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Polynomial Closure in Essential Domains and Pullbacks

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ABSTRACT. Let D be a domain with quotient field K . Let $E \subseteq K$ be a subset; the ring of D -integer-valued polynomials over E is $\text{Int}(E, D) := \{f \in K[X]; f(E) \subseteq D\}$. The polynomial closure in D of a subset $E \subseteq K$ is the largest subset $F \subseteq K$ containing E such that $\text{Int}(E, D) = \text{Int}(F, D)$, and it is denoted by $cl_D(E)$. We study the polynomial closure of ideals in several classes of domains, including essential domains and domains of strong Krull-type, and we relate it with the t -closure. For domains of Krull-type we also compute the Krull dimension of $\text{Int}(D)$.

INTRODUCTION

Let D be any integral domain with quotient field K . For each subset $E \subseteq K$, $\text{Int}(E, D) := \{f \in K[X]; f(E) \subseteq D\}$ is called the *ring of D -integer-valued polynomials over E* . As usual, when $E = D$, we set $\text{Int}(D) := \text{Int}(D, D)$. When E is “large enough”, it may happen that $\text{Int}(E, D) = D$ (for instance, $\text{Int}(S^{-1}\mathbb{Z}, \mathbb{Z}) = \mathbb{Z}$ for each nontrivial multiplicative subset S of \mathbb{Z} , [CC2, Corollary I.1.10]). This does not happen if E is a D -fractional subset of K , i.e. if there exists $d \in D \setminus (0)$ such that $dE \subseteq D$. Indeed, in this case, $dX \in \text{Int}(E, D)$. It is well known that

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$\text{Int}(D) \subseteq \text{Int}(E, D)$ if and only if $E \subseteq D$ [CC2, Corollary I.1.7]. Two subsets E and F of K may be distinct while $\text{Int}(E, D) = \text{Int}(F, D)$. When this happens, we say that E and F are *polynomially D -equivalent*. For instance, \mathbb{N} and \mathbb{Z} are polynomially \mathbb{Z} -equivalent [CC2, Corollary I.1.2]. In particular, if $\text{Int}(E, D) = \text{Int}(D)$ we say that E is a *polynomially dense* subset of D (so, \mathbb{N} is a polynomially dense subset of \mathbb{Z}). In [G1] Gilmer characterized the polynomially dense subsets of \mathbb{Z} and, in [C2] and [C4], Cahen studied the polynomial density, with special emphasis to the Noetherian domains case. McQuillan, pursuing Gilmer's work, investigated the polynomially D -equivalent subsets of a Dedekind domain D [Mc]. Among other results, he proved that two fractional ideals I and J of a Dedekind domain D are polynomially equivalent if and only if $I = J$. After noticing that, Cahen introduced the notion of *polynomial closure* (in D) of a subset E of K as follows:

$$cl_D(E) := \{x \in K; f(x) \in D, \text{ for each } f \in \text{Int}(E, D)\},$$

that is, $cl_D(E)$ is the largest subset $F \subseteq K$ such that $\text{Int}(E, D) = \text{Int}(F, D)$. Obviously, E is said to be *polynomially dense* in D if $cl_D(E) = D$ and *polynomially D -closed* if $cl_D(E) = E$.

In the first section of this paper, we consider *essential domains*, that is, domains D such that

$$(1) \quad D = \bigcap_{P \in \mathcal{P}} D_P,$$

where \mathcal{P} is a subset of $\text{Spec}(D)$ and D_P is a valuation domain. In particular, among these domains, we will focus our attention on the *strong Krull-type domains*, that is the essential domains D such that the intersection (1) is locally finite (i.e., each nonzero of D belongs to finitely many prime ideals $P \in \mathcal{P}$) and the valuation rings D_P are pairwise independent. Examples of strong Krull-type domains are *Krull domains* and *generalized Krull domains* [G1, p. 524]. We prove that if E is a fractional subset of a strong Krull-type domain D , then $cl_D(E) = \bigcap_{P \in \mathcal{P}} cl_{D_P}(E_P)$, where $cl_{D_P}(E_P)$ denotes the polynomial D_P -closure of $E_P := \{e/s; e \in E, s \in D \setminus P\}$. This yields a generalization of a result proved by Cahen for Krull domains, [C3] or [C4]. Moreover, we study the polynomial closure as a star-operation and we relate it to the t -operation. We find that if D is an essential domain, then $I_t \subseteq cl_D(I)$, for each fractional ideal I of D and, for some distinguished classes of domains, as Krull domains, Prüfer domains in which each nonzero ideal is divisorial, $I_t = cl_D(I)$.

In the second section we compute the Krull dimension of the ring of $\text{Int}(D)$, when D is a domain with a locally finite representation. By using this result we show that if D is a domain of Krull-type then $\dim(\text{Int}(D)) = \dim(D[X])$, obtaining further evidence for the validity of the conjecture about the Krull dimension of $\text{Int}(D)$ stating that $\dim(\text{Int}(D)) \leq \dim(D[X])$ for each integral domain D (cf. [Ch], [C1], and [FIKT]).

In Section 3, we study the quotient of the polynomial closure of a subset modulo a divided prime ideal, and we apply this result to some classes of domains defined by making use of pullback constructions.

1. POLYNOMIAL CLOSURE IN STRONG KRULL-TYPE AND NOETHERIAN DOMAINS

We start this section by studying the polynomial closure of the ideals in a *strong Krull-type domain*, that is a domain D having the following representation:

$$(1.0.I) \quad D = \bigcap_{P \in \mathcal{P}} D_P,$$

where $\mathcal{P} \subseteq \text{Spec}(D)$, the intersection (1.0.I) is locally finite and the rings D_P are pairwise independent valuation domains.

We prove for this class of domains some results already proved by Cahen in case of Noetherian and Krull domains [C3, Proposition 1.3], [C4, Proposition 3.2, 3.5, 3.6, Corollaries 3.7 and 3.8].

THEOREM 1.1. *With the notation above, let D be a strong Krull-type domain with quotient field K and let E be any D -fractional subset of K . Then:*

- (1) $\text{Int}(E, D)_P = \text{Int}(E, D_P)$, for each $P \in \mathcal{P}$;
- (2) $cl_D(E) = \bigcap_{P \in \mathcal{P}} cl_{D_P}(E_P)$;
- (3) if $E \subseteq D$, then E is polynomially dense in D if and only if E is polynomially dense in D_P for each $P \in \mathcal{P}$.

Proof. (1) The argument of the proof runs parallel with the one used in [C4, Proposition 3.2] (cf. also [CC2, Proposition I.2.8 and IV.2.9]). We wish to prove that $\text{Int}(E, D_{\overline{P}}) \subseteq \text{Int}(E, D)_{\overline{P}}$, for each fixed ideal $\overline{P} \in \mathcal{P}$ (the opposite inclusion holds in general, [CC2, Lemma I.2.4]).

Since E is a fractional D -subset of K , with a standard argument we can easily assume, without loss of generality, that $E \subseteq D$. Let $f \in \text{Int}(E, D_{\overline{P}})$, $f \neq 0$. It is obvious that there exists $d \in D$, $d \neq 0$, such that $df \in D[X]$. Set

$$\mathcal{P}(d) := \{Q \in \mathcal{P}; d \in Q \text{ and } Q \not\subseteq \overline{P}\}.$$

Since the given representation of D is locally finite, then $\mathcal{P}(d)$ is a finite set.

We claim that there exists $a \in D$ such that $a \in dD_Q \setminus \overline{P}$, for each $Q \in \mathcal{P}(d)$, that is, $v_Q(a) \geq v_Q(d)$, for each $Q \in \mathcal{P}(d)$, and $v_{\overline{P}}(a) = 0$, where v_P is the valuation associated to the ring D_P for each $P \in \mathcal{P}$.

The Approximation Theorem for valuations [B, Ch. VI § 7 n. 2, Corollaire 1, p. 135] states that there exists an element $b \in K$ such that $v_{\overline{P}}(b) = 0$ and $v_Q(b) \geq v_Q(d)$, for each $Q \in \mathcal{P}(d)$. Now, there exist $a, c \in D$, $a \notin \overline{P}$, such that $b = a/c$. For each $P \in \mathcal{P}$, we have that $v_P(a) = v_P(b) + v_P(c)$. Thus, if $Q \in \mathcal{P}(d)$, then $v_Q(c) \geq 0$ (since $c \in D \subseteq D_Q$) and $v_Q(a) \geq v_Q(b) \geq v_Q(d)$. Moreover, since $v_{\overline{P}}(b) = v_{\overline{P}}(c) = 0$, then also $v_{\overline{P}}(a) = 0$.

Therefore $af \in D_P[X]$ for each $P \in \mathcal{P}(d)$ and $P \in \mathcal{P}$ such that $d \notin P$. As a matter of fact, for each $P \in \mathcal{P} \setminus \mathcal{P}(d)$, with $d \notin P$, we have that d is a unit in D_P . Thus, since $df \in D[X]$, we deduce that $f \in D_P[X]$ and $af \in D_P[X]$. If $Q \in \mathcal{P}(d)$, then $ad^{-1} \in D_Q$ and $af = (ad^{-1})(df) \in D_Q[X]$. In these cases, since $E \subseteq D$ and $af \in D_P[X]$, then $af(E) \subseteq D_P$, that is, $af \in \text{Int}(E, D_P)$. On the other hand, $f \in \text{Int}(E, D_{\overline{P}})$ and $a \in D$, whence $af(E) \subseteq D_{\overline{P}}$. If $d \in P$ and $P \subseteq \overline{P}$, then $D_P \supseteq D_{\overline{P}}$. Thus $f(E) \subseteq D_{\overline{P}} \subseteq D_P$ and $f \in \text{Int}(E, D_P)$.

We conclude that $af(E) \subseteq \bigcap_{P \in \mathcal{P}} D_P = D$, that is $af \in \text{Int}(E, D)$. Hence $f \in \text{Int}(E, D)_{\overline{P}}$, because $a \in D \setminus \overline{P}$.

(2) follows from (1) and from [C4, Proposition 3.5].

(3) is a straightforward consequence of (2). \square

In order to deepen the study of the polynomial closure of fractional ideals, we recall some properties about star-operations.

Let D be an integral domain with quotient field K , let $\mathfrak{F}(D)$ denote the set of nonzero fractional ideals of D and let $\mathfrak{F}_{\text{fg}}(D)$ denote the subset of $\mathfrak{F}(D)$ of nonzero finitely generated fractional ideals of D . A mapping $I \mapsto I^*$ of $\mathfrak{F}(D)$ into $\mathfrak{F}(D)$ is called a *star-operation on D* if the following conditions hold for all $a \in K \setminus \{0\}$ and $I, J \in \mathfrak{F}(D)$:

$$(*1) \quad (aD)^* = aD;$$

$$(*2) \quad (aI)^* = aI^*;$$

$$(*3) \quad I \subseteq I^*;$$

$$(*4) \quad I \subseteq J \Rightarrow I^* \subseteq J^*;$$

$$(*5) \quad I^{**} = I^*.$$

A fractional ideal $I \in \mathfrak{F}(D)$ is called a *star-ideal* if $I = I^*$. A *star-operation $*$ on D* is said to be of *finite character* if, for each $I \in \mathfrak{F}(D)$,

$$I^* = \cup \{J^*; J \subseteq I \text{ and } J \in \mathfrak{F}_{\text{fg}}(D)\}.$$

Given a star-operation $*$, then the function $*_s$ defined as follows:

$$I \mapsto I^{*s} = \cup \{J^*; J \subseteq I \text{ and } J \in \mathfrak{F}_{\text{fg}}(D)\}$$

is a star-operation of finite character. The star-operation $*_s$ is called the *star-operation of finite character associated to $*$* . It is obvious that:

$$J^* = J^{*s}, \text{ for each } J \in \mathfrak{F}_{\text{fg}}(D),$$

$$I^{*s} \subseteq I^*, \text{ for each } I \in \mathfrak{F}(D).$$

The *v -operation*

$$I \mapsto I_v := (D : (D : I))$$

is a star-operation. The *t -operation*

$$I \mapsto I_t := \cup \{J_v; J \subseteq I \text{ and } J \in \mathfrak{F}_{\text{fg}}(D)\}$$

is the star-operation of finite character associated to the v -operation (cf. [G2, Sections 32 and 34]).

The following result is implicitly proved by Cahen [C4, Lemma 1.2].

LEMMA 1.2. *Let D be an integral domain, then the polynomial closure*

$$cl_D : \mathfrak{F}(D) \rightarrow \mathfrak{F}(D), \quad I \mapsto cl_D(I),$$

is a star-operation. \square

COROLLARY 1.3. *Let D be an integral domain. Then, for all I, J in $\mathfrak{F}(D)$ and for each subset $\{I_\alpha; \alpha \in A\}$ of $\mathfrak{F}(D)$, we have:*

- (1) $cl_D(\sum_\alpha I_\alpha) = cl_D(\sum_\alpha cl_D(I_\alpha))$, if $\sum_\alpha I_\alpha \in \mathfrak{F}(D)$;
- (2) $\cap_\alpha cl_D(I_\alpha) = cl_D(\cap_\alpha cl_D(I_\alpha))$, if $\cap_\alpha I_\alpha \neq (0)$;
- (3) $cl_D(IJ) = cl_D(Icl_D(J)) = cl_D(cl_D(I)J) = cl_D(cl_D(I)cl_D(J))$;
- (4) $cl_D(I) \subseteq I_v$;
- (5) $cl_D(I_v) = I_v$.

Proof. It is a straightforward consequence of Lemma 1.2 and of [G2, Proposition 32.2 and Theorem 34.1(4)]. \square

Note that, from Corollary 1.3(1) and (3), we recover for the fractional ideals some results proved by Cahen for subsets [C4, Lemma 2.4], in particular we obtain that $cl_D(I) + cl_D(J) \subseteq cl_D(I + J)$, $cl_D(I)cl_D(J) \subseteq cl_D(IJ)$.

We will need the following result, that is a consequence of [C4, Proposition 3.5(2)], in order to deepen the relation between the polynomial closure and the star-operations.

LEMMA 1.4. *Let D be an integral domain and let \mathcal{P} be a subset of $\text{Spec}(D)$ such that $D = \cap_{P \in \mathcal{P}} D_P$. For each $I \in \mathfrak{F}(D)$, we have:*

$$\cap_{P \in \mathcal{P}} cl_{D_P}(ID_P) \subseteq cl_D(I). \quad \square$$

It is well known from the theory of star-operations that, if $\{D_\alpha; \alpha \in A\}$ is a collection of overrings of an integral domain D such that $D = \cap_{\alpha \in A} D_\alpha$, and if $*_\alpha$ is a star-operation on D_α , for each $\alpha \in A$, then the mapping

$$I \mapsto I^{*^A} := \cap\{(ID_\alpha)^{*\alpha}; \alpha \in A\}$$

is a star-operation on D and $(I^{*^A} D_\alpha)^{*\alpha} = (ID_\alpha)^{*\alpha}$, for each $\alpha \in A$ [A, Theorem 2]. If $D = \cap_{P \in \mathcal{P}} D_P$, for some subset $\mathcal{P} \subseteq \text{Spec}(D)$, we call the \mathcal{P} -polynomial closure of $I \in \mathfrak{F}(D)$, the following fractional ideal of D :

$$\mathcal{P}\text{-}cl_D(I) := \cap_{P \in \mathcal{P}} cl_{D_P}(ID_P).$$

PROPOSITION 1.5. *Let D be an integral domain such that $D = \cap_{P \in \mathcal{P}} D_P$, for some subset \mathcal{P} of $\text{Spec}(D)$.*

- (1) *The mapping:*

$$I \mapsto \mathcal{P}\text{-}cl_D(I)$$

defines a star-operation on D , with $\mathcal{P}\text{-}cl_D(I) \subseteq cl_D(I)$ for each $I \in \mathfrak{F}(D)$.

- (2) *Let $I \in \mathfrak{F}(D)$. If $ID_P \neq D_P$ for finitely many $P \in \mathcal{P}$, then*

$$\mathcal{P}\text{-}cl_D(I)D_P = cl_{D_P}(ID_P).$$

- (3) *If $cl_{D_P}(F \cap G) = cl_{D_P}(F) \cap cl_{D_P}(G)$, for each $P \in \mathcal{P}$ and for all $F, G \in \mathfrak{F}(D_P)$, then*

$$\mathcal{P}\text{-}cl_D(I \cap J) = \mathcal{P}\text{-}cl_D(I) \cap \mathcal{P}\text{-}cl_D(J), \text{ for all } I, J \in \mathfrak{F}(D).$$

(4) If $cl_{D_P}((F:_{D_P} G)) = (cl_{D_P}(F):_{D_P} cl_{D_P}(G))$, for each $P \in \mathcal{P}$, $F \in \mathfrak{F}(D_P)$ and $G \in \mathfrak{F}_{fg}(D_P)$, then

$$\mathcal{P}\text{-cl}_D((I:_{D} J)) = (\mathcal{P}\text{-cl}_D(I):_{D} \mathcal{P}\text{-cl}_D(J)),$$

for each $I \in \mathfrak{F}(D)$ and $J \in \mathfrak{F}_{fg}(D)$.

(5) If $D = \bigcap_{P \in \mathcal{P}} D_P$ is locally finite and if, for each $P \in \mathcal{P}$, cl_{D_P} is a star-operation on D_P with finite character, then $\mathcal{P}\text{-cl}_D$ is a star-operation on D with finite character.

Proof. These results are a straightforward consequence of Lemma 1.4, of the definition of the \mathcal{P} -polynomial closure and of [A, Theorem 2]. \square

COROLLARY 1.6. If $D = \bigcap_{P \in \mathcal{P}} D_P$ is a strong Krull-type or a Noetherian domain, then, for each $I \in \mathfrak{F}(D)$, we have that

$$\mathcal{P}\text{-cl}_D(I) = cl_D(I).$$

Proof. If D is strong Krull-type, the thesis is a consequence of Theorem 1.1(2) and the definition of \mathcal{P} -polynomial closure. More precisely, if $\text{Int}(ID_P, D_P) = \text{Int}(I, D)_P$, then $cl_D(I) \subseteq cl_{D_P}(ID_P)$ [C4, Proposition 3.5(1)] and hence $cl_D(I) \subseteq \mathcal{P}\text{-cl}_D(I)$. Therefore $cl_D(I) = \mathcal{P}\text{-cl}_D(I)$ by Lemma 1.4.

If D is Noetherian, then $S^{-1}\text{Int}(I, D) = \text{Int}(S^{-1}I, S^{-1}D)$, for each multiplicative set S of D [CC2, Proposition I.2.7]. As shown above for strong Krull-type domains, also in the Noetherian domain case, for $\mathcal{P} = \text{Max}(D)$, we have that:

$$cl_D(I) = \bigcap_{P \in \mathcal{P}} cl_{D_P}(ID_P),$$

for each $I \in \mathfrak{F}(D)$, that is, $cl_D(I) = \mathcal{P}\text{-cl}_D(I)$, by definition of \mathcal{P} -polynomial closure. \square

COROLLARY 1.7. Let D be a Noetherian domain.

(1) For each $M \in \text{Max}(D)$ and for each $I \in \mathfrak{F}(D)$, we have:

$$cl_D(I)D_M = cl_{D_M}(ID_M).$$

(2) If S is a multiplicative subset of D , for each $I \in \mathfrak{F}(D)$ we have:

$$S^{-1}cl_D(I) = cl_{S^{-1}D}(S^{-1}I).$$

In particular, if I is polynomially closed in D then $S^{-1}I$ is polynomially closed in $S^{-1}D$.

(3) For each $M \in \text{Max}(D)$,

$$cl_D(M) = (MD_M)_v \cap D = M_v.$$

In particular, M is polynomially closed (respectively, polynomially dense) in D if and only if $M = M_v$ (respectively, $M_v = D$) or, equivalently, if and only if MD_M is polynomially closed (respectively, polynomially dense) in D_M .

(4) For each nonzero ideal I of D there exists a prime ideal P of D such that $I \subseteq P = cl_D(P)$.

(5) If $\dim(D) = 1$, then every nonzero prime ideal of D is polynomially closed.

Proof. (1) is an easy consequence of Proposition 1.5(2) and Corollary 1.6, since D is Noetherian and $D = \bigcap_{M \in \text{Max}(D)} D_M$.

(2) Note that $S^{-1}D = \bigcap \{D_M; M \in \text{Max}(D) \text{ and } M \cap S = \emptyset\}$ is locally finite. Hence the conclusion follows from (1) and from Corollary 1.6, since

$$\begin{aligned} S^{-1}cl_D(I) &= \bigcap \{cl_D(I)D_M; M \in \text{Max}(D) \text{ and } M \cap S = \emptyset\} = \\ &= \bigcap \{cl_{D_M}(ID_M); M \in \text{Max}(D) \text{ and } M \cap S = \emptyset\} = \\ &= cl_{S^{-1}D}(S^{-1}I). \end{aligned}$$

(3) By Corollary 1.6 we have

$$\begin{aligned} cl_D(M) &= \bigcap_{N \in \text{Max}(D)} cl_{D_N}(MD_N) = \\ &= cl_{D_M}(MD_M) \cap (\bigcap \{D_N; N \in \text{Max}(D), N \neq M\}) = cl_{D_M}(MD_M) \cap D. \end{aligned}$$

Moreover, by the proof of [C4, Proposition 2.3], we know that if (R, \mathfrak{m}) is a local Noetherian domain then $cl_R(\mathfrak{m}) = \mathfrak{m}_v$. Finally, since D is Noetherian, by [G2, Theorem 4.4(4)], we have that $(MD_M)_v = M_v D_M$ and by [G2, Theorem 4.10(3)] we have

$$M_v = (MD_M)_v \cap D.$$

The conclusion is straightforward.

(4) Since D is Noetherian, cl_D is a star-operation on D with finite character (Corollary 1.6). It is well known, in this situation, that each proper star-ideal of D is contained in a maximal proper star-ideal of D and that a maximal proper star-ideal of D is a prime ideal (cf. for example [J]).

(5) follows immediately from (4). \square

Note that Corollary 1.7(2) gives a positive answer to Question 3.10 in [C4] and Corollary 1.7(3) generalizes to the nonlocal case [C4, Proposition 2.3]. Note also that Cahen [C4, Example 3.9] has given an example of an ideal I of an integrally closed (non Noetherian) domain D such that $I = cl_D(I)$ and $S^{-1}I \neq cl_{S^{-1}D}(S^{-1}I)$, for some multiplicative set S of D .

The equality in Corollary 1.7(3) does not hold for the nonmaximal ideals, i.e. the inclusion $cl_D(I) \subseteq I_v$ may be a proper inclusion even in the Noetherian local case. In fact, it is enough to consider a local, Noetherian, one-dimensional, analitically irreducible domain D with finite residue field and a nonzero nonmaximal ideal I of D (cf. [C4, Corollary 4.8] or [CC2, Theorem IV.1.15]). For instance, let k be a finite field, $D := k[[X^3, X^4, X^5]]$ and $I := (X^3, X^4)D$. In this case $(D : I) = k[[X]]$, hence $I_v = (X^3, X^4, X^5)D$; but $I = cl_D(I)$ [C4, Corollary 4.8].

We recall some definitions. An *essential domain* is an integral domain D such that $D = \bigcap_{P \in \mathcal{P}} D_P$, where D_P is a valuation domain for P belonging to a subset \mathcal{P} of $\text{Spec}(D)$. If D is an essential domain with the valuation rings D_P pairwise independent and $D = \bigcap_{P \in \mathcal{P}} D_P$ is locally finite (i.e. each nonzero element of D belongs to finitely many prime ideals P of \mathcal{P}) then D is a domain of strong Krull-type. Obviously, each Prüfer domain is an essential domain and each Krull domain is a domain of strong Krull-type.

A relevant case is when \mathcal{P} is the set $t_m(D)$ of all t -maximal ideals of D (i.e. the maximal elements among the integral t -ideals of D). It is well known that each maximal t -ideal is a prime ideal and, for each ideal I of D , $I = \bigcap_{P \in t_m(D)} ID_P$; in particular $D = \bigcap_{P \in t_m(D)} D_P$ [Gr, Proposition 4]. A Prüfer v -multiplication domain D is an integral domain such that D_P is a valuation domain for each $P \in t_m(D)$. This class of domains was introduced by Griffin [Gr].

In order to study the polynomial closure of fractional ideals in an essential domain, we start by considering the local case, i.e. when $D = V$ is a valuation domain.

PROPOSITION 1.8. *Let V be a valuation domain with maximal ideal M .*

- (1) *If M is principal, then, for each nonzero fractional ideal I of V , $I = cl_V(I) = I_v$.*
- (2) *If M is not principal, then:*
 - a) $cl_V(M) = M_v = V$;
 - b) *for each nonzero ideal I of V , $cl_V(I) = I_v$; moreover, if $I \neq I_v$, then $cl_V(I) = I_v$ is a principal ideal of V .*

Proof. We recall that, in general, for each integral domain D and for each $I \in \mathfrak{F}(D)$ we have the following inclusions: $I \subseteq cl_D(I) \subseteq I_v$ (Corollary 1.3).

(1) If M is principal, then each nonzero fractional ideal of V is divisorial [G2, Exercise 12, p. 431]. The conclusion follows immediately from the previous tower of inclusions.

(2) If M is not principal, then $\{aM; a \in V, a \neq 0\}$ is the set of all nonzero nondivisorial (integral) ideals of V [G2, Exercise 12, p. 431]. Therefore, in this case, $M \neq M_v$, hence $M_v = V$.

(a) In order to prove that $cl_V(M) = V$, we will show that $\text{Int}(V) = \text{Int}(V, V) = \text{Int}(M, V)$.

Since M is not principal, $V[X] = \text{Int}(V)$ [CC2, Proposition I.3.16]. Let $f := c_0 + c_1X + \dots + c_nX^n \in \text{Int}(M, V)$ be a polynomial of degree n . By [CC2, Corollary I.3.3], if a_0, a_1, \dots, a_n are $n+1$ elements of M and if $d := \prod_{0 \leq i < j \leq n} (a_i - a_j)$, then $df \in V[X]$. Let \mathbf{v} be the valuation associated to V . By using the assumption that M is not finitely generated, we can choose the elements a_i 's such that $0 < \mathbf{v}(d) < |\mathbf{v}(c_i)|$, for each c_i such that $\mathbf{v}(c_i) \neq 0$. On the other hand, $df \in V[X]$ hence $\mathbf{v}(dc_i) = \mathbf{v}(d) + \mathbf{v}(c_i) \geq 0$, for each $0 \leq i \leq n$. If $f \notin V[X]$, then $\mathbf{v}(c_i) < 0$ for some i with $0 \leq i \leq n$, hence we have a contradiction. Therefore, we can conclude that $V[X] = \text{Int}(M, V)$ and thus a) holds.

b) It is obvious that, if $I = I_v$, then $cl_V(I) = cl_V(I_v) = I_v$ (Corollary 1.3(5)). If $I \neq I_v$ and $I \subseteq V$, then $I = aM$ for some nonzero element $a \in V$, hence $cl_V(I) = cl_V(aM) = acl_V(M)$ (Lemma 1.2). By point a), we deduce that $cl_V(I) = aV$ is a principal (hence, divisorial) ideal and $cl_V(I) = I_v$. If $I \neq I_v$ and I is a fractional ideal of V , then $bI \subset V$ and $bI \neq bI_v$, for some nonzero element $b \in V$. The conclusion follows easily from the previous argument. \square

THEOREM 1.9. *Let $D = \bigcap_{P \in \mathcal{P}} D_P$ be an essential domain.*

- (1) *For each $J \in \mathfrak{F}_{\text{fg}}(D)$, we have:*

$$\mathcal{P}\text{-}cl_D(J) = cl_D(J) = J_v.$$

(2) For each $I \in \mathfrak{F}(D)$, we have:

$$I_t \subseteq \mathcal{P}\text{-cl}_D(I) \subseteq \text{cl}_D(I) \subseteq I_v.$$

Proof. (1) If J is finitely generated, then

$$J_v D_P \subseteq (J D_P)_v, \text{ for each prime ideal } P \text{ of } D,$$

[B, Ch. I § 2 n. 11 (11), p. 41]. On the other hand, by Proposition 1.8,

$$\begin{aligned} J_v &\subseteq \bigcap_{P \in \mathcal{P}} J_v D_P \subseteq \bigcap_{P \in \mathcal{P}} (J D_P)_v = \\ &= \bigcap_{P \in \mathcal{P}} \text{cl}_D(J D_P) = \mathcal{P}\text{-cl}_D(J). \end{aligned}$$

The conclusion follows by recalling that, in general for each $I \in \mathfrak{F}(D)$, we have:

$$\mathcal{P}\text{-cl}_D(I) \subseteq \text{cl}_D(I) \subseteq I_v.$$

(2) Since $I_t := \cup\{J_v; J \subseteq I \text{ and } J \in \mathfrak{F}_{fg}(D)\}$ then, by Proposition 1.5(1) and by (1), we have:

$$I_t := \cup\{\mathcal{P}\text{-cl}_D(J); J \subseteq I \text{ and } J \in \mathfrak{F}_{fg}(D)\} \subseteq \mathcal{P}\text{-cl}_D(I). \quad \square$$

COROLLARY 1.10. Let $D = \bigcap_{P \in \mathcal{P}} D_P$ be an essential domain.

(1) For all $J', J'' \in \mathfrak{F}_{fg}(D)$:

$$\text{cl}_D(J' \cap J'') = \text{cl}_D(J') \cap \text{cl}_D(J'').$$

(2) For all $I', I'' \in \mathfrak{F}(D)$, then:

$$\mathcal{P}\text{-cl}_D(I' \cap I'') = \mathcal{P}\text{-cl}_D(I') \cap \mathcal{P}\text{-cl}_D(I'').$$

Proof. (1) follows from (2) and from Theorem 1.9(1).

(2) Since D_P is a valuation domain, for each $P \in \mathcal{P}$, then either $I' D_P \subseteq I'' D_P$ or $I'' D_P \subseteq I' D_P$, hence $\text{cl}_{D_P}(I' D_P \cap I'' D_P) = \text{cl}_{D_P}(I' D_P) \cap \text{cl}_{D_P}(I'' D_P)$. The conclusion follows from Proposition 1.5(3). \square

Let $D = \bigcap_{P \in t_m(D)} D_P$ and let $I \in \mathfrak{F}(D)$. In this case we set $\mathcal{P} = t_m(D)$ and

$$t\text{-cl}_D(I) := \bigcap_{P \in t_m(D)} \text{cl}_{D_P}(I D_P).$$

COROLLARY 1.11. Let $D = \bigcap_{P \in t_m(D)} D_P$ be a Prüfer v -multiplication domain. Assume that, for each maximal t -ideal P of D , $P D_P$ is a principal ideal. Then, for each $I \in \mathfrak{F}(D)$, we have:

$$I_t = t\text{-cl}_D(I).$$

If, moreover, $D = \bigcap_{P \in t_m(D)} D_P$ is locally finite and the valuation rings D_P are pairwise independent (i.e. D is an integral domain of strong Krull-type) then, for each $I \in \mathfrak{F}(D)$, we have

$$I_t = t\text{-cl}_D(I) = \text{cl}_D(I).$$

Proof. In a Prüfer v -multiplication domain D , for each $I \in \mathfrak{F}(D)$, $I_t = \bigcap_{P \in t_m(D)} I D_P$ (cf. for instance [A, Theorem 6]).

On the other hand, by Proposition 1.8(1),

$$\bigcap_{P \in t_m(D)} I D_P \subseteq t\text{-cl}_D(I) = \bigcap_{P \in t_m(D)} \text{cl}_{D_P}(I D_P) = \bigcap_{P \in t_m(D)} I D_P,$$

hence $I_t = t\text{-cl}_D(I)$. The last statement is a consequence of Corollary 1.6. \square

COROLLARY 1.12. *Let $D = \bigcap_{P \in t_m(D)} D_P$ be an integral domain of strong Krull-type. Assume that, for each $P \in t_m(D)$, there exists a finitely generated ideal J of D such that $J \subseteq P$ and $J^{-1} = P^{-1}$ and that each prime t -ideal of D is contained in a unique maximal t -ideal. Then, for each $I \in \mathfrak{F}(D)$, we have:*

$$I_t = t\text{-cl}_D(I) = cl_D(I) = I_v.$$

Proof. These assumptions characterize the Prüfer v -multiplication domains such that each t -ideal is divisorial [HZ, Theorem 3.1]. The conclusion is a straightforward consequence of Theorem 1.9(2). \square

Examples of integral domains satisfying the assumptions of Corollary 1.12 are Krull domains and the Prüfer domains in which each nonzero ideal is divisorial (cf. [H, Theorem 5.1] and [K, 127]).

REMARK 1.13. For each nonzero fractional ideal I of an integral domain D , since I^{-1} is divisorial, $I \subseteq cl_D(I) \subseteq I_v$ and $I^{-1} = I_v^{-1}$, we have:

$$(1.13.I) \quad cl_D(I^{-1}) = cl_D(I)^{-1} = I^{-1}.$$

Since $I_v = (I^{-1})^{-1}$, then the previous identity generalizes the fact that $cl_D(I_v) = I_v$ (Corollary 1.3(5)). From (1.13.I), we deduce that if $D \neq I^{-1}$, then I is not polynomially dense in D . In particular, for a maximal ideal M of D , we obtain that $D \neq M^{-1}$ implies that $M = cl_D(M)$. This statement could be obtained also as a consequence of Corollary 1.3(5), since $M = M_v$ if and only if $D \neq M^{-1}$.

2. KRULL DIMENSION OF $\text{Int}(D)$ WHEN D IS OF KRULL-TYPE

This section is devoted to the study of the Krull dimension of the ring $\text{Int}(D)$ for the integral domains D having a locally finite representation. If a domain D has a representation

$$D = \bigcap_{P \in \mathcal{P}} D_P$$

where $\mathcal{P} \subseteq \text{Spec}(D)$, the ring D_P are valuation domains for each $P \in \mathcal{P}$ and the intersection is locally finite, then D is called a *Krull-type domain*. Strong Krull-type domains studied in Section 1, are a particular case of Krull-type domains.

As a consequence of our main result we prove that, for each domain of Krull-type D , $\dim(\text{Int}(D)) = \dim(D[X])$. This improves the knowledge of the Krull dimension of the ring of integer-valued polynomials giving further evidence for the conjecture stating that $\dim(\text{Int}(D)) \leq \dim(D[X])$, for each integral domain D .

THEOREM 2.1. *Let D be an integral domain and \mathcal{P} a subset of $\text{Spec}(D)$. Assume that $D = \bigcap_{P \in \mathcal{P}} D_P$ is a locally finite representation of D . Set $\mathcal{P}_0 := \{P \in \mathcal{P} \cap \text{Max}(D); \text{Card}(D/P) < \infty\}$ and $\mathcal{M} := \text{Max}(D) \setminus \mathcal{P}_0$. Then*

$$\dim(\text{Int}(D)) = \text{Max}(\{\dim(D_M[X]); M \in \mathcal{M}\}, \{\dim(\text{Int}(D_P)); P \in \mathcal{P}_0\}).$$

Proof. We note that, for each maximal ideal M of D ,

$$(2.1.I) \quad D_M = \bigcap_{P \in \mathcal{P}} (D_P)_{(D \setminus M)} = (\bigcap_{P \in \mathcal{P}, P \subseteq M} D_P) \cap (\bigcap_{P \in \mathcal{P}, P \not\subseteq M} (D_P)_{(D \setminus M)})$$

is a locally finite representation of D_M [G2, Proposition 43.5]. Since $D = \bigcap_{M \in \text{Max}(D)} D_M$, then, by [CC1, Corollaire 3, p. 303],

$$\text{Int}(D) = \bigcap_{M \in \text{Max}(D)} \text{Int}(D_M).$$

Since $\text{Int}(D) \subseteq \text{Int}(D)_M \subseteq \text{Int}(D_M)$, for each $M \in \text{Max}(D)$, it follows that

$$(2.1.II) \quad \text{Int}(D) = \bigcap_{M \in \text{Max}(D)} \text{Int}(D)_M.$$

Now, we will show that for each $M \in \mathcal{M}$, $\text{Int}(D_M) = \text{Int}(D)_M = D_M[X]$. As a matter of fact, if $M \in \text{Max}(D) \setminus \mathcal{P}$, for each $P \in \mathcal{P}$, $P \subset M$, D/P is infinite and then $\text{Int}(D_P) = D_P[X]$. If $P \in \mathcal{P}$ with $P \not\subseteq M$, then clearly the maximal ideals of $(D_P)_{(D \setminus M)}$ contract to nonmaximal prime ideals of D . Therefore, $(D_P)_{(D \setminus M)}$ has infinite residue fields and whence $\text{Int}((D_P)_{(D \setminus M)}) = (D_P)_{(D \setminus M)}[X]$ [CC2, Corollary I.3.7]. From (2.1.I) and [CC1, Corollaire 3, p. 303], we deduce that, if $M \in \text{Max}(D) \setminus \mathcal{P}$,

$$\begin{aligned} \text{Int}(D_M) &= \bigcap_{P \in \mathcal{P}} \text{Int}((D_P)_{(D \setminus M)}) = \\ &= (\bigcap_{P \in \mathcal{P}, P \subseteq M} \text{Int}(D_P)) \cap (\bigcap_{P \in \mathcal{P}, P \not\subseteq M} \text{Int}((D_P)_{(D \setminus M)})) = \\ &= (\bigcap_{P \in \mathcal{P}, P \subseteq M} D_P[X]) \cap (\bigcap_{P \in \mathcal{P}, P \not\subseteq M} (D_P)_{(D \setminus M)}[X]) = D_M[X]. \end{aligned}$$

It is obvious that if $M \in (\text{Max}(D) \cap \mathcal{P}) \setminus \mathcal{P}_0$ then $\text{Int}(D_M) = D_M[X]$, since D/M is infinite. Since $D_M[X] \subseteq \text{Int}(D)_M \subseteq \text{Int}(D_M)$, we have that $D_M[X] = \text{Int}(D)_M = \text{Int}(D_M)$.

From the previous claim and from (2.1.II), we deduce that:

$$(2.1.III) \quad \text{Int}(D) = (\bigcap_{M \in \mathcal{M}} D_M[X]) \cap (\bigcap_{M \in \mathcal{P}_0} \text{Int}(D)_M).$$

We know that there exists $N \in \text{Max}(\text{Int}(D))$ such that $\text{ht}(N) = \dim(\text{Int}(D))$ and $N \cap D$ maximal. (In fact, if $N \cap D = P$ is a nonmaximal prime ideal of D , then $\text{Int}(D)_P = \text{Int}(D_P) = D_P[X]$ and hence $\text{ht}(N) = \dim(D_P[X]) \leq \dim(D[X]) - 1 \leq \dim(\text{Int}(D))$ [CC2, Proposition V.1.6]. Therefore $\dim(\text{Int}(D)) = \dim(D[X]) - 1$. Arguing as in the proof of [CC2, Proposition V.1.6], we can find a chain \mathcal{C} of prime ideals of $D_P[X]$ of length $n = \dim(D_P[X]) = \dim(\text{Int}(D))$,

$$\mathcal{C} : \quad (0) \subset Q_1 \subset \cdots \subset Q_{n-1} \subset Q_n,$$

such that $(Q_i \cap D_P)[X] \in \mathcal{C}$, for each $i = 1, \dots, n$. Therefore, $Q_{n-1} = PD_P[X]$. When we contract \mathcal{C} to $\text{Int}(D)$ we get a chain

$$\mathcal{C}' : \quad (0) \subset Q'_1 \subset \cdots \subset Q'_{n-1} \subset Q'_n,$$

where $Q'_{n-1} = PD_P[X] \cap \text{Int}(D)$. If M is a maximal ideal of D containing P and $a \in D$, then it is easy to see that $Q'_{n-1} \subset P_a \subseteq M_a$, where $Q_a := \{f \in \text{Int}(D); f(a) \in Q\}$ for $Q \in \{P, M\}$ and M_a is a maximal ideal of $\text{Int}(D)$ above M [CC2, Lemma V.1.3]. Therefore we reach a contradiction: $\dim(\text{Int}(D)) \geq n + 1$.

Let $M := (N \cap D) \in \text{Max}(D)$. Since $N \cap (D \setminus M) = \emptyset$, then $N\text{Int}(D)_M$ is a maximal ideal of $\text{Int}(D)_M$. The conclusion follows immediately by examining the two possible cases:

Case 1. $M \in \mathcal{M}$. In this case, $\text{Int}(D)_M = \text{Int}(D_M) = D_M[X]$, hence $\text{ht}(N) = \dim(D_M[X])$.

Case 2. $M \in \mathcal{P}_0$. In this case, $\text{Int}(D)_M = \text{Int}(D_M)$ and $\text{ht}(N) = \dim(\text{Int}(D_M))$. \square

COROLLARY 2.2. *With the same notation and hypotheses of Theorem 2.1, if $\dim(\text{Int}(D_P)) \leq \dim(D_P[X])$ for each $P \in \mathcal{P}_0$, then $\dim(\text{Int}(D)) \leq \dim(D[X])$. In particular, the previous inequality holds in the following cases:*

- (a) $D_P[X]$ is a Jaffard domain, for each $P \in \mathcal{P}_0$ (e.g. when D is a locally Jaffard domain [ABDFK]);
- (b) D_P is a P^n VD, with $n \geq 0$, for each $P \in \mathcal{P}_0$ (e.g. when D is a locally PVD domain, [F] or [DF]).

Proof. The first inequality is a straightforward consequence of Theorem 2.1. As concerns the particular cases, we proceed as follows.

- (a) We note that, for each $P \in \mathcal{P}_0$, we have:

$$\dim(\text{Int}(D_P)) \leq \dim_v(\text{Int}(D_P)) \leq \dim_v(D_P[X]) = \dim(D_P[X]).$$

- (b) follows from [FIKT, Lemma 3.1]. \square

COROLLARY 2.3. *If D is a domain of Krull-type (e.g. a generalized Krull domain [G2, p.524]), then $\dim(\text{Int}(D)) = \dim(D[X])$.*

Proof. In this case, $\mathcal{P} = t_m(D)$, $\mathcal{P}_0 = \{P \in \mathcal{P} \cap \text{Max}(D); \text{Card}(D/P) < \infty\}$ and D_P is a valuation domain, for each $P \in \mathcal{P}$. In particular $\dim(\text{Int}(D_P)) = \dim(D_P) + 1 = \dim(D_P[X])$ for each $P \in \mathcal{P}_0$. The conclusion follows from Theorem 2.1. \square

3. POLYNOMIAL CLOSURE IN PULLBACK DOMAINS

Let D be any domain and let P be a prime ideal of D with $\text{Card}(D/P) = \infty$. Let I be an ideal of D such that $P \subset I$, thus $I_P = ID_P = D_P$. From [C4, Lemma 3.4] we have

$$\text{Int}(I, D_P) = \text{Int}(I_P, D_P) = \text{Int}(D_P) = D_P[X],$$

hence $\text{Int}(I, D) \subseteq D_P[X]$. Therefore, we can consider the canonical map

$$\varphi : \text{Int}(I, D) \rightarrow \text{Int}(I/P, D/P), \quad f \mapsto \bar{f} := f + PD_P[X],$$

where it is easily seen that $\bar{f} \in \text{Int}(I/P, D/P)$.

We start this section with an observation about $\text{Int}(I, D)$ and $\text{Int}(I/P, D/P)$ when P is a divided prime ideal, i.e. $P = PD_P$.

LEMMA 3.1. *Let D be an integral domain, P a divided prime ideal of D with $\text{Card}(D/P) = \infty$ and I an ideal of D with $P \subseteq I$. Then,*

- (1) *the canonical map $\varphi : \text{Int}(I, D) \rightarrow \text{Int}(I/P, D/P)$ is a surjective homomorphism;*
- (2) $\ker(\varphi) = P[X]$;
- (3) $\text{Int}(I, D)/P[X] \cong \text{Int}(I/P, D/P)$.

Proof. (1) Let $g \in (D_P/PD_P)[X]$ such that $g(I/P) \subseteq D/P$. Then it is easy to see that $g = G + PD_P[X] = G + P[X]$, where $G \in D_P[X]$ and $G(i) + P \in D/P$, for each $i \in I$. Therefore $G \in \text{Int}(I, D)$.

- (2) It is obvious that $\ker(\varphi) = PD_P[X] \cap \text{Int}(I, D) = P[X]$.
- (3) is a straightforward consequence of (1) and (2). \square

PROPOSITION 3.2. *Let D , P and I as in Lemma 3.1. Then, the canonical homomorphism φ defines the following isomorphism:*

$$cl_D(I)/P \cong cl_{D/P}(I/P).$$

Proof. Let $x \in cl_D(I)$, then $f(x) \in D$ for each $f \in \text{Int}(I, D)$. Hence, $\overline{f(x)} = \overline{f(x)} \in D/P$ for each $f \in \text{Int}(I, D)$. Since the map $\varphi : f \mapsto \overline{f}$ is surjective (Lemma 3.1(1)), then $g(\overline{x}) \in D/P$ for each $g \in \text{Int}(I/P, D/P)$, i.e. $\overline{x} \in cl_{D/P}(I/P)$. Therefore $cl_D(I)/P \subseteq cl_{D/P}(I/P)$. Conversely, if $y = x + P \in cl_{D/P}(I/P)$ then for each $g \in \text{Int}(I/P, D/P)$, $g(y) \in D/P$. Since φ is surjective, $g = \overline{f} = f + P[X]$ for some $f \in \text{Int}(I, D)$. By the fact that $g(y) \in D/P$, for each $g \in \text{Int}(I/P, D/P)$, we deduce that $f(x) \in D$, for each $f \in \text{Int}(I, D)$, i.e. $x \in cl_D(I)$. \square

COROLLARY 3.3. *Let D be a domain with a divided prime ideal P . Suppose that D/P is a valuation domain V with nonzero principal maximal ideal. Then, each ideal of D containing P is polynomially closed.*

Proof. If I is an ideal of D and $P \subset I$, then from Proposition 3.2 $cl_D(I)/P \cong cl_{D/P}(I/P)$. But $D/P = V$ is a valuation domain with principal maximal ideal and, by Proposition 1.8, $cl_{D/P}(I/P) = I/P$. Therefore $cl_D(I)/P = I/P$ and $cl_D(I) = I$, since they both contain P . \square

Relevant examples of divided domains are the pseudo-valuation domains (PVD) or, more generally, the pseudo-valuation domains of type n ($P^n\text{VD}$). We recall that a PVD, D , is defined by a pullback of the following type:

$$(3.1) \quad \begin{array}{ccc} D := \alpha^{-1}(k) & \longrightarrow & k \\ \downarrow & & \downarrow \\ V & \xrightarrow{\alpha} & V/M \end{array}$$

where (V, M) is a valuation domain (called the *valuation overring associated to D*), $\alpha : V \twoheadrightarrow V/M$ is the canonical projection and k is a subfield of the residue field of V . From [HH, Theorem 2.13], every prime ideal of D is divisorial, hence it is polynomially closed. Moreover, if I is any nonprincipal integral ideal of D , then $I_v = IV$ [HH, Corollary 2.14]. It follows immediately that $cl_D(I) \subseteq IV$. Moreover, for a nonvaluation PVD, the t -operation and the v -operation coincide [HZ, Proposition 4.3] so that $cl_D(I) \subseteq I_t$. If $D = V$ is a valuation domain, then it is known that the t -operation and the v -operation coincide if and only if the maximal ideal M of V is principal [HZ, Remark 1.5]; in fact, in this situation, every nonzero ideal of V is divisorial.

In [C4, § 4] the author establishes some relations between the polynomial closure of a fractional subset and its \mathfrak{A} -adic closure, where \mathfrak{A} is an ideal of a *Zariski domain* D (i.e. a Noetherian domain, equipped with the \mathfrak{A} -adic topology, in which every ideal is \mathfrak{A} -adic closed). Next goal is to obtain a link between the polynomial closure and the adic closure for a special class of PVD's.

PROPOSITION 3.4. *Let D be a PVD. Assume that D possesses a height-one prime ideal P such that $P \neq P^2$. Then, the polynomial closure of each D -fractional subset E of K contains the P -adic closure of E .*

Proof. We start by proving that all ideals of D are closed in the P -adic topology. If I is any ideal of D , then its P -adic closure is given by $\overline{I} := \bigcap_{n \geq 0} (I + P^n)$. If

$I \supseteq P$, it is obvious that $\bar{I} = I$. If $I \subset P$, then $P^2 \subsetneq \sqrt{I} = P$ and I contains a power of P by [HH, Corollary 2.5]. Therefore, $\bar{I} = I$. Since each ideal of D is closed in the P -adic topology, we can use the same argument of [C4, Theorem 4.1] in order to conclude. \square

Recall that a P^n VD, D , is defined by induction on n in the following way. A P^0 VD is a PVD and a P^n VD is obtained by a pullback diagram of the following type:

$$\begin{array}{ccc} D := \alpha^{-1}(A_{n-1}) & \longrightarrow & A_{n-1} \\ \downarrow & & \downarrow \\ W_{n-1} & \xrightarrow{\alpha} & F = W_{n-1}/\mathcal{M} \end{array}$$

where W_{n-1} is a P^{n-1} VD with maximal ideal \mathcal{M} , F is its residue field, $\alpha : W_{n-1} \rightarrow F$ is the canonical projection and A_{n-1} is a PVD with quotient field F . For details about P^n VD the reader is referred to [F].

In the next proposition, we will show that also in a P^n VD all prime ideals are polynomially closed.

PROPOSITION 3.5. *Let D be a P^n VD, then all nonzero prime ideals of D are polynomially closed.*

Proof. Since in a P^n VD every prime ideal is divided [F, Theorem 1.9], then if Q is a prime ideal of D , then D_Q is a P^m VD, with $m \leq n$, and $Q = (D : D_Q)$, since $QD_Q = Q$. Then, if $Q \neq 0$, Q is a divisorial ideal, whence it is polynomially closed. \square

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