Nagata Transforms and Localizing Systems

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Abstract. The main purpose of this paper is to undertake a more in-depth study of the link between Nagata ideal transforms and the rings of fractions with respect to localizing systems. The principal result obtained here is a characterization of when the Nagata ideal transform $T(I)$ of an ideal I of an integral domain R is a ring of fractions of R with respect to a localizing system; in this case, we give also a description of the localizing system $\mathcal{F}$ of R such that $T(I) = R_{\mathcal{F}}$.

1. Introduction and preliminary results

Let R be an integral domain with quotient field K. For each ideal I of R, the following overring of R

$$(1.1) \quad T_R(I) = T(I) := \bigcup_{n \geq 0} (R : I^n) = \{z \in K : (R :_R zR) \supseteq I^n \text{ for some } n \geq 0\}$$

is called the *Nagata (ideal) transform of R with respect to I.*

Ideal transforms are a very useful tool in various aspects of commutative ring theory. A particularly important application was the treatment of Hilbert's XIV$^{\text{th}}$ Problem given by Nagata in [N1], [N2] and [N3]. Nagata transforms were also employed in the study of the catenary chain conjectures (cf. [R1] and [R2]) and Brewer, Brewer–Gilmer and Hedstrom have proved that Nagata transforms are very useful in the study of the overrings of an integral domain (cf. [Br], [BrG], [He1] and [He2]).

Other interesting applications of the ideal transforms are mentioned in [F], where an ample bibliography is listed.

Outside the Noetherian setting, the behaviour of Nagata transform is not entirely satisfactory [F]. Kaplansky in [K] considered, for each ideal I of R, the following overring of R:

$$\Omega_R(I) = \Omega(I) := \{z \in K : \ \mathrm{rad}((R :_R zR)) \supseteq I\} \tag{1.2}$$

called (in [F]) the *Kaplansky (ideal) transform of R with respect to I*. It is straightforward that, for each ideal I, $T(I) \subseteq \Omega(I)$ and, if I is finitely generated, then $T(I) = \Omega(I)$.

A *generalized multiplicative system* (or a *multiplicative system of ideals*) $\mathcal{S}$ of R is a multiplicatively closed set of ideals of R. The following overring of R:

$$R_{\mathcal{S}} := \{x \in K : xI \subseteq R \text{ for some } I \in \mathcal{S}\}$$

is called *the generalized transform* or *the (generalized) ring of fractions of R with respect to $\mathcal{S}$* (cf. [HOP], [H], [AB], [BS]). This terminology is justified by the fact that if $\mathfrak{N}(I) := \{I^n : n \geq 0\}$ then $\mathfrak{N}(I)$ is obviously a generalized multiplicative system of R and

$$T(I) = R_{\mathfrak{N}(I)} \ , \tag{1.3}$$

and by the fact that the ring of fractions $S^{-1}R$ of R with respect to a multiplicative system S of R coincides with $R_{\mathcal{S}}$, where $\mathcal{S} = \{sR \mid s \in S\}$.
Note that if $\mathcal{S}$ is a generalized multiplicative system of R then also

$$\overline{\mathcal{S}} := \{J : J \text{ ideal of } R \text{ such that } J \supseteq I \text{ for some } I \in \mathcal{S}\}$$

is a generalized multiplicative system called the *saturation of* $\mathcal{S}$ and $R_{\overline{\mathcal{S}}} = R_{\mathcal{S}}$.

A generalized multiplicative system $\mathcal{S}$ is *saturated* if $\mathcal{S} = \overline{\mathcal{S}}$.

A distinguished class of generalized multiplicative systems is given by the localizing (or topologizing) systems of ideals introduced by Gabriel [Ga] (cf. also [B, p. 157], [P], [C] and [St]). We recall that a *localizing system of ideals* $\mathcal{F}$ of R is a set of ideals of R verifying the following conditions:

(LS1) $\quad I \in \mathcal{F} \quad \text{and} \quad I \subseteq J \Rightarrow J \in \mathcal{F}$;

(LS2) $\quad I \in \mathcal{F}$, $\quad J$ ideal of R such that $(J :_R iR) \in \mathcal{F}$ for each $i \in I \Rightarrow J \in \mathcal{F}$.

For instance, for each subset Δ of prime ideals of R,

$$\mathcal{F}(\Delta) := \{I : I \text{ ideal of } R \text{ such that } I \not\subseteq P \text{ for each } P \in \Delta\}$$

is a localizing system of R. If P is a prime ideal of R, we denote simply by $\mathcal{F}(P)$ the localizing system $\mathcal{F}(\{P\})$. It is obvious that:

$$\mathcal{F}(\Delta) = \cap\{\mathcal{F}(P) : P \in \Delta\} \ .$$

A *localizing system of finite type* is a localizing system $\mathcal{F}$ such that for each $I \in \mathcal{F}$ there exists a finitely generated ideal $J \in \mathcal{F}$ with $J \subseteq I$. It is well known that if $\mathcal{F}$ is a localizing system of finite type then $\mathcal{F} = \mathcal{F}(\Phi)$, where $\Phi := \{P : P \text{ prime ideal of } R \text{ and } P \notin \mathcal{F}\}$ [FHP, Lemma 5.1.5].

The converse is not true in general, since $\mathcal{F}$ is a localizing system of finite type if and only if $\mathcal{F} = \mathcal{F}(\Delta)$ where Δ is a quasi–compact subspace of $\mathrm{Spec}(R)$ [FHP, Proposition 5.1.8].

Since a localizing system $\mathcal{F}$ is a multiplicatively closed and saturated system of ideals [FHP, Proposition 5.1.1], we can consider the generalized ring of fractions $R_{\mathcal{F}}$ of R with respect to $\mathcal{F}$. Note that it is possible to find an overring of an integral domain R which is a ring of fractions of R with respect to a generalized multiplicative system of ideals but is such that none of the multiplicative systems giving rise to this overring is a localizing system [F].

If $\mathcal{F} = \mathcal{F}(\Delta)$ for some subset Δ of prime ideals of R, then it is well known that

$$R_{\mathcal{F}} = \cap\{R_P : P \in \Delta\}$$

[FHP, Proposition 5.1.4].

For each ideal I of R, let

$$D(I) := \{P \in \ \mathrm{Spec}(R) : P \not\supseteq I\}$$

$$\mathcal{K} = \mathcal{K}(I) := \mathcal{F}(D(I)) = \{J \text{ ideal of } R \ : \ P \in \ \mathrm{Spec}(R)\ ,\ J \subseteq P \Rightarrow I \subseteq P\},$$

then Kaplansky [K] (and Hays [Ha, Theorem 1.7]) proved that

$$\Omega(I) = R_{\mathcal{K}(I)}. \tag{1.4}$$

Since, when I is finitely generated, $T(I) = \Omega(I)$, the equality (1.4) generalizes a result obtained by Brewer [Br, Theorem 1.5]. Furthermore, (1.4) provides also an extension of the following representation given by Nagata [N3, Lemma 2.4]: if I is a nonzero ideal of a Krull domain R, then

$$\begin{aligned} T_R(I) &= \cap\{R_Q :\ Q \in \mathrm{Spec}(R),\ Q \not\supseteq I \text{ and } ht(Q) = 1\} = \\ &= \cap\{R_P :\ P \in \mathrm{Spec}(R), P \not\supseteq I\} = R_{\mathcal{K}(I)}\ . \end{aligned} \tag{1.5}$$

In this situation, the ring $T_R(I)$, which is always by (1.3) a generalized ring of fractions of R with respect to the multiplicatively closed set of ideals $\mathfrak{N}(I)$, is also a ring of fractions of R with respect to a localizing system (i.e. $\mathcal{K}(I)$).

Note that the inclusion $T(I) \subseteq \Omega(I)$, that holds in general, may be also deduced from the inclusion of the system of ideals $\overline{\mathfrak{N}(I)} \subseteq \mathcal{K}(I)$. This inclusion can be proved synthetically by the fact that:

$$J \in \overline{\mathfrak{N}(I)} \Rightarrow \ \mathrm{rad}(I) = \ \mathrm{rad}(J \cap I) \tag{1.6}$$

and by the following explicit description of $\mathcal{K}(I)$:

$$\mathcal{K}(I) = \{J : J \text{ ideal of } R \text{ such that } \mathrm{rad}(I) = \ \mathrm{rad}(J \cap I)\} \tag{1.7}$$

[FHP, Remark 5.8.5 (a)].

The main purpose of this paper is to undertake a more in-depth study of the link between Nagata ideal transforms and rings of fractions with respect to localizing systems. Indeed, one of the reasons that makes Kaplansky's generalization of the ideal transform very satisfactory in the not necessarily Noetherian context is the fact that $\Omega(I) = R_{\mathcal{F}}$, where $\mathcal{F}$ is the localizing system $\mathcal{K}(I)$ (cf. also [F] and [FH]). The principal result obtained here is a characterization of when the Nagata ideal transform $T(I)$ is a ring of fractions of R with respect to a localizing system; in this case, we give also a description of the localizing system $\mathcal{F}$ of R such that $T(I) = R_{\mathcal{F}}$.

2. The Nagata localizing system

Let R be an integral domain and K its quotient field.

We call the *Nagata localizing system associated to an ideal I of R*, and we denote it by $\mathcal{N}(I)$, the smallest localizing system of R containing I. Obviously $\mathcal{N}(I)$ contains the saturation of the multiplicatively closed system of ideals $\mathfrak{N}(I)$ considered in Section 1.

Proposition 2.1. *Let R be an integral domain. For each ideal I of R, we set*

$$\mathcal{N}_0(I) := \{J : J \text{ ideal of } R\ , \quad J \supseteq I^n \text{ for some } n \geq 1\}$$

and, for each ordinal number α, we define by transfinite induction a set of ideals $\mathcal{N}_\alpha(I)$ of R in the following way:

– if α is not a limit ordinal, i.e. $\alpha = \beta + 1$ for some ordinal number $\beta \geq 0$,

$\mathcal{N}_\alpha(I) := \{J : J$ ideal of R such that there exists $J_\beta \in \mathcal{N}_\beta(I)$ with $(J :_R xR) \in \mathcal{N}_\beta(I)$, for each $x \in J_\beta\ \}$;

– if α is a limit ordinal,

$$\mathcal{N}_\alpha(I) := \cup_{\beta<\alpha}\mathcal{N}_\beta(I)\ .$$

Then, for each ordinal number $\alpha \geq 0$,

(a) *The set $\mathcal{N}_\alpha(I)$ is a saturated multiplicative system of ideals of R;*

(b) *The Nagata localizing system $\mathcal{N}(I)$ coincides with $\cup_\alpha\mathcal{N}_\alpha(I)$.*

Proof. (a) We want to show that the set $\mathcal{N}_\alpha(I)$ verifies the following properties:

(1) If $\alpha \geq \beta \geq 0$, then $\mathcal{N}_\alpha(I) \supseteq \mathcal{N}_\beta(I)$;

(2) $J \in \mathcal{N}_\alpha(I)$, J' an ideal of R with $J \subseteq J' \ \Rightarrow\ J' \in \mathcal{N}_\alpha(I)$;

(3) $J, J' \in \mathcal{N}_\alpha(I) \ \Rightarrow\ JJ' \in \mathcal{N}_\alpha(I)$.

(1) is obvious, since if $J \in \mathcal{N}_\beta(I)$, then $(J :_R xR) = R$, for each $x \in J \in \mathcal{N}_\beta(I)$.

For (2) and (3) it is sufficient to show the statements when $\alpha = 0$ and, by transfinite induction, when α is not a limit ordinal (i.e. $\alpha = \beta + 1$, for some ordinal number $\beta \geq 0$).

The case $\alpha = 0$ is obvious, since $\mathcal{N}_0(I)$ is the saturation of the multiplicative ideal system $\mathfrak{N}(I)$.

Assume that for a given ordinal number $\beta \geq 0$, $\mathcal{N}_\beta(I)$ verifies (2) and (3). It is easy to see that property (2) is verified also by $\mathcal{N}_{\beta+1}(I)$, by the definition of $\mathcal{N}_{\beta+1}(I)$.

For $1 \leq k \leq 2$, let J_k ideal of $\mathcal{N}_{\beta+1}(I)$ hence:

- there exist $J_{k,\beta} \in \mathcal{N}_\beta(I)$ and
- $(J_k :_R x_k R) \in \mathcal{N}_\beta(I)$, for each $x_k \in J_{k,\beta}$.

Since

$$(J_1 :_R x_1R)(J_2 :_R x_2R) \in \mathcal{N}_\beta(I) \text{ and } (J_1 :_R x_1R)(J_2 :_R x_2R) \subseteq (J_1J_2 :_R x_1x_2R),$$

then we have:

$$(J_1J_2 :_R x_1x_2R) \in \mathcal{N}_\beta(I), \quad \text{for all } x_1x_2 \in J_{1,\beta}J_{2,\beta}.$$

By the fact that the elements of the type x_1x_2 generate $J_{1,\beta}J_{2,\beta}$, we can conclude easily that $J_1J_2 \in \mathcal{N}_{\beta+1}(I)$.

(b) Set $\mathcal{N}_*(I) := \cup_\alpha \mathcal{N}_\alpha(I)$. Note that, for each ordinal number α, $\mathcal{N}_\alpha(I)$ satisfies **(LS1)** hence also $\mathcal{N}_*(I)$ satisfies **(LS1)**.

Furthermore, from the definitions, it follows easily that $I \in \mathcal{N}_*(I) \subseteq \mathcal{N}(I)$, in order to conclude we show that $\mathcal{N}_*(I)$ satisfies also **(LS2)**, i.e. $\mathcal{N}_*(I)$ is a localizing system of R.

Let $L \in \mathcal{N}_*(I)$ and let H be an ideal of R such that $(H :_R xR) \in \mathcal{N}_*(I)$, for each $x \in L$. Therefore $(H :_R xR) \in \mathcal{N}_{\alpha_x}(I)$ for some ordinal number α_x depending on $x \in L$, where $L \in \mathcal{N}_\beta(I)$, for some ordinal number $\beta \geq 0$. Hence, if $\overline{\alpha} := \sup\{\alpha_x : x \in L; \beta\}$, then $(H :_R xR) \in \mathcal{N}_{\overline{\alpha}}(I)$ for each $x \in L$, with $L \in \mathcal{N}_{\overline{\alpha}}(I)$, and thus $H \in \mathcal{N}_{\overline{\alpha}+1}(I) \subseteq \mathcal{N}_*(I)$. $\square$

Proposition 2.2. *With the notation introduced above,*

$$J \in \mathcal{N}(I) \Rightarrow \ rad(J \cap I) = \ rad(I)\ .$$

Proof. The statement is obvious if $J \in \mathcal{N}_0(I)$ (cf. also (1.6)). By using Proposition 2.1 and transfinite induction, it is sufficient to show that if the statement holds for each $J \in \mathcal{N}_\beta(I)$, when $0 \leq \beta \leq \alpha$, then the statement holds for each $J \in \mathcal{N}_{\alpha+1}(I)$. We can assume that $J \in \mathcal{N}_{\alpha+1}(I)$ and $J \not\supseteq I$ (because if $I \subseteq J$ then $J \in \mathcal{N}_0(I)$). Then there exists $J_\alpha \in \mathcal{N}_\alpha(I)$ with $(J :_R xR) \in \mathcal{N}_\alpha(I)$ for each $x \in J_\alpha$.

Let P be a prime ideal containing $J \cap I$. Assume that $P \not\supseteq I$, hence $P \supseteq J$. By induction $\mathrm{rad}((J :_R xR) \cap I) = \ \mathrm{rad}(I)$, for each $x \in J_\alpha$. Note that $x(J :_R xR) \subseteq J$ for each $x \in J_\alpha$, hence also $x((J :_R xR) \cap I) \subseteq J \subseteq P$ for each $x \in J_\alpha \cap I$. Two cases are possible.

Case 1: $(J :_R xR) \cap I \subseteq P$, for some $x \in J_\alpha \cap I$. In this case $I \subseteq P$ because $\mathrm{rad}((J :_R xR) \cap I) = \ \mathrm{rad}(I)$ and, hence, we reach a contradiction.

Case 2: $x \in P$, for each $x \in J_\alpha \cap I$. Since $J_\alpha \in \mathcal{N}_\alpha(I)$, then $\mathrm{rad}(J_\alpha \cap I) = \ \mathrm{rad}(I)$. In this situation, we have that $P \supseteq J_\alpha \cap I$ hence $P \supseteq I$: a contradiction.

We conclude that the only possible situation is the following: if a prime ideal contains $J \cap I$ then it contains I, hence $\mathrm{rad}(J \cap I) = \ \mathrm{rad}(I)$. $\square$

Remark 2.3. Recall that we have already noticed (cf. (1.4) and (1.7)) that the Kaplansky transform $\Omega_R(I)$ is the ring of fractions with respect to the localizing system $\mathcal{K}(I) = \{J : J \text{ ideal of } R \text{ such that } \mathrm{rad}(J \cap I) = \mathrm{rad}(I)\}$. By the previous proposition we have $\mathcal{N}(I) \subseteq \mathcal{K}(I)$ and hence we (re)obtain easily that $T_R(I) \subseteq R_{\mathcal{N}(I)} \subseteq R_{\mathcal{K}(I)} = \Omega_R(I)$.

For each ordinal number α, we can define by transfinite induction an overring of the Nagata transform $T_R(I)$ in the following way:

$$T_{0,R}(I) := T_R(I)$$

– if α is not a limit ordinal, i.e. $\alpha = \beta + 1$ for some ordinal number $\beta \geq 0$, then

$$T_{\alpha,R}(I) := T_{T_{\beta,R}(I)}(IT_{\beta,R}(I))$$

– if α is a limit ordinal, then

$$T_{\alpha,R}(I) := \cup_{\beta<\alpha} T_{\beta,R}(I)\ .$$

If no ambiguity occurs, we will denote simply by T_α the ring $T_{\alpha,R}(I)$, thus $T_{\alpha+1} = T(IT_\alpha)$.

For each ordinal number α, we can consider the following subring of K

$$R_{\mathcal{N}_\alpha(I)} := \{x \in K : (R :_R xR) \in \mathcal{N}_\alpha(I)\}\ ,$$

since, for each ordinal number $\alpha \geq 0$, $\mathcal{N}_\alpha(I)$ is a (saturated) multiplicative system of ideals of R (Proposition 2.1 (a)).

Lemma 2.4. *With the notation introduced above, then*
(a) $T_R(I) = R_{\mathcal{N}_0(I)}$;
(b) $T_{1,R}(I) = R_{\mathcal{N}_1(I)}$;
(c) *for each ordinal number* $\alpha \geq 2$, $T_{\alpha,R}(I) \subseteq R_{\mathcal{N}_\alpha(I)}$;
(d) $R_{\mathcal{N}(I)} = \cup_\alpha R_{\mathcal{N}_\alpha(I)}$.

Proof. (a). It is a clear since $\mathcal{N}_0(I)$ is the (multiplicative system of ideals of R) saturation of $\mathfrak{N}(I)$ considered in Section 1, and thus $T_R(I) = R_{N(I)} = R_{\mathcal{N}_0(I)}$.

(b). We denote simply by T_α the domain $T_{\alpha,R}(I)$, for each ordinal number $\alpha \geq 0$. Let $y \in T_1$, then $y(IT_0)^n \subseteq T_0$ and thus $yI^n \subseteq T_0$ for some $n \geq 0$.
Set $J := (R :_R yR)$. For each $x \in I^n$, we have $yx \in T_0$, whence $(J :_R xR) = (R :_R xyR) \in \mathcal{N}_0(I)$ (by (a)). Therefore $J = (R :_R yR) \in \mathcal{N}_1(R)$. Conversely, let $y \in K$ be such that $(R :_R yR) \in \mathcal{N}_1(I)$. We know that there exists $n \geq 0$ such that, for each $x \in I^n$, $((R :_R yR) :_R xR) \in \mathcal{N}_0(I)$. In particular, there exists $N_x \geq 0$ (depending on x) such that $(R :_R yxR) \supseteq I^{N_x}$, hence $yx \in T_0$, i.e. $yI^n \subseteq T_0$, and thus $y(IT_0)^n \subseteq T_0$. This fact implies that $y \in T_1$.

(c) is proved by transfinite induction. To avoid the trivial case, we can assume that α is not a limit ordinal, then $\alpha = \beta + 1$, for some ordinal number $\beta \geq 0$, and the proof is analogous to that of (b). More precisely, let $y \in T_\alpha = T_{\beta+1}$ then $yI^nT_\beta \subseteq T_\beta$ for some $n \geq 0$. Set $J := (R :_R yR)$. Since $yx \in T_\beta$ for each $x \in I^n$, then using the inductive hypothesis (i.e. $T_\beta \subseteq R_{\mathcal{N}_\beta(I)}$) we have $(J :_R xR) = (R :_R yxR) \in \mathcal{N}_\beta(I)$. Since x varies in I^n and $I^n \in \mathcal{N}_0(I) \subseteq \mathcal{N}_\beta(I)$ for each $\beta \geq 0$, then $J \in \mathcal{N}_{\beta+1}(I) = \mathcal{N}_\alpha(I)$, thus $y \in R_{\mathcal{N}_\alpha(I)}$.

(d). It is obvious that $\cup_\alpha R_{\mathcal{N}_\alpha(I)} \subseteq R_{\mathcal{N}(I)}$. Conversely, if $x \in R_{\mathcal{N}(I)}$ then there exists an ordinal number α and an ideal $J_\alpha \in \mathcal{N}_\alpha(I)$ such that $J_\alpha \subseteq (R :_R xR)$. Therefore $(R :_R xR) \in \mathcal{N}_\alpha(I)$ and hence $x \in R_{\mathcal{N}_\alpha(I)}$. $\square$

Theorem 2.5. *Let I be an ideal of an integral domain R. The following are equivalent:*

(i) *there exists a localizing system $\mathcal{F}$ of R such that $T_R(I) = R_{\mathcal{F}}$;*
(ii) $T_R(I) = R_{\mathcal{N}(I)}$;
(iii) $T_R(I) = T_{1,R}(I)$.

Proof. (i) $\Rightarrow$ (ii). The set $\mathbf{LS}_R(T_R(I)) := \{\mathcal{G} : \mathcal{G}$ localizing system of R such that $R_{\mathcal{G}} = T_R(I)\}$ ordered under set–theoretic inclusion has a first element $\overline{\mathcal{G}}$ [FHP, Lemma 5.1.19]. More precisely, $\overline{\mathcal{G}} = \cup_\alpha \mathcal{G}_\alpha$ where $\mathcal{G}_0 := \{J : J$ ideal of R such that $J \supseteq (R :_R yR)$ for some $y \in T_R(I)\}$ and if α is not a limit ordinal, i.e. $\alpha = \beta + 1$ for some ordinal β, then

$\mathcal{G}_\alpha := \{J$ ideal of R such that there exists an ideal J' of R and $J'' \in \mathcal{G}_\beta$ with $J \supseteq J'$ and $(J' :_R y''R) \in \mathcal{G}_\beta$ for each $y'' \in J''\}$;

if α is a limit ordinal, then

$$\mathcal{G}_\alpha := \cup_{\beta<\alpha} \mathcal{G}_\beta \ .$$

Note that if $y \in T_R(I)$, then $y \in (R : I^{n_y})$ for some $n_y \geq 0$, thus $(R :_R yR) \supseteq I^{n_y}$. From this fact we deduce that $J \in \mathcal{G}_0$ implies that J contains a power of the ideal I, hence $\mathcal{G}_0 \subseteq \mathcal{N}_0(I)$. Therefore, by transfinite induction and by Proposition 2.1, it follows that $\overline{\mathcal{G}} \subseteq \mathcal{N}(I)$. We can conclude that $\overline{\mathcal{G}} = \mathcal{N}(I)$ by the minimality property of $\mathcal{N}(I)$, thus $T_R(I) = R_{\overline{\mathcal{G}}} = R_{\mathcal{N}(I)}$.

(ii) $\Rightarrow$ (iii) follows from Lemma 2.4, since $T_0 \subseteq T_1 \subseteq R_{\mathcal{N}(I)}$.

(iii) $\Rightarrow$ (i). For each ordinal number α, we denote simply by T_α the ring $T_{\alpha,R}(I)$. In the present situation, for each ordinal number $\alpha \geq 0$, obviously we have

$$T_\alpha = T_0 \ .$$

Furthermore, we claim that, for each ordinal number $\alpha \geq 0$,

$$T_\alpha = R_{\mathcal{N}_\alpha(I)} = \cup\{(R_{\mathcal{N}_\alpha(I)} : J) : J \in \mathcal{N}_\alpha(I)\} \ . \tag{2.5.1}$$

For $\alpha = 0$ (or $\alpha = 1$) the first equality in (2.5.1) holds in general by Lemma 2.4 (a) (or (b)) (without the assumption $T_0 = T_1$).

It is obvious, in general, that

$$R_{\mathcal{N}_\alpha(I)} \subseteq \cup\{(R_{\mathcal{N}_\alpha(I)} : J) : J \in \mathcal{N}_\alpha(I)\}$$

since, by definition, $x \in R_{\mathcal{N}_\alpha(I)}$ is equivalent to the fact that $J := (R :_R xR) \in \mathcal{N}_\alpha(I)$ and thus $x \in (R : (R :_R xR)) \subseteq (R_{\mathcal{N}_\alpha(I)} : J)$.

In order to show that the second equality in (2.5.1) holds when $\alpha = 0$, we take $x \in K$ such that $xJ \subseteq T_R(I)$, for some $J \in \mathcal{N}_0(I)$, and we want to prove that $x \in T_R(I)$. Without loss of generality we can assume that $J = I^n$, for some $n \geq 1$. For each $y \in I^n$ we have $xy \in T_R(I)$ and, thus, $xyI^{m_y} \subseteq R$ for some $m_y \geq 0$. Therefore $((R :_R xR) :_R yR) = (R :_R xyR) \supseteq I^{m_y}$, hence $(R :_R xR) \in \mathcal{N}_1(I)$ and thus we conclude that $x \in T_{1,R}(I) = T_R(I)$.

Let $\alpha \geq 1$. We can assume by transfinite induction that (2.5.1) holds for each ordinal number $\beta < \alpha$.

If α is a limit ordinal then, by the definitions, it is obvious that (2.5.1) holds also for α.

We assume that $\alpha = \beta + 1$, for some ordinal number $\beta \geq 0$, and we want to prove that the first equality in (2.5.1) holds.
Let $x \in R_{\mathcal{N}_\alpha(I)}$, hence $(R :_R xR) \in \mathcal{N}_\alpha(I)$ and, to avoid the trivial case, we can assume that $(R :_R xR) \notin \mathcal{N}_\beta(I)$. This means that there exists an ideal $J \in \mathcal{N}_\beta(I)$ such that, for each $y \in J$, we have $(R :_R xyR) = ((R :_R xR) :_R yR) \in \mathcal{N}_\beta(I)$. By the inductive hypothesis, $xy \in T_\beta = R_{\mathcal{N}_\beta(I)}$ for each $y \in J$. Therefore $x \in (R_{\mathcal{N}_\beta(I)} : J)$ with $J \in \mathcal{N}_\beta(I)$ and thus, also by the inductive hypothesis, $x \in R_{\mathcal{N}_\beta(I)} = T_\beta \subseteq T_\alpha$. By Lemma 2.4 (c) we conclude that the first equality in (2.5.1) holds, i.e. $T_\alpha = R_{\mathcal{N}_\alpha(I)}$. For the second equality in (2.5.1), let $x \in K$ be such that $xJ_\alpha \subseteq R_{\mathcal{N}_\alpha(I)}$ for some $J_\alpha \in \mathcal{N}_\alpha(I)$ and, to avoid the trivial case, we can assume that $J_\alpha \in \mathcal{N}_\beta(I)$. By definition of $\mathcal{N}_\alpha(I)$, there exists an ideal $H_\beta \in \mathcal{N}_\beta(I)$ such that $(J_\alpha :_R hR) \in \mathcal{N}_\beta(I)$, for each $h \in H_\beta$. Since $(xJ_\alpha :_R xhR) = (J_\alpha :_R hR)$, then $xh(xJ_\alpha :_R xhR) \subseteq xJ_\alpha \subseteq T_\alpha = T_\beta = R_{\mathcal{N}_\beta(I)}$, thus by the inductive hypothesis $xh \in R_{\mathcal{N}_\beta(I)}$ for each $h \in H_\beta$. Therefore $x \in (R_{\mathcal{N}_\beta(I)} : H_\beta)$ and, again by the inductive hypothesis, $(R_{\mathcal{N}_\beta(I)} : H_\beta) \subseteq R_{\mathcal{N}_\beta(I)} = T_\beta = T_\alpha = R_{\mathcal{N}_\alpha(I)}$, hence we deduce that $x \in R_{\mathcal{N}_\alpha(I)}$.

By Lemma 2.4 (d) and (2.5.1) we can conclude immediately that $T_0 = R_{\mathcal{N}(I)}$. $\square$

Example 2.6. Let k be a field and $\{X_n : n \geq 1\}$ a countable set of indeterminates over k . Let $\mathbf{S}$ be the additive monoid of all sequences of non negative integers $\mathbf{s} = (s_n : n \geq 1)$. Set

$$\mathbf{X}^{\mathbf{s}} := \prod_{n \geq 1} X_n^{s_n}.$$

In the set $\mathbf{M} := \{\mathbf{X}^{\mathbf{s}} : \mathbf{s} \in \mathbf{S}\}$ we can define in a natural way a multiplicative monoid structure. Consider $R := k[\mathbf{M}]$ the monoid ring for $\mathbf{M}$ over k, then R is an integral domain; let K be the quotient field of R. Consider the following elements of $\mathbf{S}$:

$$\mathbf{u} := (1, 2, \ldots, n-1, n, n+1, \ldots\ldots)$$

and for each $m \geq 2$,

$$\mathbf{u}(m) := (1, 2, \ldots, m-1, 1, m+1, \ldots\ldots).$$

(For $m = 1, \mathbf{u}(1) = \mathbf{u}$.)

Let $y_m := \mathbf{X}^{\mathbf{u}(m)}$ and let I be the ideal of R generated by y_m, for $m \geq 1$.

It is not difficult to see that the element $1/X_m$, which belongs to K, is inside $T_0 = T_R(I)$, for all $m \geq 1$.

If $z := \mathbf{X}^{\mathbf{u}}$, we claim that the element $1/z$, which is in K, does not belong to T_0. As a matter of fact if, for some $h \geq 1$, $(1/z)I^h \subseteq R$ then, in particular, $(1/z)y_{h+1}^h$ would belong to R. But

$$\frac{1}{z} y_{h+1}^h = \frac{1}{X_{h+1}} \prod_{\substack{n \geq 1 \\ n \neq h+1}} X_n^{hn-n}$$

does not belong to R, hence we have a contradiction.

However, $1/z$ belongs to $T_1 = T_{T_R(I)}(IT_R(I))$ because, for each $m \geq 1$, $(1/z)y_m = 1/X_m^{m-1}$ belongs to T_0, since -as we noticed before- $1/X_m \in T_0$ and T_0 is a ring. We conclude that, in the present situation, $T_0 \neq T_1$, hence, by Theorem 2.5 we deduce that $T_R(I)$ is not a ring of fractions of R with respect to some localizing system.

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