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PRÜFER ⋆-MULTIPLICATION DOMAINS AND SEMISTAR OPERATIONS

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Starting from the notion of semistar operation, introduced in 1994 by Okabe and Matsuda [49], which generalizes the classical concept of star operation (cf. Gilmer's book [27]) and, hence, the related classical theory of ideal systems based on the works by W. Krull, E. Noether, H. Prüfer, P. Lorenzen and P. Jaffard (cf. Halter–Koch's book [32]), in this paper we outline a general approach to the theory of Prüfer ⋆-multiplication domains (or P⋆MDs), where ⋆ is a semistar operation. This approach leads to relax the classical restriction on the base domain, which is not necessarily integrally closed in the semistar case, and to determine a semistar invariant character for this important class of multiplicative domains (cf. also J. M. García, P. Jara and E. Santos [25]). We give a characterization theorem of these domains in terms of Kronecker function rings and Nagata rings associated naturally to the given semistar operation, generalizing previous results by J. Arnold and J. Brewer [10] and B. G. Kang [39]. We prove a characterization of a P⋆MD, when ⋆ is a semistar operation, in terms of polynomials (by using the classical characterization of Prüfer domains, in terms of polynomials given by R. Gilmer and J. Hoffman [28], as a model), extending a result proved in the star case by E. Houston, S. J. Malik and J. Mott [36]. We also deal with the preservation of the P⋆MD property by *ascent* and *descent* in case of field extensions. In this context, we generalize to the P⋆MD case some classical results concerning Prüfer domains and PvMDs. In particular, we reobtain as a particular case a result due to H. Prüfer [51] and W. Krull [41] (cf. also F. Lucius [43] and F. Halter-Koch [34]). Finally, we develop several examples and applications when ⋆ is a (semi)star given explicitly (e.g. we consider the case of the *standard v-, t-, b-, w*-operations or the case of semistar operations associated to appropriate families of overrings).

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1. Introduction

The theory of ideal systems is based on the classical works by W. Krull, E. Noether, H. Prüfer and P. Lorenzen; a systematic treatment of this theory can be found in the volumes by P. Jaffard [37] and F. Halter-Koch [32]. A different presentation, using the notion of star operation, is given in 1972 by R. Gilmer [27, Secs. 32–34] (cf. also for further developments [3, 5, 6, 11, 35, 38] and [48]). In 1994 Okabe and Matsuda [49] generalize the concept of star operation by introducing the more "flexible" notion of semistar operation. After that paper new developments of the multiplicative theory of ideals have been realized and successfully applied to analyze the structure of different classes of integral domains (cf. for instance [15, 21–24, 33, 46] and [50]).

Semistar operations of a special type appear naturally in relation with the general constructions of Kronecker function rings and Nagata function rings (in Sec. 2, we recall the definitions and the principal properties of these objects). More precisely, given a semistar operation $\star$ on an integral domain D with quotient field K, the Kronecker function ring $\mathrm{Kr}(D, \star)$ ($\subseteq K(X)$) (respectively, the Nagata function ring $\mathrm{Na}(D, \star)$ ($\subseteq K(X)$)) induces naturally a *distinguished* semistar operation $\star_a$ (respectively, $\tilde{\star}$) on D such that $F\,\mathrm{Kr}(D, \star) \cap K = F^{\star_a}$ (respectively, $\mathrm{Na}(D, \star) \cap K = F^{\tilde{\star}}$), for each finitely generated fractional ideal F of D. These semistar operations were intensively studied in [24], where the authors examine also the interplay of $\mathrm{Kr}(D, \star)$ and $\star_a$ with $\mathrm{Na}(D, \star)$ and $\tilde{\star}$ and show a "parallel" behaviour of these pairs of objects.

The equality of Nagata function ring with Kronecker function ring characterizes, in the classical Noetherian case, the Dedekind domains. It is natural, in the general context, to investigate on the existence of *semistar invariants* for different classes of Prüfer-like domains. A first attempt in this direction is due to F. Halter-Koch [34], who obtained a deep axiomatic approach to the theory of Kronecker function rings, with applications to the characterization of Bézout domains that are Kronecker function rings (cf. [23]). On the other hand, the study initiated in [24] leads naturally to the investigation of the class of integral domains, having a semistar operation $\star$ such that the semistar operation $\tilde{\star}$, associated to the Nagata function ring, coincides with the semistar operation $\star_a$, associated to the Kronecker function ring.

One of the aims of this paper is to characterize a distinguished class of *multiplication domains*, called *the Prüfer semistar multiplication domains* or P$\star$MD, that arises naturally in this context, having the property that $\tilde{\star} = (\tilde{\star})_a = \star_a$ (Sec. 2). This class contains as examples Prüfer domains, Krull domains and PvMD, but also integral domains, that are not integrally closed, having although an appropriate overring which is Prüfer star multiplicative domain (cf. [25, 36] and [39]). An explicit example of a *non* integrally closed Prüfer semistar multiplication domain

is given in Example 3.1 (recall that a Prüfer star multiplication domain is always integrally closed).

In Sec. **3** we show that, if $\star$ is semistar operation of finite type which is spectral and e.a.b. on an integral domain D (definitions are given in Sec. **2**), then D is a P$\star$MD. Moreover we prove that D is a P$\star$MD, for some semistar operation $\star$ on D, if and only if D is a P$\tilde{\star}$MD, where $\tilde{\star}$ is a semistar operation of finite type which is spectral and e.a.b. This result extends one of the principal results of [25], proved by using torsion theories. After this characterization, we apply our theory to some special types of semistar operations and we give new characterizations of P$\star$MDs in the *classical* star setting. In particular, we obtain also that the PwMDs studied recently by W. Fanggui and R. L. McCasland [19] coincide with the PvMDs introduced by M. Griffin [30].

In Sec. **4** we deal with the preservation of the P$\star$MD property by *ascent* and *descent*, in case of algebraic field extensions. We generalize to the P$\star$MD case some classical results concerning Prüfer domains and PvMDs. In particular, we reobtain the following generalization of a result due to H. Prüfer and W. Krull (for the *only if* case, cf. [51, §11] and [41, Satz 9]) and to F. Lucius and F. Halter-Koch (for the *if* case, cf. [43, Theorems 4.6 and 4.4] and [34, Theorem 3.6]):

> Let $K \subseteq L$ be an algebraic field extension, let T be an integral domain with quotient field L, set $D := T \cap K$. Assume that D is integrally closed and that T is the integral closure of D in L. Then D is a PvMD if and only if T is a PvMD.

We use as main reference Gilmer's book [27] and any unexplained material is as in [27] and [40]. Since many preliminary results on semistar operations and applications, that we will need in this paper, are not easily available, because the related work was presented or appeared in the Proceedings of recent Conferences (in particular, [22, 23] and [24]), we will recall the principal definitions and the statements of the main properties in Sec. **2**.

2. Background Results

Let D be an integral domain with quotient field K. Let $\bar{\mathbf{F}}(D)$ denote the set of all nonzero D-submodules of K and let $\mathbf{F}(D)$ be the set of all nonzero fractional ideals of D, i.e. all $E \in \bar{\mathbf{F}}(D)$ such that there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\mathbf{f}(D)$ be the set of all nonzero finitely generated D-submodules of K. Then, obviously $\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \bar{\mathbf{F}}(D)$.

We recall that a mapping

$$\star : \bar{\mathbf{F}}(D) \to \bar{\mathbf{F}}(D), \quad E \mapsto E^{\star}$$

is called a *semistar operation on D* if, for $x \in K$, $x \neq 0$, and $E, F \in \bar{\mathbf{F}}(D)$, the following properties hold:

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$(\star_1)$ $(xE)^\star = xE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^\star = (E^\star)^\star =: E^{\star\star}$

(cf. for instance [21, 22, 45, 46, 48] and [49]). In order to avoid trivial cases, we will assume tacitly that the semistar operations are non trivial, i.e. if $D \neq K$ then $D^\star \neq K$ (or, equivalently, the map $\star : \bar{\mathbf{F}}(D) \to \bar{\mathbf{F}}(D)$ is not constant onto K; cf. [24, Section 2]).

A semistar operation $\star$ on D is called an *e.a.b.* (= *endlich arithmetisch brauchbar*) (respectively, *a.b.* (= *arithmetisch brauchbar*)) if, for each $E \in \mathbf{f}(D)$ and for all F, $G \in \mathbf{f}(D)$ (respectively, F, $G \in \mathbf{F}(D)$):

$$(EF)^\star \subseteq (EG)^\star \Rightarrow F^\star \subseteq G^\star \,,$$

(cf. for instance [22, Definition 2.3 and Lemma 2.7]).

If $\star_1$ and $\star_2$ are two semistar **operations** on D, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \bar{\mathbf{F}}(D)$; in fact, for semistar operations $\star_1$ and $\star_2$, the following assertions are equivalent (i) $\star_1 \leq \star_2$; (ii) $(E^{\star_1})^{\star_2} = E^{\star_2}$ for each $E \in \bar{\mathbf{F}}(D)$ and (iii) $(E^{\star_2})^{\star_1} = E^{\star_2}$ for each $E \in \bar{\mathbf{F}}(D)$.

Several new semistar operations can be derived from a given semistar operation $\star$. The essential details are given in the following example.

Example 2.1. Let D be an integral domain and let $\star$ be a semistar operation on D.

(a) If $\star$ is a semistar operation such that $D^\star = D$, then the map $\star : \mathbf{F}(D) \to \mathbf{F}(D)$, $E \mapsto E^\star$, is called a *star operation on D*. Recall [27, (32.1)] that a star operation $\star$ verifies the properties $(\star_2)$, $(\star_3)$, for all E, $F \in \mathbf{F}(D)$; moreover, for each $x \in K$, $x \neq 0$ and for each $E \in \mathbf{F}(D)$, a star operation $\star$ verifies also:

$(\star\star_1)$ $(xD)^\star = xD\,, \quad (xE)^\star = xE^\star.$

If $\star$ is a semistar operation on D such that $D^\star = D$, then we will write often in the following of the paper that $\star$ is *a (semi)star operation on D*, for emphasizing the fact that the semistar operation $\star$ is an extension to $\bar{\mathbf{F}}(D)$ of a "*classical*" *star operation* $\star$, i.e. a map $\star : \mathbf{F}(D) \to \mathbf{F}(D)$, verifying the properties $(\star\star_1)$, $(\star_2)$ and $(\star_3)$ [27, Section 32]. Note that not every semistar operation is an extension of a star operation [21, Remark 1.5(b)].

(b) For each $E \in \bar{\mathbf{F}}(D)$, set

$$E^{\star_f} := \bigcup \{F^\star \mid F \subseteq E, \ F \in \mathbf{f}(D)\} \,.$$

Then $\star_f$ is also a semistar operation on D, called *the semistar operation of finite type associated to $\star$*. Obviously, $F^\star = F^{\star_f}$, for each $F \in \mathbf{f}(D)$. If $\star = \star_f$, then $\star$ is called a *semistar operation of finite type* [22, Example 2.5(4)]. For instance, if v is *the v-(semi)star operation on D* defined by $E^v := (E^{-1})^{-1}$, for each $E \in \bar{\mathbf{F}}(D)$, with $E^{-1} := (D :_K E) := \{z \in K \mid zE \subseteq D\}$) [21, Example 1.3(c)

and Proposition 1.6(5)], then the semistar operation of finite type v_f associated to v is called *the t-(semi)star operation on D* (in this case $D^v = D^t = D$).

Note that, in general, $\star_f \leq \star$, i.e. $E^{\star_f} \subseteq E^\star$ for each $E \in \bar{\mathbf{F}}(D)$. Thus, in particular, if $E = E^\star$, then $E = E^{\star_f}$. Note also that $\star_f = (\star_f)_f$.

(c) Next example of a semistar operation is connected with the constructions already in [3, 5] and [54], and with a weak version of integrality. The essential techniques and motivations for considering this weak version of integrality, using ideal systems, can be found in Jaffard's book [37]. More recently, starting from an **ideal** in [7], where the authors introduced a weak version of integrality (called *semi-integrality* and associated to the v-operation), a weak general version of integrality, depending on a star operation, was introduced and studied in [17, 31, 34] and [48]. The natural extension of this notion to the case of semistar operations was considered in [22, 23] and [49].

We start by defining a new operation on D, denoted by $[\star]$, called *the semistar integral closure of* $\star$, by setting:

$$F^{[\star]} := \bigcup \{((H^\star : H)F)^{\star_f} \mid H \in \mathbf{f}(D)\}, \quad \text{for each } F \in \mathbf{f}(D),$$

and

$$E^{[\star]} := \bigcup \{F^{[\star]} \mid F \in \mathbf{f}(D), \ F \subseteq E\}, \quad \text{for each } E \in \bar{\mathbf{F}}(D).$$

It is not difficult to see that the operation $[\star]$ defined in this manner is a semistar operation of finite type on D, that $\star_f \leq [\star]$, hence $D^\star \subseteq D^{[\star]}$, and that $D^{[\star]}$ is integrally closed [22, Definition 4.2, Propositions 4.3 and 4.5(3)]. Therefore, it is obvious that if $D^\star = D^{[\star]}$ then $D^\star$ is integrally closed. The converse is false, even when $\star$ is a (semi)star operation on D.

(c.1) There exists *an integral domain D with a semistar operation $\star$ such that $D^\star$ is integrally closed and $D^\star \subsetneq D^{[\star]}$.*

Let V be a valuation domain of the form $K + M$, where K is a field and M is the maximal ideal of V. Let k be a proper subfield of K and assume that k is algebraically closed in K. Set $D := k + M \subsetneq V$ and consider the (semi)star operation $\star := v$ on D. Then, clearly, D is integrally closed and $D^\star(= D^v) = D$. On the other hand, let $z \in K \backslash k$ and let $W := k + zk$ then W is a k-submodule of K, which obviously is not a fractional ideal of k. Then $H := W + M$ is a finitely generated fractional ideal of D and $H^v = V$ by [12, Theorem 4.3 and its proof]. Therefore $(H^v : H^v) = V$, and so $V \subseteq D^{[v]}$ (in fact, $V = D^{[v]}$ by [7, Proposition 8(ii)]).

A simple case for having that $D^\star$ is integrally closed if and only if $D^\star = D^{[\star]}$ is when $\star$ is a semistar operation of finite type on D which is stable with respect to finite intersections (i.e. $(E \cap F)^\star = E^\star \cap F^\star$, for all $E, F \in \bar{\mathbf{F}}(D)$).

(c.2) *Let $\star$ be a semistar operation of an integral domain D. Assume that $\star_f$ is stable, then $D^{[\star]} = (D')^{\star_f}$, where D' is the integral closure of D.*

Indeed,

$$D^{[\star]} = \bigcup \{(H^{\star_f} : H) \mid H \in \mathbf{f}(D)\} = \bigcup \{(H : H)^{\star_f} \mid H \in \mathbf{f}(D)\}$$
$$\subseteq (D')^{\star_f} \subseteq (D^{[\star]})^{[\star]} = D^{[\star]}\,.$$

In particular, if $D^{\star}$ is integrally closed, then $D \subseteq D' \subseteq D^{\star}$ implies $D^{[\star]} = (D')^{\star_f} = D^{\star}$.

(d) The essential constructions related to the following example of semistar oper-
     ation are due to P. Lorenzen [42] and P. Jaffard [37] (cf. also F. Halter-Koch
     [32]).

     Given an arbitrary semistar operation $\star$ on an integral domain D, it is
     possible to associate to $\star$, an e.a.b. semistar operation of finite type $\star_a$ on D,
     called *the e.a.b. semistar operation associated to* $\star$, defined as follows:

     $$F^{\star_a} := \bigcup \{((FH)^{\star} : H) \mid H \in \mathbf{f}(D)\}\,, \quad \text{for each } F \in \mathbf{f}(D)\,,$$

     and

     $$E^{\star_a} := \bigcup \{F^{\star_a} \mid F \subseteq E\,, F \in \mathbf{f}(D)\}\,, \quad \text{for each } E \in \bar{\mathbf{F}}(D)\,,$$

     [22, Definition 4.4]. Note that $[\star] \leq \star_a$, that $D^{[\star]} = D^{\star_a}$ and if $\star$ is an e.a.b.
     semistar operation of finite type then $\star = \star_a$ [22, Proposition 4.5].

(e) Let D be an integral domain and T an overring of D. Let $\star$ be a semistar
     operation on D and define $\dot{\star}^T : \bar{\mathbf{F}}(T) \to \bar{\mathbf{F}}(T)$ by setting:

     $$E^{\dot{\star}^T} := E^{\star}\,, \quad \text{for each } E \in \bar{\mathbf{F}}(T)(\subseteq \bar{\mathbf{F}}(D))\,.$$

     Then, we know [22, Proposition 2.8]:

     (e.1) *The operation $\dot{\star}^T$ is a semistar operation on T and, if $\star$ is of finite type
            on D, then $\dot{\star}^T$ is also of finite type on T.*

     (e.2) *When $T = D^{\star}$, then $\dot{\star}^{D^{\star}}$, restricted to $\mathbf{F}(D^{\star})$, defines a star operation on
            $D^{\star}$.*

     (e.3) *If $\star$ is e.a.b., then $\dot{\star}^{D^{\star}}$ is also e.a.b.*

     Conversely, let $\star$ be a semistar operation on an overring T of D and define
     $\star_D : \bar{\mathbf{F}}(D) \to \bar{\mathbf{F}}(D)$ by setting:

     $$E^{\star_D} := (ET)^{\star}\,, \quad \text{for each } E \in \bar{\mathbf{F}}(D)\,.$$

     Then, we know [22, Proposition 2.9 and Corollary 2.10]:

     (e.4) *The operation $\star_D$ is a semistar operation on D.*

     (e.5) *If $\star$ is e.a.b., then $\star_D$ is also e.a.b.*

     (e.6) *If we denote simply by $*$ the semistar operation $\star_D$, then the semistar
            operations $\dot{*}^T$ and $\star$ (both defined on T) coincide.*

     Note that the module systems approach, developed by Halter-Koch in [33],
     gives a natural and general setting for (re)considering semistar operations and,
     in particular, the semistar operations $\dot{\star}^T$ and $\star_D$.

(f) Let Δ be a nonempty set of prime ideals of an integral domain D. For each D-submodule E of K, set:

$$E^{\star_\Delta} := \bigcap \{ ED_P \mid P \in \Delta \} \,.$$

The mapping $E \mapsto E^{\star_\Delta}$, for each $E \in \bar{\mathbf{F}}(D)$, defines a semistar operation on D, moreover [21, Lemma 4.1]:

(f.1) *For each $E \in \bar{\mathbf{F}}(D)$ and for each $P \in \Delta$, $ED_P = E^{\star_\Delta} D_P$.*

(f.2) *The semistar operation $\star_\Delta$ is stable (with respect to the finite intersections), i.e. for all $E, F \in \bar{\mathbf{F}}(D)$ we have $(E \cap F)^{\star_\Delta} = E^{\star_\Delta} \cap F^{\star_\Delta}$.*

(f.3) *For each $P \in \Delta$, $P^{\star_\Delta} \cap D = P$.*

(f.4) *For each nonzero integral ideal I of D such that $I^{\star_\Delta} \cap D \neq D$, there exists a prime ideal $P \in \Delta$ such that $I \subseteq P$.*

A *semistar operation* $\star$ is called *spectral*, if there exists a nonempty set Δ of $\mathrm{Spec}(D)$ such that $\star = \star_\Delta$; in this case we say that $\star$ is *the spectral semistar operation associated with Δ*. We say that $\star$ is a *quasi-spectral semistar operation* (or that $\star$ *possesses enough primes*) if, for each nonzero integral ideal I of D such that $I^{\star_\Delta} \cap D \neq D$, there exists a prime ideal P of D such that $I \subseteq P$ and $P^\star \cap D = P$. From (f.3) and (f.4), we deduce that each spectral semistar operation is quasi-spectral.

A subset Δ of $\mathrm{Spec}(D)$ is called *stable for generizations* if $Q \in \mathrm{Spec}(D)$, $P \in \Delta$ and $Q \subseteq P$ then $Q \in \Delta$. Set $\Delta^\downarrow := \{ Q \in \mathrm{Spec}(D) \mid Q \subseteq \mathbf{P} \ \mathbf{for} \text{ some } P \in \Delta \}$ and let $\Lambda \subseteq \mathrm{Spec}(D)$, it is easy to see that:

(f.5) *If $\Delta \subseteq \Lambda \subseteq \Delta^\downarrow$, then $\star_\Delta = \star_\Lambda = \star_{\Delta^\downarrow}$.*

(g) Example (f) can be generalized as follows. Let $\mathcal{T} := \{ T_\alpha \mid \alpha \in A \}$ be a nonempty family of overrings of D and define $\star_\mathcal{T} : \bar{\mathbf{F}}(D) \to \bar{\mathbf{F}}(D)$ by setting:

$$E^{\star_\mathcal{T}} := \bigcap \{ ET_\alpha \mid \alpha \in A \} \,, \quad \text{for each } E \in \bar{\mathbf{F}}(D) \,.$$

Then we know that [22, Lemma 2.4(3), Example 2.5(6), Corollary 3.8]:

(g.1) *The operation $\star_\mathcal{T}$ is a semistar operation on D. Moreover, if*

$$\mathcal{T} = \{ D_P \mid P \in \Delta \} \,,$$

then $\star_\mathcal{T} = \star_\Delta$.

(g.2) *$E^{\star_\mathcal{T}} T_\alpha = ET_\alpha$, for each $E \in \bar{\mathbf{F}}(D)$ and for each $\alpha \in A$.*

(g.3) *If $\mathcal{T} = \mathcal{W}$ is a family of valuation overrings of D, then $\star_\mathcal{W}$ is an a.b. semistar operation on D. If $\mathcal{W}$ is the family of all the valuation overrings of D, then $\star_\mathcal{W}$ is called the b-semistar operation on D; moreover, if D is integrally closed, then $D^b = D$ [27, Theorem 19.8], and thus the operation b, restricted to $\mathbf{F}(D)$, defines a star operation on D, called the b-star operation [27, p. 398].*

Let $\star$ be a semistar operation of an integral domain D and assume that the set:

$$\Pi^\star := \{P \in \operatorname{Spec}(D) \mid P \neq 0 \text{ and } P^\star \cap D \neq D\}$$

is nonempty, then the spectral semistar operation of D defined by $\star_{\mathrm{sp}} := \star_{\Pi^\star}$ is called *the spectral semistar operation associated to* $\star$. Note that if $\star$ is quasi-spectral, then $\Pi^\star$ is nonempty and $\star_{\mathrm{sp}} \leq \star$ [21, Proposition 4.8 and Remark 4.9]. It is easy to see that $\star$ is spectral if and only if $\star = \star_{\mathrm{sp}}$.

Let $I \subseteq D$ be a nonzero ideal of D. We say that I is a *quasi-$\star$-ideal of D* if $I^\star \cap D = I$. Note that, for each nonzero integral ideal I of D, the ideal $J := I^\star \cap D$ is a quasi-$\star$-ideal of D and $I \subseteq J$. Note also that the quasi-$\star$-ideals form a weak ideal system on D, in the sense of [32]: this alternative approach can be applied for recovering some of the results mentioned next.

A *quasi-$\star$-prime* [respectively, a *quasi-$\star$-maximal*] is a quasi-$\star$-ideal which is also a prime ideal [respectively, quasi-$\star$-ideal which is a maximal element in the set of all proper quasi-$\star$-ideals of D]. It is not difficult to see that,

Lemma 2.1. [24, Lemma 2.3] *When $\star = \star_f$, then*:

(a) *each proper quasi-$\star$-ideal is contained in a quasi-$\star$-maximal*;
(b) *each quasi-$\star$-maximal is a quasi-$\star$-prime*;
(c) *the (nonempty) set $\mathcal{M}(\star)$ of all quasi-$\star$-maximals coincide with the set*:

$$\operatorname{Max}\{P \in \operatorname{Spec}(D) \mid 0 \neq P \ and \ P^\star \cap D \neq D\} = \operatorname{Max}(\Pi^\star) \,.$$

Remark 2.1. Note that, if $\star$ is a semistar operation of finite type, then $\star$ is quasi-spectral [Lemma 2.1 ((a) and (b))]. Moreover, by Lemma 2.1(c) and Example 2.1 (f.5),

$$(\star_f)_{\mathrm{sp}} = \star_{\mathcal{M}(\star_f)} \,.$$

We will simply denote by $\tilde{\star}$ the spectral semistar operation $(\star_f)_{\mathrm{sp}}$, (cf. also [21, Proposition 3.6(b) and Proposition 4.23(1)]). From the previous considerations it follows that $\tilde{\star} \leq \star_f$ and that $\tilde{\star}$ is a spectral semistar operation of finite type (cf. also [21, Proposition 3.2 and Theorem 4.12(2)]).

When $\star$ is the (semi)star v-operation, the (semi)star operation $\tilde{v}$ coincides with *the (semi)star operation w* defined as follows:

$$E^w := \bigcup \{(E : H) \mid H \in \mathbf{f}(D) \text{ and } H^v = D\} \,, \quad \text{for each } E \in \bar{\mathbf{F}}(D) \,.$$

This (semi)star operation was firstly considered by J. Hedstrom and E. Houston in 1980 [35, Section 3] under the name of F_∞-operation, starting from the F-operation introduced by H. Adams [1]. Later, from 1997, this operation was intensively studied by W. Fanggui and R. McCasland (cf. [17–19]) under the name of w-operation. Note also that the notion of w-ideal coincides with the notion of semi-divisorial ideal considered by S. Glaz and W. Vasconcelos in 1977 [29]. Finally, in 2000, for

each (semi)star operation $\star$, D. D. Anderson and S. J. Cook [6] considered the $\star_w$-operation which can be defined as follows:

$$E^{\star_w} := \bigcup \{(E : H) \mid H \in \mathbf{f}(D) \text{ and } H^\star = D\}, \quad \text{for each } E \in \bar{\mathbf{F}}(D).$$

From their theory it follows that $\star_w = \tilde{\star}$ [6, Corollary 2.10]. A deep link between the semistar operations of type $\tilde{\star}$ and the localizing systems of ideals was established in [21].

Let R be a ring and X an indeterminate over R, for each $f \in R[X]$, we denote by $\mathbf{c}(f)$ the *content of f*, i.e. the ideal of R generated by the coefficients of the polynomial f. The following ring, subring of the total ring of rational functions:

$$R(X) := \left\{ \frac{f}{g} \mid f, g \in R[X] \text{ and } \mathbf{c}(g) = R \right\}$$

is called the *Nagata ring of R* [27, Proposition 33.1].

Lemma 2.2. [24, Proposition 3.1] *Let $\star$ be a semistar operation of an integral domain D and set:*

$$N(\star) := N_D(\star) := \{h \in D[X] \mid h \neq 0 \text{ and } \mathbf{c}(h)^\star = D^\star\}.$$

(a) $N(\star) = D[X] \backslash (\bigcup \{Q[X] \mid Q \in \mathcal{M}(\star_f)\})$ *is a saturated multiplicatively closed subset of $D[X]$ and, obviously, $N(\star) = N(\star_f)$.*
(b) $\mathrm{Max}(D[X]_{N(\star)}) = \{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$.
(c) $D[X]_{N(\star)} = \bigcap \{D[X]_{Q[X]} \mid Q \in \mathcal{M}(\star_f)\} = \bigcap \{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.
(d) $\mathcal{M}(\star_f)$ *coincides with the canonical image into $\mathrm{Spec}(D)$ of the set of the maximal ideals of $D[X]_{N(\star)}$, i.e. $\mathcal{M}(\star_f) = \{M \cap D \mid M \in \mathrm{Max}(D[X]_{N(\star)})\})$.*

We set:

$$\mathrm{Na}(D, \star) := D[X]_{N_D(\star)}$$

and we call this integral domain *the Nagata ring of D with respect to the semistar operation $\star$*. Obviously, $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \star_f)$ and if $\star = d$, where d is *the identical (semi)star operation of D* (i.e. $E^d := E$, for each $E \in \bar{\mathbf{F}}(D)$), then $\mathrm{Na}(D, d) = D(X)$.

Lemma 2.3. [24, **Corollary 2.7**, Proposition 3.4, **Corollary 3.5** and **Theorem 3.8**] *Let $\star$ be a given semistar operation of an integral domain D and let $\tilde{\star} := \star_{\mathcal{M}(\star_f)} = (\star_f)_{\mathrm{sp}}$ be the spectral semistar operation of finite type canonically associated to $\star$ (cf. Remark 2.1)). Denote simply by $\dot{\tilde{\star}}$ the following (semi)star operation on $D^{\tilde{\star}}$ (Example 2.1(e)):*

$$\dot{\tilde{\star}}^{D^{\tilde{\star}}} : \bar{\mathbf{F}}(D^{\tilde{\star}}) \to \bar{\mathbf{F}}(D^{\tilde{\star}}), \quad E \mapsto E^{\tilde{\star}}.$$

Then, for each $E \in \bar{\mathbf{F}}(D)$,

(a) $E^{\star_f} = \bigcap \{E^{\star_f} D_Q \mid Q \in \mathcal{M}(\star_f)\}$;
(b) $E^{\tilde{\star}} = \bigcap \{E D_Q \mid Q \in \mathcal{M}(\star_f)\}$;

(c) $E \operatorname{Na}(D, \star) = \bigcap \{ED_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$;

(d) $E \operatorname{Na}(D, \star) \cap K = \bigcap \{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$;

(e) $E^{\tilde{\star}} = E \operatorname{Na}(D, \star) \cap K$.

(f) *For each* $Q \in \mathcal{M}(\star_f)$, *set* $Q^\diamond := QD_Q(X) \cap \operatorname{Na}(D, \star)$, *then* $Q^\diamond = Q[X]_{N_D(\star)} \in \operatorname{Max}(\operatorname{Na}(D, \star))$ *and* $\operatorname{Na}(D, \star)_{Q^\diamond} = D_Q(X)$.

(g) $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.

(h) $\mathcal{M}(\dot{\tilde{\star}}) = \{\widetilde{Q} := QD_Q \cap D^{\tilde{\star}} \mid Q \in \mathcal{M}(\star_f)\}$ *and* $D^{\tilde{\star}}_{\widetilde{Q}} = D_Q$, *for each* $Q \in \mathcal{M}(\star_f)$.

(i) $\operatorname{Na}(D, \star) = \operatorname{Na}(D, \tilde{\star}) = \operatorname{Na}(D^{\tilde{\star}}, \dot{\tilde{\star}}) \supseteq D^{\tilde{\star}}(X)$.

We recall now a notion of invertibility that generalizes the classical concepts of invertibility, v-invertibility and t-invertibility (cf. for instance [8] and [6, Section 2]). Let $\star$ be a semistar operation on an integral domain D. Let $I \in \mathbf{F}(D)$, we say I is $\star$-*invertible* if $(II^{-1})^\star = D^\star$. Note that, if $I \in \mathbf{f}(D)$, then I is $\star_f$-invertible if and only if there exists $J \in \mathbf{f}(D)$ such that $(IJ)^\star = D^\star$ and $J \subseteq I^{-1}$, [15]. The following lemma generalizes a result proved by B. G. Kang [39, Theorem 2.12] (cf. also D. D. Anderson [2, Theorem 2]).

Lemma 2.4. [15, Theorem 2.5] *Let* $\star$ *be a semistar operation on an integral domain* D. *Assume that* $\star = \star_f$. *Let* $I \in \mathbf{f}(D)$, *then the following are equivalent*:

(i) I *is* $\star$-*invertible*;

(ii) $ID_Q \in \operatorname{Inv}(D_Q)$, *for each* $Q \in \mathcal{M}(\star)$;

(iii) $I \operatorname{Na}(D, \star) \in \operatorname{Inv}(\operatorname{Na}(D, \star))$.

Let $\star$ be a semistar operation on an integral domain D. We say that D is a P$\star$MD (*Prüfer* $\star$-*multiplication domain*), if each $I \in \mathbf{f}(D)$ is $\star_f$-invertible.

It is obvious that if $\star_1 \leq \star_2$ are two semistar operations on an integral domain D and if D is a P$\star_1$MD, then D is also a P$\star_2$MD. Moreover, if $\star_1$ is equivalent to $\star_2$, then D is a P$\star_1$MD if and only if D is also a P$\star_2$MD. In particular, the notions of P$\star$MD and P$\star_f$MD coincide.

Note that if $\star$ is a semistar operation on D such that $D^\star = D$ (i.e. if $\star$ restricted to $\mathbf{F}(D)$ defines a star operation on D; cf. Example 2.1(a)), then $\star \leq v$ (where v is the v-(semi)star operation, Example 2.1(b)) [27, Theorem 34.1(4)]. In particular, if $D^\star = D$, then $\star_f \leq t$ (where t is the (semi)star operation of finite type associated to v); moreover, in the present situation, if D is a P$\star$MD, then D is also a PvMD. In the semistar case a P$\star$MD is not necessary a PvMD (see Example 3.1 below).

Recall that if d is the identical (semi)star operation on D, then obviously $d \leq \star$, for each semistar operation $\star$ on D. Moreover, the notion of PdMD coincide with the notion of a Prüfer domain [27, Theorem 21.1]. Therefore, a Prüfer domain is a P$\star$MD, for each semistar operation $\star$.

Lemma 2.5. [22, Theorem 3.11(2), Theorem 5.1, Corollary 5.2 and Corollary 5.3] *Let* $\star$ *be any semistar operation defined on an integral domain* D *with quotient field* K *and let* $\star_a$ *be the e.a.b. semistar operation associated to* $\star$ (*Example* 2.1(d)). *Consider the e.a.b. (semi)star operation* $\dot{\star}_a := \dot{\star}_a^{D^{\star_a}}$ (*defined in Example* 2.1(e)) *on the integrally closed integral domain* $D^{\star_a} = D^{[\star]}$ (*cf. Example* 2.1 ((*c*) *and* (*d*))).

Set

$$\mathrm{Kr}(D, \star) := \{f/g \mid f, g \in D[X] \backslash \{0\} \quad \text{and there exists } h \in D[X] \backslash \{0\}$$

$$\text{such that } (\mathbf{c}(f)\mathbf{c}(h))^{\star} \subseteq (\mathbf{c}(g)\mathbf{c}(h))^{\star}\} \cup \{0\} \,.$$

Then we have:

(a) $\mathrm{Kr}(D, \star)$ *is a Bézout domain with quotient field $K(X)$, called the Kronecker function ring of D with respect to the semistar operation $\star$.*

(b) $\mathrm{Na}(D, \star) \subseteq \mathrm{Kr}(D, \star)$.

(c) $\mathrm{Kr}(D, \star) = \mathrm{Kr}(D, \star_a) = \mathrm{Kr}(D^{\star_a}, \dot{\star}_a)$.

(d) *For each $F \in \mathbf{f}(D)$*:

$$F \, \mathrm{Kr}(D, \star) \cap K = \mathrm{Kr}(D, \star_a) \cap K = F^{\star_a} \,.$$

(e) *If $F := (a_0, a_1, \ldots, a_n) \in \mathbf{f}(D)$ and $f(X) := a_0 + a_1 X + \cdots + a_n X^n \in K[X]$, then*:

$$F \, \mathrm{Kr}(D, \star) = f(X) \, \mathrm{Kr}(D, \star) = \mathbf{c}(f) \, \mathrm{Kr}(D, \star) \,.$$

The notion that we recall next is essentially due to P. Jaffard [37] (cf. also [23, 31] and [34]). Let $\star$ be a semistar operation on D and let V be a valuation overring of D. We say that V is a $\star$-*valuation overring of D* if, for each $F \in \mathbf{f}(D)$, $F^{\star} \subseteq FV$ (or equivalently, $\star_f \leq \star_{\{V\}}$, where $\star_{\{V\}}$ is the semistar operation of finite type on D defined by:

$$E^{\star_{\{V\}}} := EV = \bigcup \{FV \mid F \subseteq E, \ F \in \mathbf{f}(D)\} \,,$$

for each $E \in \bar{\mathbf{F}}(D)$; cf. Example 2.1(g) and [22, Example 2.5 (1) and Example 3.6]).

Note that a valuation overring V of D is a $\star$-valuation overring of D if and only if $V^{\star_f} = V$. (The "only if" part is obvious; for the "if" part recall that, for each $F \in \mathbf{f}(D)$, there exists a nonzero element $x \in K$ such that $FV = xV$, thus $F^{\star} \subseteq (FV)^{\star_f} = xV^{\star_f} = xV = FV$.)

We collect in the following lemma the main properties of the $\star$-valuation overrings.

Lemma 2.6. [23, Proposition 3.3, Proposition 3.4 and Theorem 3.5] *Let $\star$ be a semistar operation of an integral domain D with quotient field K and let V be a valuation overring of D. Then*:

(a) *V is a $\star$-valuation overring of D if and only if V is a $\star_a$-valuation overring of D.*

(b) *V is a $\star$-valuation overring of D if and only if there exists a valuation overring W of $\mathrm{Kr}(D, \star)$ such that $W \cap K = V$; moreover, in this case, $W = V(X)$.*

(c) *$\mathrm{Kr}(D, \star) = \bigcap \{V(X) \mid V$ is a $\star$-valuation overring of $D\}$.*

(d) *Assume that $\star = \star_a$ and that $\mathcal{V}$ is the set of all the $\star$-valuation overrings of D. For each $F \in \mathbf{f}(D)$,*

$$F^{\star} = F^{\star_{\mathcal{V}}} := \bigcap \{FV \mid V \text{ is a } \star\text{-valuation overring of } D\} \,,$$

thus an e.a.b. semistar operation on D is always equivalent to an a.b. semistar operation on D.

Lemma 2.7. [24, Theorem 4.3] *Let $\star$ be any semistar operation defined on an integral domain D and let $\star_a$ be the e.a.b. semistar operation of finite type associated to $\star$. Assume that $\star = \star_f$. Then*:

(a) $\mathcal{M}(\star_a) \subseteq \{N \cap D \mid N \in \mathrm{Max}(\mathrm{Kr}(D, \star))\}$.
(b) *For each $Q \in \mathcal{M}(\star_a)$ there exists a $\star$-valuation overring (V, M) of D such that $M \cap D = Q$ (i.e., V dominates D_Q).*

Although the essential results of the theory developed in the present paper concern finite type semistar operations, we will consider general semistar operations not only in order to establish the results in a more general and natural setting, but also because one the most important example of semistar operation, the (semi)star operation v, is not, in general, of finite type. The alternative use of the (semi)star operations v and t — in our case of $\star$ and $\star_f$ — helps for a better understanding of the motivations and the applications of the theory presented in this paper.

3. Characterization of P⋆MDs

In this section we prove several characterizations for an integral domain to be a P⋆MD, when $\star$ is a semistar operation.

We start with a first theorem in which some of the statements generalize some of the classical characterizations of the PvMDs (cf. M. Griffin [30, Theorem 5], R. Gilmer [26, Theorem 2.5], J. Arnold and J. Brewer [10, Theorem 3], J. Querré [52, Théorème 3, page 279] and B. G. Kang [39, Theorem 3.5, Theorem 3.7]).

Theorem 3.1. *Let D be an integral domain and $\star$ a semistar operation on D. The following are equivalent*:

 (i) *D is a P⋆MD*;
 (ii) *D_Q is a valuation domain, for each $Q \in \mathcal{M}(\star_f)$*;
(iii) *$\mathrm{Na}(D, \star)$ is a Prüfer domain*;
(iv) *$\mathrm{Na}(D, \star) = \mathrm{Kr}(D, \tilde{\star})$*;
 (v) *$\tilde{\star}$ is an e.a.b. semistar operation*;
(vi) *$\star_f$ is stable and e.a.b.*

In particular D is a P⋆MD if and only if it is a P$\tilde{\star}$MD.

Proof. (i) $\Rightarrow$ (ii). Let $Q \in \mathcal{M}(\star_f)$ and let J be a finitely generated ideal of D_Q, then $J = ID_Q$ for some $I \in \mathbf{f}(D)$, [27, Theorem 4.4]. Since I is $\star_f$-invertible, then $J = ID_Q$ is invertible, and hence principal, in the local domain D_Q (Lemma 2.4 (i) $\Rightarrow$ (ii) and [27, Corollary 7.5]). As a consequence D_Q is a local Bézout domain, i.e., D_Q is a valuation domain.

(ii) $\Rightarrow$ (i) is a consequence of Lemma 2.4 ((ii) $\Rightarrow$ (i)), since we are assuming that D_Q is a valuation domain, for each $Q \in \mathcal{M}(\star_f)$. A direct proof is the following. Let $I \subseteq D$ be a finitely generated ideal. For each $Q \in \mathcal{M}(\star_f)$, we have:

$$(II^{-1})D_Q = (ID_Q)(I^{-1}D_Q) = (ID_Q)(ID_Q)^{-1} = D_Q \,,$$

hence $II^{-1} \not\subseteq Q$, thus $(II^{-1})^\star = D^\star$ (Lemma 2.1(a)).

(ii) $\Rightarrow$ (iii). The maximal ideals of the Nagata ring $\mathrm{Na}(D, \star)$ are of the form $\bar{Q} :=$ $Q[X]_{N_D(\star)}$, for each $Q \in \mathcal{M}(\star_f)$, and we have $\mathrm{Na}(D, \star)_{\bar{Q}} = D_Q(X)$ (Lemma 2.3(f)). If D_Q is a valuation domain, then $D_Q(X)$ is also a valuation domain and hence $\mathrm{Na}(D, \star)$ is a Prüfer domain.

(iii) $\Rightarrow$ (iv). By assumption and Lemma 2.3(i) we have that $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \tilde{\star})$ is a Prüfer domain. Moreover, from the definition of $\tilde{\star}$ and from Lemma 2.3(f), (g) and (h), we deduce that D_Q is a $\tilde{\star}$-valuation overring of D, for each $Q \in \mathcal{M}(\star_f)$. Since $\mathrm{Kr}(D, \tilde{\star}) = \bigcap \{V(X) \mid V \text{ is a } \tilde{\star}\text{-valuation overring of } D\}$ (Lemma 2.6(c)), we obtain that $\mathrm{Kr}(D, \tilde{\star}) \subseteq \bigcap \{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\} = \mathrm{Na}(D, \star)$ (Lemma 2.3(c)), and thus $\mathrm{Kr}(D, \tilde{\star}) = \mathrm{Na}(D, \star)$.

(iv) $\Rightarrow$ (v). From the equality $\mathrm{Kr}(D, \tilde{\star}) = \mathrm{Na}(D, \star)$ and from Lemma 2.3(e) and Lemma 2.5(d) we deduce that $\tilde{\star} = (\tilde{\star})_a$.

(v) $\Rightarrow$ (ii). We recall that the following statements are equivalent:

(1) D_Q is a valuation domain, for each $Q \in \mathcal{M}(\star_f)$;
(2) FD_Q is an invertible ideal of D_Q, for each $F \in \mathbf{f}(D)$ and for each $Q \in \mathcal{M}(\star_f)$;
(3) FD_Q is a *quasi-cancellation ideal* of D_Q (i.e., $FGD_Q \subseteq FHD_Q$ implies $GD_Q \subseteq HD_Q$ when $G, H \in \mathbf{f}(D)$), for each $F \in \mathbf{f}(D)$ and for each $Q \in \mathcal{M}(\star_f)$.

Note that (1) $\Leftrightarrow$ (2) since, in a local domain, for a finitely generated ideal, invertible is equivalent to principal [27, Corollary 7.5]. (2) $\Leftrightarrow$ (3): this is a consequence of a result by Kaplansky (cf. [27, Exercise 7, p. 67], [4, Theorem 1] and [32, Theorem 13.8]).

Therefore, in order to prove that D_Q is a valuation domain, for each $Q \in \mathcal{M}(\star_f)$, we show that:

$$FGD_Q \subseteq FHD_Q \Rightarrow GD_Q \subseteq HD_Q \,,$$

for all $F, G, H \in \mathbf{f}(D)$. Now, from the assumption and from Lemma 2.3(b), for all $E \in \bar{\mathbf{F}}(D)$ we have:

$$E^{\tilde{\star}} = \bigcap \{ED_Q \mid Q \in \mathcal{M}(\star_f)\}, \quad \text{and} \quad E^{\tilde{\star}}D_Q = ED_Q, \quad \text{for each } Q \in \mathcal{M}(\star_f) \,.$$

Hence, if $FGD_Q \subseteq FHD_Q$, then $FG \subseteq FHD_Q$ and so there exists $t \in D \backslash Q$ such that $tFG \subseteq FH$. In particular, $(FtG)^{\tilde{\star}} \subseteq (FH)^{\tilde{\star}}$, hence by assumption $(tG)^{\tilde{\star}} \subseteq H^{\tilde{\star}}$. From the previous remark we deduce that $tGD_Q \subseteq HD_Q$, for each $Q \in \mathcal{M}(\star_f)$, that is $GD_Q \subseteq HD_Q$, because $tD_Q = D_Q$.

(vi) $\Rightarrow$ (v). Note that $\star_f$ is always quasi-spectral (Remark 2.1) and that a semistar operation is spectral if and only if is quasi-spectral and stable [21, Theorem 4.12(3)]. Therefore

$$\star_f \text{ is stable } \Leftrightarrow \star_f \text{ is spectral}.$$

Since $\tilde{\star} = (\star_f)_{\mathrm{sp}} \leq \star_f$, then:

$$\star_f \text{ is stable } \Leftrightarrow \star_f = (\star_f)_{\mathrm{sp}} = \tilde{\star}.$$

(v) $\Rightarrow$ (vi). Assume that $\tilde{\star}$ is an e.a.b. semistar operation (of finite type) on D, hence $\tilde{\star} = \star_\Delta$, where $\Delta := \mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$ and (by (v) $\Rightarrow$ (ii)) D_Q is a valuation domain, for each $Q \in \Delta$.

Claim 3.1. *Let $\star$ be a semistar operation on an integral domain D. If $\tilde{\star}$ is e.a.b., then $\star$ and $\star_f$ are a.b. (hence, in particular, e.a.b.).*

Assume that $\tilde{\star}$ is e.a.b. (note that for a semistar operation of finite type, like $\tilde{\star}$, the notions of e.a.b. and a.b. coincide). Henceforth (by (v) $\Rightarrow$ (i)) D is a P$\star$MD, and thus each nonzero finitely generated ideal in D is $\star_f$-invertible. Let $E \in \mathbf{f}(D)$, and suppose that $(EF)^\star \subseteq (EG)^\star$, for all $F, G \in \mathbf{F}(D)$ [respectively, $F, G \in \mathbf{f}(D)$, for the e.a.b. case]. Since $E \in \mathbf{f}(D)$, then there exists a nonzero $d \in D$ such that $I := dE$ is a nonzero finitely generated ideal in D. Let $J \in \mathbf{f}(D)$ be such that $(IJ)^\star = D^\star$. Then:

$$(EF)^\star \subseteq (EG)^\star \Rightarrow d(EF)^\star \subseteq d(EG)^\star \Rightarrow (IF)^\star \subseteq (IG)^\star$$
$$\Rightarrow J(IF)^\star \subseteq J(IG)^\star \Rightarrow (J(IF)^\star)^\star \subseteq (J(IG)^\star)^\star$$
$$\Rightarrow (JIF)^\star \subseteq (JIG)^\star \Rightarrow ((JI)^\star F)^\star \subseteq ((JI)^\star G)^\star$$
$$\Rightarrow F^\star \subseteq G^\star.$$

Therefore $\star$ is **a.b.**. Since $\widetilde{(\star_f)} = \tilde{\star}$, from the above argument we deduce also that $\star_f$ is a.b.. This proves Claim 3.1.

Under the present assumption, by Claim 3.1 and by [22, Proposition 4.5(5)] we have that $\star_a = (\star_f)_a = \star_f$ is an a.b. semistar operation of finite type on D. Therefore $\star_f = \star_{\mathcal{W}}$, for some set $\mathcal{W}$ of valuation overrings of D [23, Proposition 3.4] (i.e. $F^\star = \bigcap \{FW \mid W \in \mathcal{W}\}$, for each $F \in \mathbf{f}(D)$).

Furthermore, note that, in the present situation, (by [22, Corollary 3.8] and (v) $\Rightarrow$ (iv)) we have:

$$\bigcap \{W(X) \mid W \in \mathcal{W}\} = \mathrm{Kr}(D, \star_a) = \mathrm{Kr}(D, \star)$$

$$\supseteq \mathrm{Kr}(D, \tilde{\star}) = \mathrm{Na}(D, \star) = \bigcap \{D_Q(X) \mid Q \in \Delta\}.$$

Since $\mathrm{Na}(D, \star)$ is a Prüfer domain and, by [24, Theorem 3.9],

$$\mathrm{Max}(\mathrm{Na}(D, \star)) = \{QD_Q(X) \cap \mathrm{Na}(D, \star) \mid Q \in \Delta\}$$

then, for each $W \in \mathcal{W}$, there exists a prime ideal $Q \in \Delta$ and a prime ideal H in $D_Q(X)$, such that $W(X) = (D_Q(X))_H$. Therefore, we have that

$$W = W(X) \cap K = (D_Q(X))_H \cap K \supseteq D_Q\,.$$

Since, for each $Q \in \Delta$, D_Q is a valuation domain, then there exists a prime ideal $Q' \subseteq Q$ of D such that $W = D_{Q'}$. Set $\Delta' := \{Q' \mid D_{Q'} = W, \text{ for some } W \in \mathcal{W}\}$. Therefore, we have $\star_\Delta = \tilde{\star} \leq \star_f = \star_{\Delta'}$ (note that, by construction of Δ', $\Delta' \subseteq \Delta^\downarrow$). On the other hand, $\Delta = \mathcal{M}(\star_f) = \mathcal{M}(\star_{\Delta'}) \subseteq \Delta'$ and so $\Delta^\downarrow \subseteq \Delta'^\downarrow$. From the previous remarks, we deduce that $\Delta^\downarrow = \Delta'^\downarrow$ and so we conclude that $\tilde{\star} = \star_\Delta = \star_{\Delta'} = \star_f = (\star_f)_a$.

The last statement of the theorem follows easily from the equivalence (i) $\Leftrightarrow$ (iv) and from Lemma 2.3(i). $\qquad\square$

Remark 3.1. As a consequence of the proof of Theorem 3.1, we have that:

$$D \text{ is P}\star\text{MD} \;\Leftrightarrow\; \mathrm{Na}(D, \star) = \mathrm{Kr}(D, \star)\,.$$

As a matter of fact, when D is P$\star$MD, then $\tilde{\star} = \star_f = (\star_f)_a = \star_a$ and so $\mathrm{Na}(D, \star) = \mathrm{Kr}(D, \star)$ (and conversely).

Recently, W. Fanggui and R. L. McCasland [19, Section 2] have introduced, studied and characterized the integral domains that are PwMD, where w is the (semi)star operation considered in Remark 2.1 that, in our notation, coincides with $\tilde{v} (= t_{\mathrm{sp}})$. They observed that, for a given integral domain D,

$$D \text{ is a P}w\text{MD} \Rightarrow D \text{ is a P}v\text{MD}\,.$$

The following corollary to Theorem 3.1 shows, among other properties, that this implication is in fact an equivalence, reobtaining a result proved by D. D. Anderson and S. J. Cook [6, Theorem 2.18] that a nonzero fractional ideal is t-invertible if and only if is w-invertible. This property was generalized in [21, Proposition 4.25].

Corollary 3.1. *Let D be an integral domain. The following are equivalent*:

(i) *D is a PvMD*;
(ii) $\mathrm{Na}(D, v) = \mathrm{Kr}(D, t_{\mathrm{sp}})$;
(iii) *t_{sp} is an e.a.b. semistar operation.*

In particular D is a PvMD if and only if it is a $Pt_{\mathrm{sp}}MD$.

Proof. It is a straightforward consequence of the previous theorem, after observing that $\tilde{v} = (v_f)_{\mathrm{sp}} = t_{\mathrm{sp}}$. $\qquad\square$

Remark 3.2. (1) Note that, if $\tilde{v} = (v_f)_{\mathrm{sp}} = t_{\mathrm{sp}} = \tilde{t}$ is an e.a.b. (semi)star operation on a domain D, then the v-operation is also e.a.b. operation on D, but the converse is not necessarily true [27, page 418, Theorem 34.11 and Exercise 5 page 429].

(2) Recall that if D is an integrally closed integral domain and if $D = \bigcap_\alpha V_\alpha$ can be represented as the intersection of a family of essential valuation overrings

(e.g. if D is a PvMD) then the a.b. (semi)star operation $\star_\mathcal{W}$, where $\mathcal{W} := \{V_\alpha\}$ (Example 2.1(g.3)), is equivalent to the v (semi)star operation [27, Proposition 44.13]. In particular, in a PvMD, $t_{\mathrm{sp}} = \tilde{t}$ is equivalent to v, i.e. $\tilde{t} = t$, since $\tilde{t}$ is a (semi)star operation of finite type (Remark 2.1).

Note that, in this context, Zafrullah [54, Theorem 5] has proved the following general result: *Let D be an integral domain and Δ a set of prime ideals of D such that $D = \bigcap\{D_P \mid P \in \Delta\}$. Then the (semi)star operation $\star_\Delta$ is equivalent to the v (semi)star operation on D if and only if, for each $F \in \mathbf{f}(D)$ and for each $P \in \Delta$, $FD_P = F^v D_P$.* (It is obvious that when D_P is a valuation domain, then $FD_P = F^v D_P = (FD_P)^v$, because FD_P is a principal ideal in D_P.)

(3) For Prüfer domains, J. Arnold [9, Theorem 4] has proved that, if D is an integral domain, then:

$$D \text{ is a Prüfer domain} \Leftrightarrow \mathrm{Na}(D, d) = D(X) \text{ is a Prüfer domain}$$

$$\Leftrightarrow \mathrm{Na}(D, d) = \mathrm{Kr}(D, b).$$

Note that the previous equivalence follows from Theorem 3.1 ((i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)), since if D is Prüfer then $d = \tilde{d} = b$ and if $\mathrm{Na}(D, d) = \mathrm{Kr}(D, b)$ then $d = \tilde{d} = b_a = b$ is an e.a.b. (semi)star operation.

Next result gives a positive answer to the problem of the *ascent* of the P$\star$MD property.

Proposition 3.1. *Let $\star$ be a semistar operation defined on an integral domain D and let T be an overring of D. Denote simply by $\dot{\star}$ the semistar operation $\dot{\star}^T$ on T (Example 2.1(e)). Assume that D is a P$\star$MD, then T is a P$\dot{\star}$MD.*

Proof. To avoid the trivial case, we can assume that T is different from the quotient field of D. Let H be a prime ideal of T which is a maximal element in the set of nonzero ideals of T with the property that $H^{\dot{\star}_f} \cap T = H$, i.e. H is a quasi-$\dot{\star}_f$-maximal of T. We want to show that T_H is a valuation domain (Theorem 3.1 ((ii)$\Rightarrow$(i))). If we consider the prime ideal $Q := H \cap D$ of D, then Q is nonzero, since $D_Q \subseteq T_H$, and moreover:

$$Q^{\star_f} \cap D = (H \cap D)^{\star_f} \cap D \subseteq H^{\star_f} \cap D = H^{\dot{\star}_f} \cap T \cap D$$

$$= H \cap D = Q \subseteq Q^{\star_f} \cap D,$$

and thus Q is a prime quasi-$\star_f$-ideal of D. If Q is not a quasi-$\star_f$-maximal, then there exists a prime ideal P such that $Q \subseteq P$ and $P = P^{\star_f} \cap D$ (Lemma 2.1(a)). Now we have:

$$D_P \subseteq D_Q \subseteq T_H$$

with D_P valuation domain, because D is a P$\star$MD (Theorem 3.1 ((i)$\Rightarrow$(ii))). We conclude immediately that T_H is a valuation domain. $\qquad\square$

Corollary 3.2. *Let $\star$ be a semistar operation defined on an integral domain D. Assume that D is a $P\star MD$ and denote simply by $\dot{\star}$ the (semi)star operation $\dot{\star}^{D^\star}$ on $D^\star$ (Example 2.1(e)). Then $D^\star$ is a $P\dot{\star}MD$.*

Proof. The statement is a straightforward consequence of Proposition 3.1 (taking $T = D^\star$). $\qquad\square$

Next goal is to study the "descent" of the P⋆MD property. The following lemma is required in the proof of next proposition.

Lemma 3.1. *Let T be an overring of an integral domain D and let $\star$ be a semistar operation on T. The semistar operations of finite type $(\star)_f$ and $(\star_f)$ (both defined on D) coincide. (For the sake of simplicity, we will simply denote by $\star_f$ this semistar operation.)*

Proof. Let $E \in \bar{\mathbf{F}}(D)$, then

$$E^{(\star)_f} = \bigcup \{F^\star \mid F \subseteq E,\ F \in \mathbf{f}(D)\}$$

$$= \bigcup \{(FT)^\star \mid F \subseteq E,\ F \in \mathbf{f}(D)\}$$

$$\subseteq \bigcup \{H^\star \mid H \subseteq ET,\ H \in \mathbf{f}(T)\} \quad (= (ET)^{\star_f} = E^{(\star_f)})$$

$$= \bigcup \{(FT)^\star \mid F \subseteq ET,\ F \in \mathbf{f}(D)\}$$

$$\subseteq \bigcup \{(FT)^\star \mid F \subseteq E,\ F \in \mathbf{f}(D)\} = E^{(\star)_f},$$

since, if $F \subseteq ET$ with $F \in \mathbf{f}(D)$, it is possible to find $E_0 \subseteq E$ with $E_0 \in \mathbf{f}(D)$ and $F \subseteq E_0 T$, therefore $(FT)^\star \subseteq (E_0 T)^\star$. $\qquad\square$

Proposition 3.2. *Let T be a flat overring of an integral domain D. Let $\star$ be a semistar operation on T. Assume that T is a $P\star MD$. Denote simply by $\dot{\star}$ the semistar operation $\star_D$ on D (Example 2.1(e)). Then D is a $P\dot{\star}MD$.*

Proof. Let $Q \in \mathcal{M}(\dot{\star}_f)$, then by Lemma 3.1 we have

$$Q^{\star_f} \cap D = (QT)^{\star_f} \cap D = Q.$$

In particular $QT \neq T$, hence there exists $H \in \mathcal{M}(\star_f)$ such that $H \supseteq QT$ and so $H \cap D \supseteq Q$. Note that $(H \cap D)^{\dot{\star}_f} = ((H \cap D)T)^{\star_f}$, and since $H \in \mathcal{M}(\star_f)$, then:

$$H \cap D \subseteq ((H \cap D)T)^{\star_f} \cap D \subseteq H^{\star_f} \cap D = H^{\star_f} \cap T \cap D = H \cap D.$$

Henceforth, $H \cap D$ is a quasi-$\dot{\star}_f$-prime of D and so $H \cap D = Q$. Therefore, we conclude that $\mathcal{M}(\dot{\star}_f)$ coincides with the contraction to D of the set $\mathcal{M}(\star_f)$. Since T_H is a valuation domain, for each $H \in \mathcal{M}(\star_f)$, and T is D-flat then, by [53, Theorem 2], we conclude that $D_{H \cap D} = T_H$ is also a valuation domain, and so D is a $P\dot{\star}MD$ (Theorem 3.1 ((ii) $\Rightarrow$ (i))). $\qquad\square$

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Remark 3.3. (1) Note that, in Proposition 3.2, the hypothesis that T is D-flat is essential (cf. also [32, Theorem 27.2]). For example, let (T, M) be a discrete 1-dimensional valuation domain with residue field k. Let k_0 be a proper subfield of k and assume that k is a finite field extension of k_0. Set

$$D := \varphi^{-1}(k_0) \longrightarrow k_0$$

$$T \xrightarrow{\ \varphi\ } T/M = k$$

Then D and T are local with the same maximal ideal M which is a finitely generated ideal both in D and in T [20, Theorem 2.3]. Let $\star := b \ (= d)$ be the identical (semi)star operation on the valuation domain T. Then $\star_D = \star_{\{T\}}$, i.e. $E^{\star_D} = ET$, for each $E \in \bar{\mathbf{F}}(D)$. Obviously T is a (local) Prüfer domain, but D is not a P$\star_{\{T\}}$MD, since $M \in \mathcal{M}((\star_D)_f) = \mathcal{M}((\star_{\{T\}})_f)$ but $D_M = D$ is not a valuation domain.

(2) Note that, from Proposition 3.1 and Example 2.1 (e.6), if $\star$ is a semistar operation on the overring T of D, if $\dot{\star} = \star_D$ and if D is a P$\dot{\star}$MD, then T is a P$\star$MD.

Example 3.1. *When $\star$ is a semistar operation, a P$\star$MD, is not necessarily integrally closed.* (Note that if $\star$ is a semistar operation on an integral domain D and D is a P$\star$MD, then $D^\star$ must be integrally closed by Corollary 2.3 and [27, Theorem 34.6, Proposition 34.7 and Theorem 34.11]; in particular, if $\star$ is a star operation on D, then D is integrally closed.)

Let D be a non integrally closed integral domain and let Δ be a nonempty finite set of nonzero prime ideals of D with the following properties:

(a) D_P is a valuation domain, for each $P \in \Delta$;
(b) $D_{P'}$ and $D_{P''}$ are incomparable, if $P' \neq P''$ and $P', P'' \in \Delta$.

Let $\star := \star_\Delta$ be the spectral semistar operation on D associated to Δ (Example 2.1(f)). Since

$$D^\star = \bigcap \{D_P \mid P \in \Delta\}, \quad \mathrm{Max}(D) = \{PD_P \cap D^\star \mid P \in \Delta\}$$

and $D^\star$ is a semilocal Bézout domain [40, Theorem 107], then clearly $D \subsetneq D^\star$ and $D^\star$ is flat over D [53, Theorem 2].

Let $* := \dot{\star} = \dot{\star}^{D^\star}$ denote the (semi)star operation defined on $D^\star$ induced by $\star$ (Example 2.1(e)), then $D^\star$ is trivially a P$*$MD, since $D^\star$ is a Bézout domain. Denote simply by $\underset{*}{*}$ the semistar operation $*_D$ on D induced by $*$ (Example 2.1(e)) then, by Proposition 3.1, D is a P$\underset{*}{*}$MD, but by assumption is not integrally closed. Note that it is easy to verify that, in the present situation, $\star = \underset{*}{*}$ since, for each $E \in \bar{\mathbf{F}}(D)$, we have:

$$E^{\underset{*}{*}} = (ED^\star)^* = (ED^\star)^\star = (ED)^\star = E^\star.$$

Therefore D is a P$\star$MD but, by assumption, it is not integrally closed. In particular D is not a PvMD.

The following explicit construction produces an example similar to the situation described in previous Remark 3.3(1).

Example 3.2. Let K be a field and X, Y indeterminates over K. Set $F := K(X)$ and $D := K + YF[[Y]]$. It is well known that D in an integrally closed 1-dimensional non-valuation local domain with maximal ideal $M := YF[[Y]]$ and that $V := K[X]_{(X)} + YF[[Y]]$ is a 2-dimensional valuation overring of D with maximal ideal $N := XK[X]_{(X)} + YF[[Y]]$, [27, Section 17, Exercises 11–14 and page 231]. Note that M is also an ideal inside V, and precisely M is the height 1 prime ideal of V.

Consider the semistar operation $\star := \star_{\{V\}}$ on D (cf. Example 2.1(g)). It is of finite type and induces over $V = D^\star$ the identity (semi)star operation d_V on V, i.e. $\dot\star(=\dot\star^V) = d_V$. Henceforth $D^\star$ is a P$\dot\star$MD, in fact it is a valuation domain.

Note that D is not P$\star$MD, because the only maximal (quasi)$\star$-ideal is M, since $M^\star = MV = M$, and because $D = D_M$ is not a valuation domain (Theorem 3.1, ((i) $\Leftrightarrow$ (ii))).

Keeping in mind Proposition 3.2, note also that V is not D-flat by [53, Theorem 2], because it is easy to see that $V_M = F[[Y]] \supsetneq D_M = D$. Moreover, if $\delta := d_V$ is the identical (semi)star operation on V, then the semistar operation $\dot\delta := \delta_D$ on D induced by δ, defined in Example 2.1(e), coincides with $\star$.

Note, also, that in the present situation $\tilde\star = \star_{\mathrm{sp}}$, since $\star = \star_f$; moreover $\star_{\mathrm{sp}} = d_D$ the identical (semi)star operation on D, since $\mathcal{M}(\star_f) = \{M\}$ and $D_M = D$. Furthermore, $\tilde\star = d_D$ is not an e.a.b. (semi)star operation on D (cf. also Theorem 3.1 ((i) $\Leftrightarrow$ (v))), because of the equivalence (1) $\Leftrightarrow$ (3) in the proof (v) $\Rightarrow$ (ii) of Theorem 3.1 and because $D = D_M$ is not a valuation domain.

The previous example shows that if $D^\star$ is a P$\dot\star$MD then D is not necessarily a P$\star$MD. This fact induces to strengthen the condition "$D^\star$ is P$\dot\star$MD" for characterizing D as a P$\star$MD and it suggests (in the finite type case) the use of the semistar operation $\star_{\mathrm{sp}}$ (or, equivalently, $\tilde\star$) instead of $\star$.

Proposition 3.3. *Let $\star$ be a semistar operation defined on an integral domain D. With the notation of Lemma 2.3, we have*:

$$D \text{ is a } P\star MD \;\Leftrightarrow\; D \text{ is a } P\tilde\star MD \;\Leftrightarrow\; D^{\tilde\star} \text{ is a } P\dot{\tilde\star}MD.$$

Proof. From Theorem 3.1 and Corollary 3.2 we deduce immediately that:

$$D \text{ is a } P\star MD \;\Leftrightarrow\; D \text{ is a } P\tilde\star MD \;\Rightarrow\; D^{\tilde\star} \text{ is a } P\dot{\tilde\star}MD.$$

Set $\tilde D := D^{\tilde\star}$. By Lemma 2.3(h) we know that

$$\mathcal{M}(\dot{\tilde\star}) = \{\tilde Q := QD_Q \cap \tilde D \mid Q \in \mathcal{M}(\star_f)\} \quad \text{and} \quad \tilde D_{\tilde Q} = D_Q,$$

for each $Q \in \mathcal{M}(\star_f)$. Assume that $\tilde{D}$ is a P$\dot{\tilde{\star}}$MD, then $\tilde{D}_{\tilde{Q}} = D_Q$ is a valuation domain, for each $Q \in \mathcal{M}(\star_f)$, by Theorem 3.1 ((i) $\Rightarrow$ (ii))) applied to $\tilde{D}$. We conclude that D is a P$\tilde{\star}$MD from Theorem 3.1 ((ii) $\Rightarrow$ (i)) and from Lemma 2.3(g). $\quad\square$

Next example shows that the flatness hypothesis in Proposition 3.2 is essential also outside of a pullback setting (cf. for instance Remark 3.3(1) and Example 3.2).

Example 3.3. Let T be an overring of an integral domain D and let $\star := \star_{\{T\}}$ be the semistar operation of finite type on D, defined in Example 2.1(g) with $\mathcal{T} := \{T\}$. Assume that T is integral over D and that $D \neq T$, then D is not a P$\star$MD even if T is a Prüfer domain.

Note that, as in Example 3.2, if $\delta := d_T$ is the identical (semi)star operation on T, then the semistar operation $\delta := \delta_D$ on D induced by δ, defined in Example 2.1(e), coincides with $\star$. Moreover, since T is integral over D then, by the lying-over theorem, we have:

$$\text{Max}\{P \in \text{Spec}(D) \mid 0 \neq P \text{ and } PT \cap D \neq D\} = \text{Max}(D).$$

Therefore, by [13, Chapitre II, §3, No. 3, Corollaire 4], Lemma 1.2(c) and Remark 1.3, we have:

$$\tilde{\star} = \star_{\text{sp}} = d,$$

where $d := d_D$ is the identical (semi)star operation on D, and so $D^{\tilde{\star}} = D$. By using **Theorem 3.1**, we have:

$$D \text{ is a P}\star\text{MD} \iff D \text{ is a P}d\text{MD} \quad (\text{i.e. } D \text{ is a Prüfer domain}),$$

and this is excluded if $D \neq T$.

More generally, the previous argument shows that:

Let T be a proper integral overring of an integral domain D. Assume that there exists a semistar operation $*$ on T such that T is a P$*$MD. Then D is not a P$\star$MD, for any semistar operation $\star$ on D such that $\star \leq \star_{\{T\}} (\leq *)$.

In fact, recall that if $\star_1$ and $\star_2$ are two semistar operations on an integral domain D, if $\star_1 \leq \star_2$ and if D is a P$\star_1$MD, then D is also a P$\star_2$MD. Therefore, it is sufficient to show that D is not a P$\star_{\{T\}}$MD and this fact follows from the equivalence proved above, since D is not a Prüfer domain because, by assumption, $T \neq D$ is integral over D.

In case of star operations, next goal is to characterize P$\star$MDs in terms of PvMDs. We start with few general remarks concerning the "star setting".

Remark 3.4. Let $\star$, $\star_1$ and $\star_2$ be *star operations* on an integral domain D. We denote by $\text{Spec}_\star(D)$ the set of all prime ideals P of D, such that $P^\star = P$, then obviously:

$$\star_1 \leq \star_2 \implies \text{Spec}_{\star_2}(D) \subseteq \text{Spec}_{\star_1}(D).$$

A prime ideal P of an integral domain D is called *a valued prime* if D_P is a valuation domain.

Let $\star$ be a star operation on D and assume that D is a P$\star$MD (hence, in particular, a PvMD). Then, by [47, Proposition 4.1], a prime ideal of D is valued if and only if it is t-ideal. As a consequence, under the present assumptions, the valued prime ideals of D are inside $\mathrm{Spec}_t(D)$. Moreover, since $\star_f \leq t$, for each star operation $\star$ on D [27, Theorem 34.1(4)], then $\mathrm{Spec}_t(D) \subseteq \mathrm{Spec}_{\star_f}(D)$. On the other hand, as D is P$\star$MD, then each maximal $\star_f$-ideal is valued, hence $\mathcal{M}(\star_f) \subseteq \mathrm{Spec}_t(D)$, but this means $\mathrm{Spec}_{\star_f}(D) \subseteq \mathrm{Spec}_t(D)$, which implies that $\mathrm{Spec}_{\star_f}(D) = \mathrm{Spec}_t(D)$.

In the following proposition we prove that the implication (i) $\Rightarrow$ (ii), due to Kang [39, Theorem 3.5], can be inverted, obtaining a new characterization of a P$\star$MD which is related to [25, Proposition 21] (cf. also [32, Theorem 17.1(ii)]):

Proposition 3.4. *Let $\star$ be a star operation on an integral domain D. The following statements are equivalent*:

- (i) *D is a P$\star$MD.*
- (ii) *D is a PvMD and $\tilde{\star} = t$.*
- (iii) *D is a PvMD and $\star_f = t$.*

Proof. (i) $\Rightarrow$ (ii). Since $\mathrm{Spec}_{\star_f}(D) = \mathrm{Spec}_t(D)$ (Remark 3.4), then $\tilde{\star} = \tilde{t}$. Moreover a P$\star$MD is a P$v$MD and, in a P$v$MD, $\tilde{t} = t$ [39, Theorem 3.5].

(ii) $\Leftrightarrow$ (iii). It is a consequence of Remark 3.1.

(iii) $\Rightarrow$ (i) Since $v_f = t = \star_f$, then the conclusion follows immediately from the fact that the notions of P$\star$MD and P$\star_f$MD coincide, for each semistar operation $\star$. $\blacksquare$

From the previous result it is possible to find star operations $\star$ on an integral domain D such that D is a PvMD, but D is not a P$\star$MD. For instance, if D is a Krull non Dedekind domain, then obviously D is a PvMD but not a P$\star$MD, if $\star$ coincides with d the identical star operation on D, since $d = d_{\mathrm{sp}}$ and, in a Krull domain D, $t = d$ if and only if D is a Dedekind domain, [27, Theorem 34.12 and Theorem 43.16].

Next example describes a more general situation.

Example 3.4. Let K be a field and X and Y be indeterminates over K. Let us consider two distinct maximal ideals M_1 and M_2 of $K[X,Y]$. Let $S := K[X,Y]\backslash(M_1 \cup M_2)$ be a multiplicative closed subset of $K[X,Y]$ and let $D := S^{-1}K[X,Y]$. Thus D is a Noetherian Krull domain, hence D is a PvMD. Moreover D is semilocal with maximal ideals $N_1 = S^{-1}M_1$ and $N_2 = S^{-1}M_2$ (note that D_{N_1} and D_{N_2} are not valuation domains).

Let us consider the spectral star operation $\star$ on D defined by the subset $\Delta := \mathrm{Spec}(D)\backslash\{N_2\}$, i.e. $\star = \star_\Delta$ as in Example 2.1(f). It is not difficult to show that

$\star \neq d$ (in fact $(N_2)^d = N_2 \neq D = N_2^\star$) and D is not P$\star$MD (as N_1 is a maximal $\star$-ideal and D_{N_1} is not a valuation domain).

Next result makes more precise the statement of Proposition 3.4 in case $\star$ coincides with the identical star operation (cf. also [32, Theorem 17.3]**) add text....**

Proposition 3.5. *Let D be an integral domain, then the following are equivalent*:

  (i) *D is a Prüfer domain.*
 (ii) *D is integrally closed and $d = t$.*
(iii) *D is integrally closed and has a unique star operation of finite type.*

Proof. It is obvious that (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii), since it is well known that for each star operation of finite type $\star$ of an integral domain, $d \leq \star \leq t$, [27, Theorem 34.1(4)]. Finally (ii) $\Rightarrow$ (i) because, under the present assumptions, for each nonzero ideal I of D, we have:

$$I_t = I = \bigcap \{ID_M \mid M \in \text{Max}(D)\}$$

where, obviously, $\text{Max}(D) = \mathcal{M}(d) = \mathcal{M}(t)$, and thus the conclusion follows from [39, Theorem 3.5]. $\qquad\square$

Remark 3.5. (1) From the previous result we deduce that, *in a Prüfer domain, any two star operations are equivalent* (*in fact, both are equivalent to v* (Proposition 3.5 ((i) $\Rightarrow$ (ii))) *and each star operation $\star$ is a.b.* (*in fact, $\tilde{\star}$ is a.b.* (cf. Remark 3.2)), [27, Proposition 32.18]).

The last part of the statement follows from the fact that each localization of D is a valuation domain, thus the star operation $\tilde{\star}$ is necessarily a $\star_\mathcal{W}$-operation, for some family $\mathcal{W}$ of valuation overrings of D (Example 2.1(g.3)).

In relation with the first part of the statement note that, for each star operation $\star$ on a Prüfer domain, we have $\star_f = t = d = b$ and thus $\tilde{\star} = \widetilde{(\star_f)} = \tilde{d} = d = b = \star_f = t$.

(2) Note that the statement in (1) is not a characterization of Prüfer domains, since there exists an integrally closed non-Prüfer integral domain such that any two star operations are equivalent and each star operation is a.b. [27, Section 32, Exercise 12] and [50, Proposition 24].

On the other hand, for an integral domain D, we have:

  D is Prüfer if and only if each semistar operation on D is a.b.

By an argument as in (1), we have that if D is Prüfer then each semistar operation on D is a.b. Conversely, for each prime ideal P of D, if $\star_{\{D_P\}}$ is an a.b. operation then, by the equivalence (1) $\Leftrightarrow$ (3) in the proof of Theorem 3.1 ((v) $\Rightarrow$ (ii)), we deduce that D_P is a valuation domain.

The following remark provides a *quantitative information* about the size of the set of all the semistar operations $\star$ on a given integral domain D for which D is a P$\star$MD.

Remark 3.6. Let $\boldsymbol{\mathcal{P}}(D)$ be the set of all semistar operations of finite type on D such that D is a P⋆MD and let $\boldsymbol{\mathcal{B}}(\mathrm{Spec}(D))$ be the set of all the subsets of $\mathrm{Spec}(D)$. Then, the map:

$$\mu : \boldsymbol{\mathcal{P}}(D) \to \boldsymbol{\mathcal{B}}(\mathrm{Spec}(D)), \quad \star \mapsto \mathcal{M}(\star_f),$$

defines a surjection onto the set $\boldsymbol{\mathcal{M}}(D) \, (\subseteq \boldsymbol{\mathcal{B}}(\mathrm{Spec}(D)))$ of all the subsets of $\mathrm{Spec}(D)$ that are quasi-compact and that are formed by valued incomparable prime ideals of D [21, Corollary 4.6]. Obviously $\mu(\star_1) = \mu(\star_2)$ if and only if $\tilde{\star}_1 = \tilde{\star}_2$ (Remark 2.1). Note that the map:

$$\mu' : \boldsymbol{\mathcal{M}}(D) \to \boldsymbol{\mathcal{P}}(D), \quad \mathcal{M} \mapsto \star_{\mathcal{M}},$$

is such that $\mu \circ \mu'$ is the identity.

Next goal is to give a characterization of a P⋆MD, when $\star$ is a semistar operation, in terms of polynomials, by generalizing the classical characterization of Prüfer domain in terms of polynomials given by R. Gilmer and J. Hoffman [28, Theorem 2]. Note that similar properties, in the "star setting", were already considered by J. Mott and M. Zafrullah [47, Theorem 3.4] and by E. Houston, S. J. Malik and J. Mott [36, Theorem 1.1].

Let D be an integral domain with quotient field K, recall that an ideal I of a polynomial ring $D[X]$ is called *an upper to* 0 *in* $D[X]$ if there exists a nontrivial ideal J in $K[X]$ such that $J \cap D[X] = I$. Note that a nontrivial primary ideal H of $D[X]$ is an upper to 0 if and only if $H \cap D = 0$.

Theorem 3.2. *Let* $\star$ *be a semistar operation defined on an integral domain D with quotient field K. The following statements are equivalent*:

(i) *D is a P⋆MD*;

(ii) *$D^{\tilde{\star}}$ is integrally closed (i.e. $D^{\tilde{\star}} = D^{[\tilde{\star}]}$) and, for each I upper to 0 in $D[X]$, we have $I\,\mathrm{Na}(D, \star) = \mathrm{Na}(D, \star)$ (or, equivalently, there exists $f \in I$ such that $\mathbf{c}(f)^{\star} = D^{\star}$);*

(iii) *$D^{\tilde{\star}}$ is integrally closed (i.e. $D^{\tilde{\star}} = D^{[\tilde{\star}]}$) and, for each nonzero prime ideal H of $D[X]$ such that $H \cap D = 0$, we have $H\,\mathrm{Na}(D, \star) = \mathrm{Na}(D, \star)$ (or, equivalently, there exists $f \in H$ such that $\mathbf{c}(f)^{\star} = D^{\star}$);*

(iv) *$D^{\tilde{\star}}$ is integrally closed (i.e. $D^{\tilde{\star}} = D^{[\tilde{\star}]}$) and, for all nonzero elements $a, b \in D$, the prime ideal $H := (aX + b)K[X] \cap D[X]$ of $D[X]$ is such that $H\,\mathrm{Na}(D, \star) = \mathrm{Na}(D, \star)$ (or, equivalently, there exists $f \in H$ such that $\mathbf{c}(f)^{\star} = D^{\star}$).*

Proof. (i) $\Rightarrow$ (ii). We know, from Corollary 3.2, that $D^{\star}$ is a P$\dot{\star}$MD, where $\dot{\star} = \dot{\star}^{D^{\star}}$ defines a star operation on $D^{\star}$ (when restricted to $\mathbf{F}(D^{\star})$), and hence $D^{\star}$ is integrally closed [27, Corollary 32.8 and Theorem 34.11]. The same argument can be applied to $\tilde{\star}$ and $D^{\tilde{\star}}$. Moreover, since $\tilde{\star}$ is stable by Example 2.1(f.2), then we deduce that $D^{\tilde{\star}} = D^{[\tilde{\star}]}$ or, equivalently, that $D^{\tilde{\star}}$ is integrally closed (Example 2.1(c.2)).

From Theorem 3.1 ((i) $\Rightarrow$ (iv)) we know that, in the present situation $\mathrm{Na}(D, \star) = \mathrm{Kr}(D, \tilde{\star})$. Therefore, if $I := hK[X] \cap D[X]$, with h a non constant polynomial of $K[X]$, by Lemma 2.5(e), we have:

$$I\,\mathrm{Na}(D, \star) = I\,\mathrm{Kr}(D, \tilde{\star}) \supseteq \{\mathbf{c}(f)\,\mathrm{Kr}(D, \tilde{\star}) \mid f \in I\}$$

$$= \{\mathbf{c}(hg)\,\mathrm{Kr}(D, \tilde{\star}) \mid g \in K[X],\ hg \in D[X]\}$$

$$= \{\mathbf{c}(hg)\,\mathrm{Na}(D, \star) \mid g \in K[X],\ hg \in D[X]\}\,.$$

Since D is a P$\star$MD, then there exists a finitely generated (fractional) ideal L of D such that $(\mathbf{c}(h)L)^\star = D^\star$ and $L \subseteq (D :_K \mathbf{c}(h))$.

Let $\ell \in K[X]$ be such that $\mathbf{c}(\ell) = L$. Then, by the content formula [27, Theorem 28.1], for some $m \geq 0$, we have

$$\mathbf{c}(h)\mathbf{c}(\ell)\mathbf{c}(h)^m = \mathbf{c}(h\ell)\mathbf{c}(h)^m$$

and so

$$(\mathbf{c}(h)\mathbf{c}(\ell)\mathbf{c}(h)^m L^m)^\star = (\mathbf{c}(h\ell)\mathbf{c}(h)^m L^m)^\star\,.$$

Therefore:

$$D^\star = (\mathbf{c}(h)\mathbf{c}(\ell))^\star = \mathbf{c}(h\ell)^\star\,.$$

Set $f := h\ell$, since $L \subseteq (D :_K \mathbf{c}(h))$ then $f \in I \subseteq I\,\mathrm{Na}(D, \star)$. By the fact that $\mathbf{c}(f)^\star = D^\star$, we deduce $f\,\mathrm{Na}(D^\star, \star) = \mathrm{Na}(D^\star, \star)$, and thus $I\,\mathrm{Na}(D^\star, \star) = \mathrm{Na}(D^\star, \star)$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) are trivial.

(iv) $\Rightarrow$ (i). Set $\tilde{D} := D^{\tilde{\star}}$ and, for each $Q \in \mathcal{M}(\star_f)$, $\tilde{Q} := QD_Q \cap \tilde{D}$. Note that, by Lemma 2.3(h), $\tilde{D}_{\tilde{Q}} = D_Q$. By assumption, $\tilde{D}$ (and so $\tilde{D}_{\tilde{Q}}$) is integrally closed, for each $Q \in \mathcal{M}(\star_f)$. In order to conclude we want to show that $\tilde{D}_{\tilde{Q}}$ is a valuation domain, for each $Q \in \mathcal{M}(\star_f)$ (Proposition 3.3 and Theorem 3.1 ((ii) $\Rightarrow$ (i))). Let $t := a/b \in K$ with $a, b \in D$, $b \neq 0$, and let $H := (bX - a)K[X] \cap D[X]$. By assumption, there exists a polynomial $f \in H \subseteq D[X]$ such that $\mathbf{c}(f)^\star = D^\star$. In particular we have that $\mathbf{c}(f) \in D\backslash Q$, since $Q \in \mathcal{M}(\star_f)$. Henceforth $f \in D[X]\backslash QD[X] \subseteq \tilde{D}[X]\backslash\tilde{Q}\tilde{D}[X]$. Since $f \in H$ then $f(t) = 0$, this implies that t or t^{-1} is in $\tilde{D}_{\tilde{Q}}$ [27, Lemma 19.14]. $\qquad\square$

4. Passing Through Field Extensions

In this section we deal with the preservation of the P$\star$MD property by *ascent* and *descent*, in case of field extensions. Our purpose is to generalize to the P$\star$MD case the following classical results concerning Prüfer domains (cf. [27, Theorem 22.4 and Theorem 22.3]):

(1) Let D be an integrally closed domain with quotient field K which is a subring of an integral domain T. Assume that T is integral over D and that T is a Prüfer domain, then D is also a Prüfer domain.

(2) Let D be a Prüfer domain with quotient field K and let L be an algebraic field extension of K. Then the integral closure T of D in L is a Prüfer domain.

When we study the "descent" of the P⋆MD property, we have to consider also a "natural restriction" of the semistar operation $\star$. Recall that, in 1936 W. Krull [41, Satz 9] proved that if D in an integrally closed integral domain with quotient field K, if L is an algebraic field extension of K and if T is the integral closure of D in L, then, for each nonzero fractional ideal E of D,

$$(ET)^v \cap K = E^v \,,$$

(cf. also [34, Lemma 3.7]). The same formula holds, when X is indeterminate over K and $T := D[X]$ (cf. [27, Section 34, Exercise 16]).

The following result shows that, when we assume for the "natural restriction" that a property of the previous type holds, then we have a "descent" theorem for P⋆MDs:

Proposition 4.1. *Let $K \subseteq L$ be any field extension and let T be an integral domain with quotient field L. Assume that $D := T \cap K \neq K$, that T is integral over D and that $\star$ is semistar operation on T such that T is a P⋆MD. Define $\underline{\star}_D : \bar{\mathbf{F}}(D) \to \bar{\mathbf{F}}(D)$ in the following way*:

$$E^{\underline{\star}_D} := (ET)^\star \cap K \,.$$

Then:

(1) *the operation $\underline{\star}_D$ is a semistar operation on D*;
(2) *D is a $P\underline{\star}_D MD$.*

Proof. (1) It is obvious that, if $E, F \in \bar{\mathbf{F}}(D)$, then $E \subseteq F$ implies $E^{\underline{\star}_D} \subseteq F^{\underline{\star}_D}$. Moreover, if $E \in \bar{\mathbf{F}}(D)$ and $x \in K$, $x \neq 0$, then:

$$(E^{\underline{\star}_D})^{\underline{\star}_D} = (((ET)^\star \cap K)T)^\star \cap K \subseteq (((ET)^\star)T)^\star \cap K$$

$$= (ET)^{\star\star} \cap K = (ET)^\star \cap K = E^{\underline{\star}_D} \,;$$

$$(xE)^{\underline{\star}_D} = (xET)^\star \cap K = x(ET)^\star \cap K = x((ET)^\star \cap K)$$

$$= xE^{\underline{\star}_D} \,.$$

(2) Since T is a P⋆MD, then T_H is a valuation domain, for each $H \in \mathcal{M}(\star_f)$ (Theorem 3.1 ((i) $\Rightarrow$ (ii))). By the assumption that $D \subseteq T$ is an integral extension, we know that, if we denote by P the prime ideal $H \cap D$, then $D_P = T_H \cap K$ and so D_P is a valuation domain [27, Theorem 22.4, Proposition 12.7]. To conclude we need to show that, for all $Q \in \mathcal{M}((\underline{\star}_D)_f)$, there exists $H \in \mathcal{M}(\star_f)$ such that $H \cap D = Q$.

Claim 4.1. *For each $H \in \mathcal{M}(\star_f)$, the prime ideal $P := H \cap D$ has the following property*:

$$P^{(\underline{\star}_D)_f} \cap D = P \,.$$

As a matter of fact, since $D = T \cap K$, then $D^{\pm_D} = T^\star \cap D$ and so:

$$P^{(\pm_D)_f} = (PT)^{\star_f} \cap K = (((H \cap D)T)^{\star_f}) \cap K \subseteq H^{\star_f} \cap K .$$

Therefore:

$$P^{(\pm_D)_f} \cap D \subseteq H^{\star_f} \cap D = H^{\star_f} \cap T \cap D = H \cap D = P .$$

Claim 4.2. *If $Q \in \mathcal{M}((\pm_D)_f)$, then there exists $H \in \mathcal{M}(\star_f)$ such that $Q \subseteq H \cap D$.*

Since $(QT)^{\star_f} \cap K = Q$, then also $(QT)^{\star_f} \cap D = Q$. Take $L := (QT)^{\star_f} \cap T$, then it is easy to see that $L^{\star_f} \cap T = L$ and $L \cap D = Q$. Therefore, by Lemma 2.1(a), L is contained in some $H \in \mathcal{M}(\star_f)$ with $H \cap D \supseteq L \cap D = Q$.

From the previous claims, we deduce that $\mathcal{M}((\pm_D)_f)$ coincides with the contraction of $\mathcal{M}(\star_f)$ into D. This is enough to conclude. $\square$

From Krull's result concerning the v (semi)star operation cited before Proposition 4.1, we deduce immediately that:

Corollary 4.1. *Let $K \subseteq L$ be an algebraic field extension, let T be an integral domain with quotient field L, set $D := T \cap K$. Assume that T is the integral closure of D in L and that T is a PvMD. Then D is a PvMD.*

In [51, Section 11] H. Prüfer showed that the integral closure of a Prüfer domain [respectively, a PvMD] in an algebraic field extension is still a Prüfer domain [respectively, a PvMD]. An explicit proof of a stronger form of this result with different techniques was given recently by F. Lucius [43, Theorem 4.6 and Theorem 4.4] (cf. also [34, Theorem 3.6]). A generalization to the case of P$\star$MDs, when $\star$ is a semistar operation, is proven next.

Theorem 4.1. *Let $K \subseteq L$ be an algebraic field extension. Let D be an integral domain with quotient field K. Assume that $\star$ is a semistar operation on D such that D is a P$\star$MD. Let T be the integral closure of $D^\star$ into L, and let:*

$$\mathcal{W} := \{W \text{ is valuation domain of } L \mid W \cap K = D_Q, \text{ for some } Q \in \mathcal{M}(\star_f)\} .$$

For each $E \in \bar{\mathbf{F}}(T)$, set:

$$E^{\bar{\star}^T} := \bigcap \{EW \mid W \in \mathcal{W}\} .$$

(1) *The operation $\bar{\star}^T$ is an a.b. (semi)star operation on T.*
(2) *T is a $P\bar{\star}^T MD$.*

Proof. First at all, note that $\mathcal{W}$ is nonempty (Theorem 3.1 ((i) $\Rightarrow$ (ii)) and [27, Theorem 20.1]), $T = \bigcap\{W \mid W \in \mathcal{W}\}$ ([27, Theorem 19.6 and Theorem 19.8]) and $T^{\bar{\star}^T} = T$.

(1) Is a straightforward consequence of Example 2.1(a) and (g.3).

(2) It is sufficient to show that $\mathrm{Na}(T, \overline{\star}^T)$ is Prüfer domain (Theorem 3.1 ((iii) $\Rightarrow$ (i))). Since D is a P$\star$MD, then $\mathrm{Na}(D, \star)$ is a Prüfer domain (Theorem 3.1 ((i) $\Rightarrow$ (iii))), so it is the same for its integral closure $\overline{\mathrm{Na}(D, \star)}$ in the algebraic field extension $L(X)$ of $K(X)$, [27, Theorem 22.3]. If we show that $\overline{\mathrm{Na}(D, \star)} \subseteq \mathrm{Na}(T, \overline{\star}^T)$, then we conclude [27, Theorem 26.1(1)]. In order to prove this fact it is enough to note that:

(a) $\mathrm{Na}(T, \overline{\star}^T)$ is integrally closed in $L(X)$;
(b) $\mathrm{Na}(D, \star) \subseteq \mathrm{Na}(T, \overline{\star}^T)$.

For (a), we have that

$$\mathrm{Na}(T, \overline{\star}^T) = \bigcap \{T_H(X) \mid H \in \mathcal{M}((\overline{\star}^T)_f)\}, \quad \text{(Lemma 2.3(c))},$$

and T_H is integrally closed (and, thus, $T_H(X)$ is integrally closed), for each H, since T is integrally closed.

For (b), let $z = f/g \in \mathrm{Na}(D, \star) = \mathrm{Na}(D, \tilde{\star})$, with f, $g \in D[X]$ and $D^{\tilde{\star}} = \mathbf{c}(g)^{\tilde{\star}} = \bigcap\{\mathbf{c}(g)D_Q \mid Q \in \mathcal{M}(\star_f)\}$, (Lemma 2.3 ((b) and (i))). Then:

$$T = D^{\overline{\star}^T} \subseteq (D^{\tilde{\star}})^{\overline{\star}^T} = (\mathbf{c}(g)^{\tilde{\star}})^{\overline{\star}^T} = \bigcap \{\mathbf{c}(g)W \mid W \in \mathcal{W}\} = \mathbf{c}(g)^{\overline{\star}^T},$$

and so $1 \in \mathbf{c}(g)^{\overline{\star}^T}$, i.e. $\mathbf{c}(g)^{\overline{\star}^T} = T^{\overline{\star}^T} = T$. Therefore $f/g \in \mathrm{Na}(T, \overline{\star}^T)$. $\square$

From the previous result and Corollary 4.1 we reobtain the following result (cf. for instance [43, Theorem 4.6]):

Corollary 4.2. *Let $K \subseteq L$ be an algebraic field extension, let T be an integral domain with quotient field L, set $D := T \cap K$. Assume that D is integrally closed and that T is the integral closure of D in L. Then D is a PvMD if and only if T is a PvMD.*

Proof. With the notation of Theorem 4.1, it is sufficient to remark that if v_D [respectively, v_T] is the v-operation on D [respectively, on T] and if D is a Pv_DMD, then the a.b. semistar operation $\overline{v_D}^T$ on the integrally closed domain T is equivalent to v_T (Remark 3.2(2)). $\square$

Acknowledgment

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