LOCAL–GLOBAL PROPERTIES FOR SEMISTAR OPERATIONS

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ABSTRACT. We study the "local" behavior of several relevant properties concerning semistar operations, like finite type, stable, spectral, e.a.b. and a.b. We deal with the "global" problem of building a new semistar operation on a given integral domain, by "gluing" a given homogeneous family of semistar operations defined on a set of localizations. We apply these results for studying the local–global behavior of the semistar Nagata ring and the semistar Kronecker function ring. We prove that an integral domain D is a Prüfer $\star$–multiplication domain if and only if all its localizations D_P are Prüfer $\star_P$–multiplication domains.

INTRODUCTION

Krull's theory on ideal systems and star operations was motivated by the construction of Kronecker function rings in a more general context than that of algebraic integers, originally considered by L. Kronecker. The theory developed by Krull requires some restrictions on the integral domain, which has to be integrally closed, and on the star operation, which has to be e.a.b. [9, Section 32]. Semistar operations, introduced by Okabe and Matsuda [17], lead to very general theory of Kronecker function rings, also in case of non necessarily integrally closed domains (cf. [18], [15], [11], [6], [7] and [8]).

Semistar operations are an appropriate tool for extending the theory of Prüfer domains and, more generally, of Prüfer v–multiplication domains (cf. [10], [16] and [14]) to the non necessarily integrally closed domains case. Let $\star$ be a semistar operation of finite type on an integral domain D (the formal definition is recalled in Section 1), then D is called a Prüfer $\star$–multiplication domain (for short, P$\star$MD) if each nonzero finitely generated fractional ideal I of D is $\star$–invertible (i.e. $(II^{-1})^\star = D^\star$). In the semistar case, if D is a P$\star$MD, then the semistar integral closure of D is integrally closed, thus, in the Krull's setting of e.a.b. star operations, we recover the classical situation that D has to be integrally closed [13]. Several characterizations of Prüfer $\star$–multiplication domains were obtained recently, making also use of the semistar Nagata ring $\mathrm{Na}(D, \star)$ and the semistar Kronecker functions ring $\mathrm{Kr}(D, \star)$ (cf. [5], [8], [1] and [12]).

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The starting point of this paper is the study of the localization of a P$\star$MD, D, at any prime ideal (possibly, not quasi–$\star$–prime, i.e. not a prime ideal P such that $P = P^\star \cap D$, since the localization at any prime ideal of this type is known to be a valuation domain). As a consequence of this local study, we obtain new examples of local P$\star$MDs.

One of the first results proved here is a characterization of a P$\star$MD, D, through a local property, concerning the localizations of D at a family of prime ideals P of D, and a global "arithmetical" condition, concerning a finiteness property of the ideals of the type $(aD \cap bD)^\star$, see Theorem 2.9. We apply this result to characterize P$\star$MDs as those domains such that the localization at any prime ideal P is a P$\star_P$MD (where $\star_P$ is a semistar operation canonically associated to $\star$ by "ascent" to D_P), see Theorem 3.13. This result points out the important fact that Prüfer multiplication-like properties are really local properties and it opens the way for a local–global study of Prüfer $\star$–multiplication domains.

In order to realize these results we develop, preliminarily, a study on the behavior of semistar operations properties under localizations. In particular, we show that finite type, spectral, stable, a.b. or e.a.b. properties on a semistar operation $\star$ transfer to the induced semistar operation $\star_P$, defined on D_P, for any any prime ideal $P \in \mathrm{Spec}(D)$.

At this stage, it is also natural to investigate on the relationship between the semistar Nagata ring $\mathrm{Na}(D, \star)$ [respectively, semistar Kronecker function ring $\mathrm{Kr}(D, \star)$] and the Nagata rings $\mathrm{Na}(D_P, \star_P)$ [respectively, the Kronecker function rings $\mathrm{Kr}(D_P, \star_P)$] (i.e., on the local behavior of the general Nagata ring and Kronecker function rings). In this context, we show that $\mathrm{Na}(D, \star) = \cap\{\mathrm{Na}(D_P, \star_P) \mid P \in \mathrm{Spec}(D)\}$ [respectively, $\mathrm{Kr}(D, \star) = \cap\{\mathrm{Kr}(D_P, \star_P) \mid P \in \mathrm{Spec}(D)\}$]; we observe also that the canonical inclusions $\mathrm{Na}(D, \star)_{D\setminus P} \subseteq \mathrm{Na}(D_P, \star_P)$ [respectively, $\mathrm{Kr}(D, \star)_{D\setminus P} \subseteq \mathrm{Kr}(D_P, \star_P)$] are not equalities, in general.

In the last section we deal with the "global" problem of building a semistar operation in an integral domain D by "gluing" a given family of semistar operations defined on D_P, for P varying in a subset Θ of $\mathrm{Spec}(D)$. Since the description of this semistar operation is in part folklore (at least in the star setting), we deal specially with the problem of which properties, verified by all the semistar operations defined on the localizations D_P, transfer to the "glued" semistar operation defined on D. Among the other results, we prove that the finite type and stable properties pass on, in the case the representation $D = \cap\{D_P \mid P \in \Theta\}$ has finite character. In order to glue semistar operations verifying other relevant properties, like e.a.b. and a.b., we evidentiate some obstructions; in fact, this type of semistar operations we prove giving rise to a semistar operation of the same type under an extra condition, a sort of "stability under generalizations", denoted here by ($\downarrow$), see Theorem 4.6. Finally, we have included several examples in order to better illustrate the different constructions considered here and to show the essentiality of the assumptions in the main results.

1. BACKGROUND

Let us consider a commutative integral domain D with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ [respectively, $\boldsymbol{F}(D)$ and $\boldsymbol{f}(D)$] denote the set of nonzero D–submodules of K [respectively, fractional ideals and nonzero finitely generated D–submodules of K]. Note that $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

A semistar operation on D is a map $\star\colon \overline{\boldsymbol{F}}(D) \longrightarrow \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$, such that

- **(1)** $(xE)^\star = xE^\star$;
- **(2)** $E \subseteq F$ implies $E^\star \subseteq F^\star$;
- **(3)** $E \subseteq E^\star$ and $E^\star = (E^\star)^\star =: E^{\star\star}$,

for each $0 \neq x \in K$ and for all $E, F \in \overline{\boldsymbol{F}}(D)$.
When $D^\star = D$, we say that $\star$ is *a (semi)star operation on* D. The *identical operation* d_D *on* D (simply denoted by d), defined by $E^{d_D} := E$, for each $E \in \overline{\boldsymbol{F}}(D)$, is a (semi)star operation on D. If not stated explicitly, we generally assume that $\star$ is not the *trivial semistar operation* e_D *on* D (simply denoted by e), where $E^{e_D} := K$, for each $E \in \overline{\boldsymbol{F}}(D)$. It is easy to see that $\star \neq e$ if and only if $D \neq K$ implies that $D^\star \neq K$.

A semistar operation $\star$ is of *finite type*, if for any $E \in \overline{\boldsymbol{F}}(D)$, we have:

$$E^{\star_f} := \cup\{F^\star \mid F \subseteq E \text{ and } F \in \boldsymbol{f}(D)\} = E^\star.$$

In general, for each semistar operation defined on D, $\star_f$, as defined before, is also a semistar operation on D and $\star_f \leq \star$, i.e., for any $E \in \overline{\boldsymbol{F}}(D)$ we have $E^{\star_f} \subseteq E^\star$.

There are several examples of finite type semistar operations; the most known is probably the t–operation. Indeed, we start from the v_D*–operation* on an integral domain D (simply denoted by v), defined as follows:

$$E^v := (E^{-1})^{-1} = (D\colon (D\colon E)),$$

for any $E \in \overline{\boldsymbol{F}}(D)$, and we set $t_D := (v_D)_f$ (or, simply, $t = v_f$).
Other examples can be constructed as follows: let T be an overring of an integral domain D and let $\star_{\{T\}}$ be the semistar operation on D, defined by $E^{\star_{\{T\}}} := ET$, for each $E \in \overline{\boldsymbol{F}}(D)$, then $\star_{\{T\}}$ is a finite type semistar operation on D. In particular, $\star_{\{D\}} = d$ and $\star_{\{K\}} = e$.

In the following, we will construct several new semistar operations from a given one and we will show that most of them are of finite type.

Let $\star$ be a semistar operation on D, we define the following set of prime ideals of D:

$$\Pi^\star := \{P \in \operatorname{Spec}(D) \mid P \neq (0) \text{ and } P^\star \cap D \neq D\}.$$

It may be that $\Pi^\star$ is an empty set but, in the particular case in which $\star$ is nontrivial of finite type, then $\Pi^\star \neq \emptyset$. In fact, we have that each proper ideal I is always contained in a proper *quasi–$\star$–ideal* of D, i.e. a proper ideal J of D such that such that $J^\star \cap D = J$, and moreover (in the finite type case) each proper quasi–$\star$–ideal of D is contained in a maximal element in *the set* Γ *of all proper quasi–$\star$–ideals;* finally, maximal elements in Γ are prime ideals. We will denote by

$$\mathcal{Q}(\star) := \operatorname{Spec}^\star(D) := \{Q \in \operatorname{Spec}(D) \mid Q = Q^\star \cap D\}$$

the set of all the *quasi–$\star$–prime ideals* of D.

If $\star$ is possibly not of finite type, we say that $\star$ *possesses enough primes*, if each proper quasi–$\star$–ideal is contained in a quasi–$\star$–prime ideal. As a consequence, for each semistar operation that possesses enough primes (e.g. a nontrivial finite type semistar operation), we have $\mathrm{Max}(\Pi^\star) = \mathrm{Max}(\Gamma)$ (and it is a nonempty set). We will denote simply by $\mathcal{M}(\star)$ the set of all the maximal elements in Γ.

After analyzing this situation, we ask about how to relate semistar operations and prime ideals. For any nonempty set Π of prime ideals on D, we define a semistar operation $\star_\Pi$ on D as follows:

$$E^{\star_\Pi} := \cap\{ED_P \mid P \in \Pi\},$$

for any $E \in \overline{\boldsymbol{F}}(D)$. If $\Pi = \emptyset$, we set $\star_\emptyset := e$; obviously, if $\Pi = \{(0)\}$, then $\star_\Pi := e$. Semistar operations defined by sets of prime ideals are called *spectral semistar operations*.

If $\star$ is a finite type semistar operation, we define $\star_{sp} := \star_{\mathcal{M}(\star)}$; we have always that $\star_{sp} \leq \star$. Of course, we can start from a general semistar operation $\star$, in that case we have a new spectral semistar operation $\widetilde{\star} := (\star_f)_{sp}$; this is the biggest finite type spectral semistar operation in the set of all the finite type spectral semistar operations on D, smaller or equal to $\star$.

A semistar operation $\star$ is *stable* if $(E \cap F)^\star = E^\star \cap F^\star$, for each pair E, $F \in \overline{\boldsymbol{F}}(D)$; any spectral semistar operation is stable, and stable finite type semistar operations coincide with spectral finite type semistar operations.

The semistar operation $\widetilde{\star}$ can be also defined using the general *Nagata ring* $\mathrm{Na}(D,\star)$ *associated to* $\star$. Indeed, if we define:

$$\mathrm{Na}(D,\star) := D[X]_{N(\star)},$$

where $N(\star) := \{f \in D[X] \mid \mathbf{c}(f)^\star = D^\star\}$ is a saturated multiplicative subset of $D[X]$, then, for any $E \in \overline{\boldsymbol{F}}(D)$, we have $E^{\widetilde{\star}} = E\,\mathrm{Na}(D,\star) \cap K$.

Nagata ring has a parallel behavior to the general *Kronecker function ring* associated to a semistar operation $\star$, defined as follows:

$$\mathrm{Kr}(D,\star) = \Big\{ \tfrac{f}{g} \in K(X) \mid f,\ g \in D[X] \setminus \{0\} \text{ and there exists } 0 \neq h \in D[X] \text{ such that } (\mathbf{c}(f)\,\mathbf{c}(h))^\star \subseteq (\mathbf{c}(g)\,\mathbf{c}(h))^\star \Big\} \cup \{0\}.$$

Note that $\mathrm{Na}(D,\star) \subseteq \mathrm{Kr}(D,\star)$ and a "new" finite type semistar operation can be defined on D by the Kronecker function ring by setting:

$$F^{\star_a} := F\,\mathrm{Kr}(D,\star) \cap K,$$

for any $F \in \boldsymbol{f}(D)$. This finite type semistar operation $\star_a$ on D has another more arithmetic description, as follows:

$$F^{\star_a} = \cup\{((FH)^\star \colon H^\star) \mid H \in \boldsymbol{f}(D)\},$$

for any $F \in \boldsymbol{f}(D)$. Moreover, $\star_a$ has a useful "cancellation" property: if E, F, $G \in \boldsymbol{f}(D)$ and $(EF)^\star \subseteq (EG)^\star$, then $F^\star \subseteq G^\star$. A semistar operation satisfying this property is called an *e.a.b. (= endlich arithmetisch brauchbar) semistar operation*. In the previous "cancellation" property, if we take $E \in \boldsymbol{f}(D)$ and F, $G \in \overline{\boldsymbol{F}}(D)$, a semistar operation having this modified cancellation property is called an *a.b. semistar operation*. In general, we have the following characterizations: a finite type semistar operation $\star$ is e.a.b. if and only if it is a.b. if and only if $\star = \star_a$.

A Kronecker function ring, and its "counterpart" the associated finite type a.b. semistar operation, parameterizes certain valuation overrings of D. A valuation overring $V \supseteq D$ is called a *$\star$–valuation overring* of D if $F^\star \subseteq FV$ for any $F \in \boldsymbol{f}(D)$ (or, equivalently, if $\star_f \leq \star_{\{V\}}$), then a finite type a.b. semistar operation $\star$ is characterized by the following property:

$$F^\star = \cap\{FV \mid V \text{ is a } \star\text{–valuation overring of } D\},$$

for each $F \in \boldsymbol{f}(D)$.

We refer to [5], and to the references contained in that paper, as a documented source on semistar operations and on some of their properties briefly recalled above.

The following notation shall be use throughout the text. For a nonempty subset $\Delta \subseteq \operatorname{Spec}(D)$ of prime ideals of an integral domain D we define

$$\Delta^{\downarrow} := \{Q \in \operatorname{Spec}(D) \mid Q \subseteq P\,,\ \text{for some } P \in \Delta\},$$

and we say that Δ is *closed under generizations* if $\Delta = \Delta^{\downarrow}$. In the same way, for any prime ideal $P \in \operatorname{Spec}(D)$, we define $P^{\downarrow} := \{P\}^{\downarrow}$.

2. Localizing semistar operations

Let $\star$ be a semistar operation on an integral domain D and let K be the quotient field of D. For each $P \in \operatorname{Spec}(D)$, we consider the inclusion $D \subseteq D_P$ of D into its localization D_P and the semistar operation $\dot{\star}^{D_P}$, denoted simply $\star_P$, on D_P, obtained from $\star$ by "ascent to" D_P, i.e. $E^{\dot{\star}^{D_P}} := E^{\star_P} := E^\star$, for each $E \in \overline{\boldsymbol{F}}(D_P)\,(\subseteq \overline{\boldsymbol{F}}(D)\,)$. Note that if $P = (0)$ then $D_P = K$ and so $\star_P$ coincides with d_K $(= e_K)$ on K.

Our first goal is to study the transfer of some relevant properties from $\star$ to $\star_P$.

Lemma 2.1. *Let $\star$ be a semistar operation on an integral domain D and let $P \in \operatorname{Spec}(D)$.*

- **(a)** *If $\star$ is a finite type semistar operation on D, then $\star_P$ is a finite type semistar operation on D_P.*
- **(b)** *If $\star$ is an e.a.b. [respectively, a.b.] semistar operation on D, then $\star_P$ is an e.a.b. [respectively, a.b.] semistar operation on D_P.*

Proof. (a) is a consequence of [5, Example 1 (e.1)].

(b) We give the proof in the a.b. case; a similar argument shows the e.a.b. case. Let $G, H \in \overline{\boldsymbol{F}}(D_P) \subseteq \overline{\boldsymbol{F}}(D)$ and $F \in \boldsymbol{f}(D_P)$ such that $(FG)^{\star_P} \subseteq (FH)^{\star_P}$. Since we can find $F_0 \in \boldsymbol{f}(D)$ such that $F = F_0D_P$, then we obtain $(F_0G)^\star = (F_0D_PG)^{\star_P} \subseteq (F_0D_PH)^{\star_P} = (F_0H)^\star$. Therefore, $G^{\star_P} = G^\star \subseteq H^\star = H^{\star_P}$, because $\star$ is a.b. □

If $\star$ is a finite type a.b. semistar operation on an integral domain D, then it is wellknown that $\star$ coincides with the semistar operation $\star_{\mathcal{V}}$, where $\mathcal{V} := \mathcal{V}(\star) := \{V \supseteq D \mid V \text{ is a } \star\text{–valuation overring}\}$ and $E^{\star_{\mathcal{V}}} := \cap\{EV \mid V \in \mathcal{V}\}$, for each $E \in \overline{\boldsymbol{F}}(D)$ [5, Lemma 2.8 (d)].

Corollary 2.2. *Let $\star$ be a finite type a.b. semistar operation on an integral domain D. For each prime ideal $P \in \operatorname{Spec}(D)$, we consider the (finite type a.b.) semistar operation $\star_P$ on D_P and the subset of overrings $\mathcal{V}_P := \{V \supseteq D \mid V$ is a $\star$–valuation overring and $D_P \subseteq V\}$. Then $\mathcal{V}_P$ is exactly the set $\mathcal{V}(\star_P)$ of the $\star_P$–valuation overrings of D_P, i.e. $E^{\star_P} := \cap\{EV \mid V \in \mathcal{V}(\star_P)\} = \cap\{EV \mid V \in \mathcal{V}_P\}$, for each $E \in \overline{\boldsymbol{F}}(D_P)$.*

Proof. We know already, from the previous lemma, that $\star_P$ is a finite type a.b. semistar operation on D_P. If $V \in \mathcal{V}_P$ and $F \in \boldsymbol{f}(D)$, then $(FD_P)^{\star_P} = (FD_P)^\star \subseteq (FD_P)V$, thus V is a $\star_P$–valuation overring of D_P. Conversely, if W is a $\star_P$–valuation overring of D_P, then we have $F^\star \subseteq (FD_P)^\star = (FD_P)^{\star_P} \subseteq D_PW = FW$, for each $F \in \boldsymbol{f}(D)$, thus W ($\supseteq D_P$) is a $\star$–valuation overring of D. As a consequence, for each $F \in \boldsymbol{f}(D)$, we obtain that $(FD_P)^\star = (FD_P)^{\star_P} = \cap\{FD_PW \mid W$ is a $\star_P$–valuation overring$\} = \cap\{FV \mid V \in \mathcal{V}_P\} = F^{\star_{\mathcal{V}_P}}$. The conclusion follows since $\star_P$ is a finite type semistar operation. □

Proposition 2.3. *Let $\star$ be a semistar operation on an integral domain D and let $P \in \operatorname{Spec}(D)$. If $\star = \tilde{\star}$ (is a finite type stable semistar operation on D), then $\star_P = \widetilde{\star_P}$ (is a finite type stable semistar operation on D_P).*

Proof. Note that if $\star$ is a stable semistar operation then, from the definitions of stability and of the semistar operation $\star_P$, it follows that $\star_P$ is a stable semistar operation. The conclusion follows from Lemma 2.1 (a).

We give another proof that describes explicitly the set $\mathcal{Q}(\star_P)$ of all the quasi–$\star_P$–prime ideals of D_P in relation with the set $\mathcal{Q}(\star)$ of all the quasi–$\star$–prime ideals of D.

To avoid the trivial case, we can assume that $\star \neq e_D$ and that P is a nonzero prime ideal of D. Note that $\star = \star_f$, because $\star$ is a finite type semistar operation, and so $\star_P = (\star_P)_f$ (Lemma 2.1 (a)). Let $\mathcal{Q} := \mathcal{Q}(\star)$ be the set of all the quasi–$\star$–prime ideals of D then, for each $F \in \overline{\boldsymbol{F}}(D)$, we have $F^\star = \cap\{FD_Q \mid Q \in \mathcal{Q}\}$, because $\star = \tilde{\star}$ [8, Corollary 2.11 (2)].

Assume that $P \in \mathcal{Q}$. Let $E \in \overline{\boldsymbol{F}}(D_P) \subseteq \overline{\boldsymbol{F}}(D)$, then

$$E^{\star_P} = E^\star = \cap\{ED_Q \mid Q \in \mathcal{Q}\} = \cap\{ED_Q \mid Q \in \mathcal{Q}\setminus\{P\}\} \cap E \subseteq E,$$

hence $\star_P = d_{D_P}$ is the identical (semi)star operation on D_P and so, obviously, $\star_P = d_{D_P} = \widetilde{d_{D_P}} = \widetilde{\star_P}$.

Assume that $P \notin \mathcal{Q}$. Let $E \in \overline{\boldsymbol{F}}(D_P) \subseteq \overline{\boldsymbol{F}}(D)$, then

$$\begin{aligned} E^{\star_P} &= (ED_P)^{\star_P} = (ED_P)^\star = \cap\{ED_PD_Q \mid Q \in \mathcal{Q}\} = \\ &= (\cap\{ED_PD_Q \mid Q \in \mathcal{Q}_0\}) \cap (\cap\{ED_PD_Q \mid Q \in \mathcal{Q}_1\}) \end{aligned}$$

where $\mathcal{Q}_1 := \{Q \in \mathcal{Q} \mid P \cap Q$ contains a nonzero prime ideal of $D\}$ and $\mathcal{Q}_0 := \{Q \in \mathcal{Q} \mid P \cap Q$ does not contain a nonzero prime ideal of $D\}$.

Note that if $Q \in \mathcal{Q}_0$, i.e. if $P \cap Q$ does not contain a nonzero prime ideal, then D_PD_Q coincides necessarily with K, the quotient field of D.

Assume that $Q \in \mathcal{Q}_1$. It is wellknown that there exists a natural bijective correspondence between the set of prime ideals of D_PD_Q and the set $\{H \in \operatorname{Spec}(D) \mid H \subseteq P \cap Q\}$, hence $D_PD_Q = \cap\{D_H \mid H \subseteq P \cap Q$ and $H \in \operatorname{Spec}(D)\}$. Moreover, note that the set $\mathcal{S}(P;Q)$ of all nonzero quasi–$\star$–ideals I of D contained in $P \cap Q$ is not empty (since at least $H^\star \cap D$ is in $\mathcal{S}(P;Q)$, if $H \subseteq$

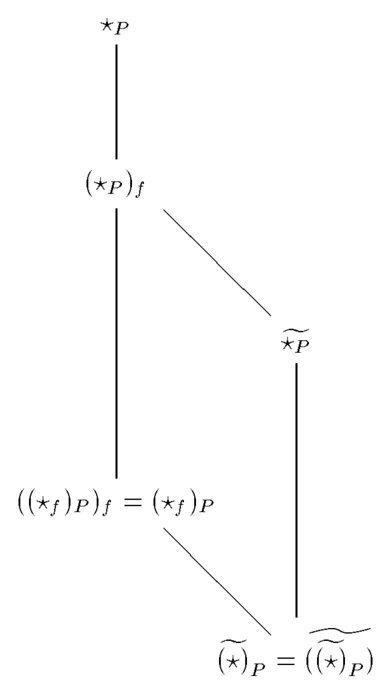
$P \cap Q$ and $H \in \mathrm{Spec}(D)$). It is easy to see that the set of maximal elements $\mathcal{S}(P;Q)_{\max}$ of $\mathcal{S}(P;Q)$ is a nonempty set of prime ideals, with $\mathcal{S}(P;Q)_{\max} \subseteq \mathcal{Q}_1 \subseteq \mathcal{Q}$ and, furthermore, each prime ideal H, with $H \subseteq P \cap Q$, is contained in some element of $\mathcal{S}(P;Q)_{\max}$. Thus, we can rewrite:

$$\begin{aligned} E^{\star_P} &= EK \cap (\cap\{ED_H \mid H \subseteq P \cap Q \text{ and } H \in \mathrm{Spec}(D), \text{ for } Q \text{ varying in } \mathcal{Q}_1\}) = \\ &= \cap\{ED_H \mid H \in \mathcal{S}(P;Q)_{\max}, \text{ for } Q \text{ varying in } \mathcal{Q}_1\} = \\ &= \cap\{ED_H \mid H \subseteq P \text{ and } H \in \mathcal{Q}\}. \end{aligned}$$

Therefore it is easy to see that the set $\{HD_P \mid H \subseteq P \text{ and } H \in \mathcal{Q}\}$ coincides with the set $\mathcal{Q}(\star_P)$ of all the quasi–$\star_P$–prime ideals of D_P, which "defines" $\widetilde{\star_P}$, i.e. $\star_P = \widetilde{(\star_P)}$. □

Remark 2.4. Note that the proof of the previous proposition shows the following statement: *If $\star$ is a finite type spectral semistar operation on D, defined by a subset $\Delta \subseteq \mathrm{Spec}(D)$ (i.e. $\star := \star_\Delta$), then $\star_P$ is also a finite type spectral semistar operation on D_P and it is defined by the set $\Delta_P := \{HD_P \mid H \subseteq P,\ H \in \Delta\}$ (i.e. $\star_P = \star_{\Delta_P}$).*

Remark 2.5. Let $\star$ be a semistar operation on D and $P \in \mathrm{Spec}(D)$, then we have the following diagram of semistar operations on D_P.



where the equalities are direct consequences of Lemma 2.1 and Proposition 2.3. As a consequence of the proof of Proposition 2.3, we obtain that:

$$\begin{aligned} P \in \mathrm{Spec}^{\star_f}(D) \quad \Rightarrow \quad & \mathrm{Spec}^{\widetilde{\star_P}}(D_P) = \mathrm{Spec}^{(\star_P)_f}(D_P) = \\ & = \mathrm{Spec}^{(\star_f)_P}(D_P) = \mathrm{Spec}^{\widetilde{(\star)}_P}(D_P)\,. \end{aligned}$$

Indeed, if $QD_P \in \mathrm{Spec}^{\widetilde{(\star)}_P}(D_P)$, then

$$\begin{aligned} QD_P &= (QD_P)^{\widetilde{(\star)}_P} \cap D_P = (QD_P)^{\tilde{\star}} \cap D_P = \\ &= (\cap\{QD_P D_H \mid H \in \mathrm{Spec}^{\star_f}(D)\}) \cap D_P = (QD_P)^{\widetilde{\star_P}} \cap D_P \subseteq \\ &\subseteq QD_P \cap D_P = QD_P\,. \end{aligned}$$

Let $\star$ be a semistar operation on an integral domain D. Assume that D is a P$\star$MD, i.e. an integral domain such that each $F \in \boldsymbol{f}(D)$ is a $\star_f$–invertible ideal, that is $(FF^{-1})^{\star_f} = D^\star$. P$\star$MDs are characterized in several different ways in [5, Theorem 1]: for instance, D_Q is a valuation domain for each $Q \in \mathcal{M}(\star_f)$, where $\mathcal{M}(\star_f)$ is the (nonempty) set of all the maximal elements of $\mathrm{Spec}^{\star_f}(D)$. A consequence of this fact is that D is a P$\star$MD if and only if it is a P$\tilde{\star}$MD, since $\mathcal{M}(\tilde{\star}) = \mathcal{M}(\star_f)$ [5, Lemma 3 (g)].

Remark 2.6. Note that, *if* $0 \neq a$, b *belong to an integral domain* D*, then the following equality holds:*

$$\frac{1}{ab}(aD \cap bD) = ((a,b)D)^{-1}.$$

Proof. Let $x \in aD \cap bD$, then $x = ax' = bx''$, for some x', $x'' \in D$, thus we obtain that $\frac{x}{ab} = \frac{1}{b}x' = \frac{1}{a}x''$. Henceforth, for each $az_1 + bz_2 \in (a,b)D$, we have $\frac{x}{ab}(az_1 + bz_2) = x''z_1 + x'z_2 \in D$. Therefore $\frac{x}{ab} \in ((a,b)D)^{-1}$. Conversely, let $y \in ((a,b)D)^{-1}$, then $ya = x'$ and $yb = x''$, for some x', $x'' \in D$. Henceforth $aby = bx' = ax''$, and so $aby \in (aD \cap bD)$. □

Lemma 2.7. *Let* $\star$ *be a semistar operation on an integral domain* D*. Assume that* D *is a P*$\star$*MD. For each pair of nonzero elements* a, $b \in D$*, there exists* $F \in \boldsymbol{f}(D)$ *such that:*

$$(aD \cap bD)^\star = F^\star$$

(in this situation, we say briefly that $(aD \cap bD)^\star$ *is an* ideal of $\star$–finite type*).*

Proof. Recall that $\frac{1}{ab}(aD \cap bD) = ((a,b)D)^{-1}$ and thus note that $(aD \cap bD)^\star$ is an ideal of $\star$–finite type if and only if $(((a,b)D)^{-1})^\star$ is *a (fractional) ideal of* $\star$*–finite type,* i.e. there exists $G \in \boldsymbol{f}(D)$ such that $G^\star = (((a,b)D)^{-1})^\star$, (Remark 2.6). Since D is a P$\star$MD, each nonzero finitely generated (fractional) ideal F of D is $\star_f$–invertible and so $(F^{-1})^\star$ is also a (fractional) ideal of $\star$–finite type. Therefore, we conclude that $(((a,b)D)^{-1})^\star$ is a (fractional) ideal of $\star$–finite type. □

The converse of this result holds if we add some extra conditions.

Theorem 2.8. *Let* $\star$ *be a spectral semistar operation on* D *defined by a set* Δ *of valuation prime ideals of* D *(or, equivalently, by a family of essential valuation overrings of* D*), i.e.* $E^\star := \cap\{ED_P \mid P \in \Delta\}$*, for each* $E \in \overline{\boldsymbol{F}}(D)$*, where* D_P *is a valuation domain for each* $P \in \Delta$*. If, for each pair of nonzero elements* a, $b \in D$*, we have that* $(aD \cap bD)^\star$ *is an ideal of* $\star$*–finite type, then* D *is a P*$\star$*MD.*

Proof. Step 1. *If* I *is a finitely generated ideal of* D *and* $a \in D$*, with* $a \neq 0$*, then* $(I \cap aD)^\star$ *is an ideal of* $\star$*–finite type.*

The proof is based on an argument from [19]. Set $I := \sum_{i=1}^n a_iD$ and $J_i := a_iD \cap aD$, for $1 \leq i \leq n$. By the hypothesis $(J_i)^\star$ is an ideal of $\star$–finite type. For each index i, $1 \leq i \leq n$, let F_i be a finitely generated ideal of D such that $(J_i)^\star = (F_i)^\star$. Since D_P is a valuation domain and a D–flat overring of D, we have:

$$\begin{aligned}(\sum_i F_i)^\star &= (\sum_i (F_i)^\star)^\star = (\sum_i (J_i)^\star)^\star = (\sum_i J_i)^\star = \\ &= \cap_{P\in\Delta} \left(\sum_i (a_i D \cap aD)\right) D_P = \cap_{P\in\Delta} \left(\sum_i (a_i D \cap aD) D_P\right) = \\ &= \cap_{P\in\Delta} \left(\sum_i (a_i D_P \cap aD_P)\right) = \cap_{P\in\Delta} \left((\sum_i a_i) D_P \cap aD_P\right) = \\ &= (\cap_{P\in\Delta} (\sum_i a_i) D_P) \cap (\cap_{P\in\Delta} aD_P) = (\sum_i a_i D)^\star \cap (aD)^\star = \\ &= I^\star \cap (aD)^\star = (I \cap aD)^\star \qquad \text{(as } \star \text{ is stable).}\end{aligned}$$

Step 2. *Any finite intersection of nonzero principal ideals of D is an ideal of $\star$–finite type.*

Let $a_1, a_2, \ldots, a_t \in D$ be a family of nonzero elements. We may assume that $t \geq 2$ and, by induction on t, we may assume that $a_1 D \cap a_2 D \cap \ldots \cap a_{t-1} D$ is an ideal of $\star$–finite type, i.e. there is a finitely generated ideal F of D such that $(a_1 D \cap a_2 D \cap \ldots \cap a_{t-1} D)^\star = F^\star$. Then, we have:

$$\begin{aligned}(a_1 D \cap a_2 D \cap \ \ldots \ \cap a_t D)^\star = &\ \ ((a_1 D \cap a_2 D \cap \ \ldots \ \cap a_{t-1} D)^\star \cap (a_t D)^\star)^\star = \\ = &\ \ (F^\star \cap (a_t D)^\star)^\star = (F \cap a_t D)^\star\end{aligned}$$

and, by Step 1, this is an ideal of $\star$–finite type.

Step 3. *If I is a nonzero finitely generated ideal of D, then I^{-1} is a (fractional) ideal of $\star$–finite type.*

The case of an ideal I generated by two elements $a, b \in D$ follows immediately from Remark 2.6, since we know already that $(aD \cap bD)^\star$ is an ideal of $\star$–finite type if and only if $(((a,b)D)^{-1})^\star$ is a (fractional) ideal of $\star$–finite type. The conclusion follows from the assumption that $(aD \cap bD)^\star$ is an ideal of $\star$–finite type. The general case of a finitely generated ideal $I := (x_1, x_2, \ldots, x_t)D$ follows from Step 2. In fact, without loss of generality, we can assume that $x_i \neq 0$, for each $1 \leq i \leq t$, thus:

$$I^{-1} = (D : I) = (D : \sum_{1 \leq i \leq t} x_i D) = \cap_{1 \leq i \leq t} (D : x_i D) = \cap_{1 \leq i \leq t} x_i^{-1} D,$$

and, if we write $x_i^{-1} := a_i/d$, with a_i and d nonzero elements in D, for $1 \leq i \leq t$, then:

$$(\cap_{1 \leq i \leq t} x_i^{-1} D)^\star = (d^{-1}(\cap_{1 \leq i \leq t} a_i D))^\star = d^{-1}(\cap_{1 \leq i \leq t} a_i D)^\star = d^{-1} F^\star = (d^{-1} F)^\star,$$

for some $F \in \boldsymbol{f}(D)$.

Let I be any nonzero finitely generated ideal of D. By Step 3, we know that I^{-1} is a (fractional) ideal of $\star$–finite type. Since D_P is a valuation domain and a D–flat overring of D, we have:

$$\begin{aligned}(I\ I^{-1})^\star &= \cap_{P\in\Delta}(I\ I^{-1}) D_P = \cap_{P\in\Delta}(ID_P\ I^{-1} D_P) = \\ &= \cap_{P\in\Delta}(ID_P\ (ID_P)^{-1}) = \cap_{P\in\Delta} D_P = D^\star.\end{aligned}$$

We conclude that D is a P$\star$MD. □

Theorem 2.9. *Let $\star$ be a semistar operation on an integral domain D. Then the following statements are equivalent:*

- **(i)** *D is a P$\star$MD;*
- **(ii)** *the following two conditions hold:*

(a) *for each subset* $\Theta \subseteq \mathrm{Spec}(D)$*, such that* $\tilde{\star} = \wedge_{\tilde{\Theta}} := \wedge\{\widetilde{\star_P} \mid P \in \Theta\}$*, where* $E^{\wedge_{\tilde{\Theta}}} := \cap\{(ED_P)^{\widetilde{\star_P}} \mid P \in \Theta\}$*, for each* $E \in \overline{\boldsymbol{F}}(D)$*, we have that* D_P *is a P*$\star_P$*MD, for each* $P \in \Theta$*;*

(b) *for any pair of nonzero elements* $a, b \in D$*, we have that* $(aD \cap bD)^\star$ *is an ideal of* $\star$*–finite type.*

Proof. (i) $\Rightarrow$ (ii). We can assume that $\star = \tilde{\star}$, since the notions of P$\star$MD and P$\tilde{\star}$MD coincide [5, Section 3, Theorem 1] and $\star_P = (\tilde{\star})_P = \widetilde{\star_P}$ (Proposition 2.3). By Lemma 2.7, we only need to show that condition (a) holds. More generally, we show that, under the assumption (i), D_P is a $P\star_P$MD, for each $P \in \mathrm{Spec}(D)$. Let ID_P be a finitely generated ideal of D_P, with I a nonzero finitely generated ideal of D. Since D is a P$\star$MD by assumption, there exists a finitely generated (fractional) ideal J of D such that $(IJ)^{\star_f} = (IJ)^\star = D^\star$.

Assume that $P \in \mathcal{Q}(\star_f) = \mathcal{Q}(\star)$ (since $\star = \tilde{\star}$ implies that $\star = \star_f$). Then $E^\star D_P = ED_P$, for every $E \in \overline{\boldsymbol{F}}(D)$ (because $\star = \tilde{\star}$) hence, in particular, $(IJ)^\star D_P = IJD_P = ID_P JD_P = D_P$ and thus ID_P is invertible in D_P (i.e. D_P is a valuation domain and so, trivially, it is a P$\ast$MD, for every semistar operation $\ast$ on D_P).

Assume that $P \in \mathrm{Spec}(D) \setminus \mathcal{Q}(\star)$. Then $(IJ)^\star = D^\star$ implies $1 \in IJD_Q$, for each $Q \in \mathcal{Q}(\star)$; in particular $1 \in IJD_Q$, for each $Q \in \mathcal{Q}(\star)$ such that $Q \subseteq P$ (this set of prime ideals is nonempty, since each minimal prime ideal of a nonzero principal ideal of D is in $\mathcal{Q}(\star)$, for any finite type semistar operation $\star$). Therefore, by the proof of Proposition 2.3, $D \subseteq \cap\{IJD_Q \mid Q \in \mathcal{Q}(\star),\ Q \subseteq P\} = (IJD_P)^{\widetilde{\star_P}} \subseteq (ID_P JD_P)^{\star_P} = (ID_P JD_P)^{(\star_P)_f}$, thus we obtain $(ID_P JD_P)^{(\star_P)_f} = D^{\star_P}$ and, hence, ID_P is a $(\star_P)_f$–invertible ideal of D_P (with $\star_P = (\star_P)_f$), i.e. D_P is a P$\star_P$MD. See also [5, Section 3, Theorem 1].

(ii) $\Rightarrow$ (i). Note that, by (a), we have that, for each $E \in \overline{\boldsymbol{F}}(D)$, $E^{\tilde{\star}} = E^{\wedge_{\tilde{\Theta}}} = \cap\{(ED_P)^{\widetilde{\star_P}} \mid P \in \Theta\} = \cap\big\{ \cap\{(ED_P)D_Q = ED_Q \mid Q \in \mathcal{M}(\widetilde{\star_P})\} \mid P \in \Theta\big\}$, where D_Q is a valuation domain, for each $Q \in \mathcal{M}(\widetilde{\star_P})$ and for each $P \in \Theta$. By Theorem 2.8, we deduce that D is a P$\tilde{\star}$MD, i.e. D is a P$\star$MD [5, Section 3, Theorem 1]. □

Remark 2.10. From the previous proof it follows that: *if* D *is a P*$\star$*MD, then* D_P *is a P*$\star_P$*MD, for each* $P \in \mathrm{Spec}(D)$. In the next section, we will show that the converse holds. Furthermore, in Section 4, we will deepen the study of the semistar operations of the type $\wedge_\Theta$; in particular, we will establish a natural relation between the semistar operation $\wedge_{\tilde{\Theta}}$ (considered in Theorem 2.9) and the finite type stable semistar operation, $\widetilde{(\wedge_\Theta)}$, canonically associated to $\wedge_\Theta$, where $\wedge_\Theta$ is defined as follows: $E^{\wedge_\Theta} := \cap\{(ED_P)^{\star_P} \mid P \in \Theta\}$, for each $E \in \overline{\boldsymbol{F}}(D)$.

3. Compatibility with localizations

Let D be an integral domain with quotient field K and let $P \in \mathrm{Spec}(D)$. On the localization D_P of D at P, we can consider the (semi)star operation v_{D_P} [respectively, the semistar operation $v_P := \dot{v}_D^{D_P}$] which denotes the (semi)star v–operation on D_P [respectively, the semistar operation on D_P induced by the the (semi)star v–operation v_D on D]. If the conductor $(D :$

D_P) is zero, then $(D_P)^{v_P} = (D_P)^{v_D} = (D:(0)) = K$, hence, in general, the (semi)star operation v_{D_P} (on D_P) does not coincide with the semistar operation v_P (on D_P).

Let us now relate v_P and v_{D_P} in some particular case.

As a special case of [14, Lemma 3.4 (2)] we have the following:

Lemma 3.1. *Let D be an integral domain. For each $F \in \boldsymbol{f}(D)$ and for each $P \in \mathrm{Spec}(D)$, we have $(FD_P)^{v_{D_P}} = (F^{v_D}D_P)^{v_{D_P}}$.* □

It is known that, if D is a PvMD, then D is a *v–coherent* domain in the sense of [3], i.e. if $I, J \in \boldsymbol{f}(D)$, then $I^v \cap J^v$ is an ideal of v–finite type. Also in [3] there is the following characterization of v–coherent domains: D is a v–coherent domain if and only if, for each $I \in \boldsymbol{f}(D)$, there exists $F \in \boldsymbol{f}(D)$ such that $I^{-1} = F^v$ (i.e. I^{-1} is an ideal of v–finite type).

Lemma 3.2. *Let D be a v–coherent domain (in particular, a PvMD). For each $F \in \boldsymbol{f}(D)$ and for each $P \in \mathrm{Spec}(D)$, we have $(FD_P)^{v_{D_P}} = F^{v_D}D_P$.*

Proof. Since we are assuming that D is a *v–coherent* domain, if $I \in \boldsymbol{f}(D)$, then there exists $F \in \boldsymbol{f}(D)$ such that $I^{-1} = F^{v_D}$ (or, equivalently, $I^{v_D} = F^{-1}$). Now we localize both sides of the previous equality at P and we obtain:

$$F^{v_D}D_P = I^{-1}D_P = (ID_P)^{-1}.$$

Since $I^{v_D} = (D:F) = (D:F)^{v_D}$, then $(FD_P)^{v_{D_P}} = (D_P:(D_P:FD_P)) = (D_P:(D:F)D_P) = (D_P:I^{v_D}D_P) \subseteq (D_P:ID_P) = (D:I)D_P = I^{-1}D_P = F^{v_D}D_P$. By the previous Lemma 3.1, we know that $(FD_P)^{v_{D_P}} = (F^{v_D}D_P)^{v_{D_P}}$, thus we conclude immediately that $(FD_P)^{v_{D_P}} = F^{v_D}D_P$. □

Proposition 3.3. *If D is a v–coherent domain (in particular, if D is a PvMD domain) then, for each $P \in \mathrm{Spec}(D)$ and for each $F \in \boldsymbol{f}(D_P)$, we have $F^{v_{D_P}} \subseteq F^{v_P}$.*

Proof. For each $F \in \boldsymbol{f}(D_P)$ ($\subseteq \overline{\boldsymbol{F}}(D)$) there exists $F_0 \in \boldsymbol{f}(D)$ such that $F = F_0D_P$, then by using Lemma 3.2 we have:

$$\begin{aligned} F^{v_{D_P}} &= (F_0D_P)^{v_{D_P}} = {F_0}^{v_D}D_P \subseteq \\ &\subseteq F^{v_D}D_P = F^{v_D} = (D:(D:F)) = F^{v_P} \end{aligned}$$

(note that the first equality in the second line is a consequence of the following general fact: if $E \in \overline{\boldsymbol{F}}(D_P)$, then $(D:E)$ is also in $\overline{\boldsymbol{F}}(D_P)$ and so E^{v_D} belongs to $\overline{\boldsymbol{F}}(D_P)$). □

Remark 3.4. Let $\star$ be a semistar operation on D and, for each $P \in \mathrm{Spec}(D)$, let $\star_P$ be the semistar operation induced on D_P, defined in the previous section. For which properties **(P)** concerning $(D,\star)$ we have that $(D_P,\star_P)$ satisfies **(P)** ?

A positive answer to this question was already given for the following properties: **(a)** $\star$ is a finite type semistar operation on D; **(b)** $\star$ is a stable semistar operation on D; **(c)** $\star$ is a finite type spectral semistar operation on D; **(d)** $\star$ is a finite type stable semistar operation on D (i.e. $\star = \tilde{\star}$); **(e)** $\star$ is an e.a.b semistar operation on D; **(f)** $\star$ is an a.b semistar operation on D; **(g)** D is a P$\star$MD, (cf. Lemma 2.1, Proposition 2.3, Remark 2.4 and Remark 2.10).

In this ambit, a natural problem is to study the behavior of the generalized Nagata ring and of the generalized Kronecker function ring in relation with the localization at any prime ideal P. We have the following:

Proposition 3.5. *Let $\star$ be a semistar operation on an integral domain D and let $P \in \mathrm{Spec}(D)$. Then the following statements hold:*

- **(1)** $\mathrm{Na}(D,\star)_{D\setminus P} \subseteq \mathrm{Na}(D_P, \star_P)$.
- **(2)** $\mathrm{Na}(D,\star) = \cap\{\mathrm{Na}(D_P,\star_P) \mid P \in \mathrm{Spec}(D)\} = $
  $= \cap\{\mathrm{Na}(D_M,\star_M) \mid M \in \mathrm{Max}(D)\}$.
- **(3)** $\mathrm{Kr}(D,\star)_{D\setminus P} \subseteq \mathrm{Kr}(D_P,\star_P)$.
- **(4)** $\mathrm{Kr}(D,\star) = \cap\{\mathrm{Kr}(D_P,\star_P) \mid P \in \mathrm{Spec}(D)\} = $
  $= \cap\{\mathrm{Kr}(D_M,\star_M) \mid M \in \mathrm{Max}(D)\}$.

Proof. (1) Set $\mathcal{Q} := \mathcal{Q}(\star_f)$. Recall from the proof of Proposition 2.3, see Remark 2.5, that $\mathcal{Q}((\star_f)_P) = \{QD_P \mid Q \subseteq P,\ Q \in \mathcal{Q}\}$. Set simply $\mathcal{Q}_P := \mathcal{Q}((\star_f)_P)$ and $\underset{\cdot}{\mathcal{Q}}_P := \{Q \in \mathrm{Spec}(D) \mid QD_P \in \mathcal{Q}_P\}$. Note also that $\mathcal{Q} = \cup\{\underset{\cdot}{\mathcal{Q}}_P \mid P \in \mathrm{Spec}(D)\}$. We know that $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star_f) = \cap\{D_Q(X) \mid Q \in \mathcal{Q}\}$. Therefore $\mathrm{Na}(D,\star)_{D\setminus P} = (\cap\{D_Q(X) \mid Q \in \mathcal{Q}\})_{D\setminus P} \subseteq \cap\{D_Q(X)_{D\setminus P} \mid Q \in \mathcal{Q}\} = \cap\{D_Q(X) \mid Q \in \mathcal{Q}$ and $Q \subseteq P\} = \cap\{D_Q(X) \mid Q \in \mathcal{Q}_P\} = \mathrm{Na}(D_P, (\star_f)_P) = \mathrm{Na}(D_P,\star_P)$.

(2) Since $\mathcal{Q} = \cup\{\underset{\cdot}{\mathcal{Q}}_P \mid P \in \mathrm{Spec}(D)\}$, then $\mathrm{Na}(D,\star) = \cap\{D_Q(X) \mid Q \in \mathcal{Q}\} = \cap\big\{ \cap\{D_Q(X) \mid Q \in \underset{\cdot}{\mathcal{Q}}_P\} \mid P \in \mathrm{Spec}(D)\big\} = \cap\{\mathrm{Na}(D_P,\star_P) \mid P \in \mathrm{Spec}(D)\}$. The proof is similar for the $\mathrm{Max}(D)$ case.

(3) We start by recalling, from Corollary 2.2, the following fact: *A $\star_P$–valuation overring W of D_P is the same as a $\star$–valuation overring W of D such that $W \supseteq D_P$.*

We know that $\mathrm{Kr}(D,\star) = \cap\{W(X) \mid W$ is a $\star$–valuation overring of $D\}$. Therefore, using Corollary 2.2, the fact that $W(X)_{D\setminus P} = W_{D\setminus P}(X)$ and that $W_{D\setminus P}$ is a valuation overring of D_P, for each valuation overring W of D, then

$$\begin{aligned}\mathrm{Kr}(D,\star)_{D\setminus P} &= (\cap\{W(X) \mid W \text{ is a } \star\text{–valuation overring of } D\})_{D\setminus P} \subseteq \\ &\subseteq \cap\{W(X) \mid W \text{ is a } \star\text{–valuation overring of } D \text{ and } W \supseteq D_P\} = \\ &= \cap\{W(X) \mid W \text{ is a } \star_P\text{–valuation overring of } D\} = \\ &= \mathrm{Kr}(D_P,\star_P).\end{aligned}$$

(4) From the Corollary 2.2 we deduce that $\{W \mid W$ is a $\star$–valuation overring of $D\} = \cup\big\{\{W \mid W$ is a $\star_P$–valuation overring of $D_P\} \mid P \in \mathrm{Spec}(D)\big\}$. Therefore, we have that:

$$\begin{aligned}\mathrm{Kr}(D,\star) &= \cap\{W(X) \mid W \text{ is a } \star\text{–valuation overring of } D\} = \\ &= \cap\big\{\cap\{W(X) \mid W \text{ is a } \star_P\text{–valuation overring of } D_P\} \mid P \in \mathrm{Spec}(D)\big\} \\ &= \cap\{\mathrm{Kr}(D_P,\star_P) \mid P \in \mathrm{Spec}(D)\}.\end{aligned}$$

The proof is similar for the $\mathrm{Max}(D)$ case. □

Next problem is to relate the Nagata ring or the Kronecker function ring, associated to a localized semistar operation, to the corresponding localization of the Nagata ring or of the Kronecker function ring, respectively. More precisely,

Problem 3.6. *Let D be an integral domain, $\star$ a semistar operation on D and P a prime ideal of D.*

(1) *Under which conditions on* D *and* P*,* $\mathrm{Na}(D,\star)_{D\setminus P} = \mathrm{Na}(D_P, \star_P)$ *?*
(2) *Under which conditions on* D *and* P*,* $\mathrm{Kr}(D,\star)_{D\setminus P} = \mathrm{Kr}(D_P, \star_P)$ *?*
(3) *In case of a Prüfer–*$\star$*–multiplication domain is the answer to both questions positive ?*

We deal first with the Nagata ring. Without loss of generality, we may assume that $\star = \widetilde{\star}$. For each prime ideal $P \in \mathrm{Spec}(D)$, we have the following picture of finite type stable semistar operations:

$$\begin{array}{ccccc} D & \longrightarrow & D[X] & \xrightarrow{\quad \varphi_P \quad} & D_P[X] \\ \star & \longmapsto & \eta\ (= N(\star)) & \longmapsto & \left\{ \begin{array}{l} \eta_P\ (= N(\star_P)) \\ \varphi_P(\eta) \end{array} \right. \end{array}$$

where η [respectively, η_P] is the semistar operation on $D[X]$ [respectively, on $D_P[X]$] "defined by the saturated multiplicative subset" $N(\star)$ [respectively, $N(\star_P)$], i.e. $E^\eta := ED[X]_{N(\star)} = E\,\mathrm{Na}(D,\star)$, for each $E \in \overline{\boldsymbol{F}}(D[X])$ [respectively, $E^{\eta_P} := ED_P[X]_{N(\star_P)} = E\,\mathrm{Na}(D_P, \star_P)$, for each $E \in \overline{\boldsymbol{F}}(D_P[X])$], and $\varphi_P(\eta)$ (or, simply, $\varphi(\eta)$) is the semistar operation on $D_P[X]$ defined as follows:

$$E^{\varphi(\eta)} := \left\{ z \in K(X) \mid \varphi^{-1}\left(E :_{D_P[X]} zD_P[X]\right) \cap N(\star) \neq \emptyset \right\},$$

for each $E \in \overline{\boldsymbol{F}}(D_P[X])$. Note that η , η_P and $\varphi(\eta)$ are finite type stable semistar operations. In general, we have $\varphi(\eta) \leq \eta_P$. Indeed, given an ideal $J \subseteq D_P[X]$ satisfying $\varphi^{-1}(J) \cap N(\star) \neq \emptyset$, then there exists $f \in \varphi^{-1}(J)$ such that $\mathbf{c}(f)^\star = D^\star$, hence $(\mathbf{c}(\varphi(f)))^{\star_P} = (\mathbf{c}(f)D_P)^{\star_P} = (\mathbf{c}(f)D_P)^\star = (\mathbf{c}(f)^\star D_P)^\star = (D^\star D_P)^\star = D_P^\star = D_P^{\star_P}$, thus $\varphi(f) \in N(\star_P)$. From this remark, we deduce immediately that $E^{\varphi(\eta)} \subseteq E^{\eta_P}$, for each $E \in \overline{\boldsymbol{F}}(D_P[X])$.

With this background, now we use some wellknown fact of hereditary torsion theories [2] or, equivalently, of localizing systems associated to semistar operations [4]. More precisely, we know that applying a finite type stable semistar operation is exactly the same as doing the localization with respect to the associated finite type hereditary torsion theory or with respect to the associated localizing system of ideals and, moreover, it is wellknown that it is possible to "interchange", in a natural way, two subsequent localizations of the previous type. Therefore:

$$\begin{aligned} \mathrm{Na}(D,\star)_{D\setminus P} &= \left(D[X]^\eta\right)_{D\setminus P} = \left(D[X]_{D\setminus P}\right)^{\varphi(\eta)} = (D_P[X])^{\varphi(\eta)} \subseteq \\ &\subseteq D_P[X]^{\eta_P} = \mathrm{Na}(D_P, \star_P), \end{aligned}$$

since the localizing system of ideals of $D[X]$ associated to η is the set $\mathcal{F}^\eta := \{I \text{ ideal of } D[X] \mid I^\eta = D[X]^\eta\} = \{I \text{ ideal of } D[X] \mid I \cap N(\star) \neq \emptyset\}$ and the localizing system of ideals of $D_P[X]$ associated to $\varphi(\eta)$ [respectively, η_P] is the set $\mathcal{F}^{\varphi(\eta)} := \{ID_P[X] \mid I \text{ ideal of } D[X],\ (ID_P[X])^{\varphi(\eta)} = D_P[X]^{\varphi(\eta)}\} = \{ID_P[X] \mid I \text{ ideal of } D[X],\ ID_P[X] \cap N(\star) \neq \emptyset\}$ [respectively, $\mathcal{F}^{\eta_P} := \{J \text{ ideal of } D_P[X] \mid J^{\eta_P} = D_P[X]^{\eta_P}\} = \{J \text{ ideal of } D_P[X] \mid J \cap N(\star_P) \neq \emptyset\}$] .

To show that $D_P[X]^{\varphi(\eta)}$ and $D_P[X]^{\eta_P}$ are either equals or different we only need to compare the prime spectra associated to the finite type stable semistar operations $\varphi(\eta)$ and η_P, defined on $D_P[X]$. More precisely,

$$\begin{aligned}\operatorname{Spec}^{\eta_P}(D_P[X]) &= \{Q \in \operatorname{Spec}(D_P[X]) \mid Q \cap N(\star_P) = \emptyset\} = \\ &= \{Q \in \operatorname{Spec}(D_P[X]) \mid \text{ for all } g \in Q\,,\ \mathbf{c}(g)^\star \neq (D_P)^\star\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Spec}^{\varphi(\eta)}(D_P[X]) &= \{Q \in \operatorname{Spec}(D_P[X]) \mid \varphi^{-1}(Q) \cap N(\star) = \emptyset\} = \\ &= \{Q \in \operatorname{Spec}(D_P[X]) \mid \text{ for all } f \in \varphi^{-1}(Q), \mathbf{c}(f)^\star \neq D^\star\} = \\ &= \{Q \in \operatorname{Spec}(D_P[X]) \mid \text{ for all } f \in \varphi^{-1}(Q), \text{ there exists} \\ &\qquad H \in \operatorname{Spec}^\star(D) \text{ with } \mathbf{c}(f) \subseteq H\}.\end{aligned}$$

Recall also that the prime ideals in $\operatorname{Na}(D,\star)_{D\setminus P} = D_P[X]^{\varphi(\eta)}$ [respectively, in $\operatorname{Na}(D_P,\star_P) = D_P[X]^{\eta_P}$] are in a natural bijective correspondence with prime ideals in $\operatorname{Spec}^{\varphi(\eta)}(D_P[X])$ [respectively, $\operatorname{Spec}^{\eta_P}(D_P[X])$].
Finally, observe that, in general, $\operatorname{Spec}^{\eta_P}(D_P[X]) \subseteq \operatorname{Spec}^{\varphi(\eta)}(D_P[X])$, since if $\star_1 \leq \star_2$ are two semistar operations on an integral domain R, then $\operatorname{Spec}^{\star_2}(R) \subseteq \operatorname{Spec}^{\star_1}(R)$.

Remark 3.7. Let $\star$ be a semistar operation defined on an arbitrary integral domain D. Note that: *if* $P \in \operatorname{Spec}^{\star_f}(D)$, *then* $\operatorname{Na}(D,\star)_{D\setminus P} = \operatorname{Na}(D_P,\star_P)$. As a matter of fact, without loss of generality, we can assume that $\star = \widetilde{\star}$ and, in this case, PD_P belongs to $\operatorname{Spec}^{(\star_P)_f}(D_P)$ and so [5, Lemma 2.5 (f)]:

$$\operatorname{Na}(D,\star)_{D\setminus P} = (\cap\{D_Q(X) \mid Q \in \operatorname{Spec}^{\star_f}(D)\})_{D\setminus P} = D_P(X) = \operatorname{Na}(D_P,\star_P)\,.$$

Proposition 3.8. *If D is a Bézout domain then, for each $P \in \operatorname{Spec}(D)$ and for each semistar operation $\star$ on D, we have $\operatorname{Na}(D,\star)_{D\setminus P} = \operatorname{Na}(D_P,\star_P)$ (and $\operatorname{Kr}(D,\star)_{D\setminus P} = \operatorname{Kr}(D_P,\star_P)$).*

Proof. If D is a Bézout domain, then $\mathbf{c}(g)D$ is a nonzero principal ideal, for any nonzero polynomial $g \in D[X]$. Let $Q \in \operatorname{Spec}^{\varphi(\eta)}(D_P[X])$, if $Q \not\subseteq PD_P[X]$ there exists $f \in Q \setminus PD_P[X]$, and, without loss of generality, we may also assume that $f \in \operatorname{Im}(\varphi)$ (i.e. $f = \frac{f}{1}$, with $f \in D[X]$), such that $\mathbf{c}_{D_P}(f) = \mathbf{c}_D(f)D_P = D_P$. Henceforth, $\mathbf{c}_D(f) = sD$ for some $s \in D \setminus P$. Therefore, $\frac{f}{s} \in D[X]$ and $\mathbf{c}_D(\frac{f}{s}) = D$. On the other hand, $f \in Q$ and thus $\frac{f}{s} \in \varphi^{-1}(Q)$, which is a contradiction. As a consequence, we have $Q \subseteq PD_P[X]$ and we conclude that $\operatorname{Spec}^{\varphi(\eta)}(D_P[X]) = \operatorname{Spec}^{\eta_P}(D_P[X])$. As we have noticed above, this fact implies that $\operatorname{Na}(D,\star)_{D\setminus P} = \operatorname{Na}(D_P,\star_P)$. The parenthetical equality follows from the fact that, for any Prüfer domain D and for any semistar operation $\star$ on D, $\operatorname{Na}(D,\star) = \operatorname{Kr}(D,\star)$ [5, Remark 3.2], and so, in particular, $\operatorname{Na}(D_P,\star_P) = \operatorname{Kr}(D_P,\star_P)$, for each $P \in \operatorname{Spec}(D)$. □

Next two examples show that the identity $\operatorname{Na}(D,\star)_{D\setminus P} = \operatorname{Na}(D_P,\star_P)$ does not hold in general.

Example 3.9. *Let $D := \mathbb{Z}[Y]$ and let $\star := d$ be the identical semistar operation on D, i.e., $E^\star := E$, for any $E \in \overline{\boldsymbol{F}}(D)$. We consider the prime* *ideal $P := 2\mathbb{Z}[Y] = 2D = (2)$. With the notation introduced in the previous Problem 3.6, in case of the local domain $D_{(2)}$, we have:*

$$\begin{aligned}\mathrm{Spec}^{\eta_{(2)}}(D_{(2)}[X]) &= \{Q \in \mathrm{Spec}(\mathbb{Z}[Y]_{(2)}[X]) \mid Q \subseteq 2\mathbb{Z}[Y]_{(2)}[X]\} \subsetneqq \\ &\subsetneqq \mathrm{Spec}^{\varphi(\eta)}((D_{(2)})[X]) = \\ &= \{Q \in \mathrm{Spec}(\mathbb{Z}[Y]_{(2)}[X]) \mid \text{ for all } f \in \varphi^{-1}(Q)\,,\ \text{there exists} \\ &\qquad\qquad H \in \mathrm{Spec}(D) \text{ with } \mathbf{c}(f) \subseteq H\}.\end{aligned}$$

As a matter of fact, if we take $f := YX + 3 \in D[X]$, then $\mathbf{c}(f) = (Y, 3) \neq \mathbb{Z}[Y] = D$, hence there exists a maximal ideal H of D such that $f \in H[X]$. In addition, $\varphi(f)$ is not invertible in $D_{(2)}[X]$ and $\varphi(f) \notin 2D_{(2)}[X]$, hence there exists $Q \in \mathrm{Spec}^{\varphi(\eta)}(D_{(2)}[X])$, with $Q \neq 2D_{(2)}[X]$, such that $\varphi(f) \in Q$; in particular, $Q \notin \mathrm{Spec}^{\eta_{(2)}}(D_{(2)}[X])$. As a consequence, we have $\mathrm{Na}(D,\star)_{D\setminus P} = \mathrm{Na}(D,d)_{D\setminus(2)} \neq \mathrm{Na}(D_{(2)}, d_{(2)}) = \mathrm{Na}(D_P, \star_P)$.

It is also possible to give a direct arithmetic proof of the previous fact, that is $\mathrm{Na}(D,d)_{D\setminus(2)} \neq \mathrm{Na}(D_{(2)}, d_{(2)})$. We consider as before $f := YX + 3 \in D[X]$, then $\varphi(f) \in D_{(2)}[X]$ satisfies $\mathbf{c}_{D_{(2)}}(\varphi(f)) = (Y,3)D_{(2)} = D_{(2)}$, hence $\varphi(f)$ is invertible in $\mathrm{Na}(D_{(2)}, d_{(2)})$. If f is invertible in $\mathrm{Na}(D,d)_{D\setminus(2)}$, then there exist $h \in D[X]$, $k \in N(d)$ and $b \in D \setminus (2)$ such that $fh = kb$. Since D is a factorial domain, let $b = p_1p_2\cdots p_t$ be a factorization in prime elements of b in D. For any p_i we have either $p_i \mid f$ or $p_i \mid h$, hence we may find an identity of the following type: $fh' = kp_1p_2\cdots p_s$, where $h' \in D[X]$, $s \leq t$, $p_i \nmid h'$, for any $i = 1, 2, \ldots, s$.

Case 1. If $\mathbf{c}(h') = D$, then $\mathbf{c}(f) = \mathbf{c}(fh') = \mathbf{c}(k)p_1p_2\cdots p_s = p_1p_2\cdots p_s$, which is a contradiction, since $\mathbf{c}(f)$ is not a principal ideal of D.

Case 2. If $\mathbf{c}(h') \neq D$, then there exists a maximal ideal H in D such that $\mathbf{c}(h') \subseteq H$. From $fh' = kp_1p_2\cdots p_s$, with $p_i \nmid h'$, for any $i = 1, 2, \ldots, s$, we have $p_1\cdots p_s f'h' = kp_1\cdots p_s$, for some $f' \in D[X]$. Therefore, $\mathbf{c}(f'h') = \mathbf{c}(k) = D$. On the other hand, $\mathbf{c}(f'h') \subseteq \mathbf{c}(h') \subseteq H \neq D$, which is a contradiction.

From the previous argument we deduce that f is not invertible in the ring $\mathrm{Na}(D,d)_{D\setminus(2)}$, and so $\mathrm{Na}(D,d)_{D\setminus(2)} \neq \mathrm{Na}(D_{(2)}, d_{(2)})$.

Example 3.10. *Let D be an integral domain and assume that D possesses two incomparable prime ideals P_1 and P_2. Set $P := P_1$ and $\star := \star_{\{D_{P_2}\}}$ (i.e. $E^\star := ED_{P_2}$, for each $E \in \overline{\boldsymbol{F}}(D)$), then we have:*

$$\begin{aligned}\mathrm{Na}(D,\star)_{D\setminus P} &= \left(D[X]_{N(\star)}\right)_{D\setminus P_1} = (D_{P_1}[X])_{N(\star)} \\ \mathrm{Na}(D_P,\star_P) &= D_{P_1}[X]_{N(\star_{P_1})}.\end{aligned}$$

(For simplicity of notation, we have identified $D[X]$, and $N(\star)$, with its canonical image in $D_{P_1}[X]$). We claim that $D_{P_1}[X]_{N(\star)} \neq D_{P_1}[X]_{N(\star_{P_1})}$.

We compare the two multiplicative sets $N(\star)$ and $N(\star_{P_1})$; more precisely, we show that there is a canonical injective map $(D_{P_1}[X])_{N(\star)} \longrightarrow D_{P_1}[X]_{N(\star_{P_1})}$, which is not surjective.

Note that if $f \in N(\star)$ $(\subseteq D[X])$, then $\mathbf{c}(f)^\star = D^\star$, hence $(\mathbf{c}(f)D_{P_1})^{\star_{P_1}} = (\mathbf{c}(f)D_{P_1})^\star = (D^\star D_{P_1})^\star = D^\star_{P_1} = D^{\star_{P_1}}_{P_1}$, i.e. $f \in N(\star_{P_1})$.

Let $f \in D_{P_1}[X]$, then there is $s' \in D \setminus P_1$ such that $f = s'f'$ with $f' \in D[X]$, then — without loss of generality — we may assume $f \in D[X]$.

If $f \in N(\star_{P_1}) \cap D[X]$, then $\mathbf{c}(f)^{\star_{P_1}} = (\mathbf{c}(f)D_{P_1})^\star = (D_{P_1})^\star$. On the other hand, the finite type stable semistar operation $\star$ and the localization at P_1

commute, as they are defined by hereditary torsion theories of finite type. Therefore, we have $\mathbf{c}(f)^\star D_{P_1} = (\mathbf{c}(f)D_{P_1})^\star = (D_{P_1})^\star = (D^\star)_{D\setminus P_1}$.
It is obvious that, in general, the previous equality does not imply $\mathbf{c}(f)^\star = D^\star$, i.e. $f \in N(\star)$. In fact, if $f \in P_2[X] \setminus P_1[X]$, then $\mathbf{c}(f) \subseteq P_2 \setminus P_1$ and $\mathbf{c}(f)D_{P_1} = D_{P_1}$ hence, in particular, $\mathbf{c}(f)^\star D_{P_1} = (D^\star)_{D\setminus P_1}$. On the other hand, $\mathbf{c}(f)^\star = \mathbf{c}(f)D_{P_2} \subseteq P_2 D_{P_2} \subsetneq D_{P_2} = D^\star$, hence $f \notin N(\star)$.
Since $f \in N(\star_{P_1})$, then $1/f \in (D_{P_1}[X])_{N(\star_{P_1})}$. If $1/f = h/k$, for some $h/k \in D_{P_1}[X]_{N(\star)}$, where $h \in D_{P_1}[X]$ and $k \in N(\star)$, then $hf = k \in N(\star)$. Therefore, $\mathbf{c}(hf)^\star = \mathbf{c}(k)^\star = D^\star$, which is a contradiction, as $\mathbf{c}(hf)^\star \subseteq \mathbf{c}(f)^\star \subseteq P_2 D_{P_2} \neq D_{P_2} = D^\star$.

Let us now consider the second question considered in Problem 3.6, concerning the Kronecker function rings. Also in this case the answer is negative in general, as the following Example 3.12 proves. First we give a positive example.

Example 3.11. Let V a valuation overring of an integral domain D, with maximal ideal M, which is not essential (i.e. $D_P \subsetneq V$, where $P := M \cap V$). Set $\star := \star_{\{V\}}$ (i.e. $E^\star := EV$, for each $E \in \overline{\boldsymbol{F}}(D)$). In this situation, $\star$ is a finite type a.b. semistar operation on D and P is the only maximal quasi–$\star$–ideal of D.

In this particular case, the description of the Kronecker function ring $\mathrm{Kr}(D,\star)$ is rather easy. Let $a,\ b \in V$, we set as usual $a \mid_V b$ if there exists $v \in V$ such that $av = b$ and, for each $f \in D[X]$, we denote by $a(f)$ a generator (in V) of the principal ideal $\mathbf{c}(f)V$. Then, using also [6, Example 3.6], we have:

$$\begin{aligned}\mathrm{Kr}(D,\star) &= \left\{ \tfrac{f}{g} \mid f, g \in D[X] \setminus \{0\}\,, \ \text{ with } a(g) \mid_V a(f) \right\} \cup \{0\} = \\ &= V(X)\,.\end{aligned}$$

Therefore $\mathrm{Kr}(D,\star)_{D\setminus P} = \mathrm{Kr}(D,\star)$, because each element $b \in D \setminus P$ is a unit in V (i.e. $\mathbf{c}(b)^\star = (bD)^\star = bV = V$).
On the other hand, $\mathrm{Kr}(D_P,\star_P)$ has a similar description:

$$\begin{aligned}\mathrm{Kr}(D_P,\star_P) &= \left\{ \tfrac{bf}{cg} \mid f, g \in D[X] \setminus \{0\},\ b, c \in D \setminus P \text{ with } a(cg) \mid_V a(bf) \right\} = \\ &= V(X)\,,\end{aligned}$$

and we conclude immediately that $\mathrm{Kr}(D_P,\star_P) = \mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star)_{D\setminus P}$.

Next we give a negative example for the Kronecker function rings case.

Example 3.12. Let $D := \mathbb{Z}[\sqrt{-5}]$. Since D is a Dedekind domain, if d is the identical semistar operation on D, then $\mathrm{Kr}(D,d) = \mathrm{Na}(D,d) = D(X)$. We take $f := (1+\sqrt{-5}) + (1-\sqrt{-5})X \in D[X]$ and we consider, for instance, the prime ideal $P := (3, 1+\sqrt{-5})D \not\supseteq Q := (1+\sqrt{-5}, 1-\sqrt{-5})D$ $(= \mathrm{rad}(2D))$. Then, arguing for f as in Example 3.9, we obtain that $D(X)_{D\setminus P} = \mathrm{Kr}(D,d)_{D\setminus P} \neq \mathrm{Kr}(D_P,d_P) = D_P(X)$.

Note that this example produces also a negative answer to Problem 3.6 (3).

We finish this section with a local characterization of P$\star$MDs. Recall that P$\star$MDs were characterized in [5], using quasi–$\star_f$–prime ideals; here we extend this characterization by using the whole prime spectrum. In particular, next result provides new examples of nontrivial local P$\star$MDs, by taking the localizations of a P$\star$MD at its prime non quasi–$\star_f$–ideals.

Theorem 3.13. *Let $\star$ be a semistar operation on an integral domain D. Then the following statements are equivalent:*

(i) *D is a P$\star$MD;*
(ii) *D_P is a P$\star_P$MD, for each $P \in \mathrm{Spec}(D)$;*
(iii) *D_M is a P$\star_M$MD, for each $M \in \mathrm{Max}(D)$;*
(iv) *D_N is a P$\star_N$MD, for each $N \in \mathcal{M}(\star_f)$;*
(v) *D_Q is a P$\star_Q$MD, for each $Q \in \mathcal{Q}(\star_f)$.*

Proof. We already proved that (i) $\Rightarrow$ (ii) (Remark 2.10). Obviously, (ii) $\Rightarrow$ (iii), (v); and (v) $\Rightarrow$ (iv).

(iii) (or (ii)) $\Rightarrow$ (i). Recall from [5, Section 3, Theorem 1 and Remark 2] that D is a P$\star$MD if and only if $\mathrm{Na}(D,\star) = \mathrm{Kr}(D,\star)$ and this last equality follows from Proposition 3.5 (2) and (4).

(iv) $\Rightarrow$ (i). We have already observed (Remark 3.7) that $\mathrm{Na}(D,\star)_{D\setminus N} = \mathrm{Na}(D_N,\star_N)$ and we know that $\mathrm{Kr}(D,\star)_{D\setminus N} \subseteq \mathrm{Kr}(D_N,\star_N)$, for each $N \in \mathcal{M}(\star_f)$ (Proposition 3.5 (3)). On the other hand, by assumption, $\mathrm{Na}(D_N,\star_N) = \mathrm{Kr}(D_N,\star_N)$, for each $N \in \mathcal{M}(\star_f)$. From the previous relations and from the fact that $\mathrm{Na}(D,\star) \subseteq \mathrm{Kr}(D,\star)$, we deduce immediately that $\mathrm{Na}(D,\star)_{D\setminus N} = \mathrm{Kr}(D,\star)_{D\setminus N}$, for each $N \in \mathcal{M}(\star_f)$ The conclusion follows immediately, since $\mathrm{Na}(D,\star) = \cap\{\mathrm{Na}(D,\star)_{D\setminus N} \mid N \in \mathcal{M}(\star_f)\}$ and $\mathrm{Kr}(D,\star) = \cap\{\mathrm{Kr}(D,\star)_{D\setminus N} \mid N \in \mathcal{M}(\star_f)\}$ (Proposition 3.5 (2) and (4)). □

Remark 3.14. M. Zafrullah, in [20], proves a different local characterization of P$\star$MDs in the particular case where $\star = v$. More precisely, he obtains that a domain D is a PvMD if and only if (a) D_P is a Pv_{D_P}MD for every prime ideal P of D and (b) for every prime t_D–ideal Q of D, QD_Q is a t_{D_Q}–ideal (about condition (b) see also Remark 4.12). As we have already observed at the beginning of this section, recall that v_{D_P} is different, in general, from v_P.

4. Inducing semistar operations

In this section, we deal with the converse of the problem considered in the first part of this paper, i.e., we start from a family of "local" semistar operations on the localized rings D_P, where P varies in a nonempty set of prime ideals of an integral domain D, and the goal is the description of a gluing process for building a new "global" semistar operation on the ring D.

Let D be an integral domain. Let P be a prime ideal of D and let $*_P$ be a semistar operation on the localization D_P of D at P. Then we may consider $\dot{*}_P$, the induced semistar operation on D defined as follows, for each $E \in \overline{\boldsymbol{F}}(D)$:

$$E^{\dot{*}_P} = (ED_P)^{*_P} .$$

Let Θ be a given nonempty subset of $\mathrm{Spec}(D)$ and let $\{*_P \mid P \in \Theta\}$ be a family of semistar operations, where $*_P$ is a semistar operation on the localization D_P of D at P. We define $\wedge := \wedge_{\Theta,\{*_P\}} := \wedge_\Theta := \wedge\{\dot{*}_P \mid P \in \Theta\}$ as the semistar operation on D defined as follows, for each $E \in \overline{\boldsymbol{F}}(D)$,

$$E^{\wedge} := \cap\{(ED_P)^{*_P} \mid P \in \Theta\}.$$

If Θ is the empty set, then we set $\wedge := \wedge_\emptyset := e_D$. Given a semistar operation $\star$ on D, for each prime ideal P of D, we denote as usual by $\star_P$ the semistar operation $\dot{\star}^{D_P}$ on D_P, deduced from $\star$ by ascent to D_P (i.e. $E^{\star_P} := E^\star$, for each $E \in \overline{\boldsymbol{F}}(D_P)$ $(\supseteq \overline{\boldsymbol{F}}(D))$; in particular if $\star$ coincides with the semistar operation $\wedge := \wedge_{\Theta,\{*_P\}}$ defined on D, we can consider a semistar operation $\wedge_P$ on D_P, for each $P \in \Theta$.

Note that:

(a) for each $P \in \Theta$, $(E^{\wedge} D_P)^{*_P} = (ED_P)^{*_P}$;
(b) for each $P \in \Theta$, $\wedge_P \leq *_P$;
(c) $\wedge = \wedge\{\dot{\wedge}_P \mid P \in \Theta\} = \wedge\{\dot{*}_P \mid P \in \Theta\}$;
(d) for each semistar operation $\star$ on D, $\star \leq \underset{\star}{\wedge} := \wedge\{\dot{\star}_P \mid P \in \operatorname{Spec}(D)\}$.

(a) We remark that: $E^{\wedge} = \cap\{(ED_{P'})^{*_{P'}} \mid P' \in \Theta\} \subseteq (ED_P)^{*_P}$, for each $P \in \Theta$. Therefore, we deduce that:

$$(E^{\wedge} D_P)^{*_P} \subseteq ((ED_P)^{*_P} D_P)^{*_P} = (ED_P D_P)^{*_P} = (ED_P)^{*_P}.$$

The opposite inclusion is trivial.

(b), (c) and (d) are straightforward.

Theorem 4.1. *Let $\star$ be a semistar operation on an integral domain D. For each $P \in \operatorname{Spec}(D)$, denote as usual by $\star_P$ the semistar operation $\dot{\star}^{D_P}$ on D_P, induced from $\star$ by ascent to D_P. Set $\underset{\star}{\wedge} := \wedge\{\dot{\star}_P \mid P \in \operatorname{Spec}(D)\}$. If $\star$ is a spectral semistar operation on D then $\star = \underset{\star}{\wedge}$.*

Proof. Let $\Delta \subseteq \operatorname{Spec}(D)$ be such that $\star = \star_\Delta$. For each $E \in \overline{\boldsymbol{F}}(D)$, then:

$$\begin{aligned} E^\star &= (E^\star)^\star = (\cap\{ED_Q \mid Q \in \Delta\})^\star \supseteq \cap\{(ED_Q)^\star \mid Q \in \Delta\} = \\ &= \cap\{(ED_Q)^{\star_Q} \mid Q \in \Delta\} = \cap\{E^{\dot{\star}_Q} \mid Q \in \Delta\} \supseteq \\ &\supseteq \cap\{E^{\dot{\star}_P} \mid P \in \operatorname{Spec}(D)\} = E^{\underset{\star}{\wedge}} = \cap\{(ED_P)^{\star_P} \mid P \in \operatorname{Spec}(D)\} \supseteq E^\star. \end{aligned}$$

As for any $P \in \Delta$ and any $E \in \overline{\boldsymbol{F}}(D)$ we have $(ED_P)^\star := \cap\{ED_P D_Q \mid Q \in \Delta\} \subseteq ED_P D_P = ED_P$. $\square$

Lemma 4.2. *Let D be an integral domain and let P be a prime ideal of D. If $*_P$ is a spectral semistar operation on D_P, defined by a nonempty subset $\Delta_P \subseteq \operatorname{Spec}(D_P)$, then $\dot{*}_P$ is a spectral semistar operation on D defined by the (nonempty) set $\dot{\Delta}_P := \{Q \in \operatorname{Spec}(D) \mid \ QD_P \in \Delta_P\}$.*

Proof. For each $E \in \overline{\boldsymbol{F}}(D)$:

$$\begin{aligned} E^{\dot{*}_P} &= (ED_P)^{*_P} = \cap\{(ED_P)_H \mid H \in \Delta_P\} = \\ &= \cap\{ED_Q \mid QD_P =: H \in \Delta_P\} = \cap\{ED_Q \mid Q \in \dot{\Delta}_P\} = E^{\star_{\dot{\Delta}_P}}. \end{aligned}$$

$\square$

Corollary 4.3. *Let D be an integral domain and let Θ be a nonempty subset of $\operatorname{Spec}(D)$. If $*_P$ is a spectral semistar operation on D_P, defined by a subset $\Delta_P \subseteq \operatorname{Spec}(D_P)$, for each $P \in \Theta$, and if $\wedge := \wedge_{\Theta,\{*_P\}} := \wedge\{\dot{*}_P \mid \ P \in \Theta\}$ is the semistar operation on D defined as above, then $\wedge$ is a spectral semistar* *operation on D defined by the subset $\Delta := \cup\{\Delta_P \mid P \in \Theta\} \subseteq \mathrm{Spec}(D)$ (i.e. $\wedge = \star_\Delta$).*

Proof. This statement is a straightforward consequence of the previous Lemma 4.2. ☐

Lemma 4.4. *Let D be an integral domain with quotient field K. Let Θ be a given nonempty subset of $\mathrm{Spec}(D)$ and let $\{*_P \mid P \in \Theta\}$ be a family of semistar operations, where $*_P$ is a semistar operation on the localization D_P of D at P. Assume that $*_P$ is a semistar operation of finite type and that the family $\{D_P \mid P \in \Theta\}$ has the finite character (i.e. for each non zero element $x \in K$, $xD_P = D_P$ for almost all the D_P's). Then the semistar operation $\wedge := \wedge_{\Theta,\{*_P\}} := \wedge\{*_P \mid P \in \Theta\}$ is a finite type semistar operation on D.*

Proof. Let $E \in \overline{\boldsymbol{F}}(D)$, recall that $E^\wedge := \cap\{(ED_P)^{*_P} \mid P \in \Theta\}$. We want to show that if $x \in E^\wedge$ then there exists $F \subseteq E$, with $F \in \boldsymbol{f}(D)$, such that $x \in F^\wedge$. By the finite character condition, we may assume that $xD_P = D_P$, for all $P \in \Theta \setminus \{P_1, P_2, \dots, P_r\}$. Since $x \in \cap\{(ED_P)^{*_P} \mid P \in \Theta\}$, by the finiteness condition on the $*_P$'s, we can find $F_i \subseteq E$, with $F_i \in \boldsymbol{f}(D)$, such that $xD_{P_i} \subseteq (F_iD_{P_i})^{*_{P_i}} \subseteq (ED_{P_i})^{*_{P_i}}$, for $1 \leq i \leq r$, and $(F_iD_P)^{*_P} = (D_P)^{*_P}$, for each $P \in \Theta \setminus \{P_1, P_2, \dots, P_r\}$ and each $1 \leq i \leq r$.
If we set $F := F_1 + F_2 + \dots + F_r$, then we have that $F \subseteq E$, with $F \in \boldsymbol{f}(D)$, such that $FD_P = D_P = xD_P$, for each $P \in \Theta \setminus \{P_1, P_2, \dots, P_r\}$, and $(FD_{P_i})^{*_{P_i}} \supseteq (F_iD_{P_i})^{*_{P_i}} \supseteq xD_{P_i}$, for $1 \leq i \leq r$. We conclude that $x \in F^\wedge = \cap\{(FD_P)^{*_P} \mid P \in \Theta\}$. ☐

Corollary 4.5. *Let D be an integral domain, let Θ be a nonempty subset of $\mathrm{Spec}(D)$ and let $\{*_P \mid P \in \Theta\}$ be a family of semistar operations, where $*_P$ is a semistar operation on the localization D_P of D at $P \in \Theta$. We can associate to the semistar operation $\wedge_\Theta := \wedge_{\Theta,\{*_P\}} := \wedge\{*_P \mid P \in \Theta\}$ (defined on D) two semistar operations (both defined on D): $\widetilde{\wedge_\Theta}$ and $\wedge\{\widetilde{*_P} \mid P \in \Theta\} =: \wedge_{\Theta,\{\widetilde{*_P}\}} := \wedge_{\widetilde{\Theta}}$. Assume that the family $\{D_P \mid P \in \Theta\}$ has the finite character, then:*

$$\widetilde{\wedge_\Theta} = \wedge_{\widetilde{\Theta}}.$$

Proof. By the previous Lemma 4.4, $\wedge_{\widetilde{\Theta}}$ is a finite type semistar operation on D, since $\widetilde{*_P}$ is a finite type semistar operation on D, because $\widetilde{*_P}$ is a finite type semistar operation on D_P, for each $P \in \Theta$, and the family $\{D_P \mid P \in \Theta\}$ has the finite character.

Note that $\widetilde{*_P}$ is a spectral semistar operation on D_P, defined by the subset $\Delta_P := \mathrm{Spec}^{(*_P)_f}(D_P)$. In this situation, we know from Corollary 4.3 and Lemma 4.2 that $\wedge_{\widetilde{\Theta}}$ is a spectral semistar operation on D defined by the set $\Delta := \cup\{\Delta_P \mid P \in \Theta\}$ ($\subseteq \mathrm{Spec}(D)$), i.e. $\wedge_{\widetilde{\Theta}} = \star_\Delta$. Therefore, we deduce that $\widetilde{\wedge_{\widetilde{\Theta}}} = \wedge_{\widetilde{\Theta}}$, since $\wedge_{\widetilde{\Theta}}$ is a finite type stable semistar operation [4, Corollary 3.9 (2)]. On the other hand, $\wedge_{\widetilde{\Theta}} \leq \wedge_\Theta$ (because $\widetilde{*_P} \leq *_P$, for each $P \in \Theta$), thus we have also that $\wedge_{\widetilde{\Theta}} = \widetilde{\wedge_{\widetilde{\Theta}}} \leq \widetilde{\wedge_\Theta}$. Moreover, for each $Q \in \Delta$, $QD_P \subseteq (QD_P)^{(*_P)_f} \cap D_P \neq D_P$, for some $P \in \Theta$. Since $(\wedge_\Theta)_f \leq (*_P)_f$, then $Q \subseteq Q^{(\wedge_\Theta)_f} \cap D \neq D$. From this fact, we deduce that $\widetilde{\wedge_\Theta} \leq \star_\Delta = \wedge_{\widetilde{\Theta}}$ and so we conclude that $\widetilde{\wedge_\Theta} = \wedge_{\widetilde{\Theta}}$. ☐

Theorem 4.6. *Let D be an integral domain, let Θ be a nonempty subset of $\mathrm{Spec}(D)$ and let $\{*_P \mid P \in \Theta\}$ be a family of spectral semistar operations, where $*_P$ is a semistar operation on the localization D_P of D at $P \in \Theta$, defined by a subset $\Delta_P \subseteq \mathrm{Spec}(D_P)$. Set $\underset{\cdot}{\Delta}_P := \{Q \in \mathrm{Spec}(D) \mid QD_P \in \Delta_P\}$ and set $\wedge := \wedge_{\Theta,\{*_P\}} :=\wedge_\Theta$. Assume that the family of spectral semistar operations $\{*_P \mid P \in \Theta\}$ satisfies the following condition:*

(↓) *for each pair of prime ideals P, $P' \in \Theta$, with $P' \neq P$, then*
$$\underset{\cdot}{\Delta}_{P'} \cap P^{\downarrow} \subseteq \underset{\cdot}{\Delta}_P .$$
Set $\Delta := \{Q \in \mathrm{Spec}(D) \mid QD_P \in \Delta_P,$ for some $P \in \Theta\}$. Then, the spectral semistar operation $\star := \star_\Delta$ on D verifies the following properties:

- **(a)** *for each $P \in \Theta$, $\star_P = *_P$ (where, as usual, $\star_P := \dot{\star}^{D_P}$);*
- **(b)** *$(\star_\Delta =)\ \star = \wedge\ (= \wedge_\Theta)$ (hence, in particular, $\wedge_P = \star_P = *_P$, for each $P \in \Theta$).*

Proof. (a) Fix $P \in \Theta$ and, to avoid the trivial case, assume that $P \neq (0)$. Set

$$\begin{aligned} \Theta_0 &:= \{P' \in \Theta \mid P' \cap P \text{ does not contain a nonzero prime ideal}\}, \quad \text{and} \\ \Theta_1 &:= \{P'' \in \Theta \mid P'' \cap P \text{ contains a nonzero prime ideal}\}. \end{aligned}$$

Note that if $P' \in \Theta_0$, then $D_P D_{P'}$ coincides necessarily with K, the quotient field of D; note also that P belongs to Θ_1.

Assume that $P'' \in \Theta_1$. We know that there is a bijective correspondence between prime ideal of $D_P D_{P''}$ and the set $\{H \in \mathrm{Spec}(D) \mid H \subseteq P'' \cap P\}$, hence $D_P D_{P''} = \cap\{D_H \mid H \subseteq P'' \cap P$ and $H \in \mathrm{Spec}(D)\} \subsetneq K$. Therefore, by assumption, for each $P'' \in \Theta_1$, the set $\underset{\cdot}{\Delta}_{P''} \cap P^{\downarrow} \subseteq \underset{\cdot}{\Delta}_P$ and so, for each $G \in \overline{\boldsymbol{F}}(D_P)$, $(GD_{P''})^{*_{P''}} \supseteq (GD_P)^{*_P} = G^{*_P}$. Henceforth, for each nontrivial $G \in \overline{\boldsymbol{F}}(D_P)$, we have $K \supsetneq \cap\{(GD_{P''})^{*_{P''}} \mid P'' \in \Theta_1\} = (GD_P)^{*_P} = G^{*_P}$; therefore:

$$\begin{aligned} G^{\star_P} &= G^{\star} = G^{\star_\Delta} = \\ &= \cap\{\cap\{GD_Q \mid QD_H \in \Delta_H\} \mid H \in \Theta\} = \\ &= \cap\{\cap\{GD_Q \mid Q \in \Delta, Q \subseteq H\} \mid H \in \Theta\} = \\ &= \cap\{(GD_H)^{*_H} \mid H \in \Theta\} = \\ &= (\cap\{(GD_{P'})^{*_{P'}} \mid P' \in \Theta_0\}) \cap (\cap\{(GD_{P''})^{*_{P''}} \mid P'' \in \Theta_1\}) = \\ &= K \cap (\cap\{(GD_{P''})^{*_{P''}} \mid P'' \in \Theta_1\}) = \\ &= G^{*_P}. \end{aligned}$$

(b) If $E \in \overline{\boldsymbol{F}}(D)$, then

$$\begin{aligned} E^{\star_\Delta} &= \cap\{\cap\{ED_Q \mid QD_P \in \Delta_P\} \mid P \in \Theta\} = \\ &= \cap\{\cap\{(ED_P)D_Q \mid QD_P \in \Delta_P\} \mid P \in \Theta\} = \\ &= \cap\{(ED_P)^{*_P} \mid P \in \Theta\} = \\ &= E^{\wedge}. \end{aligned}$$

□

Next example shows that condition (↓) does not hold in general. Later (Example 4.13), we will give an example for which condition (↓) holds.

Example 4.7. *Let D be an integral domain and let Θ be a nonempty subset of $\mathrm{Spec}(D)$ and let $\{*_P \mid P \in \Theta\}$ be a family of semistar operations, where $*_P$ is a semistar operation on the localization D_P of D at $P \in \Theta$. Let $\Delta_P := \mathcal{Q}((*_P)_f) = \mathrm{Spec}^{(*_P)_f}(D_P)$ be the set of all the quasi–$(*_P)_f$–prime* *ideals of* D_P, *for each* $P \in \Theta$. *The family of spectral semistar operations* $\{\widetilde{*_P} \mid P \in \Theta\}$ *does not verify condition* $(\downarrow)$.

For instance, let D be a domain with two incomparable prime ideals P_1 and P_2 containing a common nonzero prime ideal Q. Let $*_{P_1} := d\ (= d_{D_{P_1}})$ be the identical semistar operation on D_{P_1}, and let $*_{P_2} := e\ (= e_{D_{P_2}})$ be the trivial semistar operation on D_{P_2}. We have that $*_{P_1}$ and $*_{P_2}$ are both finite type stable semistar operations (i.e. $*_{P_1} = \widetilde{*_{P_1}}$ and $*_{P_2} = \widetilde{*_{P_2}}$), with $\underset{\cdot}{\Delta}_{P_1} = \{P \in \mathrm{Spec}(D) \mid P \subseteq P_1\}$ and $\underset{\cdot}{\Delta}_{P_2} = \{(0)\}$. The ideal Q produces a counterexample to condition $(\downarrow)$. Indeed $Q \in \underset{\cdot}{\Delta}_{P_1} \cap P_2^{\downarrow}$ and $Q \notin \underset{\cdot}{\Delta}_{P_2}$.

Moreover, set $\Theta := \{P_1, P_2, Q\}$ and $\Theta' := \{P_1, P_2\}$. Let $*_{P_1}$ and $*_{P_2}$ be as above and let $*_Q := e\ (= e_{D_Q})$ (thus $*_Q = \widetilde{*_Q}$ is also a finite type stable semistar operation). Note that $\Delta_{P_1} = \{PD_{P_1} \in \mathrm{Spec}(D_{P_1}) \mid P \subseteq P_1\}$, $\Delta_{P_2} = \{(0) \in \mathrm{Spec}(D_{P_2})\}$ and $\Delta_Q = \{(0) \in \mathrm{Spec}(D_Q)\}$. In this situation, $\Delta := \underset{\cdot}{\Delta}_{P_1} \cup \underset{\cdot}{\Delta}_{P_2} \cup \underset{\cdot}{\Delta}_Q = \{P \in \mathrm{Spec}(D) \mid P \subseteq P_1\}$. Therefore, it is easy to see that $\wedge_\Theta = \wedge_{\Theta'}$ and it coincides with the finite type spectral semistar operation $\star := \star_\Delta$, but $(\wedge_\Theta)_Q = \star_Q = d_Q \lneq e_Q = *_Q$.

Theorem 4.8. *Let* D *be an integral domain, let* Θ *be a nonempty subset of* $\mathrm{Spec}(D)$ *and let* $\{*_P \mid P \in \Theta\}$ *be a family of spectral semistar operations, where* $*_P$ *is a semistar operation on the localization* D_P *of* D *at* $P \in \Theta$, *defined by a subset* $\Delta_P \subseteq \mathrm{Spec}(D_P)$. *Set* $\wedge := \wedge_{\Theta,\{*_P\}}$. *Assume that* $\{*_P \mid P \in \Theta\}$ *satisfies the condition* $(\downarrow)$ *and that* $*_P$ *is an e.a.b. [respectively, a.b.] semistar operation on* D_P. *Then the spectral semistar operation* $\wedge$ *(Theorem 4.6 (b)) is also an e.a.b. [respectively, a.b.] semistar operation on* D.

Proof. Note that from the previous Theorem 4.6, we have that $\wedge_P = *_P$, for each $P \in \Theta$. Let $F, G, H \in \boldsymbol{f}(D)$ and suppose that $(FG)^\wedge \subseteq (FH)^\wedge$. Then, for each $P \in \Theta$, we have $(FD_PGD_P)^{*_P} = (FGD_P)^{*_P} = ((FG)^\wedge D_P)^{*_P} \subseteq ((FH)^\wedge D_P)^{*_P} = (FD_PHD_P)^{*_P}$. Therefore, for each $P \in \Theta$, from the e.a.b. hypothesis on $*_P$ we have $(GD_P)^{*_P} \subseteq (HD_P)^{*_P}$. We conclude immediately, since we have that $G^\wedge = \cap\{(GD_P)^{*_P} \mid P \in \Theta\} \subseteq \cap\{(HD_P)^{*_P} \mid P \in \Theta\} = H^\wedge$. A similar argument shows the a.b. case. □

We apply next the previous theory to the case of the finite type stable (semi)star operation $w := \widetilde{v}$ canonically associated to the (semi)star operation v.

Corollary 4.9. *Let* D *be an integral domain. For each* $P \in \mathrm{Spec}(D)$, *let* $w_{D_P} := \widetilde{v_{D_P}}$ *be the finite type spectral (semi)star operation on* D_P, *defined by the set* $\mathrm{Spec}^{t_{D_P}}(D_P)$ *of all the* t*–prime ideals of* D_P. *If*

$$\underset{w}{\bar{\wedge}} := \wedge\{\underset{\cdot}{w}_{D_P} \mid P \in \mathrm{Spec}(D)\}$$

then $\underset{w}{\bar{\wedge}}$ *is a spectral (semi)star operation on* D *defined by the following set of prime ideals of* D:

$$\Upsilon := \cup\big\{\,\{Q \in \mathrm{Spec}(D) \mid QD_P \in \mathrm{Spec}^{t_{D_P}}(D_P)\,\}\ \mid\ P \in \mathrm{Spec}(D)\big\}\,,$$

i.e. $\underset{w}{\bar{\wedge}} = \star_\Upsilon$.

Proof. This statement is a particular case of Corollary 4.3. □

At this point, it is natural to investigate the relationship between the spectral (semi)star operation $\overline{\wedge}_w$ (considered in the previous Corollary 4.9) and the finite type spectral (semi)star operation, $w_D := \widetilde{v_D}$, on D defined by the set $\operatorname{Spec}^{t_D}(D)$ of all the t–prime ideals of D. We will see that, in general, they are different.

Lemma 4.10. *Let D be an integral domain. For any prime ideal P of D, we denote by t_P the semistar operation $(t_D)_P$ of D_P (defined by $E^{t_P} := E^{t_D} = \cup\{F^{v_D} = (D:(D:F)) \mid F \subseteq E$ and $F \in \boldsymbol{f}(D)\}$, for each $E \in \overline{\boldsymbol{F}}(D_P)$). Let P be a prime ideal of D, we have that $P \in \operatorname{Spec}^{t_D}(D)$ if and only if $PD_P \in \operatorname{Spec}^{t_P}(D_P)$.*
In addition, for any prime ideal P, we have $\operatorname{Spec}^{t_{D_P}}(D_P) \subseteq \operatorname{Spec}^{t_P}(D)$.

Proof. ($\Rightarrow$). Assume that, for some $P \in \operatorname{Spec}^{t_D}(D)$, we have $(PD_P)^{t_P} \supsetneq PD_P$. Then there exists $z \in (PD_P)^{t_P} = (PD_P)^{t_D} = \cup\{F^{v_D} \mid F \subseteq PD_P$ and $F \in \boldsymbol{f}(D)\}$, but $z \notin PD_P$. Hence, for some $F \subseteq PD_P$ and $F \in \boldsymbol{f}(D)$, $z \in F^{v_D} \setminus PD_P$. Since F is finitely generated and $F \subseteq PD_P$ then, for some $b \in D \setminus P$, we have that $bF \subseteq PD_P \cap D = P$. Therefore, $bz \in bF^{v_D} = (bF)^{v_D} \subseteq P^{t_D} = P$, thus $z \in b^{-1}P \subseteq PD_P$, which is a contradiction.

($\Leftarrow$). Assume that $(PD_P)^{t_P} = PD_P$. Note that $PD_P = (PD_P)^{t_P} = (PD_P)^{t_D} \supseteq P^{t_D}$. Henceforth, $P = PD_P \cap D \supseteq P^{t_D} \cap D = P^{t_D}$, hence $P = P^{t_D}$.

For the final statement we proceed as follows. Let $PD_P \in \operatorname{Spec}^{t_{D_P}}(D_P)$ and let F be a finitely generated ideal of D contained in P, then $F^{v_D} \subseteq (FD_P)^{v_D} \subseteq (FD_P)^{v_{D_P}} \subseteq (PD_P)^{t_{D_P}} = PD_P$. Therefore $P^{t_D} \subseteq PD_P$ and so $P^{t_D} = P$. □

Remark 4.11. The same proof given above (Lemma 4.10) shows the following general statement: *Let P, Q be two prime ideals of an integral domain D, then $PD_Q \in \operatorname{Spec}^{t_Q}(D_Q)$, for each prime ideal Q, with $P \subseteq Q$, if and only if $P \in \operatorname{Spec}^{t_D}(D)$.*

Remark 4.12. We emphasize that, in general, the semistar operation t_P does not coincide with the (semi)star operation t_{D_P}, i.e. *t_P is not the t–operation on D_P*. For a prime t_D–ideal P of D, the question of when the extended ideal PD_P is a t_{D_P}–ideal was studied by M. Zafrullah in [20] and [21] (where the t_D–primes P of D such that PD_P is a t_{D_P}–ideal were called *well behaved prime t–ideals*).

For instance, if P is not a well behaved prime t–ideal of D, then necessarily $PD_P = (PD_P)^{t_P} \subsetneq (PD_P)^{t_{D_P}}$.

Using the same argument of the proof of the last statement of Lemma 4.10, note that, if PD_Q is a t_{D_Q}–ideal, for some prime ideal Q containing P) then P is a t–ideal of D. Therefore, using Remark 4.11, we have: if $Q \in \operatorname{Spec}(D)$ satisfies $Q \supseteq P$ and $PD_Q \in \operatorname{Spec}^{t_{D_Q}}(D_Q)$, then $P \in \operatorname{Spec}^{t_D}(D)$, and this happens if and only if for any $Q \in \operatorname{Spec}(D)$, such that $Q \supseteq P$, we have $PD_Q \in \operatorname{Spec}^{t_Q}(D_Q)$.

Example 4.13. *The set of all the t–prime ideals of an integral domain D induces a "natural" example for which condition ($\downarrow$) of Theorem 4.6 holds.* For each $P \in \operatorname{Spec}(D)$, we consider on D_P the set $\Omega_P := \operatorname{Spec}^{t_P}(D_P)$. Let ω_P be the spectral semistar operation on D_P, defined by Ω_P, i.e. $\omega_P := \star_{\Omega_P}$. From Remark 4.11, we deduce immediately that $\underset{\cdot}{\Omega}_{P'} \cap P^{\downarrow} \subseteq \underset{\cdot}{\Omega}_P$, for each pair P, $P' \in \mathrm{Spec}(D)$ such that $P \neq P'$. Therefore, the family of spectral semistar operations $\{\omega_P \mid P \in \mathrm{Spec}(D)\}$ verifies condition (↓).

Corollary 4.14. *Let D be an integral domain. Let $w_D := \widetilde{v_D}$ be the finite type spectral (semi)star operation on D, defined by the set $\mathrm{Spec}^{t_D}(D)$ of all the t–prime ideals of D. For each $P \in \mathrm{Spec}(D)$, set as usual $w_P := (w_D)_P$ and let $w_{D_P} := \widetilde{v_{D_P}}$ [respectively, ω_P] be the spectral semistar operation on D_P, defined by the set $\mathrm{Spec}^{t_{D_P}}(D_P)$ [respectively, $\mathrm{Spec}^{t_P}(D_P)$]. Then:*

$$\begin{aligned} w_D = {} & \wedge\{\dot{\omega}_P \mid P \in \mathrm{Spec}^{t_D}(D)\} = \wedge\{\dot{w}_P \mid P \in \mathrm{Spec}^{t_D}(D)\} = \\ = {} & \underset{w}{\wedge} \ (:= \wedge\{\dot{w}_P \mid P \in \mathrm{Spec}(D)\}) \leq \underset{w}{\bar{\wedge}} \ (:= \wedge\{\dot{w}_{D_P} \mid P \in \mathrm{Spec}(D)\}) \,. \end{aligned}$$

Proof. From Theorem 4.6 (and Example 4.13), we have that $\wedge\{\dot{\omega}_P \mid P \in \mathrm{Spec}^{t_D}(D)\}$ is the spectral semistar operation $\star_\Omega$, where $\Omega := \{Q \in \mathrm{Spec}(D) \mid QD_P \in \mathrm{Spec}^{t_P}(D_P)$, for some $P \in \mathrm{Spec}^{t_D}(D)\}$. It is easy to see that $\Omega = \mathrm{Spec}^{t_D}(D)$, hence $w_D = \star_\Omega = \wedge\{\dot{\omega}_P \mid P \in \mathrm{Spec}^{t_D}(D)\}$. Moreover, again from Theorem 4.6, we have that $w_P = (\star_\Omega)_P = \omega_P$, for each $P \in \mathrm{Spec}^{t_D}(D)$, hence $\wedge\{\dot{w}_P \mid P \in \mathrm{Spec}^{t_D}(D)\} = \wedge\{\dot{\omega}_P \mid P \in \mathrm{Spec}^{t_D}(D)\} = w_D$. The last inequality in the statement is a consequence of Corollary 4.9, since by Lemma 4.10 $\Upsilon \subseteq \Omega$ and thus $w_D = \star_\Omega \leq \star_\Upsilon = \underset{w}{\bar{\wedge}}$. □

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