Kronecker function rings: a general approach

Marco Fontana and K. Alan Loper

Abstract

In 1994, Matsuda and Okabe introduced the notion of semistar operation. This concept extends the classical concept of star operation (cf. for instance, Gilmer's book [4]) and, hence, the related classical theory of ideal systems based on the works by W. Krull, E. Noether, H. Prüfer and P. Lorenzen from 1930's.

The purpose of this paper is to outline a general approach to the theory of Kronecker function rings with coefficients in an integral domain D, by using semistar operations defined on D. This approach leads to relax the classical restrictions on D (not necessarily integrally closed) and on the (semi)star operations $\star$ (not necessarily *endlich arithmetisch brauchbar*) and it establishes a natural bridge with the "abstract" theory of Kronecker function rings recently developed by Halter-Koch [7].

1 Introduction

The classical theory of star operations on the fractional ideals of an integral domain arises from the work of W. Krull [10], [11] (see [4], [8] and [6] for different aspects regarding this theory). One of the primary applications of Krull's theory is to construct the Kronecker function rings (abbreviated here as Kfr) associated to an integral domain in a more general context than the original one considered by L. Kronecker in 1882 [9] (see [2] for a modern presentation of Kronecker's theory). There are restrictions in the development of Krull's theory of Kronecker function rings. The "abstract" star operation used to define a Kronecker function ring for an integral domain D is assumed to have a "cancellation property" known as *e.a.b.* (= endlich arithmetisch brauchbar). Moreover, the integral domain D is assumed to be integrally closed (this actually follows if one assumes the existence of an *e.a.b.* star operation on D). In 1994, Okabe and Matsuda [16] introduced the notion of a semistar operation as a generalization of the classical notion of a star operation (cf. also [13] and [14]). The aim of this paper is to study various properties of semistar operations with the goal of describing a general approach to Kronecker function rings, pursuing the work of Okabe-Matsuda [17] and Matsuda [12]. In particular, this approach will allow us to define a Kronecker function ring associated to any semistar operation $\star$ defined on an integral domain D, without assuming that D is integrally closed or that $\star$ verifies some "cancellation property" of the type *e.a.b.*.

In the second section of the paper we give some definitions and some simple background results concerning semistar operations. Then, in the third section, we generalize the standard Kfr construction to *e.a.b.* semistar operations. Then we give an "abstract" definition of a Kfr recently introduced by Halter-Koch [7] along with some results showing that these rings have the desirable properties of the classical Kronecker function rings. In Section 4 we investigate several mechanisms for producing new semistar operations from old ones and study some relevant properties of the new semistar operations. Then, in the final section, we use the machinery developed in Section 4 in order to generalize the classical Kfr construction and to cover any semistar operation defined on an arbitrary domain. In particular, we prove that the new general construction actually yields rings in the defined class of "abstract" Kronecker function rings and we show how this "new type" of Kfr can also be viewed as a Kronecker function ring built up in a completely classical fashion, i.e. by using an e.a.b. star operation on a suitable integrally closed domain.

Many of the main results of this paper are very similar to results obtained by F. Halter-Koch in [7], but we feel that the clarity of the semistar approach taken here makes this presentation valuable.

2 Semistar operations: Preliminary results

Let D be an integral domain with quotient field K. Let $\overline{\mathbf{F}}(D)$ denote the set of all nonzero D-submodules of K and let $\mathbf{F}(D)$ be the set of all nonzero fractional ideals of D, i.e., all $E \in \overline{\mathbf{F}}(D)$ such that there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\mathbf{f}(D)$ be the set of all nonzero finitely generated

D-submodules of K. Then, obviously:

$$\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \overline{\mathbf{F}}(D).$$

Definition 2.1 A mapping

$$\star : \overline{\mathbf{F}}(D) \to \overline{\mathbf{F}}(D), \quad E \mapsto E^\star,$$

is called *a semistar operation on* D if, for all $x \in K, x \neq 0$, and $E, F \in \overline{\mathbf{F}}(D)$, the following properties hold:

($\star_1$) $(xE)^\star = xE^\star$;

($\star_2$) $E \subseteq F \ \Rightarrow\ E^\star \subseteq F^\star$;

($\star_3$) $E \subseteq E^\star$ and $E^\star = E^{\star\star}$.

Lemma 2.2 *A mapping* $\star : \overline{\mathbf{F}}(D) \to \overline{\mathbf{F}}(D)$ *is a semistar operation on an integral domain* D *if and only if* $\star$ *satisfies* ($\star_1$) *and* ($\star_2$) *and, for all* $E, F \in \overline{\mathbf{F}}(D)$,

($\star_{3'}$) $E \subseteq E^\star$ *and* $E \subseteq F^\star \ \Rightarrow\ E^\star \subseteq F^\star$.

Proof: $(\star_{3'}) \Rightarrow (\star_3)$ follows from the fact that $E^\star \subseteq E^\star$ implies that $(E^\star)^\star = E^{\star\star} \subseteq E^\star$. Since $E^\star \subseteq (E^\star)^\star$, we conclude that $E^\star = E^{\star\star}$.

$(\star_2), (\star_3) \Rightarrow (\star_{3'})$: If $E \subseteq F^\star$, then, by $(\star_2)$, $E^\star \subseteq F^{\star\star}$ and, by $(\star_3)$, $F^{\star\star} = F^\star$, and hence $E^\star \subseteq F^\star$. □

Definition 2.3 A *semistar operation* $\star$ on D is said to be

- *e.a.b. (= endlich arithmetisch brauchbar)* if for all $E, F, G \in \mathbf{f}(D)$,

$$(EF)^\star \subseteq (EG)^\star \ \Rightarrow\ F^\star \subseteq G^\star;$$

- *a.b. (= arithmetisch brauchbar)* if for all $F, G \in \overline{\mathbf{F}}(D)$ and for all $E \in \mathbf{f}(D)$:

$$(EF)^\star \subseteq (EG)^\star \ \Rightarrow\ F^\star \subseteq G^\star.$$

In the following statement we collect some properties of the semistar operation that follow directly from the definition.

Lemma 2.4 **(1)** *Let $\star$ be a semistar operation on D. Then for all $E, F \in \overline{\mathbf{F}}(D)$ and for every family $\{E_i \mid i \in I\}$ of elements of $\overline{\mathbf{F}}(D)$:*

(a) $(\sum_{i\in I} E_i)^\star = (\sum_{i\in I} E_i^\star)^\star$;

(b) $\bigcap_{i\in I} E_i^\star = (\bigcap_{i\in I} E_i^\star)^\star$, *if* $\bigcap_{i\in I} E_i^\star \neq (0)$;

(c) $(EF)^\star = (E^\star F^\star)^\star = (E^\star F)^\star = (EF^\star)^\star$;

(d) $(E : F)^\star \subseteq (E^\star : F^\star) = (E^\star : F)$.

(2) *If S is an overring of D, then $S^\star$ is an overring of D containing $D^\star$. In particular, $D^\star$ is an overring of D.*

(3) *Let $\mathcal{S} := \{S_\alpha \mid \alpha \in A\}$ be a family of overrings of D. Then the map*

$$E \mapsto E^{\star_{\mathcal{S}}} := \bigcap_{\alpha\in A} ES_\alpha, \text{ for each } E \in \overline{\mathbf{F}}(D),$$

defines a semistar operation on D, denoted by $\star_{\mathcal{S}}$. Moreover, for each $\alpha \in A$, $E^{\star_{\mathcal{S}}} S_\alpha = ES_\alpha$.

Proof: [3, Theorem 1.2] and property $(\star_5)$ after [3, Proposition 1.6]. $\square$

We list some examples of semistar operations.

Example 2.5 **(1)** Let S be an overring of an integral domain D with quotient field K. Then the map

$$E \mapsto E^{\star_{\{S\}}} := ES, \text{ for each } E \in \overline{\mathbf{F}}(D),$$

is a semistar operation. In case $S = D$, we denote by d the *trivial (identity) semistar operation* $\star_{\{D\}}$. In case $S = K$, we denote by e the *trivial (constant) semistar operation* $\star_{\{K\}}$, which associates K to each $E \in \overline{\mathbf{F}}(D)$.

(2) If $\star$ is *a star operation on* D (i.e. a map $\star : \mathbf{F}(D) \to \mathbf{F}(D)$, verifying $(\star_1), (\star_2), (\star_3)$ for all $E, F \in \mathbf{F}(D)$ and $x \in K, x \neq 0$ and, also, $(\star_0)$: $(xD)^\star = xD$, for each $x \in K, x \neq 0$), then we can associate to $\star$ a semistar operation $\star_e$ on D, by setting:

$$E^{\star_e} := \begin{cases} E^\star, \text{ for all } E \in \mathbf{F}(D), \\ K, \text{ if } E \in \overline{\mathbf{F}}(D) - \mathbf{F}(D). \end{cases}$$

The semistar operation $\star_e$ is called the *trivial extension of the star operation* $\star$. The mapping $\star \mapsto \star_e$ determines a canonical embedding from $\mathbf{Star}(D)$, *the set of all star operations on* D, into **SStar**(D), *the set of all semistar operations on* D.

An example of a semistar operation which is not trivially extended by a star operation is given by $\star_{\{S\}}$, where S is a proper overring of D such that $(D :_K S) = (0)$ and $S \neq K$.

(3) If $\star$ is a semistar operation on D and $D^\star = D$, then $\star$, restricted to $\mathbf{F}(D)$, defines a star operation on D since, for each $E \in \mathbf{F}(D)$, there exists $d \in D - \{0\}$ such that $dE \subseteq D$, thus $(dE)^\star = dE^\star \subseteq D^\star = D$, i.e. $E^\star \in \mathbf{F}(D)$.

(4) Let $\star$ be a semistar operation on D. For each $E \in \overline{\mathbf{F}}(D)$, set

$$E^{\star_f} := \bigcup\{F^\star \mid F \subseteq E \text{ and } F \in \mathbf{f}(D)\}.$$

Then $\star_f$ is a semistar operation called *the finite semistar operation associated to* $\star$. A *semistar operation* $\star$ is said to be *of finite type* if $\star = \star_f$. Since $(\star_f)_f = \star_f$, then $\star_f$ is a semistar operation of finite type.

(5) For each $E \in \overline{\mathbf{F}}(D)$, set

$$E^{-1} := (D :_K E) := \{x \in K \mid xE \subseteq D\}$$

and $E_v := (E^{-1})^{-1}$. The map $E \mapsto E_v$ defines a semistar operation on D such that $D_v = D$. By (3), this semistar operation, called the *v-semistar operation*, restricted to $\mathbf{F}(D)$, defines a star operation on D which coincides with the classical (star) v-operation introduced by E. Artin (cf. for instance [4, p. 396]). The finite semistar operation associated to the v-semistar operation is called the *t-semistar operation.*

Note that

$$E \in \overline{\mathbf{F}}(D) - \mathbf{F}(D) \quad \Leftrightarrow \quad E^{-1} = (0) \quad \Leftrightarrow \quad E_v = K\,.$$

(6) If $\mathcal{W}$ is a family of valuation overrings of D, then $\star_{\mathcal{W}}$ is called a *w-semistar operation* (associated to the given family $\mathcal{W}$ of valuation overrings of D). If $\mathcal{W}$ is the family of *all* the valuation overrings of D, then $\star_{\mathcal{W}}$ is called *the b-semistar operation on* D. If D is integrally closed then, by the classical (Krull's) theory, it is well known that $D^b = D$ [4, Theorem 19.8] and, thus, the b-semistar operation, restricted to $\mathbf{F}(D)$ (cf. (3)), defines the classical (star) b-operation [4, p. 398].

On the set **SStar**(D) of all semistar operations defined on D we can define, in a natural way, a partial ordering:

$$\star_1 \leq \star_2 \;\;:\Leftrightarrow\;\; E^{\star_1} \subseteq E^{\star_2}, \text{ for each } E \in \overline{\mathbf{F}}(D)\,,$$

and an equivalence relation:

$$\star_1 \sim \star_2 \;\;:\Leftrightarrow\;\; (\star_1)_f = (\star_2)_f\,.$$

Obviously:

$$\star_f \leq \star, \quad \star_f \sim \star, \quad \star_1 \leq \star_2 \;\Rightarrow\; (\star_1)_f \leq (\star_2)_f\,.$$

Lemma 2.6 *Let $\star_1, \star_2 \in$ **SStar**(D). The following are equivalent:*
(i) $\star_1 \leq \star_2$;
(ii) $(E^{\star_1})^{\star_2} = E^{\star_2}$*, for each* $E \in \overline{\mathbf{F}}(D)$*;*
(iii) $(E^{\star_2})^{\star_1} = E^{\star_2}$*, for each* $E \in \overline{\mathbf{F}}(D)$*.*

Proof: straightforward, cf. for instance [3, Proposition 1.6 (4)]. □

We call a D-submodule L of K *$\star$-finite* provided $L = F^\star$, for some $F \in \mathbf{f}(D)$. It is obvious that if L is $\star$-finite then $L^\star = L$. The following lemma gives some useful characterizations of the semistar e.a.b. operations (cf. also [6, Section 13.3]).

Lemma 2.7 *Let $\star$ be a semistar operation defined on an integral domain D with quotient field K. The following statements are equivalent:*
(i) *$\star$ is e.a.b.;*
(ii) *Let A, B, C be $\star$-finite D-submodules of K. Then*

$$(AB)^\star \subseteq (AC)^\star \;\Rightarrow\; B \subseteq C;$$

(iii) *Let A, C be $\star$-finite D-submodules of K. Then*

$$A \subseteq (AC)^\star \;\Rightarrow\; 1 \in C;$$

(iv) *For all A, B, $\star$-finite D-submodules of K,*

$$((AB)^\star : A) \subseteq B;$$

(v) *Let A, B, C be $\star$-finite D-submodules of K. Then*

$$(AB)^\star = (AC)^\star \;\Rightarrow\; B = C.$$

Proof: (i) $\Rightarrow$ (ii) is obvious, since $B = B^\star$ and $C = C^\star$.

(ii) $\Rightarrow$ (i). Let $(EF)^\star \subseteq (EG)^\star$, with $E, F, G \in \mathbf{f}(D)$. Set $A := E^\star$, $B := F^\star$ and $C := G^\star$. Then we have $(AB)^\star \subseteq (AC)^\star$, whence we conclude that $B \subseteq C$.

(ii) $\Rightarrow$ (iii). Let $B := D$. Then $(AB)^\star = (AD)^\star = A \subseteq (AC)^\star$, hence $D \subseteq C$.

(iii) $\Rightarrow$ (iv). Let $x \in ((AB)^\star : A)$, with $x \neq 0$, then $xA \subseteq (AB)^\star$ or, equivalently, $A \subseteq (Ax^{-1}B)^\star$. Therefore, $1 \in x^{-1}B$, thus $x \in B$.

(iv) $\Rightarrow$ (v). If $(AB)^\star = (AC)^\star$, then $AB \subseteq (AB)^\star = (AC)^\star$, hence $B \subseteq ((AC)^\star : A) \subseteq C$. On the other hand, $AC \subseteq (AC)^\star = (AB)^\star$, hence $C \subseteq ((AB)^\star : A) \subseteq B$.

(v) $\Rightarrow$ (ii). If $(AB)^\star \subseteq (AC)^\star$, then $(AC)^\star = (AB)^\star + (AC)^\star = ((AB)^\star + (AC)^\star)^\star = (AB + AC)^\star = (A(B+C))^\star$. Therefore, $C = B + C$, thus $B \subseteq C$. $\square$

We close this section by reworking from [16] some "ascent" and "descent" type properties which relate semistar operations on D with semistar operations on S, where S is an overring of D. The particular example of interest will be the case where S coincides with $D^\star$ (cf. also Lemma 2.4 (2)).

Proposition 2.8 *Let D be an integral domain and S an overring of D. Let $\star$ be a semistar operation on D and define $\alpha_S(\star) : \overline{\mathbf{F}}(S) \rightarrow \overline{\mathbf{F}}(S)$ by setting:*

$$E^{\alpha_S(\star)} := E^\star, \text{ for each } E \in \overline{\mathbf{F}}(S) (\subseteq \overline{\mathbf{F}}(D)).$$

Then

(1) *$\alpha_S(\star)$ is a semistar operation on S and, if $\star$ is of finite type on D, then $\alpha_S(\star)$ is also of finite type on S.*

(2) *$\alpha_{D^\star}(\star)$, restricted to $\mathbf{F}(D^\star)$, defines a star operation on $D^\star$.*

(3) *If $\star$ is an e.a.b. (respectively, a.b.) semistar operation on D, then $\alpha_{D^\star}(\star)$ is an e.a.b. (respectively, a.b.) semistar operation on $D^\star$.*

Proof: (1) is straightforward.

(2) follows from (1) and from Example 2.5 (3), since $(D^\star)^\star = D^\star$.

To prove (3) we choose $E, F, G \in \mathbf{f}(D^\star)$. Assume that $(EF)^{\alpha_{D^\star}(\star)} \subseteq (EG)^{\alpha_{D^\star}(\star)}$. Note that $E = E_0 D^\star$, $F = F_0 D^\star$, $G = G_0 D^\star$, for some $E_0, F_0, G_0 \in \mathbf{f}(D)$. Then:

$$\begin{aligned}(E_0F_0)^\star &= (E_0F_0D)^\star = (E_0D^\star F_0D^\star)^\star = (EF)^\star = (EF)^{\alpha_{D^\star}(\star)} \subseteq \\ &\subseteq (EG)^{\alpha_{D^\star}(\star)} = (EG)^\star = (E_0D^\star G_0D^\star)^\star = (E_0G_0D)^\star = (E_0G_0)^\star\,.\end{aligned}$$

Since $\star$ is e.a.b. on D, we deduce that $F^{\alpha_{D^\star}(\star)} = (F_0D^\star)^{\alpha_{D^\star}(\star)} = (F_0D^\star)^\star = (F_0D)^\star = F_0^\star \subseteq G_0^\star = (G_0D)^\star = (G_0D^\star)^\star = (G_0D^\star)^{\alpha_{D^\star}(\star)} = (G)^{\alpha_{D^\star}(\star)}$.
A similar argument proves the a.b. statement. □

Proposition 2.9 *Let D be an integral domain and S an overring of D. Let $\star$ be a semistar operation on S and define $\delta_D(\star) : \overline{\mathbf{F}}(D) \to \overline{\mathbf{F}}(D)$ by setting:*

$$E^{\delta_D(\star)} := (ES)^\star, \quad \textit{for each } E \in \overline{\mathbf{F}}(D)\,.$$

Then

(1) *$\delta_D(\star)$ is a semistar operation on D.*

(2) *If $\star$ is an e.a.b. (respectively, a.b.) semistar operation on S, then $\delta_D(\star)$ is an e.a.b. (respectively, a.b.) semistar operation on D.*

Proof: (1) is straightforward.

(2): Choose $E \in \mathbf{f}(D)$ and $F, G \in \mathbf{f}(D)$ (respectively, $F, G \in \overline{\mathbf{F}}(D)$). Assume that $(EF)^{\delta_D(\star)} \subseteq (EG)^{\delta_D(\star)}$. Then, we deduce that $(ESFS)^\star \subseteq (ESGS)^\star$. Since $ES \in \mathbf{f}(S)$ and $FS, GS \in \mathbf{f}(S)$ (respectively, $FS, GS \in \overline{\mathbf{F}}(S)$), the conclusion follows easily from the hypothesis on $\star$. □

Corollary 2.10 *Let D be an integral domain and S an overring of D. Consider the maps:*

$$\alpha : \mathbf{SStar}(D) \to \mathbf{SStar}(S)\,, \quad \star \mapsto \alpha_S(\star)\,,$$

$$\delta : \mathbf{SStar}(S) \to \mathbf{SStar}(D)\,, \quad \star \mapsto \delta_D(\star)\,.$$

Then $\alpha \circ \delta$ is the identity map. Moreover, the following statements are equivalent:

(i) *δ is bijective;*

(ii) *α is bijective;*

(iii) *$D = S$.*

Proof: (iii) $\Rightarrow$ (i) $\Leftrightarrow$ (ii) are obvious. (i) $\Rightarrow$ (iii): Since δ is surjective, there exists $\star \in \mathbf{SStar}(S)$ such that $\delta_D(\star)$ coincides with the d-semistar operation on D (Example 2.5 (1)). Therefore, $D = D^d = D^{\delta_D(\star)} = (DS)^\star = S^\star$. □

3 Abstract Kronecker function rings

We begin by demonstrating that the generalization of the Kfr construction from the e.a.b. star operation case to the e.a.b. semistar operation case is very straightforward.

Definition 3.1 Let X be an indeterminate over an integral domain D and let $f := \sum_{i=0}^{n} a_i X^i \in D[X]$. We denote by $c_D(f)$ (or simply $c(f)$) *the content of* f *in* D, that is the ideal of D generated by the coefficients of f:

$$c(f) := c_D(f) = \sum_{i=0}^{n} a_i D$$

If S is an overring of D, we set

$$c_S(f) := \sum_{i=0}^{n} a_i S = c(f)S\,.$$

Definition 3.2 Let $\star$ be an e.a.b. semistar operation defined on an integral domain D with quotient field K and let X be an indeterminate over K. Consider the following subset of the field of rational functions $K(X)$:

$$\mathrm{Kr}(D,\star) := \{f/g \mid f,g \in D[X]-\{0\} \text{ and } c(f)^\star \subseteq c(g)^\star\} \cup \{0\}\,.$$

We will see in a moment (Proposition 3.3 and Corollary 3.4) that this is a ring which we will call *the Kronecker function ring of* D *with respect to the e.a.b. semistar operation* $\star$.

It is very easy to see that, if $D = D^\star$, then an e.a.b. semistar operation $\star$ restricts to an e.a.b. star operation on $\mathbf{F}(D)$ and the above definition coincides with the classical definition of the Kronecker function ring of D with respect to an e.a.b. star operation. It is reasonable to ask whether such a correspondence also occurs when $D \neq D^\star$.

Proposition 3.3 *Let* D *be an integral domain and let* $\star$ *be an e.a.b semistar operation on* D*. We denote simply by* $\alpha(\star)$ *the associated e.a.b. (semi)star operation* $\alpha_{D^\star}(\star)$ *defined on* $D^\star$ *(cf. Proposition 2.8). Then:*

$$\mathrm{Kr}(D,\star) = \mathrm{Kr}(D^\star, \alpha(\star))\,.$$

Proof: Suppose that $f, g \in D[X]$ and $g \neq 0$. Then:

$$\begin{aligned} &(c_{D^\star}(f))^{\alpha(\star)} \subseteq (c_{D^\star}(g))^{\alpha(\star)} \;\Leftrightarrow\; (c_{D^\star}(f))^\star \subseteq (c_{D^\star}(g))^\star \;\Leftrightarrow \\ &\quad\Leftrightarrow\; (c_D(f)D^\star)^\star \subseteq (c_D(g)D^\star)^\star \;\Leftrightarrow (c_D(f)D)^\star \subseteq (c_D(g)D)^\star \;\Leftrightarrow \\ &\quad\Leftrightarrow\; (c_D(f))^\star \subseteq (c_D(g))^\star \;\Leftrightarrow\; f/g \in \mathrm{Kr}(D,\star)\,. \end{aligned}$$

□

Corollary 3.4 *Let D be an integral domain with quotient field K and let $\star$ be an e.a.b. semistar operation on D. Then:*

(1) $\mathrm{Kr}(D,\star)$ *is an integral domain with quotient field $K(X)$ such that $\mathrm{Kr}(D,\star) \cap K = D^\star$.*

(2) $\mathrm{Kr}(D,\star)$ *is a Bezout domain.*

(3) *For each $F \in \mathbf{f}(D)$, we have $F\mathrm{Kr}(D,\star) \cap K = F^\star$.*

(4) *$D^\star$ is integrally closed.*

Proof: These all follow easily from Proposition 3.3 and the corresponding classical results on Kfr's (cf. for instance [4, Section 32]). □

Remark 3.5 Note that if $\star_1$ and $\star_2$ are two e.a.b. semistar operations defined on D, then:

(a) $\star_1 \leq \star_2 \;\Rightarrow\; \mathrm{Kr}(D,\star_1) \subseteq \mathrm{Kr}(D,\star_2)$;

(b) $\star_1 \sim \star_2 \;\Leftrightarrow\; \mathrm{Kr}(D,\star_1) = \mathrm{Kr}(D,\star_2)$.

In particular, if $\star$ is an e.a.b. semistar operation then $\star \sim \star_f$ and hence $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_f)$.

For the proof of (a), recall that, for each $E \in \overline{\mathbf{F}}(D)$, $(E^{\star_1})^{\star_2} = E^{\star_2}$ (Lemma 2.6). (b, $\Rightarrow$) is obvious since for each $g \in D[X] - \{0\}, c(g) \in \mathbf{f}(D)$ and thus $c(g)^{\star_1} = c(g)^{\star_2}$. (b, $\Leftarrow$). By Corollary 3.4 (3), for each $F \in \mathbf{f}(D)$, we have:

$$F^{\star_1} = F\mathrm{Kr}(D,\star_1) \cap K = F\mathrm{Kr}(D,\star_2) \cap K = F^{\star_2}\,.$$

Example 3.6 Let V be a valuation overring of an integral domain D with quotient field K. Then it is easy to see that the w-semistar operation $\star_{\{V\}}$ (defined by $E^{\star_{\{V\}}} = EV$, for each $E \in \overline{\mathbf{F}}(D)$) is an a.b. semistar operation on D. Moreover,

$$\mathrm{Kr}(D,\star_{\{V\}}) = \{f/g \mid f,g \in D[X] - \{0\} \text{ and } c(f)V \subseteq c(g)V\} \cup \{0\}\,.$$

Since $D^{\star_{\{V\}}} = V$ then, by Proposition 3.3, we have that $\mathrm{Kr}(D, \star_{\{V\}}) = \mathrm{Kr}(V, \alpha(\star_{\{V\}}))$, where $\alpha(\star_{\{V\}}) := \alpha_V(\star_{\{V\}})$ is the d-(semi)star operation on V (which, in this case, coincides with the b-(semi)star operation on V). This latter Kfr is well known to be equal to the trivial extension $W = V(X)$ of the valuation domain V to the field $K(X)$ (i.e. $W = V(X) := \{f/g \mid f, g \in V[X] - \{0\}$ and $c(g) = V\}$; cf. for instance [4, p. 218 and Theorem 33.4]).

Proposition 3.7 *Let $\mathcal{S} := \{S_\alpha \mid \alpha \in A\}$ be a family of overrings of D and let $\star_{\mathcal{S}}$ be the semistar operation defined on D associated to $\mathcal{S}$ (Lemma 2.4 (3)).*

(1) *If $\star_{\{S_\alpha\}}$ is an e.a.b. (respectively, an a.b.) semistar operation for each $\alpha \in A$ (where $\star_{\{S_\alpha\}}$ is the semistar operation defined on D as in Example 2.5 (1)), then $\star_{\mathcal{S}}$ is an e.a.b. (respectively, an a.b.) semistar operation on D.*

(2) *If $\star_{\{S_\alpha\}}$ is e.a.b. for each $\alpha \in A$, then:*

$$\mathrm{Kr}(D, \star_{\mathcal{S}}) = \bigcap_{\alpha \in A} \mathrm{Kr}(D, \star_{\{S_\alpha\}}) .$$

Proof: (1) Let $E, F, G \in \overline{\mathbf{F}}(D)$, then

$$(EF)^{\star_S} = \bigcap_{\alpha \in A} EFS_\alpha, \qquad (EG)^{\star_S} = \bigcap_{\alpha \in A} EGS_\alpha .$$

Assume that, for each $E \in \mathbf{f}(D)$ and for each $\alpha \in A$,

$$EFS_\alpha \subseteq EGS_\alpha \;\Rightarrow\; FS_\alpha \subseteq GS_\alpha .$$

Then:

$$(EF)^{\star_S} \subseteq (EG)^{\star_S} \;\Rightarrow\; F^{\star_S} \subseteq G^{\star_S}$$

because $(EF)^{\star_S} S_\alpha = EFS_\alpha$ and $(EG)^{\star_S} S_\alpha = EGS_\alpha$, for each $\alpha \in A$ (Lemma 2.4 (3)).

(2) It is easy to see that, for $f, g \in D[X] - \{0\}$,

$$c(f)^{\star_S} \subseteq c(g)^{\star_S} \;\Leftrightarrow\; c(f)^{\star_{S_\alpha}} \subseteq c(g)^{\star_{S_\alpha}} \;\text{ for each }\; \alpha \in A.$$

□

Corollary 3.8 *Let D be an integral domain with field of quotients K and $\mathcal{W} := \{V_\lambda \mid \lambda \in \Lambda\}$ a family of valuation overrings of D. Then the w-semistar operation associated to $\mathcal{W}$, $\star_{\mathcal{W}}$, is a.b. on D and*

$$\mathrm{Kr}(D, \star_{\mathcal{W}}) = \bigcap_{\lambda \in \Lambda} \mathrm{Kr}(D, \star_{\{V_\lambda\}}) = \bigcap_{\lambda \in \Lambda} W_\lambda \,,$$

where $W_\lambda := V_\lambda(X)$ is the valuation domain trivial extension of V_λ to $K(X)$.

Proof: Easy consequence of Proposition 3.7 (2) and Example 3.6. □

We have now demonstrated that the classical Kfr construction introduced by Krull generalizes well to the case where the semistar operation $\star$ is e.a.b.. To deal with the case of a semistar operation which is *not necessarily e.a.b.*, we first examine the question of what it means for a domain R to be a "Kronecker function ring" of a given domain D. The objective is to remove this determination (i.e., the e.a.b. hypothesis) from the realm of star/semistar operations. To accomplish this goal, we follow Halter-Koch [7] and give the following definition of an "abstract" Kronecker function ring.

Definition 3.9 Let D be an integral domain with quotient field K, let X be an indeterminate over K and let R be a subring of the field $K(X)$ of the rational functions with coefficients in K. If R satisfies the following properties:

(Kr1) $R \cap K$ is an overring of D,

(Kr2) $X,\ 1/X \in R$,

(Kr3) $f \in K[X] \ \Rightarrow \ f(0) \in fR$,

then R is called an *(abstract) Kronecker function ring of D*.

The set $\mathbf{Kr}(D) := \{R \subseteq K(X) \mid R$ is a Kronecker function ring of $D\}$ is called *the Kronecker space of D*.

Note that another approach to "abstract" Kronecker function rings was described by D.F. Anderson, D. Dobbs and M. Fontana in [1], where Bezout domains that can arise as Kronecker funtion rings were characterized.

Now we give some straightforward properties of abstract Kronecker function rings, relating them to the classical Kfr's (cf. [4, Theorem 32.11, Corollary 32.14, Theorem 32.15]).

Lemma 3.10 *Let D be an integral domain and $\mathbf{Kr}(D)$ its Kronecker space.*

(1) $R \in \mathbf{Kr}(D)$, *S is an overring of R* $\Rightarrow$ $S \in \mathbf{Kr}(D)$.

(2) *If $\{R_i \mid i \in I\}$ is a family of Kronecker function rings of D then $\bigcap_{i \in I} R_i$ is also a Kronecker function ring of D. Moreover, if $R_i \cap K = D$ for each $i \in I$, then $(\bigcap_{i \in I} R_i) \cap K = D$.*

(3) *Each $R \in \mathbf{Kr}(D)$ is contained in some proper maximal element of $\mathbf{Kr}(D)$ and contains a (unique) smallest element of $\mathbf{Kr}(D)$.* □

The following result is essentially a restatement of a result proved by Halter-Koch [7, Theorem 2.2].

Theorem 3.11 *Let R be a Kronecker function ring of a given integral domain D. Set $D_R := R \cap K$ (by definition, $D \subseteq D_R$).*

(1) *The mapping*

$$F \mapsto F^{\star_R} := FR \cap K\,, \quad \textit{for each} \quad F \in \mathbf{f}(D)\,,$$

defines uniquely *a semistar operation of finite type on D, called* the semistar operation induced by $R \in \mathbf{Kr}(D)$ on D, *by setting:*

$$E^{\star_R} := \bigcup \{F^{\star_R} \mid F \in \mathbf{f}(D)\,,\ F \subseteq E\,\}\,, \quad \textit{for each } E \in \overline{\mathbf{F}}(D)\,.$$

(2) $f = \sum_{i=0}^{n} a_i X^i \in K[X] \ \Rightarrow \ fR = (a_0, a_1, \ldots, a_n)R = c_R(f)$.

(3) *R is a Bezout domain and, hence, D_R is integrally closed.*

(4) $\star_{\{D_R\}} \leq \star_R$ *and $\star_R$ is an e.a.b. semistar operation of finite type on D.*

(5) $R = \mathrm{Kr}(D, \star_R)$.

Proof: (1) Let $\star'_R$ be the star operation of finite type on D_R associated to the finitary ideal system considered by Halter-Koch [7, Theorem 2.2 (3)] and defined "uniquely" by setting:

$$F^{\star'_R} := FR \cap D_R\,, \quad \text{for each finitely generated ideal} \quad F \subseteq D\,.$$

Since $R \cap K = D_R$, it is easy to see that $\star_R = \delta_D(\star'_R)$ and hence, by Proposition 2.9 (1), $\star_R$ is a semistar operation on D.

(2) and (3) are translations of [7, Theorem 2.2 (1) and (2)].

(4) follows from the proof of (1), from Proposition 2.9 (2) and from the fact that $\star'_R$ is an e.a.b. star operation on D_R by [7, Theorem 2.2 (5)].

(5) If $f, g \in D[X] - \{0\}$ and $c(f)^{\star_R} \subseteq c(g)^{\star_R}$ then, by (1), $f \in fR \cap D_R = fR \cap K = c(f)^{\star_R} \subseteq c(g)^{\star_R} = gR \cap K = gR \cap D_R \subseteq gR$, hence $f/g \in R$. Conversely, if $f/g \in R$ with $f, g \in D[X] - \{0\}$, then $f \in gR$. Since, by (2), $fR = c(f)R$, then $c(f) \subseteq gR \cap D \subseteq gR \cap D_R = c(g)^{\star_R}$ and thus $c(f)^{\star_R} \subseteq c(g)^{\star_R}$. □

4 Semistar integrality and some new semistar operations

As was noted in the introduction, we seek to generalize the notion of Kronecker function ring to abitrary star operations on arbitrary integral domains. Recall that in the Krull's classical theory, Kronecker function rings are only defined for integral domains D which are integrally closed. One reason for turning to the theory of semistar operations for a proper generalization is that D can fail to be integrally closed while $D^\star$ may be an integrally closed overring. We have noted that star operations can be easily and naturally extended to semistar operations. So, one goal is to associate to a given semistar operation $\star$ a suitable new semistar operation $[\star]$ such that $D^{[\star]}$ is integrally closed. Moreover, we want this "$\star$-integral closure" to arise naturally with respect to the properties of $\star$. To achieve this goal we lean on the notion of semistar integral closure introduced by Okabe and Matsuda in [16] (cf. also [8] and [15]) and on an ideal system construction of Halter-Koch from [5]. Note now that the integral closure of $D^{[\star]}$ may not be sufficient to meet our needs, since $[\star]$ still may fail to have the semistar analogue of the *e.a.b.* property. So, we again adapt a construction described by Halter-Koch in [6, Chapter 19] to associate to a given semistar operation $\star$ a new semistar operation $\star_a$ which is *e.a.b.*. The combination of these two procedures allow us to associate an *e.a.b.* semistar operation to a given semistar operation on any domain, which can then be used to construct the desired Kronecker function ring. We then conclude by showing that the generalization we have produced is a "natural one", by proving that the Kronecker function rings we produce are identical to those produced by an abstract method, inspired by a Halter-Koch construction [7], based only on the original star operation $\star$ and the original domain D.

We begin with the Okabe-Matsuda notion of semistar integral closure.

Definition 4.1 Let D be an integral domain and let $\star$ be a semistar operation on D. An element $x \in K$ is called *$\star$-integral over D* if $x \in (I^\star : I^\star)$ for some $I \in \mathbf{f}(D)$. The following set:

$$D^{[\star]} := \bigcup \{(I^\star : I^\star) \mid I \in \mathbf{f}(D)\}$$

is called *the semistar integral closure of D with respect to $\star$* or, simply, *the $\star$-integral closure of D*. If $D = D^{[\star]}$, then D is called *$\star$-integrally closed.*

We now extend to a semistar operation on D, which we will denote by $[\star]$, the above definition of $D^{[\star]}$, using the following construction inspired by Halter-Koch [5, Section 3].

Definition 4.2 Let D be an integral domain and let $\star$ be a semistar operation on D. Then we define a new operation on D, denoted by $[\star]$, by setting:

$$H^{[\star]} := \bigcup \{((F^\star : F^\star)H)^{\star_f} \mid F \in \mathbf{f}(D)\}, \quad \text{for each } \ H \in \mathbf{f}(D),$$

and

$$E^{[\star]} := \bigcup \{H^{[\star]} \mid H \in \mathbf{f}(D),\ H \subseteq E\}, \quad \text{for each } E \in \overline{\mathbf{F}}(D).$$

It is not difficult to see that the operation $[\star]$ defined in this manner is a semistar operation of finite type on D.

We follow with some properties of the semistar integral closure.

Proposition 4.3 *Let D be an integral domain and let $\star$ be a semistar operation on D. Then:*

(1) $D^{[\star]}$ *is an overring of D.*

(2) $D^{[\star]}$ *is integrally closed.*

Proof: The proof of (1) is straightforward. (2) is in [16, Proposition 34]. □

As noted above, the semistar operation $[\star]$ is "attractive" because it arises naturally from $\star$ and because $D^{[\star]}$ is integrally closed, for any choice of D and $\star$. We will develop the properties of $[\star]$ further, but we need first to introduce the other new construction mentioned, which associates an *e.a.b.* semistar operation to any given semistar operation.

Definition 4.4 Let D be an integral domain with quotient field K and let $\star$ be a semistar operation on D. Then define the function $\star_a : \overline{\mathbf{F}}(D) \to \overline{\mathbf{F}}(D)$ by first setting

$$F^{\star_a} := \bigcup \{((FH)^\star : H^\star) \mid H \in \mathbf{f}(D)\}, \quad \text{for each} \quad F \in \mathbf{f}(D),$$

and then, for each $E \in \overline{\mathbf{F}}(D)$,

$$E^{\star_a} := \bigcup \{F^{\star_a} \mid F \in \mathbf{f}(D), \quad F \subseteq E\}.$$

Next we prove a result which gives some new properties of $[\star]$, some properties of $\star_a$ and some ways in which the two interrelate.

Proposition 4.5 *Let $\star$ be a semistar operation on an integral domain D with quotient field K. Then*

(1) $\star_a$ *defines a semistar operation of finite type;*
(2) $\star_a$ *is e.a.b.;*
(3) $\star_f \leq [\star] \leq \star_a$ *;*
(4) $\star_a = (\star_f)_a = (\star_a)_f$ *;*
(5) $\star_a = \star_f \ \Leftrightarrow \ \star_f$ *is an e.a.b. semistar operation;*
(6) $\star_1 \leq \star_2 \ \Rightarrow \ (\star_1)_a \leq (\star_2)_a$ *;*
(7) $\star_1 \leq \star_2 \ \Rightarrow \ [\star_1] \leq [\star_2]$ *;*
(8) $(\star_a)_a = \star_a$ *;*
(9) $[\star]_a = \star_a$*;*
(10) $D^{\star_a} = D^{[\star]} = D^{[\star_a]} = D^{[[\star]]} = D^{[\star]_a} = D^{[\star_f]}$.

Proof: (1) In order to show that $\star_a$ verifies property $(\star_1)$, it is sufficient to note that, for each $x \in K, x \neq 0$ and for each $F \in \mathbf{f}(D)$,

$$\begin{aligned}(xF)^{\star_a} &= \bigcup \{((xFH)^\star : H^\star) \mid H \in \mathbf{f}(D)\} = \\ &= \bigcup \{x((FH)^\star : H^\star) \mid H \in \mathbf{f}(D)\} = \\ &= x(\bigcup \{((FH)^\star : H^\star) \mid H \in \mathbf{f}(D)\} = xF^{\star_a}.\end{aligned}$$

It is straightforward that $\star_a$ satisfies property $(\star_2)$.

In order to conclude that $\star_a$ is a semistar operation, we want to show that $\star_a$ satisfies property $(\star'_3)$ (Lemma 2.2).

It is obvious that, for each $F \in \mathbf{f}(D)$,

$$F \subseteq \bigcup \{((FH) : H) \mid H \in \mathbf{f}(D)\} \subseteq \bigcup \{((FH)^\star : H^\star) \mid H \in \mathbf{f}(D)\} = F^{\star_a}.$$

Assume that $F \subseteq E^{\star_a}$ for some $E \in \overline{\mathbf{F}}(D)$, with $F = y_1 D + y_2 D + \ldots + y_n D \in \mathbf{f}(D)$. Then, for each y_i, there exist $E_i, H_i \in \mathbf{f}(D)$, with $E_i \subseteq E$ such that $y_i \in ((E_i H_i)^\star : H_i^\star)$. Set

$$E_0 := \sum_{i=1}^{n} E_i \quad \text{and} \quad H_0 := \prod_{i=1}^{n} H_i \,.$$

Then we claim that $y_i \in ((E_0 H_0)^\star : H_0^\star)$, for each i. As a matter of fact:

$$y_i H_i^\star \prod_{j \neq i} H_j^\star \subseteq (E_i H_i)^\star \prod_{j \neq i} H_j^\star \subseteq (E_0 H_i)^\star \prod_{j \neq i} H_j^\star \subseteq (E_0 H_i)^\star (\prod_{j \neq i} H_j)^\star \subseteq (E_0 H_0)^\star.$$

Therefore, we deduce that $F H_0^\star \subseteq (E_0 H_0)^\star$. On the other hand, if $z \in F^{\star_a}$, then $zL^\star \subseteq (FL)^\star$ for some $L \in \mathbf{f}(D)$. This fact implies that

$$zL^\star H_0^\star \subseteq (FL)^\star H_0^\star \subseteq (F H_0^\star L)^\star \subseteq ((E_0 H_0)^\star L)^\star = (E_0 L H_0)^\star \,,$$

hence $z(LH_0)^\star \subseteq (E_0 L H_0)^\star$, thus $z \in E_0^{\star_a} \subseteq E^{\star_a}$.

(2) By using Lemma 2.7 ((i) $\Leftrightarrow$ (iv)), we need to show that, for all A and B that are $\star_a$-finite D-submodules of K, we have:

$$((AB)^{\star_a} : A) \subseteq B \,.$$

Let $z \in K$, $z \neq 0$, such that $zA \subseteq (AB)^{\star_a}$ and let $A_0 := x_1 D + x_2 D + \ldots + x_n D$, with $(A_0)^{\star_a} = A$. For each i, $1 \leq i \leq n$, there exists $H_i \in \mathbf{f}(D)$ such that $z x_i \in ((ABH_i)^\star : H_i^\star)$ (Definition 4.4). Set $H_0 := \prod_{i=1}^{n} H_i$. Then:

$$zA_0 H_0 = \sum_{i=1}^{n} z x_i H_0 \subseteq \sum_{i=1}^{n} ((A_0 B H_i)^\star (\prod_{j \neq i} H_j)^\star) \subseteq (A_0 H_0 B)^\star \,,$$

hence $z \in ((B(A_0 H_0))^\star : (A_0 H_0)^\star) \subseteq B^{\star_a} = B$.

(3) By Definition 4.2, it is obvious that, for all $H, F \in \mathbf{f}(D)$,

$$H^\star \subseteq ((F^\star : F^\star) H^\star)^{\star_f} = ((F^\star : F^\star) H)^{\star_f} \subseteq H^{[\star]} \,,$$

hence we have that $\star_f \leq [\star]$ $(= [\star]_f)$.
Moreover, if $z \in H^{[\star]}$, i.e. if $z \in ((F^\star : F^\star) H)^{\star_f}$ for some $F, H \in \mathbf{f}(D)$, then

$$zF \subseteq zF^\star \subseteq ((F^\star : F^\star) H)^{\star_f} F^\star \subseteq (F^\star H)^\star = (FH)^\star$$

hence $z \in ((HF)^\star : F) \subseteq H^{\star_a}$. This fact implies that $[\star] \leq \star_a$.

(4), (6) and (7) are obvious consequences of the definitions.

$(5, \Rightarrow)$ follows from (2).

$(5, \Leftarrow)$. Since $\star_f$ is e.a.b. then, by Lemma 2.7 ((i) $\Leftrightarrow$ (iv)), we have $((FH)^\star : H) \subseteq F^\star = F^{\star_f}$, for all $F, H \in \mathbf{f}(D)$. Hence $F^{\star_a} \subseteq F^{\star_f}$, i.e. $\star_a \leq \star_f$. The conclusion follows from (3).

(8) follows from (6) and (5), since $\star_a = (\star_a)_f = (\star_a)_a$.

(9) is a consequence of (3), (4) and (8).

(10) By definition $[\star_f] = [\star]$ hence, obviously, $D^{[\star_f]} = D^{[\star]}$. Moreover:

$$D^{\star_a} = \bigcup \{((DH)^\star : H^\star) \mid H \in \mathbf{f}(D)\} = \bigcup \{(H^\star : H^\star) \mid H \in \mathbf{f}(D)\} = D^{[\star]}.$$

On the other hand:

$$D^{[\star_a]} = \bigcup \{(F^{\star_a} : F) \mid F \in \mathbf{f}(D)\},$$

and

$$\begin{aligned}(F^{\star_a} : F) &= ((\bigcup \{((FH)^\star : H) \mid H \in \mathbf{f}(D)\}) : F) = \\ &= \bigcup \{(((FH)^\star : H) : F) \mid H \in \mathbf{f}(D)\} = \\ &= \bigcup \{((FH)^\star : (FH)) \mid H \in \mathbf{f}(D)\}.\end{aligned}$$

Since $FH \in \mathbf{f}(D)$, we deduce that $D^{[\star_a]} \subseteq D^{[\star]} = D^{\star_a}$. On the other side, from (3), we have that $\star_a \leq [\star_a]$, hence $D^{\star_a} \subseteq D^{[\star_a]}$. Finally, since $[\star] = [\star]_f$ and $(\star_a)_a = \star_a$ then, by (3) and (6), we obtain:

$$[\star] \leq [[\star]] \leq [\star]_a \leq \star_a.$$

The conclusion is now immediate, because we already proved that $D^{[\star]} = D^{\star_a}$. $\square$

5 Kronecker function ring associated to any semistar operation

In this section we achieve the goal of defining a Kronecker function ring for any semistar operation on any integral domain and demonstrating that it is a natural generalization of the classical case by showing that we obtain

- domains that are abstract Kronecker function rings (Definition 3.9);

- domains which can be also obtained in the classical manner by using a "suitable" e.a.b. star operation on an "appropriate" integrally closed domain.

Theorem 5.1 *Let $\star$ be any semistar operation defined on an integral domain D with quotient field K. Set*

$$\mathrm{Kr}(D,\star) := \{f/g \ \mid \ f,g \in D[X]-\{0\} \ \textit{and there exists} \ h \in D[X]-\{0\} \ \textit{such that} \ (c(f)c(h))^\star \subseteq (c(g)c(h))^\star\} \cup \{0\}\,.$$

Then we have:

(1) $\mathrm{Kr}(D,\star)$ *is an integral domain with quotient field $K(X)$.*

(2) $\mathrm{Kr}(D,\star)$ *is a (abstract) Kronecker function ring of D.*

(3) $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,[\star]_a) = \mathrm{Kr}(D,\star_a)$.

Proof: It follows immediately from the definitions that $\mathrm{Kr}(D,\star_f) = \mathrm{Kr}(D,\star)$ and $[\star_f] = [\star]$. Therefore we can assume, *without loss of generality*, that $\star$ is a semistar operation of finite type on D.

Case 1: *Assume that $\star$ is an e.a.b. semistar operation of finite type.* In this case, for $f,g,h \in D[X]-\{0\}$ we have:

$$(c(f)c(h))^\star \subseteq (c(g)c(h))^\star \ \Leftrightarrow \ c(f)^\star \subseteq c(g)^\star\,.$$

Therefore, $\mathrm{Kr}(D,\star)$ –as defined above– coincides with the Kronecker function ring of an e.a.b. semistar operation, as defined in Definition 3.2. Moreover, in this case $\star = [\star] = \star_a$ (Proposition 4.5 (3), (5) and (9)). Hence, in the present situation, (1) is straightforward (cf. Corollary 3.4 (1)) and (3) is trivial.

(2) It is easy to see that axioms **(Kr1)** and **(Kr2)** of the definition of an abstract Kfr are satisfied by $\mathrm{Kr}(D,\star)$. To prove that $\mathrm{Kr}(D,\star)$ is a (abstract) Kronecker function ring, it is only necessary then to show that it satisfies **(Kr3)** .

Let $f \in D[X]-\{0\}$, then $f(0) \in c(f)$, hence obviously $f(0)/f \in \mathrm{Kr}(D,\star)$, that is $f(0) \in f\mathrm{Kr}(D,\star)$, for every $\star$. If $f \in K[X]-\{0\}$, then, cleaning up denominators, for some $a \in D-\{0\}$, $af \in D[X]-\{0\}$. Therefore, as above, $af(0) \in af\mathrm{Kr}(D,\star)$, whence $f(0) \in f\mathrm{Kr}(D,\star)$.

Case 2: *General case.* Let $\star$ be a semistar operation of finite type of D.

We start by showing that (3) holds, i.e. $\mathrm{Kr}(D,\star)$ coincides with the Kronecker function ring associated to the e.a.b. semistar operation of finite type $[\star]_a(=\star_a$, by Proposition 4.5 (9).

By definition, it is easy to see that, given two semistar operations on D with $\star_1 \leq \star_2$, then $\mathrm{Kr}(D,\star_1) \subseteq \mathrm{Kr}(D,\star_2)$. By Proposition 4.5 (3) we know that $\star = \star_f \leq [\star] \leq \star_a$ and, hence, also that:

$$[\star] = [\star]_f \leq [\star]_a \,.$$

Therefore, $\mathrm{Kr}(D,\star) \subseteq \mathrm{Kr}(D,[\star]) \subseteq \mathrm{Kr}(D,[\star]_a)$.

Conversely, let $\varphi \in \mathrm{Kr}(D,[\star]_a)$. Then, by Case 1, $\varphi = f/g$ with $f,g \in D[X]-\{0\}$ and $c(f)^{[\star]_a} \subseteq c(g)^{[\star]_a}$. Set $A := c(f)$ and $B := c(g)$. Then $A \subseteq ((BC)^{[\star]} : C)$, for some $C \in \mathbf{f}(D)$, because A is finitely generated and $B^{[\star]_a} = \bigcup\{((BC)^{[\star]} : C) \mid C \in \mathbf{f}(D)\}$, for some $C \in \mathbf{f}(D)$. Hence, $AC \subseteq (BC)^{[\star]}$. Since $AC \in \mathbf{f}(D)$, then $AC \subseteq ((F^\star : F)BC)^{\star_f}$, for some $F \in \mathbf{f}(D)$ (Definition 4.2). We deduce that:

$$ACF \subseteq F((F^\star : F)BC)^{\star_f} \subseteq (F(F^\star : F)BC)^\star \subseteq (FBC)^\star \,.$$

It is easy to see that, without loss of generality, we can assume that $CF \subseteq D$. If we take a polynomial $h \in D[X]-\{0\}$ such that $c(h) = CF$, then $c(f)c(h) \subseteq (c(g)c(h))^\star$ and thus $\varphi = f/g \in \mathrm{Kr}(D,\star)$. This proves (3). Statements (1) and (2) then follow from Case 1.

□

We close with two corollaries which demonstrate that the generalization we have obtained is a "good one." The Kronecker function ring, $R := \mathrm{Kr}(D,\star)$, defined in Theorem 5.1 using just D and any semistar operation $\star$ on D, induces the new semistar operation $\star_R$ on the domain D (Theorem 3.11), which coincides with $\star_a$ (Corollary 5.2) and the construction is in fact equivalent to a classical construction when viewed in the proper context (Corollary 5.3).

Corollary 5.2 *Let $\star$ be a semistar operation defined on an integral domain D with quotient field K. For each $E \in \overline{\mathbf{F}}(D)$, we define a map $\star_{\mathrm{Kr}(D,\star)} : \overline{\mathbf{F}}(D) \to \overline{\mathbf{F}}(D)$ by setting:*

$$E^{\star_{\mathrm{Kr}(D,\star)}} := \bigcup\{F\mathrm{Kr}(D,\star) \cap K \mid \ F \in \mathbf{f}(D), \ \ F \subseteq E\}\,.$$

Then $\star_{\mathrm{Kr}(D,\star)}$ is an e.a.b. semistar operation of finite type on D and, in fact, $\star_{\mathrm{Kr}(D,\star)} = \star_a$.

Proof: This follows easily from Corollary 3.4 (3), Theorem 3.11 (1) and Theorem 5.1 (3) since, for each $F \in \mathbf{f}(D)$:

$$F\mathrm{Kr}(D,\star) \cap K = F\mathrm{Kr}(D,\star_a) \cap K = F^{\star_a}.$$

□

Corollary 5.3 *Let $\star$ be a semistar operation on an integral domain D with quotient field K. Then there exist an integrally closed overring T of D and an e.a.b. (semi)star operation $\star_i$ on T such that $T^{\star_i} = T$ and* $\mathrm{Kr}(D,\star) = \mathrm{Kr}(T,\star_i)$ *. (In particular, $T = D^{[\star]_a}$ and $\star_i = \alpha_{D^{[\star]_a}}([\star]_a) = \alpha([\star]_a)$.)*

Proof: This also follows easily from Theorem 5.1 (3) and Proposition 3.3.□

References

[1] D.F. Anderson, D.E. Dobbs and M. Fontana, *When is a Bezout domain a Kronecker function ring ?* C.R. Math. Rep. Acad. Sci. Canada **9** (1987), 25-30.

[2] H.M. Edwards, Divisor Theory. Birkhäuser, 1990.

[3] M. Fontana and J. Huckaba, *Localizing systems and semistar operations,* in "Non Noetherian Commutative Ring Theory" (S. Chapman and S. Glaz, Eds.), Kluwer Academic Publishers (in print). Chapter 8, 169-187.

[4] R. Gilmer, Multiplicative Ideal Theory. M. Dekker, New York, 1972.

[5] F. Halter-Koch, *Generalized integral closures,* in "Factorization in Integral Domains" (D.D. Anderson, Ed.). M. Dekker (1997), 349-358.

[6] F. Halter-Koch, Ideal Systems: An Introduction to Multiplicative Ideal Theory. M. Dekker, New York, 1998.

[7] F. Halter-Koch, *Kronecker function rings and generalized integral closures*, preprint 1999.

[8] P. Jaffard, Les Systèmes d'Idéaux. Dunod, Paris, 1960.

[9] L. Kronecker, *Grundzüge einer arithmetischen Theorie der algebraischen Grössen.* J. Reine Angew. Math., **92** (1882), 1-122; *Werke* **2**, 237-387.

[10] W. Krull, Idealtheorie. Springer-Verlag, Berlin, 1935.

[11] W. Krull, *Beiträge zur Arithmetik kommutativer Integritätsbereiche,* I - II. Math. Z. **41** (1936), 545-577 and 665-679.

[12] R. Matsuda, *Kronecker function rings of semistar operations on rings.* Algebra Colloquium **5** (1998), 241-254.

[13] R. Matsuda and I. Sato, *Note on star operations and semistar operations.* Bull. Fac. Sci. Ibaraki Univ. Ser. A **28** (1996), 155-161.

[14] R. Matsuda and T. Sugatani, *Semistar operations on integral domains,* II. Math. J. Toyama Univ. **18** (1995), 155-161.

[15] A. Okabe and R. Matsuda, *Star operations and generalized integral closures.* Bull. Fac. Sci. Ibaraki Univ. Ser. A **24** (1992), 7-13.

[16] A. Okabe and R. Matsuda, *Semistar operations on integral domains.* Math. J. Toyama Univ. **17** (1994), 1-21.

[17] A. Okabe and R. Matsuda, *Kronecker function rings of semistar operations.* Tsukuba J. Math., **21** (1997), 529-540.

Marco Fontana
Dipartimento di Matematica
Università degli Studi Roma Tre
Largo San Leonardo Murialdo, 1
00146 Roma, Italy
`fontana@mat.uniroma3.it`

K. Alan Loper
Department of Mathematics
Ohio State University - Newark
Newark, Ohio 43055
U.S.A.
`lopera@math.ohio-state.edu`