $\Omega_R(I)$ of an ideal I in a integral domain R as the ring of fractions of R with respect to the spectral localizing system associated to the open set $D(I) = \{P \in \mathrm{Spec}(R) : P \not\supseteq I\}$. After verifying that this definition is equivalent to the classical one, given above [K2, p. 57], we recover the principal properties of the Kaplansky transform and its links with the Nagata transform. Following some ideas of Hays [Hy] (also developed later by Rhodes [Rh]), we study the Ω-ideal I^Ω (i.e. the unique maximal element in the set of the ideals J of R such that $\Omega(J) = \Omega(I)$). Among other properties, we give in Proposition 3.17 a "topological" interpretation of I^Ω, when considering the Zariski topological space $\mathrm{Loc}(R)$ of the localizations of R [ZS]. From this topological point of view, we are led to consider the following radical ideal of R, associated to each R-submodule E of the quotient field of R:

$$\Omega^-(E) := \bigcap_{z \in E} \mathrm{rad}((R :_R zR)) \ .$$

We show that the operators Ω and Ω^- establish a "sort of duality" between radical ideals and overrings of R (Theorem 3.26). In particular, we reobtain, for the integral domain case, some of the results proved by Rhodes [Rh] in the commutative ring case.

In the last section, we start by recalling a "geometric" interpretation of the Nagata transform. More precisely, if R is a Noetherian domain, $\mathrm{Spec}(T_R(I))$ is canonically homeomorphic to the open subspace $D(I) = \{P \in \mathrm{Spec}(R) : P \not\supseteq I\}$ if and only if $IT_R(I) = T_R(I)$ (Proposition 4.1). If R is not Noetherian, $D(I)$ may be an affine space even if $IT_R(I) \neq T_R(I)$. The main result of this section is a generalization of the characterization of the affineness of the open subspace $D(I)$ in the non (necessarily) Noetherian setting, by using the Kaplansky transform $\Omega_R(I)$ instead of the Nagata transform $T_R(I)$ (Theorem 4.4).

1 SPECTRAL LOCALIZING SYSTEMS

Let R be an integral domain with quotient field K. A *multiplicative system (of ideals) of* R is a set $\mathcal{S}$ of integral ideals of R closed under multiplication. The overring

(1.1) $$R_{\mathcal{S}} := \{x \in K : xI \subseteq R \text{ for some } I \in \mathcal{S}\} = \cup\{(R : I) : I \in \mathcal{S}\}$$

is called the *generalized transform* or the *(generalized) ring of fractions of* R *with respect to* $\mathcal{S}$ (cf. [HOP], [H], [AB], [BS]). Note that, if $\mathcal{S}$ is a multiplicative system of

R, then also

$$\overline{\mathcal{S}} := \{J : J \text{ an ideal of } R \text{ such that } J \supseteq I \text{ for some } I \in \mathcal{S}\} \tag{1.2}$$

is a multiplicative system of R, called the *saturation of* $\mathcal{S}$, and, moreover

$$R_{\overline{\mathcal{S}}} = R_{\mathcal{S}} \ . \tag{1.3}$$

Obviously, a *saturated multiplicative system* is a multiplicative system $\mathcal{S}$ such that $\mathcal{S} = \overline{\mathcal{S}}$. We will concentrate our attention on the case where $\mathcal{S}$ is *non-trivial* i.e. $\mathcal{S} \neq \emptyset$ and $(0) \notin \mathcal{S}$ (or, equivalently, $\overline{\mathcal{S}}$ is a non–trivial subset of ideals of R).

Given a multiplicative system of ideals $\mathcal{S}$ of R and a fractional ideal I of R, we set

$$I_{\mathcal{S}} := \cup\{(I : J) : J \in \mathcal{S}\} \ . \tag{1.4}$$

It is easy to see that $I_{\mathcal{S}}$ is a fractional ideal of $R_{\mathcal{S}}$ and

$$IR_{\mathcal{S}} \subseteq I_{\mathcal{S}} \ . \tag{1.5}$$

LEMMA 1.1. *Let $\mathcal{S}$ be a multiplicative system of ideals of R and $\overline{\mathcal{S}}$ the saturation of $\mathcal{S}$. Set $T := R_{\overline{\mathcal{S}}}$ and $\mathcal{T} := \{JT : J \in \mathcal{S}\}$, then $\mathcal{T}$ is a multiplicative system of ideals of T. Let $\overline{\mathcal{T}}$ be the saturation of $\mathcal{T}$ and set:*

$$\nabla(\overline{\mathcal{S}}) := \{P \in \mathrm{Spec}(R) : P \notin \overline{\mathcal{S}}\} \ , \quad \nabla(\overline{\mathcal{T}}) := \{Q \in \mathrm{Spec}(T) : Q \notin \overline{\mathcal{T}}\} \ .$$

Then

(1) *If I is an ideal of R*

$$I_{\mathcal{S}} = R_{\mathcal{S}} \Leftrightarrow I \in \overline{\mathcal{S}} \ .$$

(2) *The map*

$$\nabla(\overline{\mathcal{S}}) \longrightarrow \nabla(\overline{\mathcal{T}}) \ , \quad P \mapsto P_{\mathcal{S}}$$

is an order-preserving bijection, with inverse map $Q \mapsto Q \cap R$.

Moreover, $R_P = T_{P_{\mathcal{S}}}$ for each $P \in \nabla(\overline{\mathcal{S}})$.

Proof: [Ga, Lemma 1.1] and [AB, Theorem 1.1].

□

We will say that a *multiplicative system of ideals* $\mathcal{S}$ of R is *finitely generated* if, for each $I \in \mathcal{S}$, there exists a finitely generated ideal J of R such that $J \subseteq I$ and $J \in \mathcal{S}$.

A distinguished class of multiplicative systems of ideals is given by the localizing (or topologizing) system of ideals introduced by Gabriel [Gb] (cf. also [B, Ch. 2 p.157], [P] and [St]). We recall that a *localizing system (of ideals) $\mathcal{F}$ of R* is a set of integral ideals of R verifying the following conditions:

(LS1) $I \in \mathcal{F}$ and $I \subseteq J \Rightarrow J \in \mathcal{F}$;
(LS2) $I \in \mathcal{F}$, J an ideal of R such that $(J :_R iR) \in \mathcal{F}$, for each $i \in I, \Rightarrow J \in \mathcal{F}$.

To avoid uninteresting cases, we will consider in general only non trivial localizing systems, i.e. localizing systems $\mathcal{F}$ such that $\mathcal{F} \neq \emptyset$ and $(0) \notin \mathcal{F}$.

Since a localizing system $\mathcal{F}$ of R is a saturated multiplicative system of ideals of R [FHP, Proposition 5.1.11], we can consider the (generalized) ring of fractions $R_{\mathcal{F}}$ of R with respect to $\mathcal{F}$. It can be shown that an overring of R can be a (generalized) ring of fractions with respect to a multiplicative system, but not with respect to a localizing system (Example 1.18; cf. also [FP3, Theorem 2.5]).

If P is a prime ideal of R, we set

$$\mathcal{F}(P) := \{I : I \text{ an ideal of } R,\ I \nsubseteq P\}\ , \tag{1.6}$$

then it is easy to see that $\mathcal{F}(P)$ is a localizing system of R and $R_{\mathcal{F}(P)} = R_P$ [FHP, (5.1d)]. More generally, for each nonempty set Δ of prime ideals of R, we can consider

$$\mathcal{F}(\Delta) := \cap\{\mathcal{F}(P) : P \in \Delta\}\ . \tag{1.7}$$

Since the intersection of localizing systems is a localizing system, we have [FHP, Proposition 5.1.4]:

LEMMA 1.2. *For each nonempty set Δ of prime ideals of an integral domain R, $\mathcal{F}(\Delta)$ is a localizing system of R and $R_{\mathcal{F}(\Delta)} = \cap\{R_P : P \in \Delta\}$.*

□

A localizing system $\mathcal{F}$ of R is called a *spectral localizing system of R* if $\mathcal{F} = \mathcal{F}(\Delta)$ for some set Δ of prime ideals of R. Not every localizing system is spectral (Example 1.10).

LEMMA 1.3. *Let Δ be a nonempty subset of* Spec(R). *If we denote by*

$$\Delta^{\downarrow} := \{Q \in \mathrm{Spec}(R) : Q \subseteq P \text{ for some } P \in \Delta\} \tag{1.8}$$

the closure under generizations of Δ inside Spec(R), *then for each set Λ, with $\Delta \subseteq \Lambda \subseteq \Delta^{\downarrow}$, we have*

$$\mathcal{F}(\Lambda) = \mathcal{F}(\Delta) \quad (\textit{in particular, } R_{\mathcal{F}(\Lambda)} = R_{\mathcal{F}(\Delta)})\ . \tag{1.9}$$

Proof: Since $\Delta \subseteq \Lambda$, then $\mathcal{F}(\Lambda) \subseteq \mathcal{F}(\Delta)$. If $I \in \mathcal{F}(\Delta)$, then $I \nsubseteq P$ for each $P \in \Delta$, thus *a fortiori* $I \nsubseteq Q$ for each $Q \in \Lambda$, because $\Lambda \subseteq \Delta^{\downarrow}$. Therefore, $I \in \mathcal{F}(\Lambda)$.

□

By standard arguments on partially ordered sets [AM, Proposition 6.1], we have:

LEMMA 1.4. *The following conditions are equivalent:*
(i) *each nonempty subset Δ of* Spec(R) *has a maximal element;*
(ii) *R satisfies the* acc *on prime ideals.*

□

LEMMA 1.5. *Let R be an integral domain satisfying the* acc *on prime ideals. If Δ is a nonempty subset of* Spec(R) *and if Δ_0 is the (nonempty) subset consisting of the maximal elements of Δ, then*

$$\mathcal{F}(\Delta) = \mathcal{F}(\Delta_0)\ .$$

Proof: Since $\Delta^{\downarrow} = \Delta_0^{\downarrow}$, the conclusion follows from Lemma 1.3.

□

If we start from a nontrivial localizing system $\mathcal{F}$ of R, we can associate to $\mathcal{F}$ the following nonempty subset of Spec(R):

$$\nabla(\mathcal{F}) := \{P \in \mathrm{Spec}(R) : P \notin \mathcal{F}\}\ . \tag{1.10}$$

It is easy to see that $\nabla(\mathcal{F}) = \nabla(\mathcal{F})^{\downarrow}$ and [FHP, (5.1e)]:

LEMMA 1.6. *For each nontrivial localizing system $\mathcal{F}$ of R,*

$$\mathcal{F} \subseteq \mathcal{F}(\nabla(\mathcal{F})) .$$

□

Conversely, if we start from a nonempty subset Δ of $\mathrm{Spec}(R)$, we have:

LEMMA 1.7. *Let $\mathcal{F} = \mathcal{F}(\Delta)$ be a spectral localizing system of an integral domain R, then*
(1) $\nabla(\mathcal{F}) = \Delta^{\downarrow}$;
(2) $\mathcal{F} = \mathcal{F}(\nabla(\mathcal{F}))$.

Proof: (1) If we show that $\Delta \subseteq \nabla(\mathcal{F}) \subseteq \Delta^{\downarrow}$, then the conclusion will follow because $\nabla(\mathcal{F}) = \nabla(\mathcal{F})^{\downarrow}$. Since, for each prime ideal P, $P \notin \mathcal{F}(P)$, then it is clear that

$$P \in \Delta \Rightarrow P \notin \mathcal{F}(\Delta) \Rightarrow P \in \nabla(\mathcal{F}) .$$

Moreover, if $Q \in \nabla(\mathcal{F})$ then $Q \notin \mathcal{F} = \cap\{\mathcal{F}(P) : P \in \Delta\}$, hence there exists $P \in \Delta$ such that $Q \notin \mathcal{F}(P)$. Therefore $Q \subseteq P$ for some $P \in \Delta$.
(2) follows from (1) and Lemma 1.3.

□

COROLLARY 1.8. *For each integral domain R, the map*

$$\{\mathcal{F} : \mathcal{F} \text{ is a spectral localizing system of } R\} \to \{\Delta \subseteq \mathrm{Spec}(R) : \Delta = \Delta^{\downarrow}\},\ \mathcal{F} \mapsto \nabla(\mathcal{F})$$

is an order–reversing bijection.

□

Given a spectral localizing system $\mathcal{F} = \mathcal{F}(\Delta)$, we say that Δ *is irredundant for* $\mathcal{F}$ if $\mathcal{F}(\Delta) \neq \mathcal{F}(\Delta \setminus P)$ for each $P \in \Delta$.

We say that $\mathcal{F}$ is an *irredundant spectral localizing system* if there exists $\Delta \subseteq \mathrm{Spec}(R)$ such that $\mathcal{F} = \mathcal{F}(\Delta)$ and Δ is irredundant for $\mathcal{F}$. A nonempty subset Δ of $\mathrm{Spec}(R)$ is called 0-*dimensional* if, for each pair of distinct prime ideals P and Q in Δ, $P \not\subseteq Q$.

EXAMPLE 1.9. *A spectral localizing system which is not irredundant.*

Let V be a valuation domain, having the following prime spectrum:

$$(0) = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots \subset M = \cup\{P_n : n \geq 0\} .$$

Let $\mathcal{F} = \cap\{\mathcal{F}(P_n) : n \geq 0\} = \{M, V\}$. Clearly $\mathcal{F}$ is a spectral localizing system of V and $\nabla(\mathcal{F}) = \mathrm{Spec}(V) \setminus \{M\}$. It is easy to see that $\mathcal{F}$ is not irredundant.

EXAMPLE 1.10. *A localizing system which is not spectral.*

Let V be a n–dimensional valuation domain, with $n \geq 2$. Suppose that the maximal ideal M of V is idempotent. Then $\mathcal{F} = \{M, V\}$ is a non–spectral localizing system of V, since $\mathcal{F}(M) = \{V\}$ and $\mathcal{F}(P) \supsetneqq \{M, V\}$ for each prime ideal $P \neq M$.

The following result is due to S. Gabelli [Ga]:

LEMMA 1.11. *Let Δ be a nonempty subset of* Spec(R)*, ordered under the set theoretical inclusion. Then Δ is irredundant for $\mathcal{F}(\Delta)$ if and only if Δ is 0–dimensional.*

□

COROLLARY 1.12. *Let $\mathcal{F}$ be a nontrivial localizing system of R. Then $\mathcal{F}$ is an irredundant spectral localizing system if and only if each prime ideal of $\nabla(\mathcal{F})$ is contained in a maximal element (of $\nabla(\mathcal{F})$).*

Proof: By Lemma 1.11, $\mathcal{F}$ is an irredundant spectral localizing system if and only if there exists a 0–dimensional subset Δ of Spec(R) such that $\mathcal{F} = \mathcal{F}(\Delta)$. Since $\Delta^{\downarrow} = \nabla(\mathcal{F})$ (Lemma 1.7), then each $Q \in \nabla(\mathcal{F})$ is contained in some $P \in \Delta \subseteq \nabla(\mathcal{F})$. Therefore Δ necessarily coincides with the set of maximal elements of $\Delta(\mathcal{F})$. Conversely, if ∇_0 is the set of maximal elements of $\nabla(\mathcal{F})$, ∇_0 is obviously a 0–dimensional subset of Spec(R) and $\mathcal{F}(\nabla_0) = \mathcal{F}(\nabla(\mathcal{F})) = \mathcal{F}$ (Lemma 1.7 (2)).

□

COROLLARY 1.13. *If R is an integral domain with* acc *on prime ideals, then each spectral localizing system is irredundant.*

Proof: This is an easy consequence of Corollary 1.12 and Lemma 1.4.

□

A relevant class of irredundant spectral localizing systems is the class of the finitely generated localizing systems. We recall that a *localizing system $\mathcal{F}$ is finitely generated* if, for each $I \in \mathcal{F}$, there exists a finitely generated ideal $J \in \mathcal{F}$ with $J \subseteq I$. Every finitely generated localizing system is spectral, in fact:

LEMMA 1.14. *Let $\mathcal{F}$ be a localizing system of R. The following statements are equivalent:*

(i) *$\mathcal{F}$ is a spectral localizing system of R;*

(ii) *$\mathcal{F} = \cap\{\mathcal{F}_\alpha : \alpha \in A\}$, where $\mathcal{F}_\alpha$ is a finitely generalized localizing system of R, for each $\alpha \in A$;*

(iii) *for each ideal I of R with $I \notin \mathcal{F}$, there exists a prime ideal P of R such that $I \subseteq P$ and $P \notin \mathcal{F}$.*

Proof: [FHP, Proposition 5.1.7].

□

From the previous lemma we can deduce the following:

COROLLARY 1.15. *If $\mathcal{F}$ is a finitely generated localizing system, then $\mathcal{F}$ is a spectral irredundant localizing system.*

Proof: From Lemma 1.14 ((ii) $\Rightarrow$ (i)), it is obvious that $\mathcal{F}$ is spectral and, from Lemma 1.14 ((ii) $\Rightarrow$ (iii)), we reobtain that $\mathcal{F} = \mathcal{F}(\nabla(\mathcal{F}))$ (cf. also Lemma 1.7). The conclusion will follow from Corollary 1.12, if we show that every chain of prime ideals of $\nabla(\mathcal{F})$ has an upper bound (in $\nabla(\mathcal{F})$). In fact, if $\{P_\lambda : \lambda \in \Lambda\}$ is a chain in $\nabla(\mathcal{F})$ and if $\cup_\lambda P_\lambda \in \mathcal{F}$, then there exists a finitely generated ideal $J \subseteq \cup_\lambda P_\lambda$ with $J \in \mathcal{F}$. It follows easily that $J \subseteq P_{\tilde{\lambda}}$ for some $\tilde{\lambda} \in \Lambda$, and hence $P_{\tilde{\lambda}} \in \mathcal{F}$: a contradiction.

□

We note that not every irredundant spectral localizing system is finitely generated. This fact is a consequence of the following:

LEMMA 1.16. *Let $\mathcal{F}$ be a localizing system of an integral domain R. The following are equivalent:*

(i) *$\mathcal{F}$ is a finitely generated localizing system;*

(ii) *there exists a quasi-compact subspace Δ (i.e., every open covering of Δ has a finite sub-covering) of* $\mathrm{Spec}(R)$ *such that $\mathcal{F} = \mathcal{F}(\Delta)$;*

(iii) *each prime ideal of $\nabla(\mathcal{F})$ is contained in a prime ideal of $\nabla_0 := \{P \in \nabla(\mathcal{F}) :$ P is a maximal element of $\nabla(\mathcal{F})\}$ and ∇_0 is a quasi-compact subspace of* $\mathrm{Spec}(R)$.

Proof: [FHP, Proposition 5.1.8].

□

EXAMPLE 1.17. *An irredundant spectral localizing system which is not finitely generated.*

Let R be a 1-dimensional Prüfer domain with $\mathrm{Max}(R) = \{M_n : n \geq 0\}$, where M_n is principal for each $n \geq 1$ and M_0 is not the radical of a finitely generated ideal (for an explicit example cf. [FHP, Theorems 8.2.1, 8.2.2 and 4.1.6]). Let $\Delta := \mathrm{Max}(R) \setminus \{M_0\}$. Then, clearly $\mathcal{F} := \mathcal{F}(\Delta)$ is an irredundant localizing system of R, since Δ is 0-dimensional (Lemma 1.11). But $\mathcal{F}$ is not finitely generated, since $M_0 \in \mathcal{F}$ because $M_0 \not\subseteq M_n$ for each $n \geq 1$; moreover, if $I \in \mathcal{F}$ and I is finitely generated with $I \subseteq M_0$, then $I \not\subseteq M_n$ for each $n \geq 1$, because $\mathcal{F} = \cap\{\mathcal{F}(M_n) : n \geq 1\}$, hence $M_0 = \mathrm{rad}(I)$ and we reach a contradiction. In a different terminology, we can say that Δ coincides with $\nabla(\mathcal{F}) \setminus \{0\}$ and it is a 0-dimensional non quasi-compact subspace of $\mathrm{Spec}(R)$, because each point $M_n \in \Delta$ is open and closed in the Zariski topology of $\mathrm{Spec}(R)$, for $n \geq 1$.

For each overring T of R, we can consider

$$\Lambda = \Lambda(T) := \{N \cap R : N \in \mathrm{Max}(T)\} \tag{1.11}$$

which is a quasi-compact subspace of $\mathrm{Spec}(R)$, being the continuous image of a quasi-compact space. Therefore we can consider the finitely generated (irredundant spectral) localizing system $\mathcal{F}(\Lambda)$ of R. Since $R_{N\cap R} \subseteq T_N$, for each $N \in \mathrm{Max}(T)$, it follows easily that:

$$R_{\mathcal{F}(\Lambda)} \subseteq T\,. \tag{1.12}$$

It is natural to ask when $R_{\mathcal{F}(\Lambda)} = T$. This is a particular case of the question of when an overring is an intersection of localizations. We introduce some terminology.

A *QR-overring* (respectively: *QQR-overring*; *GQR-overring*; *$\mathcal{F}QR$-overring*) T of an integral domain R is an overring such that $T = R_S$ (respectively: $T = \cap\{R_P : P \in Y\}$; $T = R_{\mathcal{S}}$; $T = R_{\mathcal{F}}$) for some multiplicative set S of elements of R (respectively: for some subset Y of $\mathrm{Spec}(R)$; for some multiplicative system $\mathcal{S}$ of ideals of R; for some localizing system $\mathcal{F}$ of R). We call a **P**-*domain* an integral domain for which every overring is a **P**-overring, where $\mathbf{P} \in \{QR, QQR, GQR, \mathcal{F}QR\}$. It is rather obvious that:

$$QR\text{-domain} \Rightarrow QQR\text{-domain} \Rightarrow \mathcal{F}QR\text{-domain} \Rightarrow GQR\text{-domain}$$

and it is well known that:

$$\text{Bézout domain} \Rightarrow QR\text{-domain} \Rightarrow \text{Prüfer domain} \Rightarrow QQR\text{-domain}$$

(cf. [G4], [GO], [R], [GH], [M], [Pe] and [D]). Moreover, in the integrally closed case, Prüfer domains, QQR-domains, $\mathcal{F}QR$-domains and GQR-domains coincide (cf. [H, Theorem 2.4], [FP2, Corollary 2.7] and [He3]).

In [FP2, Proposition 2.10], it is proved that each overring of a $\mathcal{F}QR$–domain is still a $\mathcal{F}QR$–domain. The following example, due to Heinzer [H, Example 2.9], shows not only that an overring of a GQR–domain is not necessarily a GQR–domain, as Heinzer proved, but also that there may exist an overring T of an integral domain R such that $T = R_{\mathcal{S}}$, for some multiplicative system of ideals $\mathcal{S}$ of R, but $T \neq R_{\mathcal{F}}$ for all localizing systems $\mathcal{F}$ of R.

EXAMPLE 1.18. *A (generalized) ring of fractions $R_{\mathcal{S}}$ of an integral domain R with respect to a multiplicative system of ideals $\mathcal{S}$ of R, which is not a (generalized) ring of fractions $R_{\mathcal{F}}$ of R with respect to some localizing system $\mathcal{F}$ of R.*

Let $k \subset K$ be a proper minimal extension of fields and let Y be an indeterminate over K. Pick a countable family of elements $x_1 = Y, x_2, x_3, \dots$ in $K[[Y]]$ algebraically independent over K and set $F := K(x_1, x_2, x_3, \dots, x_n, \dots)$. We can consider

$$V_0 := K[[Y]] \cap F = K + M_0, \quad \text{with } M_0 := YK[[Y]] \cap F$$

which is a 1–dimensional discrete valuation domain with quotient field F. Set

$$\begin{aligned} K_n &:= K(x_1, x_2, \dots, x_n) \\ F_{\check{n}} &:= K(\{x_i : i \neq n \ \ i \geq 1\}) \\ W_n &:= K[[Y]] \cap F_{\check{n}} = V_0 \cap F_{\check{n}} \,. \end{aligned}$$

Let V_n be the 1–dimensional valuation domain of F, associated to the valuation v_n, obtained extending to F the valuation w_n (associated to W_n) by setting $v_n(x_n) = \pi$, where π is a positive irrational number. It is easy to see that $V_0 \notin \{V_n : n \geq 1\}$ and $V_0 \cap K_n = V_m \cap K_n$ if $m > n$. Let

$$R_0 := k + M_0\,, \quad S := \cap\{V_n : n \geq 1\}\,, \quad R := R_0 \cap S\,, \quad T := V_0 \cap S\,.$$

It is not difficult to prove that the integral closure of R is T, which is a 1–dimensional Prüfer domain. Furthermore, $M_0 = (R_0 : V_0)$, $V_0 = (M_0 : M_0) = (R_0 : M_0)$ and if $\mathcal{S}_0 := \{M_0^k : k \geq 1\}$ then $(R_0)_{\mathcal{S}_0} = F$, since $\cap_k M_0^k = (0)$. Therefore, V_0 is not a GQR–overring of R_0, since every proper saturated multiplicative system of ideals of R_0 contains $\mathcal{S}_0$. However, Heinzer proved that R is a GQR–domain, even though $R_0 = R_{M_0 \cap R}$ is not GQR–domain. By [FP2, Proposition 2.10], we obtain that R is not a $\mathcal{F}QR$–domain because $R_0 = R_{M_0 \cap R}$ is not a $\mathcal{F}QR$–domain. More explicitly, set $P_0 := M_0 \cap R$, then

Claim. *$V_0 = R_{\mathcal{S}}$, where $\mathcal{S}$ is the multiplicative system of ideals of R given by $\{P_0^k I :$ I ideal of R with $I \not\subseteq P_0$ and $k \geq 0\}$, but $V_0 \neq R_{\mathcal{F}}$ for each localizing system $\mathcal{F}$ of R.*

The first part of the claim is proved in [H, p. 147]. Suppose that $V_0 = R_{\mathcal{F}}$. Note that $P_0 \in \mathcal{F}$, otherwise we would have $R_{P_0} = V_0$ (Lemma 1.1 (2)). Let

$$\mathcal{G} := \{J \text{ is an ideal of } R_0 : J \supseteq IR_0 \text{ for some } I \in \mathcal{F}\}\,.$$

It is easy to prove that $\mathcal{G}$ is a localizing system of R_0. In fact, if H is an ideal of R_0 and $I,\ I' \in \mathcal{F}$ are such that $(H :_{R_0} iR_0) \supseteq I'R_0$ for each $i \in I$, then $((H \cap R) :_R iR) = (H :_{R_0} iR_0) \cap R \supseteq I'R_0 \cap R \supseteq I'$, hence $H \cap R \in \mathcal{F}$ and thus $H \in \mathcal{G}$. Moreover, $(R_0)_{\mathcal{G}} \subseteq R_{\mathcal{F}} = V_0$, because if $xIR_0 \subseteq R_0$ for some $I \in \mathcal{F}$, then $xI \subseteq (xIR_0) \cap R \subseteq R$

and hence $x \in R_{\mathcal{F}}$. On the other hand, since $P_0 \in \mathcal{F}$, we have that $P_0R_0 = M_0 \in \mathcal{G}$ and thus $(R_0)_{\mathcal{G}} = F$ and this fact leads to a contradiction.

We return to the problem of when the equality holds in (1.12). For this purpose, we consider the following relevant localizing system of R associated to an overring T of R:

$$\mathcal{F}_0 = \mathcal{F}_0(T) := \{I : I \text{ ideal of } R \text{ such that } IT = T\} \ . \tag{1.13}$$

It is easy to see that $\mathcal{F}_0$ is a finitely generated (spectral irredundant) localizing system of R and, if $\Lambda := \Lambda(T)$, $\mathcal{F}_0 \subseteq \mathcal{F}(\Lambda)$, hence in particular

$$R_{\mathcal{F}_0} \subseteq R_{\mathcal{F}(\Lambda)} \subseteq T \ . \tag{1.14}$$

LEMMA 1.19. *Let $\mathcal{F}$ be a given localizing system of R. Set $T := R_{\mathcal{F}}$, $\Lambda := \Lambda(T)$, $\nabla := \nabla(\mathcal{F})$ and $\mathcal{F}_0 := \mathcal{F}_0(T)$.*
(1) *If $\mathcal{F}$ is a spectral localizing system of R, then*

$$\mathcal{F}(\Lambda) \subseteq \mathcal{F} \quad (\textit{in particular, } \mathcal{F}_0 \subseteq \mathcal{F}) \ .$$

(2) *If $\mathcal{F} = \mathcal{F}_0$ then $\mathcal{F}(\Lambda) = \mathcal{F}$.*

Proof: By Lemma 1.1, each $P \in \nabla$ is contained in some $Q := N \cap R \in \Lambda$, where $N \in \operatorname{Max}(T)$. Therefore $\nabla \subseteq \Lambda^{\downarrow}$ and thus, by Lemma 1.3 and 1.7,

$$\mathcal{F}(\Lambda) = \mathcal{F}(\Lambda^{\downarrow}) \subseteq \mathcal{F}(\nabla) = \mathcal{F} \ .$$

Since we have already observed that, in general, $\mathcal{F}_0 \subseteq \mathcal{F}(\Lambda)$, then clearly $\mathcal{F}_0 \subseteq \mathcal{F}$ and, if $\mathcal{F}_0 = \mathcal{F}$, we obviously have $\mathcal{F}(\Lambda) = \mathcal{F}$.

□

From the previous results, we easily deduce a sufficient condition for equality in (1.12):

COROLLARY 1.20. *Let T be an overring of an integral domain R. Set $\Lambda := \Lambda(T)$ and $\mathcal{F}_0 := \mathcal{F}_0(T)$. The following conditions are equivalent:*
(i) *$R_{\mathcal{F}_0} = T$;*
(ii) *T is R–flat;*
(iii) *$\mathcal{F}_0 = \mathcal{F}(\Lambda)$ and $R_{\mathcal{F}(\Lambda)} = T$.*

Proof: The equivalence (i) $\Leftrightarrow$ (ii) is well known (cf. [FHP, Remark 5.1.11 (b)] and also [Ak1] and [AB]).

(iii) $\Rightarrow$ (i) is trivial.

(ii) $\Rightarrow$ (iii). By [R, Theorem 2], we deduce that $R_{\mathcal{F}(\Lambda)} = T$. Since we already know that (i) $\Leftrightarrow$ (ii), then $T = R_{\mathcal{F}_0} = R_{\mathcal{F}(\Lambda)}$. The conclusion follows by applying Lemma 1.19 to the (spectral) localizing system $\mathcal{F}_0$.

□

In order to make the references easier, in the next result due to Gabelli [Ga], we will collect some equivalent conditions, each of them will imply, in particular, the equality in (1.12).

PROPOSITION 1.21. *Let $\mathcal{S}$ be a nontrivial multiplicative system of ideals of an integral domain R. Set $T := R_{\mathcal{S}}$, $\Lambda := \Lambda(T)$, $\mathcal{F}_0 := \mathcal{F}_0(T)$, $\nabla := \nabla(\overline{\mathcal{S}})$ and let ∇_0 be the set of maximal elements inside ∇. The following conditions are equivalent:*

(i) *$IT = I_{\mathcal{S}}$, for each ideal I of R;*
(ii) *$JT = T$, for each ideal $J \in \mathcal{S}$;*
(iii) *$\overline{\mathcal{S}} = \mathcal{F}_0$;*
(iv) *$\overline{\mathcal{S}} = \mathcal{F}(\Lambda)$;*
(v) *$\nabla_0^{\downarrow} = \nabla$ and $\mathrm{Max}(T) = \{QT : Q \in \nabla_0\}$;*
(vi) *$\nabla_0 = \Lambda$;*
(vii) *for each ideal H of T, $H = (H \cap R)T = (H \cap R)_{\mathcal{S}}$;*
(viii) *$\mathrm{Spec}(T) = \{PT : P \in \nabla\}$.*

In particular, when these conditions are satisfied T is R-flat.

Proof: In [Ga, Proposition 1.2] the equivalence of all the conditions except (vi) is proved. It is obvious that (v) $\Rightarrow$ (vi) (cf. also Lemma 1.1).

(vi) $\Rightarrow$ (iv). We have already noticed that $I \in \overline{\mathcal{S}}$ if and only if $I_{\mathcal{S}} = T$ (Lemma 1.1 (1)). Let I be an ideal of R with $I \notin \overline{\mathcal{S}}$. Then $I_{\mathcal{S}} \subseteq N$ for some $N \in \mathrm{Max}(T)$, whence $I \subseteq N \cap R$ and thus $I \notin \mathcal{F}(\Lambda) = \mathcal{F}(\nabla_0)$. Conversely, if $I \notin \mathcal{F}(\nabla_0) = \mathcal{F}(\Lambda)$ then $I \subseteq N \cap R$ for some $N \in \mathrm{Max}(T)$. This fact implies that $I_{\mathcal{S}} \subseteq (N \cap R)_{\mathcal{S}}$. Since $N \cap R \in \Lambda = \nabla_0$, $(N \cap R)_{\mathcal{S}}$ is a prime ideal of T (Lemma 1.1 (2)). Therefore $I_{\mathcal{S}} \neq T$, whence $I \notin \overline{\mathcal{S}}$ (Lemma 1.1 (1)).

□

EXAMPLE 1.22. *A (non finitely generated) localizing system $\mathcal{F}$ and a prime ideal P in an integral domain R such that $PR_{\mathcal{F}} \neq P_{\mathcal{F}}$.*

Let V, M and $\mathcal{F}$ be as in Example 1.10. Then, in this case, $V_{\mathcal{F}} = (V : M) = V = (M : M) = M_{\mathcal{F}}$ and $MV_{\mathcal{F}} = MV = M$. Note that $M \notin \nabla(\mathcal{F})$ (Lemma 1.1).

COROLLARY 1.23. *Let $\mathcal{F}$ be a localizing system of an integral domain R. Set $T := R_{\mathcal{F}}$, $\Lambda := \Lambda(T)$, $\mathcal{F}_0 := \mathcal{F}_0(T)$. The following conditions are equivalent:*

(i) *$\mathcal{F}_0 = \mathcal{F}$;*
(ii) *$\mathcal{F}(\Lambda) = \mathcal{F}$;*
(iii) *$\mathcal{F}_0 = \mathcal{F}(\Lambda) = \mathcal{F}$.*

Proof: This is an easy consequence of Proposition 1.21.

□

We will see later (Example 4.7) that $\mathcal{F}_0$ may be equal to $\mathcal{F}(\Lambda)$, but $\mathcal{F}_0 \subsetneqq \mathcal{F}$.

The following corollary is also due to Gabelli [Ga, Theorem 1.3]:

COROLLARY 1.24. *Let R be a Prüfer domain and let $\mathcal{S}$ be a multiplicative system of ideals of R. Set $T := R_{\mathcal{S}}$ and $\mathcal{F}_0 := \mathcal{F}_0(T)$. Then:*

$$\mathcal{S} \textit{ is finitely generated if and only if } \overline{\mathcal{S}} = \mathcal{F}_0 .$$

Therefore, in the Prüfer case, conditions (i)-(viii) *of Proposition 1.21 are equivalent to the following:*

(ix) *$\mathcal{S}$ is finitely generated.*

□

As a consequence of the previous corollary, we recover the following result proved in [FHP, Theorem 5.1.15]:

COROLLARY 1.25. *Let R be a Prüfer domain. For each overring T of R there exists a unique non trivial finitely generated localizing system $\mathcal{F}$ of R such that $T = R_{\mathcal{F}}$.*

□

2 FLATNESS, FINITENESS AND LOCALIZING SYSTEMS

In some relevant cases, the link between flatness and the finitely generated property for overrings can be studied by using localizing systems. This point of view was developed by E.L. Popescu [Po], in order to extend some results of Schentzel [S3]. We begin by recalling the following characterizations of flat overrings:

LEMMA 2.1. *Let T be an overring of an integral domain R. The statements* (i)-(iii) *of Corollary 1.20 are equivalent to the following statements:*
(iv) *$(R :_R yR)T = T$, for each $y \in T$;*
(v) *for each $Q \in \mathrm{Spec}(T)$, $R_{Q \cap R} = T_Q$;*
(vi) *$P \in \mathrm{Supp}_R(T/R) \Rightarrow PT = T$.*

Proof: The equivalences (ii) $\Leftrightarrow$ (iv) $\Leftrightarrow$ (v) are well known and proved in [R, Theorems 1 and 2].

(v) $\Rightarrow$ (vi). Let $P \in \mathrm{Supp}_R(T/R)$ and assume that $PT \neq T$. Let $\overline{Q} \in \mathrm{Spec}(T)$ such that $\overline{Q} \supseteq PT$ and let $\overline{P} := \overline{Q} \cap R$. By assumption, we deduce that $R_{\overline{P}} = T_{R \smallsetminus \overline{P}} = T_{\overline{Q}}$ and thus $(T/R)_{\overline{P}} = 0$, hence $\overline{P} \notin \mathrm{Supp}_R(T/R)$. This leads to a contradiction since $P \subseteq \overline{P}$ and $(T/R)_P \neq 0$.

(vi) $\Rightarrow$ (v). Assume that there exists a prime ideal Q of T such that $R_P \neq T_Q$, where $P := Q \cap R$. This fact implies that $R_P \neq T_{R \smallsetminus P}$ and thus $P \in \mathrm{Supp}_R(T/R)$. By assumption it follows that $PT = T$ and this is a contradiction, since $PT \subseteq Q$.

□

COROLLARY 2.2. *Let T be an overring of an integral domain R. Then*

$$T \textit{ is } R\textit{-flat} \Rightarrow \mathrm{Supp}_R(T/R) = \{P \in \mathrm{Spec}(R) : PT = T\} .$$

Proof: The inclusion $\mathrm{Supp}_R(T/R) \subseteq \{P \in \mathrm{Spec}(R) : PT = T\}$ follows from Lemma 2.1 ((ii) $\Rightarrow$ (vi)). Let $P \in \mathrm{Spec}(R)$ be such that $PT = T$. Assume that $P \notin \mathrm{Supp}_R(T/R)$. Then $R_P = T_{R \smallsetminus P}$ and thus $PR_P = PT_{R \smallsetminus P} \neq T_{R \smallsetminus P}$ and this contradicts the assumption $PT = T$.

□

COROLLARY 2.3. *Let R be a Noetherian domain and T an overring of R.*

$$T \textit{ is } R\textit{-flat} \Leftrightarrow \mathrm{Ass}_R(T/R) \subseteq \{P \in \mathrm{Spec}(R) : PT = T\} .$$

Proof: ($\Rightarrow$) is a consequence of Corollary 2.2 and of the fact that $\mathrm{Ass}_R(T/R) \subseteq \mathrm{Supp}_R(T/R)$, since R is Noetherian [B, Ch. IV § 1 N.3 Corollaire 1].

($\Leftarrow$). In order to prove that T is R-flat, we show that condition (vi) of Lemma 2.1 holds. Let $P \in \mathrm{Supp}_R(T/R)$. Since R is Noetherian, there exists a prime ideal $\overline{P} \subseteq P$ such that $\overline{P} \in \mathrm{Ass}_R(T/R)$ [B, Ch. IV § 1 N.3 Proposition 7]. By assumption, $\overline{P}T = T$ and thus $PT = T$.

□

REMARK 2.4. Corollary 2.3 can be generalized outside the Noetherian setting, by using the notion of weak associated prime [B, Ch. IV § 1 Exercise 17], since it is known that if E is a R–module, a prime ideal P of R belongs to $\mathrm{Supp}_R(E)$ if and only if contains a prime ideal $\overline{P}$ inside $\mathrm{Ass}^w_R(E)$, the set of all the weakly associated primes of E.

Mutatis mutandis, it can be shown that *if R is an integral domain and T is an overring of R, then*

$$T \textit{ is } R\textit{–flat} \Leftrightarrow \mathrm{Ass}^w_R(T/R) \subseteq \{P \in \mathrm{Spec}(R) : PT = T\}\ .$$

COROLLARY 2.5. *If R is a Noetherian domain and T an overring of R, then*

$$T \textit{ is } R\textit{–flat} \Rightarrow T \textit{ is Noetherian}\ .$$

Proof: From the equivalence (ii) $\Leftrightarrow$ (v) of Lemma 2.1, it follows that for each prime ideal Q of T, $(Q \cap R)T = Q$. Since R is Noetherian, $Q \cap R$ is finitely generated. Thus Q is finitely generated, hence T is Noetherian [K1, Theorem 8].

□

We have shown that a flat overring of a Noetherian domain is Noetherian, even if, in general, it is not finitely generated (e.g. the quotient field of an integral Noetherian domain which is not a *G–domain*, i.e. an integral domain R for which (0) is different from its *pseudo–radical* $(0)^* := \cap\{P : P \in \mathrm{Spec}(R),\ P \neq 0\}$ [K1, Theorem 18]). Our next goal is to characterize, in the general setting, the flat overrings that are finitely generated in terms of localizing systems.

Let $\mathcal{I} := \{I_\alpha : \alpha \in A\}$ be a given nonempty family of ideals of a domain R. Set

$$\mathcal{K}(\mathcal{I}) := \{J : J \text{ ideal of } R \text{ such that } \mathrm{rad}(J) \supseteq \prod_{k=1}^{n} I_{\alpha_k}, \quad \text{where } \{\alpha_1, \dots, \alpha_n\} \text{ is a finite subset of } A\}\ .$$

LEMMA 2.6. *Let $\mathcal{I} = \{\mathcal{I}_\alpha : \alpha \in A\}$ be a nonempty family of ideals of a domain R. Then*

(1) *$\mathcal{K}(\mathcal{I})$ is a saturated multiplicative system of ideals of R.*

(2) *If $\mathcal{I}$ is a finite family or if each ideal in $\mathcal{I}$ is finitely generated, then $\mathcal{K}(\mathcal{I})$ is a localizing system of R.*

(3) *If each ideal in $\mathcal{I}$ is finitely generated, then $\mathcal{K}(\mathcal{I})$ is a localizing system of R and*

$$\mathcal{K}(\mathcal{I}) = \{J : J \text{ ideal of } R \text{ such that } J \supseteq \prod_{k=1}^{n} I^e_{\alpha_k} \text{ where } \{\alpha_1, \dots, \alpha_n\} \text{ is a finite subset of } A \text{ and } e \geq 1\}\ .$$

Proof: (1) is an easy consequence of the fact that $\mathrm{rad}(J'J'') = \mathrm{rad}(J') \cap \mathrm{rad}(J'') \supseteq \mathrm{rad}(J')\mathrm{rad}(J'')$.

(2). It is obvious that $\mathcal{K}(\mathcal{I})$ satisfies condition **(LS1)** of the definition of localizing system. Let H be an ideal of R such that $(H :_R jR) \in \mathcal{K}(\mathcal{I})$, for each $j \in J$, where $J \in \mathcal{K}(\mathcal{I})$, we claim that $H \in \mathcal{K}(\mathcal{I})$.

Case 1: Each ideal in $\mathcal{I}$ is finitely generated.

If $\mathrm{rad}(J) \supseteq \prod_{k=1}^{n} I_{\alpha_k}$ and if $i_{\alpha_k} \in I_{\alpha_k}$, then by the finiteness assumption there exists $N \gg 0$ such that

$$\prod_{k=1}^{n} i_{\alpha_k}^{N} \in J \ .$$

Therefore, $\mathrm{rad}\left((H :_R (\prod_{k=1}^{n} i_{\alpha_k}^{N}) R)\right) \supseteq \prod_{h=1}^{m} I_{\beta_h}$ for some finite subset $\{\beta_1, \ldots, \beta_m\}$ of A. By the finiteness assumption, we can take the same set $\{\beta_1, \ldots, \beta_m\}$ for all elements $\prod_{k=1}^{n} i_{\alpha_k}^{N}$. If $i_{\beta_h} \in I_{\beta_h}$, then as above, there exists $M \gg 0$ such that

$$\prod_{h=1}^{m} i_{\beta_h}^{M} \prod i_{\alpha_k}^{N} \in H \ .$$

This fact implies that $\prod_{h=1}^{m} I_{\beta_h} \prod_{k=1}^{n} I_{\alpha_k} \subseteq \mathrm{rad}(H)$ and thus $H \in \mathcal{K}(\mathcal{I})$.

Case 2: $\mathcal{I} = \{I_1, I_2, \ldots, I_t\}$.

If $J \in \mathcal{K}(\mathcal{I})$ then $\mathrm{rad}(J) \supseteq \prod_k I_k$. Let $i_k \in I_k$ and let H be an ideal of R such that $\mathrm{rad}((H :_R \prod_k i_k R)) \supseteq \prod_k I_k$. We can find an integer $N \geq 0$ (depending on $\prod_k i_k$) such that $(\prod_k i_k)^N \in (H :_R \prod_k i_k R)$. We conclude that $(\prod_k i_k)^{N+1} \subseteq H$ and thus $\prod_k I_k \subseteq \mathrm{rad}(H)$.

(3) If $\mathrm{rad}(J) \supseteq \prod_{k=1}^{n} I_{\alpha_k}$ and I_{α_k} is a finitely generated ideal of R for each α_k, then there exists an integer $e \gg 0$ such that $J \supseteq (\prod_{k=1}^{n} I_{\alpha_k})^e$. Conversely, if $J \supseteq \prod_{k=1}^{n} I_{\alpha_k}^e$ then $\mathrm{rad}(J) \supseteq \mathrm{rad}\left(\prod_{k=1}^{n} I_{\alpha_k}^e\right) = \mathrm{rad}\left(\prod_{k=1}^{n} I_{\alpha_k}\right) \supseteq \prod_{k=1}^{n} I_{\alpha_k}$.

□

If $\mathcal{I}$ is the set of all the maximal ideals of R, the ring of fractions of R with respect to the saturated multiplicative system of ideals $\mathcal{K}(\mathcal{I})$ is the *global transform of* R. This ring was introduced in the case of Noetherian rings (not necessarily domains) by Matijevic [Ma] (cf. also [A] and [Zö]).

If $\mathcal{I} = \{I\}$, where I is an ideal of R, we denote simply by $\mathcal{K}(I)$ the localizing system $\mathcal{K}(\{I\})$ of R.

COROLLARY 2.7. *Let I be an ideal of an integral domain R.*
(1) $\mathcal{K}(I) = \{J : J$ is an ideal of R such that $\mathrm{rad}(I) = \mathrm{rad}(J \cap I)\}$.
(2) *If I is finitely generated, then:*

$$\mathcal{K}(I) = \{J : J \text{ ideal of } R \text{ such that } J \supseteq I^e \text{ for some } e \geq 1\} \ .$$

Proof: (1) follows from the fact that

$$\mathrm{rad}(I) = \mathrm{rad}(I \cap J) = \mathrm{rad}(I) \cap \mathrm{rad}(J) \Leftrightarrow \mathrm{rad}(I) \subseteq \mathrm{rad}(J)$$
$$\Leftrightarrow I \subseteq \mathrm{rad}(J)$$

(2) is a particular case of Lemma 2.6 (3).

□

PROPOSITION 2.8. *Let T be an overring of an integral domain R and let $\mathcal{I} := \{(R :_R yR) : y \in T\}$. Then*

$$T \text{ is } R\text{-flat} \Leftrightarrow \mathcal{F}_0(T) = \mathcal{K}(\mathcal{I}) \ .$$

Proof: $(\Rightarrow)$. Note that if J is an ideal of R then

$$\mathrm{rad}(J)T = T \Leftrightarrow JT = T \ .$$

Since T is R–flat, by Lemma 2.1 ((ii) $\Rightarrow$ (iv)) it follows that the saturated multiplicative system $\mathcal{K}(\mathcal{I})$ is contained in $\mathcal{F}_0(T)$. The conclusion follows from Proposition 1.21 ((ii) $\Rightarrow$ (iii)).

($\Leftarrow$). Since, for each $y \in T$, $(R :_R yR) \in \mathcal{K}(\mathcal{I})$ then, by assumption, $(R :_R yR) \in \mathcal{F}_0(T)$ and this is equivalent to $(R :_R yR)T = T$. The conclusion follows from Lemma 2.1 ((iv) $\Rightarrow$ (ii)).

□

The following result is due essentially to E.L. Popescu [Po, Theorem 5]:

PROPOSITION 2.9. *Let T be an overring of an integral domain R. Then the following statements are equivalent:*

(i) *T is flat and finitely generated over R;*

(ii) *there exists a (finiteiy generated) ideal I of R such that $\mathcal{F}_0(T) = \mathcal{K}(I)$ and $T = R_{\mathcal{K}(I)}$.*

Proof: (i) $\Rightarrow$ (ii). Let $T = R[y_1, \dots, y_n]$ and set

$$I := (R :_R y_1R)(R :_R y_2R)\cdots(R :_R y_nR)\ .$$

Since T is R–flat, then $IT = T$ (Lemma 2.1 ((ii) $\Rightarrow$ (iv))) thus $I \in \mathcal{F}_0(T)$ and, hence, $\mathcal{K}(I) \subseteq \mathcal{F}_0(T)$. Conversely if $JT = T$, then $\sum_{k=1}^{r} j_k t_k = 1$ with $j_k \in J$ and $t_k \in T$. Since each $t_k \in R[y_1, \dots, y_n]$ then there exists $N \geq 0$ such that $t_k I^N \subseteq R$ for each k, $1 \leq k \leq r$. Therefore

$$I^N = I^N \cdot 1 = I^N \sum_{k=1}^{r} j_k t_k = \sum_{k=1}^{r} j_k t_k I^N \subseteq J$$

and whence $J \in \mathcal{K}(I)$ (Corollary 2.7 (2)).

Furthermore, since we have proved that $\mathcal{K}(I) = \mathcal{F}_0(T)$ and T is R-flat then, by Corollary 1.20, $T = R_{\mathcal{F}_0(T)} = R_{\mathcal{K}(I)}$.

If I is not a finitely generated ideal of R, we can find a finitely generated ideal I_* such that $I_* \subseteq I$ and $I_* \in \mathcal{K}(I) = \mathcal{F}_0(T)$. This fact implies that $\mathcal{K}(I) \subseteq \mathcal{K}(I_*)$. On the other hand, since $I_* \in \mathcal{F}_0(T)$, it is easy to see that $\mathcal{K}(I_*) \subseteq \mathcal{F}_0(T)$. We conclude immediately that $\mathcal{K}(I) = \mathcal{K}(I_*) = \mathcal{F}_0(T)$.

(ii) $\Rightarrow$ (i). By the previous argument, we can assume that I is finitely generated. Let $I = (i_1, \dots, i_s)$. Since $I \in \mathcal{K}(I)$, then by assumption $IT = T$. Therefore $1 = \sum_{h=1}^{s} i_h y_h$ for $y_h \in T$. We claim that $R[y_1, \dots, y_s] = T$. We observe that, from the assumption, we have $T = R_{\mathcal{K}(I)} = R_{\mathcal{F}_0(T)}$ and hence T is R–flat (Corollary 1.20). Let $t \in T = R_{\mathcal{K}(I)}$, then $(R :_R tR) \in \mathcal{K}(I)$ and hence there exists $e \geq 1$ such that $I^e \subseteq (R :_R tR)$ (Corollary 2.7 (2)). On the other hand we know that $IR[y_1, \dots, y_s] = R[y_1, \dots, y_s]$. Thus also $I^e R[y_1, \dots, y_s] = R[y_1, \dots, y_s]$, hence

$$1 = \sum_{k=1}^{r} \iota_k z_k \quad \text{with } \iota_k \in I^e \text{ and } z_k \in R[y_1, \dots, y_s], \quad r \geq 1\ ,$$

and so

$$t = \sum_{k=1}^{r} (\iota_k t) z_k \in R[y_1, \dots, y_s]\ ,$$

because $\iota_k \in I^e$ and $I^e t \subseteq R$.

□

In the case of Noetherian domains, we can describe every localizing system in terms of localizing systems of type $\mathcal{K}(\mathcal{I})$, for some family of (prime) ideals $\mathcal{I}$.

LEMMA 2.10. *Let R be a Noetherian domain and $\mathcal{F}$ a localizing system of R. Then, there exists a quasi–compact subspace Δ of* $\mathrm{Spec}(R)$ *such that, if $\mathcal{I} := \{Q \in$* $\mathrm{Spec}(R) \setminus \Delta\}$*, then*

$$\mathcal{F} = \mathcal{F}(\Delta) = \mathcal{K}(\mathcal{I}) \ .$$

Proof: Since every localizing system of a Noetherian domain is finitely generated, then $\mathcal{F} = \mathcal{F}(\Delta)$ where $\Delta := \{P \in \mathrm{Spec}(R) : P \notin \mathcal{F}\}$ (Lemma 1.16 and Lemma 1.7). It is obvious that if $Q \in \mathcal{I} = \mathrm{Spec}(R) \setminus \Delta$ then $Q \in \mathcal{F}(\Delta) = \mathcal{F}$ and thus $\mathcal{K}(\mathcal{I}) \subseteq \mathcal{F} = \mathcal{F}(\Delta)$ (Lemma 2.6 (3)). Conversely, if $J \in \mathcal{F}$ is a proper ideal then the minimal primes of J do not belong to Δ. The conclusion follows since, in a Noetherian ring, every ideal contains a power of its radical, so that there exists $e \geq 1$ and $Q_1, \ldots, Q_t \in \mathcal{I}$ such that $Q_1^e \cdot \ldots \cdot Q_t^e \subseteq J$.

□

COROLLARY 2.11. *Let T be an overring of a Noetherian integral domain. Then the following statements are equivalent:*
(i) *T is R–flat;*
(ii) $\mathcal{F}_0(T) = \mathcal{K}(\mathrm{Supp}_R(T/R))$.

Proof: (ii) $\Rightarrow$ (i). Since every $P \in \mathrm{Supp}_R(T/R)$ belongs to $\mathcal{K}(\mathrm{Supp}_R(T/R))$ and, in the Noetherian case, $\mathrm{Ass}_R(T/R) \subseteq \mathrm{Supp}_R(T/R)$ [B, Ch. IV § 1 N.3 Corollaire 1], then every $P \in \mathrm{Ass}_R(T/R)$ belongs to $\mathcal{F}_0(T)$. Therefore T is R–flat by Corollary 2.3.

(i) $\Rightarrow$ (ii) is a consequence of (the proof of) Lemma 2.10, applied to the localizing system $\mathcal{F}_0(T)$, and Corollary 2.2.

□

Now, we are in condition to recover a result proved by Schenzel [S3, Theorem 1] (cf. also [Po, Theorem 8]).

THEOREM 2.12. *Let R be a Noetherian domain and T a flat overring of R. Then, the following statements are equivalent:*
(i) *T is finitely generated over R;*
(ii) *there exists a nonzero element $x \in R$ such that $R \subseteq T \subseteq R_x$.*

Proof: It is obvious that (i) $\Rightarrow$ (ii), since $T = R[x_1/x, \ldots, x_n/x]$ for some $x_1, \ldots, x_n, x \in R$ and $x \neq 0$.

(ii) $\Rightarrow$ (i). We claim that there exists a (finitely generated) ideal I of R such that $\mathcal{F}_0(T) = \mathcal{K}(I)$. We note that $\mathrm{Ass}_R(T/R)$ is finite, since $\mathrm{Ass}_R(T/R) \subseteq \mathrm{Ass}_R(R_x/R) = \mathrm{Ass}_R(R/x^nR) = \mathrm{Ass}_R(R/xR)$, for each $n \geq 1$, and $\mathrm{Ass}_R(R/xR)$ is finite [B, Ch. IV § 1 N.4 Corollaire p. 137]. Let I be the product of all the (finitely many) minimal primes of $\mathrm{Ass}_R(T/R)$. By [B, Ch. IV § 1 N.3 Corollaire 1] I is also the product of all the (finitely many) minimal prime ideals of $\mathrm{Supp}_R(T/R)$. Thus, by Corollary 2.7 (2), $P \in \mathcal{K}(I)$ if and only if P contains a minimal prime ideal of $\mathrm{Supp}_R(T/R)$. Therefore from Lemma 2.6 (3), we deduce that $\mathcal{K}(I) = \mathcal{K}(\mathrm{Supp}_R(T/R))$. By Corollary 2.11, since T is R–flat we have $\mathcal{K}(I) = \mathcal{F}_0(T)$. The conclusion follows immediately from Proposition 2.9.

□

3 THE KAPLANSKY TRANSFORM

The spectral localizing systems of an integral domain R are parameterized by the subsets of $\mathrm{Spec}(R)$ stable under generizations (Corollary 1.8). A relevant class of subsets stable under generizations are the open subspaces of Spec(R). Let I be a nonzero ideal of R and let

$$D(I) := \{P \in \mathrm{Spec}(R) : P \not\supseteq I\} .$$

We can consider

$$\mathcal{F} = \mathcal{F}(D(I)) = \cap\{\mathcal{F}(P) : P \in D(I)\} . \tag{3.1}$$

By [K2, Theorem 277] (or [Hy, Theorem 1.7] and [FHP, Proposition 3.2.2])

$$\begin{aligned} R_{\mathcal{F}} = \Omega_R(I) &:= \{z \in K : \forall\, a \in I,\ za^n \in R \text{ for some } n \geq 1\} = \\ &= \{z \in K : \mathrm{rad}(R :_R zR) \supseteq I\} \end{aligned} \tag{3.2}$$

where $\Omega_R(I)$ (or, simply, $\Omega(I)$) is called the *Kaplansky transform of R with respect to I*.

Note that, when $I = R$, $\Omega(R) = R$. In this case $D(I) = \mathrm{Spec}(R)$ and $\mathcal{F}(\mathrm{Spec}(R)) = \{R\}$. If $I = (0)$, then $\Omega((0))$ coincides with the quotient field K of R. On the other hand, $D((0)) = \emptyset$ and we assume that $\mathcal{F}(\emptyset)$ is the trivial localizing system consisting of all the ideals of R, whence $R_{\mathcal{F}(\emptyset)} = K$.

We collect in the following lemma some easy facts concerning $\mathcal{F}(D(I))$ and $\Omega(I)$ (cf. also [Hy]).

LEMMA 3.1. *Let I be an ideal of an integral domain R.*

(a) $\mathcal{F}(D(I)) = \{J : J \text{ an ideal of } R \text{ and } \mathrm{rad}(J) \supseteq \mathrm{rad}(I)\} =$
$= \{J : J \text{ an ideal of } R \text{ and } \mathrm{rad}(J) \supseteq I\}$.

(b) $\Omega(I) = \cup\{(R : J) : \mathrm{rad}(J) \supseteq I\}$.

(c) $\Omega(I) = \Omega(\mathrm{rad}(I))$.

(d) *For each $a \in R$, $a \neq 0$, $\Omega(aR) = R_a$.*

(e) *If J is an ideal of R and $I \subseteq J$, then $\Omega(I) \supseteq \Omega(J)$.*

(f) *If $\{I_\alpha : \alpha \in A\}$ is a nonempty family of ideals of R, then*

$$\Omega(\Sigma_\alpha I_\alpha) = \cap_\alpha \Omega(I_\alpha) ;$$
$$\Omega(\cap_\alpha I_\alpha) \supseteq \Sigma_\alpha \Omega(I_\alpha) .$$

(g) *If $I \neq (0)$, then*

$$\Omega(I) = \cap\{\Omega(aR) : a \in I,\ a \neq 0\} .$$

(h) *If J is another ideal of R,*

$$\Omega(I \cap J) = \Omega(IJ) \supseteq \Omega(I)\Omega(J) \supseteq \Omega(I) + \Omega(J) .$$

(i) *If J is an ideal of R and $I \subseteq J$ (or, more generally, if $\Omega(J) \subseteq \Omega(I)$) then*

$$\Omega(IJ) = \Omega(I)\Omega(J) = \Omega(I) + \Omega(J) .$$

(j) *If J is an invertible ideal of R and I is finitely generated then*

$$\Omega(IJ) = \Omega(I)\Omega(J) .$$

(k)

$$\begin{aligned}\Omega_R(I) &= \cap\{R_P : P \not\supseteq I\} = \\ &= \cap\{\Omega_{R_P}(IR_P) : P \in \mathrm{Spec}(R)\} = \cap\{\Omega_{R_M}(IR_M) : M \in \mathrm{Max}(R)\}\ .\end{aligned}$$

(l) *If S is an overring of R, with $R \subseteq S \subseteq \Omega_R(I)$, then*

$$\Omega_S(IS) = \Omega_R(I)\ .$$

Proof: (a) is proved in [FHP, Remark 5.8.5 (a)].

(b) follows from (a) and from (3.2).

(c) is an easy consequence of (b).

(d) follows directly from the definition of $\Omega(aR)$.

(e), (f) and (h) are straightforward, since the fractional ideal $\Omega(I)\Omega(J)$ is the smallest overring of R containing $\Omega(I)$ and $\Omega(J)$.

(g) follows from the first equality of (f).

(i) is a consequence of (h) and of the fact that $\Omega(J) \subseteq \Omega(I)$ implies that $\Omega(I) + \Omega(J) = \Omega(I)$.

(j). From (h), we only need to prove that $\Omega(IJ) \subseteq \Omega(I)\Omega(J)$. Let $z \in \Omega(IJ)$ and let $I = (i_1, i_2, \ldots, i_n)R$ and $J = (j_1, j_2, \ldots, j_m)R$. For each $i_h j_k \in IJ$ there exists an integer $r = r(h,k) \geq 0$ such that $z(i_h j_k)^r \in R$. Henceforth, for a suitable $\overline{r} \geq \sup\{r(h,k) :\ 1 \leq h \leq n,\ 1 \leq k \leq m\}$, $z(IJ)^{\overline{r}} = zJ^{\overline{r}}I^{\overline{r}} \subseteq R$, and so $zJ^{\overline{r}} \subseteq \Omega(I)$, i.e. $z \in J^{-\overline{r}}\Omega(I) \subseteq \Omega(J)\Omega(I)$.

(k) follows easily from (3.1), (3.2) and the properties of the localizations, since, by Lemma 1.2,

$$\Omega(I) = R_{\mathcal{F}(D(I))} = \cap\{R_P : P \in \mathrm{Spec}(R), P \not\supseteq I\}\ .$$

(l) is an easy consequence of the definition (cf. also [FHP, Theorem 3.3.2]).

□

Note that the inclusions in Lemma 3.1 (f) and (h) may be strict. An example of a Prüfer domain R with two ideals I and J such that $\Omega(IJ) \subsetneq \Omega(I)\Omega(J) = \Omega(I) + \Omega(J)$ is given in [FHP, Example 8.2.4 and Remark 8.2.5].

The Kaplansky transform is intimately related to the ideal transform introduced by Nagata [N1], [N4].

We recall that the *Nagata (ideal) transform of* an integral domain R *with respect to an ideal I of R* is the following overring of R:

(3.3) $$T_R(I) = T(I) := \cup\{(R : I^n) : n \geq 0\}\ .$$

With the terminology, introduced in Section 1, if $\mathfrak{N}(I)$ is the multiplicative system $\{I^n : n \geq 0\}$ of the powers of the ideal I, then $T_R(I)$ is the (generalized) ring of fractions of R with respect to $\mathfrak{N}(I)$:

(3.4) $$T_R(I) = R_{\mathfrak{N}(I)}\ .$$

From the fact that $\mathfrak{N}(I) \subseteq \mathcal{F}(D(I))$, it follows immediately that the Nagata transform is a subring of the Kaplansky transform. More precisely:

LEMMA 3.2. *Let I be an ideal of an integral domain R.*
(1) $\mathcal{F}(D(I)) = \mathcal{K}(I) = \{J : J \text{ ideal of } R \text{ such that } \mathrm{rad}(I) = \mathrm{rad}(J \cap I)\}$.
(2) $J \in \overline{\mathfrak{N}(I)} \Rightarrow \mathrm{rad}(I) = \mathrm{rad}(I \cap J)$.
(3) *If I is a finitely generated ideal, then*

$$\overline{\mathfrak{N}(I)} = \mathcal{F}(D(I)) \ .$$

Proof: (a) follows easily from Corollary 2.7 (1) and Lemma 3.1 (a), since $\mathrm{rad}(I \cap J) = \mathrm{rad}(I) \cap \mathrm{rad}(J)$.

(b) is obvious because, if $I^n \subseteq J$, then $\mathrm{rad}(I^n) = \mathrm{rad}(I) \subseteq \mathrm{rad}(J)$.

(c). By (a) and (b), $\overline{\mathfrak{N}(I)} \subseteq \mathcal{F}(D(I))$. Reciprocally, if $J \in \mathcal{F}(D(I))$ then $I \subseteq \mathrm{rad}(J)$ (Lemma 3.1 (a)). Therefore, since I is finitely generated, we have $I^n \subseteq J$ for some $n \geq 0$.

□

COROLLARY 3.3. *Let I be an ideal of an integral domain R.*
(1) $T(I) \subseteq \Omega(I)$.
(2) *If there exists a finitely generated ideal J of R and an integer $n \geq 1$ such that $I^n \subseteq J \subseteq I$ (in particular, if I is finitely generated), then $T(I) = \Omega(I)$.*

Proof: (1) follows from Lemma 3.2 (1) and (2).

(2). It is easy to see (by Lemma 3.1 (c) and Lemma 3.2 (3)) that

$$\Omega(I) = \Omega(J) = T(J) \subseteq T(I^n) = T(I) \ .$$

The conclusion follows from (1).

□

Given an ideal I of an integral domain R, we set as usual

$$I_v := (R : (R : I)) \ ,$$
$$I_t := \cup\{J_v : J \text{ is finitely generated, } J \subseteq I\} \ .$$

It is obvious that $I \subseteq I_t \subseteq I_v$. The next result is due to Anderson and Bouvier [AnB].

PROPOSITION 3.4. *Let I be an ideal of an integral domain R.*
(1) $T(I) = T(I_v)$.
(2) $\Omega(I) = \Omega(I_t)$.

Proof: (1) is obvious, since $(R : I^n) = (R : I_v^n)$ for each $n \geq 1$.

(2). For each finitely generated ideal J of R, we have

$$\Omega(J_v) \subseteq \Omega(J) = T(J) = T(J_v) \subseteq \Omega(J_v) \ ,$$

hence $\Omega(J_v) = \Omega(J)$. Therefore, by Lemma 3.1 (e) and (f), we have:

$$\begin{aligned} \Omega(I_t) &= \Omega(\cup\{J_v : J \text{ finitely generated, } J \subseteq I\}) = \\ &= \cap\{\Omega(J_v) : J \text{ finitely generated, } J \subseteq I\} = \\ &= \cap\{\Omega(J) : J \text{ finitely generated, } J \subseteq I\} \supseteq \Omega(I) \ . \end{aligned}$$

The conclusion follows immediately, since in general (Lemma 3.1 (e)) $\Omega(I_t) \subseteq \Omega(I)$.

□

From the previous result and from the fact that, in a Krull domain (respectively, a UFD), for each nonzero ideal I there exists a finitely generated (respectively, principal) ideal J such that $I_v = J_v$ [G3, Corollaries 44.3 and 44.5], we recover in particular a result due to Nagata [N4, Lemma 2.4].

COROLLARY 3.5. *Let I be an ideal in a Krull domain R. Then:*
(1) $T(I) = \Omega(I)$;
(2) *if, moreover, R is a UFD and I is a nonzero ideal, then there exists $f \in I_v$ such that*

$$T(I) = \Omega(I) = R_f \, .$$

Proof: (1). Let J be a finitely generated ideal of R such that $J_v = I_v$. Note that $I_t = J_v = I_v$, since the t-ideals coincide with the v-ideals in a Krull domain [Bo, Lemma 3 (a)]. Therefore:

$$T(I) = T(I_v) = \Omega(J_v) = \Omega(I_t) = \Omega(I) \, .$$

(2) follows from Lemma 3.1 (d) and from the previous statement (1) (and its proof). □

EXAMPLE 3.6. *An integral domain R for which there exists an ideal I such that $\Omega(I) \neq \Omega(I_v)$.*

Let (V, M) be a 1–dimensional valuation domain with $M = M^2$. In this case, $M_v = V$ [FHP, Corollary 3.1.3], therefore $\Omega(M_v) = \Omega(V) = T(V) = V = T(M_v) = T(M)$. On the other hand, by (Lemma 3.1 (k)), $\Omega(M) = V_{(0)} = qf(V)$, hence $\Omega(M_v) \subsetneqq \Omega(M)$.

EXAMPLE 3.7. *An integral domain R for which there exists an ideal I such that $T(I) \subsetneqq \Omega(I)$. Moreover, $T(I)$ is not a QQR-overring of R.*

Let (V, M) be a 1-dimensional valuation domain such that $M = M^2$ and $V = K + M$ where $K \cong V/M$. Let k be a proper subfield of K and set $R := k + M$. It is easy to see that M is the conductor of $R \subset V$ and, hence $V = (M : M) = (R : M) = T_R(M)$. On the other hand, by (3.2), $\Omega_R(M) = R_{(0)} = qf(R) = qf(V)$, hence $T_R(M) \subsetneqq \Omega_R(M)$. Note also that $\mathrm{Spec}(R) = \{(0), M\}$ and hence $T_R(M)$ is not an intersection of localizations of R.

We have already observed that, given an ideal I of R, there are several ideals J of R such that $\Omega(J) = \Omega(I)$. Set $\mathcal{J} = \mathcal{J}(\Omega_R(I)) := \{J : J \text{ ideal of } R \text{ such that } \Omega(J) = \Omega(I)\}$. By Lemma 3.1 (f), if $\{J_\alpha : \alpha \in A\} \subseteq \mathcal{J}$ then $\Sigma_\alpha J_\alpha \in \mathcal{J}$. Therefore, as already observed by Hays [Hy, Theorem 2.3], and by Rhodes [Rh, § 2] in the non integral domain case, in $\mathcal{J}$ there exists a unique maximal element that we denote by I^Ω. We say that an ideal H of R is a *Ω-ideal* if $H = I^\Omega$, for some ideal I of R. It is obvious that $R^\Omega = R$. We will see in a moment that, $(0)^\Omega$ may be larger than (0).

Our next goal is to deepen the study of Ω-ideals.

PROPOSITION 3.8. *Let I and J be two ideals of an integral domain R. Then*
(1) $(I^\Omega)^\Omega = I^\Omega$;
(2) $I^\Omega = (\mathrm{rad}(I))^\Omega = \mathrm{rad}(I^\Omega)$;
(3) $I^\Omega = (I_t)^\Omega = (I^\Omega)_t$;
(4) $I^\Omega = \{x \in R : R_x \supseteq \Omega(I)\}$;
(5) $I \subseteq J \Rightarrow I^\Omega \subseteq J^\Omega$;
(6) $(aR)^\Omega = \mathrm{rad}(aR)$, *for each nonzero element $a \in R$.*

Proof: (1), (2) and (3) are straightforward consequences of the definition of a Ω-ideal, Lemma 3.1 (c) and Proposition 3.4 (2).

(4). If $x \in I^\Omega$, $x \neq 0$, then $R_x = \Omega(xR) \supseteq \Omega(I^\Omega) = \Omega(I)$ (Lemma 3.1 (d) and (e)). If $x = 0$, then assuming that $R_x = K$ (because the saturated multiplicative set containing 0 is R), trivially $R_x = K \supseteq \Omega(I)$.

On the other hand if $x \notin I^\Omega$, then $\Omega(I) = \Omega(I^\Omega) \supsetneq \Omega(xR + I^\Omega) = \Omega(xR) \cap \Omega(I)$ (Lemma 3.1 (f)), whence $R_x = \Omega(xR) \not\supseteq \Omega(I)$.

(5) follows immediately from (4) and Lemma 3.1 (e).

(6). Clearly, by (2), $\mathrm{rad}(aR) \subseteq (aR)^\Omega$. Let $x \in (aR)^\Omega$, $x \neq 0$. Then $R_x = \Omega(xR) \supseteq \Omega(aR) = R_a$, hence $1/a = r/x^n$ for some $r \in R$ and for some integer $n \geq 1$, and so $x^n \in aR$.

□

For each ideal I of R, we call the Ω-*radical* of I the following radical ideal of R:

$$\mathrm{rad}^\Omega(I) := \cap\{P \in \mathrm{Spec}(R) : R_P \not\supseteq \Omega(I)\} .$$

It is obvious that $\mathrm{rad}^\Omega(I) = \mathrm{rad}^\Omega(I^\Omega)$. The following result is due to Hays:

THEOREM 3.9. *For each ideal I of R, we have*

$$I^\Omega = \mathrm{rad}^\Omega(I) .$$

Proof: From the definition of Kaplansky transform (3.2), for each ideal J of R and for each prime ideal Q of R, we have

$$R_Q \not\supseteq \Omega(J) \Rightarrow Q \supseteq J . \tag{3.9.1}$$

We deduce immediately that

$$I^\Omega = \mathrm{rad}(I^\Omega) = \cap\{P \in \mathrm{Spec}(R) : P \supseteq I^\Omega\} \subseteq$$
$$\subseteq \cap\{P \in \mathrm{Spec}(R) : R_P \not\supseteq \Omega(I^\Omega) = \Omega(I)\} = \mathrm{rad}^\Omega(I)$$

and hence

$$\Omega(I) = \Omega(I^\Omega) \supseteq \Omega(\mathrm{rad}^\Omega(I)) .$$

The conclusion will follow if we show that $\Omega(I) \subseteq \Omega(\mathrm{rad}^\Omega(I))$. Suppose that Q is a prime ideal of R with $Q \not\supseteq \mathrm{rad}^\Omega(I)$.

Then it is easy to see that $R_Q \supseteq \Omega(I)$. Henceforth

$$\Omega(I) \subseteq \cap\left\{R_Q : Q \not\supseteq \mathrm{rad}^\Omega(I)\right\} = \Omega(\mathrm{rad}^\Omega(I)) ,$$

as desired.

□

For each prime ideal P of R we denote by P^* the *pseudo–radical of* P, i.e. $P^* := \cap\{Q \in \mathrm{Spec}(R) : Q \supsetneq P\}$ [G1].

COROLLARY 3.10. *Let R be an integral domain.*
(1) $(0)^\Omega = (0)^*$.
(2) *If M is a maximal ideal,*

$$M = M^\Omega \Leftrightarrow R \subsetneq \Omega(M) \Leftrightarrow R_M \not\supseteq \cap\{R_P : P \in \mathrm{Spec}(R) \setminus \{M\}\} .$$

Proof: (1). Let P be a prime ideal of R. Note that $R_P \not\supseteq K = \Omega((0))$ if and only if $P \neq (0)$.

(2). Note that $\Omega(M) = \cap\{R_P : P \in \mathrm{Spec}(R) \setminus \{M\}\}$.

□

Theorem 3.9 shows that there exists a link between the Ω-ideals of an integral domain R and the space $\mathrm{Loc}(R) := \{R_P : P \in \mathrm{Spec}(R)\}$, endowed with the Zariski topology [ZS, Ch. VI § 17]. Recall that the Zariski topology on $\mathrm{Loc}(R)$ is defined as follows. For each R-submodule E of K, we can consider

$$\mathcal{L}(E) := \{R_P \in \mathrm{Loc}(R) : R_P \supseteq E\} .$$

It is easy to prove the following:

LEMMA 3.11. *Let E_α, E and F be R-submodules of K where α belongs to a nonempty set A.*
(1) $E \subseteq F \Rightarrow \mathcal{L}(F) \subseteq \mathcal{L}(E)$;
(2) $\mathcal{L}(\Sigma_\alpha E_\alpha) = \cap_\alpha \mathcal{L}(E_\alpha)$;
(3) $\mathcal{L}(E) = \cap\{\mathcal{L}(zR) : z \in E\}$;
(4) $\mathcal{L}(\cap_\alpha E_\alpha) \supseteq \cup_\alpha \mathcal{L}(E_\alpha)$.

□

By the previous lemma, it follows in particular that the subsets $\mathcal{L}(E)$, where E is a finitely generated R-submodule of K, form a basis for the open sets in $\mathrm{Loc}(R)$. The induced topology is called *the Zariski topology of* $\mathrm{Loc}(R)$.

It is easy to prove that $\mathrm{Spec}(R)$ and $\mathrm{Loc}(R)$ (with their Zariski topologies) are canonically homeomorphic. More precisely:

LEMMA 3.12. *Let $\varphi : \mathrm{Loc}(R) \to \mathrm{Spec}(R)$ be the canonical bijection defined by $R_P \mapsto PR_P \cap R = P$.*
(1) *For each element $x \in R$, $\varphi^{-1}(D(xR)) = \mathcal{L}(R_x)$.*
(2) *For each element $z \in K$,*

$$\varphi(\mathcal{L}(zR)) = \{P \in \mathrm{Spec}(R) : P \not\supseteq (R :_R zR)\} = D((R :_R zR)).$$

(3) *φ is a homeomorphism.*

□

REMARK 3.13 (a) Let E be a R-submodule of K. If we denote by $R[E]$ the smallest overring of R generated by E, then it is clear that

$$\mathcal{L}(E) = \mathcal{L}(R[E]) \ .$$

Obviously, if E is a finitely generated R–module then $R[E]$ is a finitely generated R–algebra. Conversely, if S is an overring of R and S is a finitely generated R–algebra, then there exist $y_1, y_2, \ldots, y_n \in S$ such that $S = R[y_1, y_2, \ldots, y_n]$. Let $E := y_1R + y_2R + \cdots + y_nR$, then E is a finitely generated R-module and $S = R[E]$.

(b) If x is a nonzero element of R, then $R_x = R[1/x]$ and $(R :_R (1/x)R) = xR$, and hence $\mathcal{L}((1/x)R) = \varphi^{-1}(D(xR))$ (Lemma 3.12 (1)).

(c) If $E = (z_1, \ldots, z_t)R$ is a nonzero finitely generated R-submodule of K then, by Lemma 3.11 (2) and 3.12 (2), the basic open set $\mathcal{L}(E)$ of $\mathrm{Loc}(R)$ is homeomorphic to the open set $D(I_E)$ of $\mathrm{Spec}(R)$, where

$$I_E := \bigcap_{i=1}^{t} (R :_R z_iR) = \left(R :_R \sum_{i=1}^{t} z_iR\right) = E^{-1} \cap R \ .$$

Since $D(I_E) = D(\mathrm{rad}(I_E))$ and $\mathrm{rad}(I_E)$ is the largest ideal containing I_E with this property, it is natural to associate to the basic open set $\mathcal{L}(E)$ of $\mathrm{Loc}(R)$ the radical ideal $\mathrm{rad}(I_E)$ of R. If $E = zR$, then $\mathrm{rad}(I_E) = \mathrm{rad}((R :_R zR))$. By using Lemma 3.11 (1) and 3.12 ((1) and (3)) it is easy to see that:

$$x \in \mathrm{rad}(I_E) \Leftrightarrow R_x \supseteq E \ .$$

Motivated by Lemma 3.12 (2) and Remark 3.13 (c), in an integral domain R with quotient field K we introduce, for each element $z \in K$ and for each R-submodule E of K, the following radical ideals of R:

$$\Omega_R^-(zR) = \Omega^-(zR) := \operatorname{rad}((R :_R zR)) \ ,$$

$$\Omega_R^-(E) = \Omega^-(E) := \cap\{\Omega^-(zR) : z \in E\} \ .$$

We note that, if $x \in R$ and $x \neq 0$, then

$$\Omega^-((1/x)R) = \operatorname{rad}(xR) = (xR)^\Omega \ .$$

Moreover, $\Omega^-(0R) = \Omega^-(R) = R$. The following properties are easy consequence of the definitions:

LEMMA 3.14. *Let E and F be two R-submodules of K, $\{E_\alpha : \alpha \in A\}$ a family of R-submodules of K and I an ideal of R. Then:*

(1) $\Omega^-(E) = \{x \in R : R_x \supseteq E\} = \{x \in R : \Omega(xR) \supseteq E\}$.

(2) $E \subseteq F \Rightarrow \Omega^-(F) \subseteq \Omega^-(E)$.

(3) $\Omega^-(I) = R$.

(4) $\Omega^-(K) = (0)^*$.

(5) $\operatorname{rad}(E^{-1} \cap R) \subseteq \Omega^-(E)$ *and, if E is finitely generated, then*

$$\operatorname{rad}(E^{-1} \cap R) = \Omega^-(E) \ .$$

(6) $\Omega^-(\Omega(I)) = I^\Omega$ *and* $\operatorname{rad}((R : \Omega(I))) \subseteq I^\Omega$*; moreover, if $\Omega(I)$ is finitely generated, then* $\operatorname{rad}((R : \Omega(I))) = I^\Omega$.

(7) $\Omega^-(\Sigma_\alpha E_\alpha) = \cap_\alpha \Omega^-(E_\alpha)$.

(8) $\Omega^-(\cap_\alpha E_\alpha) \supseteq \Sigma_\alpha \Omega^-(E_\alpha)$.

□

The following result gives another useful representation of $\Omega^-(E)$:

COROLLARY 3.15. *Let E be a R-submodule of K, then*

$$\Omega^-(E) = \cap\{P \in \operatorname{Spec}(R) : R_P \not\supseteq E\} \ .$$

Proof: For each $z \in K$, from (3.9.2) we have:

$$\Omega^-(zR) = \cap\{P \in \operatorname{Spec}(R) : R_P \not\ni z\}$$

and thus

$$\Omega^-(E) = \cap\{\Omega^-(zR) : z \in E\} = \cap\{P \in \operatorname{Spec}(R) : R_P \not\supseteq E\} \ .$$

□

COROLLARY 3.16. *Let R be an integral domain.*

(1) *For each ideal I of R,*

$$I \subseteq \Omega^-(\Omega(I)) \quad \text{and} \quad \Omega(I) = \Omega(\Omega^-(\Omega(I))) \ .$$

(2) *For each R-submodule E of K,*

$$E \subseteq \Omega(\Omega^-(E)) \quad \text{and} \quad \Omega^-(E) = \Omega^-(\Omega(\Omega^-(E))) \ .$$

Proof: (1) follows from Lemma 3.14 (6).

(2). For each $z \in E$, $zR \subseteq (R : (R :_R zR)) \subseteq \Omega((R :_R zR)) = \Omega(\operatorname{rad}((R :_R zR))) = \Omega(\Omega^-(zR)) \subseteq \Omega(\Omega^-(E))$ and thus $E \subseteq \Omega(\Omega^-(E))$. Moreover, by Lemma 3.14 (2), $\Omega^-(E) \supseteq \Omega^-((\Omega(\Omega^-(E))))$. The conclusion follows, since by (1) for $I := \Omega^-(E)$ we have $\Omega^-(E) \subseteq \Omega^-(\Omega(\Omega^-(E)))$.

□

As a consequence of the previous results we obtain a deeper understanding of the homeomorphism between $\operatorname{Loc}(R)$ and $\operatorname{Spec}(R)$:

PROPOSITION 3.17. *Let R be an integral and $\varphi : \mathrm{Loc}(R) \to \mathrm{Spec}(R)$ the canonical homeomorphism* (Lemma 3.12). *Then*

(1) *For each R-submodule E of K, $\varphi^{-1}(D(\Omega^{-}(E)))$ coincides with the interior of the subspace $\mathcal{L}(E)$ of* $\mathrm{Loc}(R)$, *i.e.*
$$\varphi^{-1}(D(\Omega^{-}(E))) = \mathrm{Int}(\mathcal{L}(E)) .$$
(2) *For each ideal I of R,*
$$\varphi^{-1}(D(I^{\Omega})) = \mathrm{Int}(\mathcal{L}(\Omega(I))) .$$
(3) *If E is a finitely generated R-submodule of K,*
$$\varphi^{-1}(D(\Omega^{-}(E))) = \mathcal{L}(E) .$$
(4) *If I is an ideal of R such that $\Omega(I)$ is a finitely generated R-algebra, then*
$$\varphi^{-1}(D(I^{\Omega})) = \mathcal{L}(\Omega(I)) .$$

Proof: (1). Note that $D(\Omega^{-}(E)) \subseteq \varphi(\mathcal{L}(E))$, since
$$P \not\supseteq \Omega^{-}(E) \Rightarrow R_P \supseteq E \qquad \text{(Corollary 3.15)} .$$
On the other hand, the closure $\mathrm{cl}(Y)$ of a subspace Y of $\mathrm{Spec}(R)$ coincides with $V(J_Y)$ where $J_Y := \cap\{Q \in Y\}$. Therefore,
$$\mathrm{cl}(\mathrm{Spec}(R) \smallsetminus \varphi(\mathcal{L}(E))) = V(J)$$
where
$$\begin{aligned} J &:= \cap\,\{P \in \mathrm{Spec}(R) \smallsetminus \varphi(\mathcal{L}(E))\} = \cap\{P \in \varphi(\mathrm{Loc}(R) \smallsetminus \mathcal{L}(E))\} = \\ &= \cap\,\{P \in \mathrm{Spec}(R) : R_P \not\supseteq E\} = \Omega^{-}(E) . \end{aligned}$$
We deduce that $\mathrm{Int}(\varphi(\mathcal{L}(E))) = D(\Omega^{-}(E))$.

(2). By Lemma 3.14 (6), this statement is a particular case of (1) for $E = \Omega(I)$.

(3) is a consequence of (1) and of the fact that, in the present situation, $\mathcal{L}(E)$ is an open subspace of $\mathrm{Loc}(R)$.

(4) is a consequence of (2) and (3) (cf. also Remark 3.13 (a)).

□

COROLLARY 3.18. *Let I be an ideal of R such that $\Omega(I)$ is a finitely generated R-algebra. Then*

(3.18.1) $$R_P \supseteq \cap\{R_x : x \in I\} \Rightarrow R_P \supseteq R_y \text{ for some } y \in I^{\Omega} .$$

Proof: In general, by Proposition 3.17 (2), for each ideal I of R, we have

(3.18.2) $$\varphi^{-1}(D(I^{\Omega})) \subseteq \mathcal{L}(\Omega(I)) .$$

Moreover, by Lemma 3.1 (g) and Lemma 3.12, equality in (3.18.2) holds if and only if (3.18.1) is verified. The conclusion is a consequence of Proposition 3.17 (4).

□

REMARK 3.19. If I is a nonzero principal ideal of R, $I = xR$, then $\Omega(I) = R_x = R[1/x]$ is a finitely generated R-algebra and hence, by Proposition 3.17 (4), we reobtain that $\varphi^{-1}(D(xR)) = \mathcal{L}(R_x)$ (Lemma 3.12 (1)). It is also easy to see that if $x, y \in R$ are nonzero elements, then
$$\mathcal{L}(R_x) \cap \mathcal{L}(R_y) = \varphi^{-1}(D(xR) \cap D(yR)) = \mathcal{L}(R_{xy}) .$$
Therefore, the family $\{\mathcal{L}(R_x) : x \in R\ x \neq 0\}$ is a basis for the Zariski topology of $\mathrm{Loc}(R)$.

From the previous results we can obtain a characterization of the Ω-ideals of R:

COROLLARY 3.20. *Let R be an integral domain and I an ideal of R. The following statements are equivalent:*
(i) *I is a Ω-ideal;*
(ii) *$I = I^{\Omega}$;*
(iii) *$I = \Omega^{-}(E)$, for some R-submodule E of K;*
(iv) *$I = \Omega^{-}(T)$, for some overring T of R.*

Proof: (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv) $\Rightarrow$ (iii) are obvious (cf. also Lemma 3.14 (6)).
(iii) $\Rightarrow$ (ii). By Corollary 3.16 (2), we have

$$I = \Omega^{-}(E) = \Omega^{-}(\Omega(\Omega^{-}(E))) = \Omega^{-}(\Omega(I)) = I^{\Omega} .$$

□

We collect in the following proposition some properties of Ω-ideals.

PROPOSITION 3.21. *Let $\{I_\alpha : \alpha \in A\}$ be a family of ideals of an integral domain R. Then:*
(1) $(\Sigma_\alpha I_\alpha)^{\Omega} = (\Sigma_\alpha I_\alpha^{\Omega})^{\Omega}$.
(2) $\cap_\alpha I_\alpha^{\Omega} = (\cap_\alpha I_\alpha^{\Omega})^{\Omega}$.
(3) *If each I_α is a Ω-ideal, for $\alpha \in A$, then $\cap_\alpha I_\alpha$ is also a Ω-ideal.*

Proof: (1). From the following easy inclusions (Proposition 3.8 (5)):

$$\Sigma_\alpha I_\alpha \subseteq \Sigma_\alpha I_\alpha^{\Omega} \subseteq (\Sigma_\alpha I_\alpha)^{\Omega} \subseteq \left(\Sigma_\alpha I_\alpha^{\Omega}\right)^{\Omega}$$

we deduce the statement.
(2). It is obvious that

$$\cap_\alpha I_\alpha^{\Omega} \subseteq (\cap_\alpha I_\alpha^{\Omega})^{\Omega}$$

and, by Proposition 3.8 (1),

$$(\cap_\alpha I_\alpha^{\Omega})^{\Omega} \subseteq (I_\alpha^{\Omega})^{\Omega} = I_\alpha^{\Omega} , \quad \text{for each } \alpha \in A .$$

The conclusion follows immediately.
(3) is a consequence of (2).

□

COROLLARY 3.22. *Let I and J be two nonzero ideals of an integral domain R. Then*

$$(IJ)^{\Omega} = (I \cap J)^{\Omega} .$$

Proof: Recall that $\Omega(IJ) = \Omega(I \cap J)$ (Lemma 3.1 (h)). The conclusion follows from Lemma 3.14 (6).

□

For each overring T of R, we set

$$T_{\Omega} := \Omega(\Omega^{-}(T)) = \cap\{R_P : P \not\supseteq \Omega^{-}(T)\} ,$$

where T_Ω is an overring of T and $\Omega^{-}(T_\Omega) = \Omega^{-}(T)$ (Corollary 3.16 (2)). We say that an overring S of R is a *Ω-overring of R* if $S = T_\Omega$ for some overring T of R.

PROPOSITION 3.23. *Let R be an integral domain and T on overring of R. The following conditions are equivalent:*
(i) *T is a Ω-overring of R;*
(ii) *$T = T_\Omega$;*
(iii) *$T = \Omega(I)$, for some ideal I of R.*

Proof: It is obvious that (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) (cf. also Corollary 3.16 (1)).
(i) $\Rightarrow$ (iii). If $T = S_\Omega$ for some overring S of R, then $T = \Omega(I)$ where $I := \Omega^-(S) = \Omega^-(S_\Omega) = \Omega^-(T)$.

□

Let $\{T_\alpha : \alpha \in A\}$ be a family of overrings of an integral domain R. We denote by $\vee_\alpha T_\alpha$ the smallest overring of R containing $\cup_\alpha T_\alpha$. In the case of a finite family of overrings $\{T_i : 1 \leq i \leq n\}$, it is obvious that

$$\vee_{i=1}^n T_i = \prod_{i=1}^n T_i = \left\{ \sum_{k=1}^r t_1^{(k)} t_2^{(k)} \cdot \ldots \cdot t_n^{(k)} : t_i^{(k)} \in T_i,\ r \geq 1 \right\} .$$

PROPOSITION 3.24. *Let $\{T_\alpha : \alpha \in A\}$ be a family of overrings of R. Then*
(1) $(\vee_\alpha T_\alpha)_\Omega = (\vee_\alpha (T_\alpha)_\Omega)_\Omega$.
(2) $\cap_\alpha (T_\alpha)_\Omega = (\cap_\alpha (T_\alpha)_\Omega)_\Omega$.
(3) *If each T_α is a Ω-overring of R, then $\cap_\alpha T_\alpha$ is also a Ω-overring of R.*

Proof: *Mutatis mutandis* (in particular using Corollary 3.16 (2) instead of Corollary 3.16 (1) and the fact that $T \subseteq T_\Omega$, $(T_\Omega)_\Omega = T_\Omega$ and $S_\Omega \subseteq T_\Omega$ for each pair of overrings S, T of R with $S \subseteq T$), the proof is analogous to that of Proposition 3.21.

□

COROLLARY 3.25. *Let S and T be two overrings of an integral domain R. Then*

$$(ST)_\Omega = (S_\Omega T)_\Omega = (ST_\Omega)_\Omega = (S_\Omega T_\Omega)_\Omega .$$

Proof: We have already observed that $S \vee T = ST$, hence the statement follows easily from Proposition 3.24 (1).

□

From a "global" point of view, some of the results proved above can be restated in the following way:

THEOREM 3.26. *Let R be an integral domain. Set $\mathcal{I}^\Omega(R) := \{I$ ideal of $R : I^\Omega = I\}$ and $\mathcal{O}_\Omega(R) := \{T$ overring of $R : T_\Omega = T\}$. The map*

$$\Omega : \mathcal{I}^\Omega(R) \longrightarrow \mathcal{O}_\Omega(R) , \quad I \mapsto \Omega(I)$$

is an order-reversing bijection, with inverse map

$$\Omega^- : \mathcal{O}_\Omega(R) \longrightarrow \mathcal{I}^\Omega(R) , \quad T \mapsto \Omega^-(T) .$$

Proof: The statement is a consequence of Proposition 3.8 (5), Lemma 3.14 ((2) and (6)), Corollary 3.20 and Proposition 3.23.

□

If R is not a field, it is not possible that every ideal of R is a Ω-ideal. In fact, if x is a nonzero element of R, and if every ideal of R is a Ω-ideal, then

$$x^2R = (x^2R)^\Omega = (\mathrm{rad}(x^2R))^\Omega = (\mathrm{rad}(xR))^\Omega = (xR)^\Omega = xR$$

and thus x is invertible. It would be interesting to study the integral domains such that every nonzero radical ideal is a Ω-ideal.

On the other hand, it is possible that each overring of R is a Ω-overring. For instance, if V is a finite dimensional valuation domain and if

$$(0) = P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n = M$$

is the chain of all its prime ideals, then

$$\begin{aligned} V_{P_k} &= \Omega(P_{k+1}) \ , \qquad \text{for } 0 \leq k \leq n-1 \ , \\ V &= \Omega(uV) \ , \qquad \text{where } u \in V \setminus M \ , \end{aligned}$$

and thus every overring of V is a Ω-overring. In this case, every nonzero prime (radical) ideal of V is a Ω-ideal, since $P_{k+1} = \Omega^-(V_{P_k})$ for $0 \leq k \leq n-1$. Note also that the inclusion in Lemma 3.14 (6) may be proper

$$\mathrm{rad}((V : \Omega(P_{k+1}))) = \mathrm{rad}((V : V_{P_k})) = P_k \subsetneqq P_{k+1}^\Omega = P_{k+1} \ , \quad \text{for } 0 \leq k \leq n-1 \ .$$

Integral domains with the property that each overring is a Nagata ideal transform were studied in [Br], [BrG], [He1] and [He2].

The integral domains with the property that each overring is a Ω-overring are characterized in [FH]; in particular, a domain of this type is always semilocal with Prüfer integral closure. Note that, if T is an overring of an integral domain R, then:

$$T \subseteq \cap\{R_P : R_P \supseteq T\} \subseteq T_\Omega$$

since if $P \not\supseteq \Omega^-(T)$ then $R_P \supseteq T$ (Corollary 3.15).

PROPOSITION 3.27. *If R is an integral domain and if $S \subseteq T$ are two overrings of R, then*

$$\Omega_R^-(T) = \Omega_R^-(S) \cap \Omega_S^-(T) \ .$$

Proof: It is obvious, by Lemma 3.14 (1), that

$$\Omega_R^-(T) \subseteq \Omega_R^-(S) \cap \Omega_S^-(T) \ .$$

Conversely, if $x \in \Omega_R^-(S) \cap \Omega_S^-(T)$ then $R_x \supseteq S$ and $S_x \supseteq T$. On the other hand $R_x = S_x$, since $R_x \supseteq S$ with $x \in R \subseteq S$, thus $R_x \supseteq T$ and hence $x \in \Omega_R^-(T)$.

□

The previous result was proved, in a more general setting, by Rhodes [Rh].

4 A GEOMETRIC INTERPRETATION OF THE KAPLANSKY TRANSFORM

An interesting property of the Nagata (ideal) transform is its "geometric" interpretation. For an ideal I in a Noetherian integral domain R, the Nagata transform $T_R(I)$ is the ring of global sections over the open subspace $D(I)$ of $\mathrm{Spec}(R)$. More precisely, from classical results by Chevalley [EGA, I.6.7.1], Nagata [N1] and Hartshorne [Ha] (cf. also Arezzo and Ramella [AR] and Theorem 2.11], the following can be shown:

PROPOSITION 4.1. *Let R be a Noetherian integral domain and I an ideal of R. Set $X := \operatorname{Spec}(R)$, $Y := D(I) = \{P \in X : P \not\supseteq I\}$ and $Z := \operatorname{Spec}(T(I))$. The following statements are equivalent:*
(i) *Y is an affine open subspace of X;*
(ii) *the canonical morphism:*

$$(Z, \mathcal{O}_Z) \longrightarrow (Y, \mathcal{O}_Y)\ , \quad Q \mapsto Q \cap R\ ,$$

is a scheme–isomorphism;
(iii) *$IT(I) = T(I)$.*
In particular, when the previous statements hold, $T(I)$ is flat and finitely generated over R.

□

REMARK 4.2. (a) If R is a Noetherian domain, it is easy to see that the image of the canonical map

$$\operatorname{Spec}(T(I)) \longrightarrow \operatorname{Spec}(R)\ , \quad Q \mapsto Q \cap R$$

contains $D(I)$, and if $T(I)$ is R–flat, then this image coincides with $D(I_v)$ [L, Proposition 4.4]; therefore if $T(I)$ is R-flat then $D(I_v)$ is an affine open subspace of $\operatorname{Spec}(R)$.

Furthermore, it is known that the statements (i)–(iii) of Proposition 4.1 are equivalent to the following:
(iv) *$T(I)$ is R–flat and there exists a divisorial ideal J such that $D(J) = D(I)$;*
(v) *$T(I)$ is R–flat and $D(I) = D(I_v)$;*
(vi) *$T(I)$ is R–flat and* $\operatorname{rad}(I) = \operatorname{rad}(I)_v$.
[L, Proposition 4.3].

(b) Note that for each nonzero ideal I of R and for each nonzero element $x \in I$, we have:

$$R \subseteq T(I) \subseteq \Omega(I) \subseteq \Omega(xR) = R_x\ .$$

When R is Noetherian and $T(I) = \Omega(I)$ is R-flat then, by Theorem 2.12, $T(I)$ is also finitely generated over R.

Reciprocally, Schenzel proved that if $R \subseteq T \subseteq R_x$ for some nonzero element x of a Noetherian ring R and if T is R-flat, then there exists an ideal I of R such that $T = T(I)$ and $IT(I) = T(I)$ [S3, Corollary 3].

(c) We will see in Example 4.11 that $T(I)$ may be finitely generated over R without being R-flat. The finiteness of $T(I)$ was studied in several papers (cf., for instance, [Bro], [EHKR] and [Ki]).

If R is not a Noetherian domain, $D(I)$ may be affine with $IT(I) \neq T(I)$ and $\operatorname{Spec}(T(I))$ non isomorphic to $D(I)$. Take, for instance, $R = V$ to be a 2–dimensional valuation domain with idempotent maximal ideal M. Set $I := M$; in this case $D(M) = \operatorname{Spec}(V) \setminus V(M)$ is canonically isomorphic to $\operatorname{Spec}(V_f)$ with $f \in M \setminus P$, where P is the height 1 prime ideal of V. On the other hand, $T(M) = (V : M) = (M : M) = V$, since $M = M^2$, whence $MT(M) = M \neq T(M) = V$. Furthermore, $\operatorname{Spec}(T(M)) = \operatorname{Spec}(V)$ is obviously not isomorphic to $D(M)$.

Our next goal is to generalize Proposition 4.1 to a not necessarily Noetherian context.

LEMMA 4.3. *Let I be an ideal of an integral domain R and let $\Omega(I)$ be the Kaplansky transform of R with respect to I. Set $X := \operatorname{Spec}(R)$ and $Y := D(I)$. The ring of global sections over the open subspace Y of X, $\Gamma(Y, \mathcal{O}_X|_Y)$ coincides with $\Omega(I)$.*

Proof: It follows immediately from (3.1) and (3.2) that

$$\Omega(I) = R_{\mathcal{F}(D(I))} = \cap\{R_P : P \not\supseteq I\} = \Gamma(Y, \mathcal{O}_X|_Y) .$$

□

THEOREM 4.4. *Let R be an integral domain and I an ideal of R. Set $X :=$ $\operatorname{Spec}(R)$, $Y := D(I)$ and $W := \operatorname{Spec}(\Omega(I))$.*

The following statements are equivalent:

(i) *Y is an affine open subspace of X;*

(ii) *the canonical morphism*

$$(W, \mathcal{O}_W) \longrightarrow (Y, \mathcal{O}_Y) , \qquad Q \mapsto Q \cap R ,$$

is a scheme-isomorphism;

(iii) *$I\Omega(I) = \Omega(I)$;*

(iv) *$\Omega(I)$ is R-flat and, for each $P \in \operatorname{Spec}(R)$ with $P \supseteq I$, $R_P \not\supseteq \Omega(I)$;*

(v) *$\Omega(I)$ is R-flat and, for each $P \in \operatorname{Spec}(R)$ with $P \supseteq I$, $P\Omega(I) = \Omega(I)$;*

(vi) *$\mathcal{F}_0(\Omega(I)) = \mathcal{F}(D(I))$.*

In particular, if the previous conditions are verified then $\Omega(I)$ is a finitely generated R-algebra.

Proof: Let $Y' := D_W(I\Omega(I))$. We need the following fact proved in [FHP, Theorem 3.3.2],

(4.4.1) *the canonical map $Y' \to Y$, $Q \mapsto Q \cap R$, is a bijection and $R_{Q\cap R} = \Omega(I)_Q$, for each $Q \in Y'$.*

(i) ⇔ (ii). By Lemma 4.3, Y is an affine subspace of X if and only if the canonical map $Y \to \operatorname{Spec}(\Gamma(Y, \mathcal{O}_X|_Y)) = W$ [EGA, I.2.3.2] defines a scheme isomorphism.

(ii) ⇔ (iii). It is a straightforward consequence of (4.4.1), since $I\Omega(I) = \Omega(I)$ if and only if $Y' = W$.

(iii) ⇒ (iv). The flatness of $\Omega(I)$ follows immediately from [R, Theorem 2] and (4.4.1). Moreover if, for some prime ideal P of R with $P \supseteq I$, it happens that $R_P \supseteq \Omega(I)$, then $PR_P \cap \Omega(I)$ is a prime ideal of $\Omega(I)$ containing $I\Omega(I)$. This contradicts (iii).

(iv) ⇒ (iii). If there exists a maximal ideal Q of $\Omega(I)$ such that $Q \supseteq I\Omega(I)$ then $Q \cap R \supseteq I$ and the hypothesis then yields $\Omega(I) \not\subseteq R_{Q\cap R}$. On the other hand, by the flatness of $\Omega(I)$ over R, we have that $R_{Q\cap R} = \Omega(I)_Q$ and this leads to a contradiction.

(iv) ⇔ (v). Under the assumption that $\Omega(I)$ is R-flat, we claim that, for each prime ideal P of R with $P \supseteq I$,

$$\Omega(I) \not\subseteq R_P \Leftrightarrow P\Omega(I) = \Omega(I) . \tag{4.4.2}$$

In fact, if $P\Omega(I) \subsetneqq \Omega(I)$ then there exists a maximal ideal Q of $\Omega(I)$ with $Q \supseteq P\Omega(I)$, so that $\Omega(I)_Q = R_{Q\cap R} \subseteq R_P$, and $\Omega(I) \subseteq R_P$. Conversely, if $\Omega(I) \subseteq R_P$ then $P\Omega(I) \subseteq PR_P \cap \Omega(I) \subsetneqq \Omega(I)$.

(vi) ⇒ (iii). It is sufficient to note that $I \in \mathcal{F}(D(I))$.

(iii) $\Rightarrow$ (vi). Denote simply by $\mathcal{F}_0$ the localizing system $\mathcal{F}_0(\Omega(I)) = \{J : J \text{ ideal of } R$ such that $J\Omega(I) = \Omega(I)\}$ and by $\mathcal{K}$ the localizing system $\mathcal{F}(D(I))$. Since we know already that (iii) implies that $\Omega(I)$ is R-flat, then we have $R_{\mathcal{K}} = \Omega(I) = R_{\mathcal{F}_0}$ and $\mathcal{F}_0 \subseteq \mathcal{K}$ (Lemma 1.19 (1), Corollary 1.20 and (3.2)). Moreover, if $J \in \mathcal{K}$ then $I \subseteq \operatorname{rad}(I) = \operatorname{rad}(J \cap I) \subseteq \operatorname{rad}(J)$ (Lemma 3.2 (1)) and thus $\operatorname{rad}(J) \in \mathcal{F}_0$, because $I \in \mathcal{F}_0$ by assumption. On the other hand, it is easy to see that $\operatorname{rad}(J)\Omega(I) = \Omega(I)$ if and only if $J\Omega(I) = \Omega(I)$, whence $\operatorname{rad}(J) \in \mathcal{F}_0$ is equivalent to $J \in \mathcal{F}_0$.

The last statement follows from Proposition 2.9 ((ii) $\Rightarrow$ (i)) (cf. also [FHP, Proposition 3.3.5]).

□

COROLLARY 4.5. *Let I be an ideal in an integral domain R such that $I\Omega(I) = \Omega(I)$, and write*

$$1 = \sum_{h=1}^{s} i_h y_h \quad \textit{with } i_h \in I \textit{ and } y_h \in \Omega(I) .$$

Set $I_* := \sum_{h=1}^{s} i_h R$ and $E := \sum_{h=1}^{s} y_h R$. Then
(1) $\Omega(I) = \Omega(I_*) = T(I_*)$.
(2) $I^{\Omega} = \operatorname{rad}(I)$.
(3) $I^{\Omega} = I_*^{\Omega} = \operatorname{rad}(I_*) = \operatorname{rad}(I_*)_t = \operatorname{rad}(E^{-1} \cap R)$.
(4) $D(I)$ is a quasi-compact affine (open) subspace of $\operatorname{Spec}(R)$.

Proof: (1). As in the proof of Proposition 2.8, we have that $\Omega(I) = R[E]$ and $\mathcal{K}(I) = \mathcal{K}(I_*)$ and thus $\Omega(I) = \Omega(I_*)$ (Lemma 3.2 (1), (3.1) and (3.2)). Since I_* is finitely generated $T(I_*) = \Omega(I_*)$ (Corollary 3.3 (2)).

(2). From Theorem 4.4 ((iii) $\Rightarrow$ (iv)) we deduce that $\operatorname{rad}(I) = \operatorname{rad}^{\Omega}(I) = I^{\Omega}$.

(3). Since $\Omega(I) = \Omega(I_*)$, $I^{\Omega} = I_*^{\Omega}$ and, by (2), $I_*^{\Omega} = \operatorname{rad}(I_*)$. Moreover, by (1), $\Omega(I_*) = R[E]$ thus $I_*^{\Omega} = \Omega^{-}(E) = \operatorname{rad}(E^{-1} \cap R)$ (Lemma 3.14 (5)). Since I^{Ω} is a t-ideal (Proposition 3.8 (3)), then necessarily $\operatorname{rad}(I_*) = \operatorname{rad}(I_*)_t$.

(4). In this situation, by Theorem 4.4 ((iii) $\Rightarrow$ (ii)), $D(I_*) \cong \operatorname{Spec}(\Omega(I_*)) = \operatorname{Spec}(\Omega(I)) \cong D(I)$, whence $\operatorname{rad}(I) = \operatorname{rad}(I_*)$ with I_* finitely generated.

□

COROLLARY 4.6. *If I is an ideal of an integral domain R such that* $\operatorname{rad}(I)$ *is locally the radical of a nonzero principal ideal (e.g. I is invertible), then $D(I)$ is a quasi-compact affine (open) subspace of* $\operatorname{Spec}(R)$.

Proof: Note that $I\Omega(I) = \Omega(I)$ if (and only if) $IR_P\Omega(IR_P) = \Omega(IR_P)$, for each prime ideal P of R.

As a matter of fact, if $I\Omega(I) \neq \Omega(I)$ then there exists a prime ideal Q of $\Omega(I)$ such that $I\Omega(I) \subseteq Q$. Let $P := Q \cap R$. Then $\Omega_R(I)_Q = \Omega_{R_P}(IR_P)$ because $R_P \subseteq \Omega(I)_Q$ (Lemma 3.1 (l)). By the choice of Q, we have $I\Omega(I)_Q \neq \Omega(I)_Q$. This fact leads to a contradiction since, by assumption, $IR_P\Omega_{R_P}(IR_P) = \Omega_{R_P}(IR_P)$.

The conclusion will follow from Corollary 4.5 if we show that if, given a prime ideal P of R, $\operatorname{rad}(IR_P) = \operatorname{rad}(xR_P)$ for some $x \in R_P$, $x \neq 0$, then $IR_P\Omega_{R_P}(IR_P) = \Omega_{R_P}(IR_P)$. Note that $\Omega_{R_P}(IR_P) = \Omega_{R_P}(\operatorname{rad}(xR_P)) = \Omega_{R_P}(xR_P) = R_P[1/x]$, then $xR_P[1/x] = R_P[1/x]$ and thus $\operatorname{rad}(IR_P)\Omega_{R_P}(IR_P) = \operatorname{rad}(xR_P)R_P[1/x] = R_P[1/x] = \Omega_{R_P}(IR_P)$. Therefore, $IR_P\Omega_{R_P}(IR_P) = \Omega_{R_P}(IR_P)$.

□

The previous result was proved, in the Noetherian setting, by Hartshorne [Ha, § 2 Example 2], making use of cohomological techniques, (cf. also [Se, Theorem 1] and [N1, Theorem 5]).

COROLLARY 4.7. (1) *If I is a nonzero finitely generated ideal in a Prüfer domain, then $D(I)$ is affine.*

(2) *For each nonzero ideal I in a Dedekind domain, $D(I)$ is affine.*

Proof: (1). Note that, in a Prüfer domain, a nonzero finitely generated ideal is invertible [G3, Theorem 22.1].

(2) follows from (1), since a Dedekind domain is a Noetherian Prüfer domain.

□

Note that it is not difficult to prove that statement (2) of Corollary 4.7 holds more generally for each *nonzero* ideal in a 1-dimensional domain with the property that each nonzero element lies in only finitely many maximal ideals.

We close with some examples.

EXAMPLE 4.8. Let $\mathcal{F}$ be a localizing system of an integral domain R. Set $\mathcal{F}_0 := \mathcal{F}_0(R_{\mathcal{F}}) = \{I : I \text{ ideal of } R \text{ and } IR_{\mathcal{F}} = R_{\mathcal{F}}\}$, $\Lambda := \Lambda(R_{\mathcal{F}}) = \{N \cap R : N \in \operatorname{Max}(R_{\mathcal{F}})\}$ and $\mathcal{F}(\Lambda) = \cap\{\mathcal{F}(Q) : Q \in \Lambda\}$.

We give an *example of an integral domain R having a localizing system $\mathcal{F}$ such that $\mathcal{F}_0 = \mathcal{F}(\Lambda)$, but $\mathcal{F}_0 \neq \mathcal{F}$.*

Claim. *If I is an ideal of an integral domain R such that $\Omega(I)$ is R-flat, then $\mathcal{F}_0(\Omega(I)) = \mathcal{F}(\Lambda(\Omega(I)))$.*

In fact, by flatness $\Omega(I) = R_{\mathcal{F}_0(\Omega(I))}$ (Corollary 1.20) and, by Lemma 1.19 (2), we conclude that $\mathcal{F}(\Lambda(\Omega(I))) = \mathcal{F}_0(\Omega(I))$.

Let $R = V$ be a valuation domain having an ascending chain of prime ideals of the following type:

$$P_1 \subset P_2 \subset \cdots \subset P_n \subset \cdots \subset P\ , \quad \text{with } \bigcup_{n \geq 1} P_n = P\ .$$

In this case, $\Omega(P) = \bigcap_{n \geq 1} V_{P_n} = V_P$ is trivially V-flat, hence $\mathcal{F}_0 = \mathcal{F}_0(\Omega(P)) = \mathcal{F}(\Lambda(\Omega(P))) = \mathcal{F}(P)$. On the other hand, if $\mathcal{K} = \mathcal{K}(P) = \mathcal{F}(D(P))$, then we know that $R_{\mathcal{K}} = \Omega(P)$.

Note that $\mathcal{F}_0 \subsetneqq \mathcal{K}$, because if $I \in \mathcal{F}_0$ then $I \supsetneqq P$ and thus $I \in \mathcal{K}$; moreover $P \in \mathcal{K}$ but $P \notin \mathcal{F}_0$. Furthermore, $V_{\mathcal{F}_0} = \Omega(P) = V_{\mathcal{K}}$ and thus $\mathcal{K}$ is a nonfinitely generated localizing system (Corollary 1.25).

PROPOSITION 4.9. *Let I be an ideal of an integral domain R. The localizing system $\mathcal{F}(D(I))$ is a finitely generated localizing system of R if and only if $D(I)$ is a quasi-compact (open) subspace of* $\operatorname{Spec}(R)$.

Proof: ($\Rightarrow$). Since $I \in \mathcal{F}(D(I))$ and $\mathcal{F}(D(I))$ is finitely generated, there exists a finitely generated ideal $J \subseteq I$ with $J \in \mathcal{F}(D(I))$. This fact implies that $\operatorname{rad}(J) = \operatorname{rad}(J \cap I) = \operatorname{rad}(I)$ (Corollary 2.7 (1) or Lemma 3.2 (1)) and hence $D(J) = D(I)$ is quasi-compact [EGA I.1.1.4].

($\Leftarrow$). This implication is a particular case of Lemma 1.16.

□

EXAMPLE 4.10. *An ideal I of an integral domain R such that $\mathcal{F}(D(I))$ is a finitely generated localizing system of R, but $\mathcal{F}_0(\Omega(I)) \neq \mathcal{F}(D(I))$ (in other words, $D(I)$ is quasi–compact but not affine; cf. Theorem 4.4 and Proposition 4.9).*

Let (R, M) be a local integrally closed Noetherian domain of dimension 2. In this case, since M is finitely generated, $D(M)$ is obviously a quasi-compact open subspace of Spec(R). On the other hand, $\Omega(M) = T(M) = \cap\{R_P : P \in \mathrm{Spec}(R)\ P \neq M\} = R$, because R is a Krull domain. Therefore $M\Omega(M) = MR = M \neq \Omega(M)$ and thus $D(M)$ is not affine.

Note that, in this case, $\Omega(M)$ is trivially finitely generated over R, but $M\Omega(M) \neq \Omega(M)$.

EXAMPLE 4.11. *An ideal I of an integral domain R such that $\Omega(I)$ is a finitely generated proper overring of R, but $\Omega(I)$ is not R-flat.*

Let K be a field and X, Y two indeterminates over K. Let $(V, M = \pi V)$ be a 1-dimensional discrete valuation domain of $F := K(X,Y)$ dominating the local ring $K[X,Y]_{(X,Y)}$ and with residue field K. Let $W := K[X,Y]_{(X-1,Y)}$, let N be the maximal ideal of W and set

$$\overline{R} := V \cap W .$$

It is easy to see that $\overline{R}$ is a semilocal Noetherian integrally closed domain with two maximal ideals $\boldsymbol{m} := M \cap \overline{R}$ and $\boldsymbol{n} := N \cap \overline{R}$ such that $\overline{R}_{\boldsymbol{m}} = V$, $\overline{R}_{\boldsymbol{n}} = W$. Let $J(\overline{R}) = M \cap N = \pi N$ be the Jacobson radical of $\overline{R}$ and set

$$R := K + J(\overline{R}) .$$

Then, R is a 2-dimensional local subring of $\overline{R}$ such that $\overline{R}$ is the integral closure of R, $\overline{R}$ is finitely generated over R (hence R is Noetherian) and $(R : \overline{R}) = J(\overline{R})$ [N6, E2.1].

It is easy to see that

$$T_R(J(\overline{R})) = \Omega_R(J(\overline{R})) \supseteq (J(\overline{R}) : J(\overline{R})) \supseteq \overline{R} ,$$

and thus (Lemma 3.1 (e)):

$$\Omega_R(J(\overline{R})) = \Omega_{\overline{R}}(J(\overline{R})) = \Omega_{\overline{R}}(\pi N) = \overline{R}[1/\pi] = \overline{R}_\pi = W .$$

Therefore, $R \subset \overline{R} \subset \Omega_R(J(\overline{R})) = \overline{R}[1/\pi]$ is finitely generated, but $J(\overline{R})\Omega_R(J(\overline{R})) = J(\overline{R})W = N \neq \Omega_R(J(\overline{R})) = W$, whence $\Omega_R(J(\overline{R}))$ is not R-flat (cf. also Theorem 4.4).

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