A Krull-type theorem for the semistar integral closure of an integral domain

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Abstract

The goal of this paper is to further investigate the structure of the Kronecker function ring $\mathrm{Kr}(D, \star)$ constructed in [4] when $\star$ is a semistar operation, pursuing the work initiated by Halter-Koch [7]. In particular, we investigate the set of valuation overrings of $\mathrm{Kr}(D, \star)$ and the contractions of these valuation domains to K, the quotient field of D.

1 Introduction

One of the primary applications of Krull's theory of ideal systems and star operations is to construct the Kronecker function rings, in a more general context than the original one considered by L. Kronecker in 1882 [9] (see [2] for a modern presentation of Kronecker's theory). There are some restrictions in Krull's theory. In particular, the Kronecker function ring of an integral domain D with respect to a star operation is constructed only in the case where the star operation $\star$ being considered has a particular "cancellation property" known as *e.a.b. (endlich arithmetisch brauchbar)* and D is integrally closed [5, Section 32]. In [4] we presented a generalization of these results by considering the more general notion of *semistar operations,* introduced by Okabe and Matsuda [12], and making use of some recent results of Halter-Koch on the notion of an abstract Kronecker function ring [7]. The main result was a natural construction of a Kronecker function ring $\mathrm{Kr}(D,\star)$ corresponding to an arbitrary integral domain D and an arbitrary semistar operation $\star$ on D.

The goal of this paper is to further investigate the structure of the Kronecker function ring $\mathrm{Kr}(D,\star)$ constructed in [4] when $\star$ is a semistar operation, pursuing the work by Halter-Koch [7]. In particular, we investigate the set of valuation overrings of $\mathrm{Kr}(D,\star)$ and the contractions of these valuation domains to K, the quotient field of D. We start by defining the notion of a $\star$-valuation overring of D, when $\star$ is a semistar operation. Our main results then show that:

- the valuation overrings of $\mathrm{Kr}(D,\star)$ are precisely the trivial extensions of the $\star$-valuation overrings of D to $K(X)$;
- the intersection of the $\star$-valuation overrings of D is equal to the domain $D^{[\star]}$ which was introduced by Okabe and Matsuda in [12] as a $\star$-integral closure of D.

In the second section we give some relevant definitions and background results concerning semistar operations and Kronecker function rings. The third section contains new results concerning the $\star$-valuation overrings.

2 Background Results

Let D be an integral domain with quotient field K. Let $\overline{\mathbf{F}}(D)$ denote the set of all nonzero D-submodules of K and let $\mathbf{F}(D)$ be the set of all nonzero fractional ideals of D, i.e. all $E \in \overline{\mathbf{F}}(D)$ such that there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\mathbf{f}(D)$ be the set of all nonzero finitely generated D-submodules of K. Then, obviously:

$$\mathbf{f}(D) \subseteq \mathbf{F}(D) \subseteq \overline{\mathbf{F}}(D).$$

We recall that a mapping

$$\star : \overline{\mathbf{F}}(D) \to \overline{\mathbf{F}}(D), \quad E \mapsto E^{\star}$$

is called a *semistar operation on* D if, for all $x \in K, x \neq 0$, and $E, F \in \overline{\mathbf{F}}(D)$, the following properties hold:

$(\star_1)$ $(xE)^{\star} = xE^{\star}$;

$(\star_2)$ $E \subseteq F \Rightarrow E^{\star} \subseteq F^{\star}$

$(\star_3)$ $E \subseteq E^{\star}$ and $E^{\star} = (E^{\star})^{\star} =: E^{\star\star}$

cf. for instance [11], [3] and [4].

A semistar operation $\star$ on D is called an *e.a.b.* (*endlich arithmetisch brauchbar*) [respectively, *a.b.* (*arithmetisch brauchbar*)] if for each $E \in \mathbf{f}(D)$ and for all $F, G \in \mathbf{f}(D)$ [respectively, $F, G \in \overline{\mathbf{F}}(D)$]:

$$(EF)^{\star} \subseteq (EG)^{\star} \Rightarrow F^{\star} \subseteq G^{\star},$$

[4, Definition 2.3 and Lemma 2.7].

A key element in [4] and in the current paper are several new semistar operations which can be derived from a given semistar operation $\star$. The essential details are given in the following example.

Example 2.1 **(1)** If $\star$ is a semistar operation on an integral domain D, for each $E \in \overline{\mathbf{F}}(D)$, set

$$E^{\star_f} := \bigcup \{F^{\star} \mid F \subseteq E, \ F \in \mathbf{f}(D)\}.$$

Then $\star_f$ is also a semistar operation on D, called *the semistar operation of finite type associated to* $\star$. Obviously, $F^{\star} = F^{\star_f}$, for each $F \in \mathbf{f}(D)$. If $\star = \star_f$, then $\star$ is called a *semistar operation of finite type* [4, Example 2.5(4)].

Note that $\star_f \leq \star$, i.e. $E^{\star_f} \subseteq E^\star$ for each $E \in \overline{\mathbf{F}}(D)$. Thus, in particular, if $E = E^\star$, then $E = E^{\star_f}$.

(2) Let D be an integral domain and let $\star$ be a semistar operation on D. Then we define a new operation on D, denoted by $[\star]$, called *the semistar integral closure of* $\star$, by setting:

$$F^{[\star]} := \bigcup \{((H^\star : H^\star)F)^{\star_f} \mid H \in \mathbf{f}(D)\}, \text{ for each } F \in \mathbf{f}(D),$$

and

$$E^{[\star]} := \bigcup \{F^{[\star]} \mid F \in \mathbf{f}(D),\ F \subseteq E\}, \text{ for each } E \in \overline{\mathbf{F}}(D).$$

It is not difficult to see that the operation $[\star]$ defined in this manner is a semistar operation of finite type on D, that $\star_f \leq [\star]$ and that $D^{[\star]}$ is integrally closed, called *the semistar integral closure of D with respect to $\star$*, [4, Definition 4.2, Proposition 4.3(1) and Proposition 4.5(3)].

(3) Let $\star$ be a semistar operation on an integral domain D. Then it is possible to associate to $\star$ an e.a.b. semistar operation of finite type $\star_a$ on D, called *the e.a.b. semistar operation associated to* $\star$, defined as follows:

$$F^{\star_a} := \bigcup \{((FH)^\star : H^\star) \mid H \in \mathbf{f}(D)\}, \text{ for each } F \in \mathbf{f}(D),$$

and

$$E^{\star_a} := \bigcup \{F^{\star_a} \mid F \subseteq E, F \in f(D)\}, \text{ for each } E \in \overline{\mathbf{F}}(D),$$

[4, Definition 4.4]. Note that $[\star] \leq \star_a$ and that $D^{[\star]} = D^{\star_a}$ [4, Proposition 4.5].

Regarding Kronecker function rings, if $\star$ is a semistar operation on D, then in [4, Section 5] we defined *the Kronecker function ring of D with respect to* $\star$ as follows:

$$\begin{aligned}\mathrm{Kr}(D,\star) := \{f/g \mid\ & f, g \in D[X] - \{0\} \text{ and there exists } h \in D[X] - \{0\} \\ & \text{such that } (c(f)c(h))^\star \subseteq (c(g)c(h))^\star\} \cup \{0\}.\end{aligned}$$

In [4] we proved, among other properties, the following:

Proposition 2.2 *Let $\star$ be a semistar operation on an integral domain D with quotient field K. Then*

1. $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,[\star]) = \mathrm{Kr}(D,[\star]_a) = \mathrm{Kr}(D,\star_a)$;

2. $\mathrm{Kr}(D,\star)$ *is a Bezout domain with quotient field* $K(X)$*;*

3. *for each* $F \in \mathbf{f}(D)$,

$$F^{\star_a} = F\mathrm{Kr}(D,\star) \cap K\,.$$

Proof: [4, Theorem 5.1 and proof of Corollary 5.2]. □

3 Main Results

Let $\star$ be a semistar operation on D and let V be a valuation overring of D. We say that V is a *$\star$-overring of D,* if for each $F \in \mathbf{f}(D)$,

$$F^\star \subseteq FV$$

(or equivalently, $\star_f \leq \star_{\{V\}}$, where $\star_{\{V\}}$ is the semistar operation of finite type on D defined by

$$E^{\star_{\{V\}}} := EV = \bigcup\{FV \mid F \subseteq E,\ F \in \mathbf{f}(D)\}$$

for each $E \in \overline{\mathbf{F}}(D)$ [4, Example 2.5(1) and Example 3.6]).

Proposition 3.1 *Let $\star$ be a semistar operation on an integral domain D, let V be a valuation overing of D and let v be the valuation corresponding to V. Then V is a $\star$-valuation overring of D if and only if, for each $F \in \mathbf{f}(D)$, there exists $x \in F$ with*

$$v(x) = \inf\{v(y) \mid \ y \in F^\star\}\,.$$

Proof: If V is a $\star$-valuation overring of D, then $F^\star \subseteq FV = xV$, for some $x \in F$, hence $v(x) \leq v(y)$, for all $y \in F^\star$. Conversely if $x \in F$ and $v(x) \leq v(y)$, for all $y \in F^\star$, then, in particular, $xV = FV$ and, hence, $F^\star \subseteq FV$. □

The previous result shows that, when $\star$ is a semistar operation, V is a $\star$-overring if and only if the valuation v is a $\star$-valuation in Jaffard's sense [8, p. 48].

Proposition 3.2 *Let $\star$ be a semistar operation on an integral domain D and let $[\star]$ be the semistar integral closure of $\star$ (cf. Example 2.1(2)).*

1. *For each $\star$-valuation overring V of D, we have $D^{[\star]} \subseteq V$.*
2. *Let V be a valuation overring of D. Then V is a $\star$-valuation overring of D if and only if V is a $[\star]$-overring of D.*

Proof: 1. If $y \in D^{[\star]}$, then $yJ \subseteq J^\star$ for some $J \in \mathbf{f}(D)$. Let V be a valuation overring of D and let $x \in J$ such that $JV = xV$. Since $J^\star \subseteq JV$, then $J^\star V = JV$ and:

$$yxV = yJV \subseteq J^\star V = JV = xV$$

We conclude that $y \in V$.

2. Let V be a $\star$-valuation overring of D. Note that, for each $F \in \mathbf{f}(D)$ and for each $H \in \mathbf{f}(D)$:

$$(F(H^\star : H^\star))^{\star_f} \subseteq F(H^\star : H^\star)V \subseteq FD^{[\star]}V = FV$$

thus $F^{[\star]} \subseteq FV$.

Conversely, if V is a $[\star]$-valuation overring of D, since $F^\star \subseteq F^{[\star]}$ and $F^{[\star]} \subseteq FV$, for each $F \in \mathbf{f}(D)$, the conclusion is obvious. □

Proposition 3.3 *Let $\star$ be a semistar operation of an integral domain D and let V be a valuation overring of D. Then V is a $\star$-valuation overring of D if and only if V is a $\star_a$-valuation overring of D.*

Proof: Since $\star_f \leq \star_a$ [4, Proposition 4.5(3)], it is obvious that, if V is a $\star_a$-valuation overring of D, then $F^\star \subseteq F^{\star_a} \subseteq FV$, for each $F \in \mathbf{f}(D)$.

Conversely, for each $F \in \mathbf{f}(D)$ and for each $H \in \mathbf{f}(D)$ we can consider $((FH)^\star : H^\star)$. Let $z \in H$ be such that $HV = zV$, where V is a $\star$-valuation overring of D. Then:

$$\begin{aligned}((FH)^\star : H^\star) = ((FH)^\star : H) &\subseteq (FHV : H) \subseteq \\ &\subseteq (FzV : zD) = (FV : D) = FV\,,\end{aligned}$$

thus $F^{\star_a} \subseteq FV$.

$\square$

Note that the previous proposition is analogous to a result proved by Jaffard in [8, Proposition 5, p. 48].

The following result is implicitly proved, but not explicitly stated in [4].

Proposition 3.4 *Let $\star$ be an e.a.b. semistar operation of finite type on an integral domain D with quotient field K. Let $\mathcal{W} := \{W \mid W$ is a valuation overring of $\mathrm{Kr}(D,\star)\}$ and let $\mathcal{V} := \{V := W \cap K \mid W \in \mathcal{W}\}$. Then, for each $F \in \mathbf{f}(D)$,*

$$F^{\star} = F^{\star_{\mathcal{V}}} := \bigcap \{FV \mid V \in \mathcal{V}\}.$$

Proof: By [4, Proposition 4.5(3)], since $\star = \star_f$ we have:

$$\star \leq [\star] \leq \star_a,$$

hence, also $[\star] \leq [[\star]] \leq [\star]_a \leq (\star_a)_a = \star_a$ [4, Proposition 4.5(8)]. Since $\star = \star_a$ [4, Proposition 4.5(5)], then $\star = [\star]_a$. We recall that the semistar operation $\star_{\mathcal{V}}$ of D defined as follows:

$$E^{\star_{\mathcal{V}}} := \bigcap \{EV \mid V \in \mathcal{V}\}$$

is an a.b. semistar operation on D and

$$\mathrm{Kr}(D,\star_{\mathcal{V}}) = \bigcap \{V(X) \mid V \in \mathcal{V}\} = \bigcap \{W \mid W \in \mathcal{W}\} = \mathrm{Kr}(D,\star)$$

[4, Proposition 3.7 and Corollary 3.8]. The conclusion follows since, for each $F \in \mathbf{f}(D)$, we have:

$$F^{\star} = F^{\star_a} = F\mathrm{Kr}(D,\star) \cap K = F\mathrm{Kr}(D,\star_{\mathcal{V}}) \cap K,$$

and

$$F\mathrm{Kr}(D,\star_{\mathcal{V}}) \bigcap K = F^{\star_{\mathcal{V}}}$$

by Proposition 2.2(3), because $\mathrm{Kr}(D,\star_{\mathcal{V}}) = \mathrm{Kr}(D,(\star_{\mathcal{V}})_f)$ and $(\star_{\mathcal{V}})_f$ is an e.a.b. semistar operation of finite type. $\square$

Theorem 3.5 *Let $\star$ be a semistar operation on an integral domain D with quotient field K. Then V is a $\star$-valuation overring of D if and only if there exists a valuation overring W of* $\mathrm{Kr}(D,\star)$ *such that $W \cap K = V$.*

Proof: Since the $\star$-valuation overrings of D coincide with the $[\star]_a$-valuation overrings of D (Propositions 3.2 and 3.3) and since $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,[\star]_a)$ Proposition 8(1)), we can assume that $\star = [\star]_a$ is an e.a.b. semistar operation of finite type on D.

It is not difficult to see that if V is a $\star$-valuation overring of D then the trivial extention $W := V(X)$ of V in $K(X)$ is a valuation overring of $\mathrm{Kr}(D,\star)$ such that $W \cap K = V$. As a matter of fact, let $f/g \in \mathrm{Kr}(D,\star)$ with

$$f := \sum_{k=0}^{n} a_k X^k\,, \quad g := \sum_{h=0}^{m} b_h X^h \in D[X]$$

and $g \neq 0$. Since $\star$ is e.a.b. and V is a $\star$-valuation overring of D, then:

$$\begin{aligned}(a_0, a_1, \ldots, a_n)V &\subseteq (a_0, a_1, \ldots, a_n)^\star = c(f)^\star \subseteq \\ &\subseteq c(g)^\star = (b_0, b_1, \ldots, b_m)^\star \subseteq (b_0, b_1, \ldots, b_m)V.\end{aligned}$$

If we denote by v [respectively, w] the valuation of K [respectively, $K(X)$] associated to V [respectively, $W = V(X)$], then

$$v(a_k) \geq \inf\{v(b_h) \mid 0 \leq h \leq m, b_h \neq 0\}, \text{ for each } \ 0 \leq k \leq n \text{ with } \ a_k \neq 0\,,$$

therefore,

$$\begin{aligned} w(f) &= \inf\{v(a_k) \mid 0 \leq k \leq n, a_k \neq 0\} \geq \\ &\geq \inf\{v(b_h) \mid 0 \leq h \leq m, b_h \neq 0\} = w(g)\,,\end{aligned}$$

and thus, $f/g \in W$. It is well known that $V(X) \cap K = V$.

Conversely, assume that $V = W \cap K$, where W is a valuation overring of $\mathrm{Kr}(D,\star)$. Let $H := (a_0, a_1, \ldots, a_n)D \in \mathbf{f}(D)$ and set

$$h(X) := a_0 + a_1 X + \ldots + a_n X^n \in K[X].$$

Then, for each $z \in H^\star$, we have $(zD)^\star \subseteq H^\star = c(h)^\star$, and hence we obtain $z/h \in \mathrm{Kr}(D,\star) \subseteq W$. Therefore, if w is the valuation of $K(X)$ associated to W, then

$$0 \leq w(z/h) = w(z) - w(h) \leq w(z) - \inf\{w(a_0), w(a_1), \ldots, w(a_n)\},$$

hence

$$z \in (a_0, a_1, \ldots, a_n)(W \cap K) = (a_0, a_1, \ldots, a_n)V = HV\,,$$

i.e. $H^\star \subseteq HV$. □

Corollary 3.6 *Let $\star$ be a semistar operation on an integral domain D. Then*

$$D^{[\star]} = \bigcap\{V \mid V \text{ is a } \star\text{-valuation overring of } D\}\,.$$

Proof: Obviously, $D^{[\star]} \subseteq \bigcap\{V \mid V$ is a $\star$-valuation overring of $D\}$, because of Proposition 3.2(1). To prove that the equality holds, we note that:

$$D^{[\star]} = D^{[\star]_a} \;\; [4, \text{Proposition } 4.5(10)]$$

and that, by Propositions 2.2(1) and 3.4,

$$D^{[\star]_a} = \bigcap\{V \mid V = W \cap K\,,\; W \text{ is a valuation overring of } \mathrm{Kr}(D, \star)\}\,.$$

The conclusion follows from Theorem 3.5. □

References

[1] D. F. Anderson, E. G. Houston, and M. Zafrullah, *Pseudo-integrality* Canad. Math. Bull. **34** (1991), 15-22.

[2] H.M. Edwards, Divisor Theory. Birkhäuser, 1990.

[3] M. Fontana and J. Huckaba, *Localizing systems and semistar operations,* in "Non Noetherian Commutative Ring Theory" (S. Chapman and S. Glaz, Eds.), Kluwer Academic Publishers 2000, Chapter 8, 169-197.

[4] M. Fontana and K. A. Loper, *Kronecker function rings: a general approach,* M. Dekker Lecture Notes Pure Appl. Math. (in print).

[5] R. Gilmer, Multiplicative Ideal Theory. M. Dekker, New York, 1972.

[6] F. Halter-Koch, *Generalized integral closures,* in "Factorization in integral domains" (D.D. Anderson, Ed.), M. Dekker (1997), 349-358.

[7] F. Halter-Koch, *Kronecker function rings and generalized integral closures*, preprint 1999.

[8] P. Jaffard, Les Systèmes d'Idéaux. Dunod, Paris, 1960.

[9] L. Kronecker, *Grundzüge einer arithmetischen Theorie der algebraischen Grössen.* J. Reine Angew. Math., **92** (1882), 1-122; *Werke* **2**, 237-387.

[10] R. Matsuda and T. Sugatani, *Semistar operations and generalized integral closures*, II. Math. J. Toyama Univ. **18** (1995), 155-161.

[11] A. Okabe and R. Matsuda, *Star operations and generalized integral closures.* Bull. Fac. Sci. Ibaraki Univ. Ser. A **24** (1992), 7-13.

[12] A. Okabe and R. Matsuda, *Semistar operations on integral domains.* Math. J. Toyama Univ. **17** (1994), 1-21.

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