NAGATA RINGS, KRONECKER FUNCTION RINGS AND RELATED SEMISTAR OPERATIONS

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Abstract. In 1994, Matsuda and Okabe introduced the notion of semistar operation. This concept extends the classical concept of star operation (cf. for instance, Gilmer's book [20]) and, hence, the related classical theory of ideal systems based on the works by W. Krull, E. Noether, H. Prüfer and P. Lorenzen from 1930's.

In [17] and [18] the current authors investigated properties of the Kronecker function rings which arise from arbitrary semistar operations on an integral domain D. In this paper we extend that study and also generalize Kang's notion of a star Nagata ring [30] and [31] to the semistar setting. Our principal focuses are the similarities between the ideal structure of the Nagata and Kronecker semistar rings and between the natural semistar operations that these two types of function rings give rise to on D.

1. Introduction

A principal use of the classical star operations has been to construct Kronecker function rings associated to an integral domain, in a more general context than the original one considered by L. Kronecker in 1882 [33] (cf. [34], [35], and [11] for a modern presentation of Kronecker's theory). In this setting, one begins with an integrally closed domain D and a star operation $\star$ on D with the cancellation property known as e.a.b. (*endlich arithmetisch brauchbar*). Then the Kronecker function ring is constructed as follows:

$$\mathrm{Kr}(D,\star) := \{f/g \mid f,g \in D[X]\setminus\{0\} \ \text{ and } \ \boldsymbol{c}(f)^\star \subseteq \boldsymbol{c}(g)^\star\} \cup \{0\}\,,$$

(where $\boldsymbol{c}(h)$ denotes the content of a polynomial $h \in D[X]$). This domain turns out to be a Bézout overring of the polynomial ring $D[X]$ such that $\mathrm{Kr}(D,\star)\cap K = D$ (where K is the quotient field of D), cf. [20, Section 32].

In 1994, Okabe and Matsuda [41] introduced the more "flexible" notion of semistar operation $\star$ of an integral domain D, as a natural generalization of the notion of star operation, allowing $D \neq D^\star$ (the definition is given in Section 2); cf. also [38], [39] and [42]. In several recent papers, the classical construction introduced by Kronecker has been further generalized so that we can begin with *any* integral domain (not necessarily integrally closed) D and *any* semistar operation (not necessarily e.a.b.) $\star$ of D and, in a natural manner, construct a Kronecker function ring, still denoted here by $\mathrm{Kr}(D, \star)$, which preserves the main properties of the "classical" Kronecker function ring (axiomatized by Halter-Koch [25], see also [5]), cf. the works by Okabe and Matsuda [43], by Matsuda [37], and by Fontana and Loper [17], [18]. One evidence of the "naturalness" of this construction is that this general Kronecker function ring gives rise to an e.a.b. semistar operation $\star_a$ (which can be "restricted" to an e.a.b. star operation, denoted by $\dot{\star}_a$, of the integrally closed overring $D^{\star_a}$ of D) and then $\mathrm{Kr}(D,\star)$ can be viewed in the classical star e.a.b. setting by use of $\star_a$ (or, more precisely, by use of $\dot{\star}_a$).

Another overring of $D[X]$ which has been much studied is the Nagata ring of D, i.e. $D(X) := \{f/g \mid f, g \in D[X] \text{ and } \boldsymbol{c}(g) = D\}$, cf. [36], [45, page 27], [40, page 18], [20, Section 33], [28, Chapter IV]. The interest in $D(X)$ is due to the fact that this ring has some "nice" properties that D itself need not have, mantaining in any case a strict relation with the ideal structure of D, (for instance, for each ideal I of D, we have $ID(X) \cap D = I$ and $D(X)/ID(X) \cong (D/ID)(X)$). Among the "new" properties acquired by $D(X)$ we mention (a) the residue field at each maximal ideal of $D(X)$ is infinite; (b) an ideal contained in a finite union of ideals is contained in one of them [44]; (c) each finitely generated locally principal ideal is principal [3]. Furthermore, the canonical map $\mathrm{Spec}(D(X)) \to \mathrm{Spec}(D)$ is a homeomorphism if and only if the integral closure of D is a Prüfer domain [6]. The relation between the Nagata ring and the Kronecker function ring was investigated by Arnold [1], by Gilmer [19] and by Arnold and Brewer [2]. In particular, if b denotes the star operation of D defined on the fractionary ideals I of D by $I \mapsto I^b := \cap\{IV \mid V \text{ is a valuation overring of } D\}$, then Arnold [1, Theorem 7] proved that D is a Prüfer domain if and only if $D(X) = \mathrm{Kr}(D, b)$.

A generalization of the Nagata ring construction was considered by Kang [30], [31]: for each star operation $\star$ of D he studied the ring $\{f/g \mid f, g \in D[X]$ and $\boldsymbol{c}(g)^\star = D\}$. In particular, Kang proved that, *mutatis mutandis*, many properties of the "classical" Nagata ring still hold in this more general context.

In the present paper we further generalize the previous construction so that, given any domain D and any semistar operation $\star$ on D, we define the semistar Nagata ring as follows:

$$\mathrm{Na}(D,\star) := \{f/g \mid f, g \in D[X] \text{ and } \boldsymbol{c}(g)^\star = D^\star\}.$$

We then study the ideal structure of $\mathrm{Na}(D,\star)$ and compare it to that of $\mathrm{Kr}(D,\star)$. We also show how $\mathrm{Na}(D,\star)$ gives rise to a very natural semistar operation, denoted by $\tilde{\star}$, which plays a role analogous to that of the semistar operation $\star_a$ in the Kronecker setting. In the star operation case, $\tilde{\star}$, coincides with the operation $\star_w$ considered recently by D.D. Anderson and S.J. Cook [4].

In Section 2 we give some background information concerning semistar operations and some preliminary results concerning the class of quasi–$\star$–ideals (i.e. ideals I such that $I^\star \cap D = I$), a more general class than that of $\star$–ideals (i.e. ideals I such that $I^\star = I$), which plays an important role when $\star$ is a semistar operation.

In the third section we define and study the semistar Nagata rings. For instance, we show that there is a natural 1-1 correspondence between the maximal ideals of $\mathrm{Na}(D,\star)$ and the maximal elements in the set of all proper quasi–$\star$–ideals of D; in particular, the t–maximal ideals of an integral domain D are all obtained as contractions to D of the maximal ideals of $\mathrm{Na}(D,v)$. We prove also that $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star})$. Furthermore, we show that there is a strict link between the semistar operation $\tilde{\star}$, the maximal elements P in the set of all proper quasi–$\star$–ideals of D and the valuation overrings of D_P. More precisely, if we say that a $\star$–valuation overring of D is a valuation overring of D such that $F^\star \subseteq FV$ for each finitely generated fractionary ideal F of D, then we show that a valuation overring V of D is a $\tilde{\star}$–valuation overring of D if and only if V is an overring of D_P, for some P maximal in the set of all proper quasi–$\star$–ideals of D.

In the fourth section we recall from [17] some results concerning $\mathrm{Kr}(D,\star)$ and $\star_a$ and examine the interplay with $\mathrm{Na}(D,\star)$ and $\tilde{\star}$. In particular, we show that each maximal element Q in the set of all proper quasi–$\star_a$–ideals of D is determined uniquely by a $\star$–valuation overring of D (dominating D_Q) and there is a natural 1-1 correspondence between the maximal ideals of $\mathrm{Kr}(D,\star)$ and the minimal $\star$–valuation overrings of D.

In the final section we examine more closely the relationship between $\tilde{\star}$ and $\star_a$ and we show that it is hopeless to try to attain an equality by applying $\widetilde{(\text{-})}$ and $(\text{-})_a$ in different orders to an arbitrary semistar operation.

We use Gilmer's book [20] as our main reference. Any unexplained material is as in [20] and [32]. Many preliminary results on semistar operations and applications appear in conference proceedings (in particular, [17] and [18]), and hence are not easily available. Because of this possible hindrance, we briefly restate the principal definitions and statements of the main properties that we will need so that the present work will be self–contained.
Note that the "module systems" approach, developed very recently by Halter-Koch in [26], provides a general setting for (re)considering semistar operations and, in particular, many of the constructions related to the semistar operations considered in the present paper. However, since many background results of our paper are proved in an earlier work by Fontana and Huckaba [16], which has inspired and provided the foundation also of [26], we maintain the level of generality of this paper within the more classical "semistar" setting.

2. Background and preliminary results

For the duration of this paper D will represent an integral domain with quotient field K. Let $\overline{\boldsymbol{F}}(D)$ represent the set of all nonzero D–submodules of K. Let $\boldsymbol{F}(D)$ represent the nonzero fractionary ideals of D (i.e. $E \in \overline{\boldsymbol{F}}(D)$ such that $dE \subseteq D$ for some nonzero element $d \in D$). Finally, let $\boldsymbol{f}(D)$ represent the finitely generated D-submodules of K.

A mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$ is called *a semistar operation of D* if, for all $z \in K$, $z \neq 0$ and for all $E, F \in \overline{\boldsymbol{F}}(D)$, the following properties hold:

$(\star_1)$ $(zE)^\star = zE^\star$;

$(\star_2)$ $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;

$(\star_3)$ $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Remark 2.1. Let $\star$ be a semistar operation of D.

(a) If $\star$ is a semistar operation such that $D^\star = D$, then the map $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $E \mapsto E^\star$, is called a *star operation of* D. Recall from [20, (32.1)] that a star operation $\star$ verifies the properties $(\star_2)$, $(\star_3)$, for all $E, F \in \boldsymbol{F}(D)$; moreover, the property $(\star_1)$ can be restated as follows: for each $z \in K$, $z \neq 0$ and for each $E \in \boldsymbol{F}(D)$,

$(\star\star_1)$ $(zD)^\star = zD$, $(zE)^\star = zE^\star$.

If $\star$ is a semistar operation of D such that $D^\star = D$, then we will write often in the sequel that $\star$ is *a (semi)star operation of* D, to emphasize the fact that the semistar operation $\star$ is an extension to $\overline{\boldsymbol{F}}(D)$ of *a "classical" star operation* $\star$, i.e. a map $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, verifying the properties $(\star\star_1)$, $(\star_2)$ and $(\star_3)$ [20, Section 32]. Note that not every semistar operation is an extension of a star operation [16, Remark 1.5 (b)].

(b) The *trivial semistar operation on* D is the semistar operation constant onto K, i.e. the semistar operation $\star$ such that $E^\star = K$, for each $E \in \overline{\boldsymbol{F}}(D)$. Note that $\star$ is the trivial semistar operation on D if and only if $D^\star = K$. (As a matter of fact, if $D^\star = K$, then for each $E \in \overline{\boldsymbol{F}}(D)$ and for each $e \in E$, $e \neq 0$, we have $eD \subseteq E$ and thus $K = eK = eD^\star \subseteq E^\star \subseteq K$.)

(c) Let D be an integral domain and T an overring of D. Let $\star$ be a semistar operation of D and define $\dot{\star}^T : \overline{\boldsymbol{F}}(T) \to \overline{\boldsymbol{F}}(T)$ by setting:

$$E^{\dot{\star}^T} := E^\star, \text{ for each } E \in \overline{\boldsymbol{F}}(T) \ (\subseteq \overline{\boldsymbol{F}}(D)).$$

Then, we know [17, Proposition 2.8]:

(c.1) The operation $\dot{\star}^T$ is a semistar operation of T.

(c.2) When $T = D^\star$, then $\dot{\star}^{D^\star}$ defines a (semi)star operation of $D^\star$.

(d) If $\star_1$ and $\star_2$ are two semistar operation of D, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(D)$; in this case, $(E^{\star_1})^{\star_2} = E^{\star_2}$.

We refer to the collection $\overline{\boldsymbol{F}}^\star(D) := \{E^\star \mid E \in \overline{\boldsymbol{F}}(D)\}$ [respectively, $\boldsymbol{F}^\star(D) := \{I \in \boldsymbol{F}(D) \mid I = H^\star \text{ with } H \in \boldsymbol{F}(D)\}$; $\boldsymbol{f}^\star(D) := \{J \in \boldsymbol{F}(D) \mid J = F^\star \text{ with } F \in \boldsymbol{f}(D)\}$] as *the $\star$–D–submodules of K* [respectively, *the (fractionary) $\star$–ideals of D*; *the (fractionary) $\star$–ideals of D of finite type*].

These labels seem natural, but can be problematic. As a matter of fact, if $I \in \boldsymbol{F}(D)$, then $I^\star$ is not necessarily a fractionary ideal of D and so it does not necessarily belong to $\boldsymbol{F}^\star(D)$ (e.g. if $(D : D^\star) = 0$, then $D^\star \notin \boldsymbol{F}^\star(D)$). For instance if T is an overring of an integral domain D such that the conductor $(D : T) = 0$ and if $\star := \star_{\{T\}}$ is the semistar operation of D defined by $E^{\star_{\{T\}}} := ET$ for each $E \in \overline{\boldsymbol{F}}(D)$, then it is easy to see that $\boldsymbol{F}^\star(D)$ is empty. So we need a more general notion than $\star$–ideal, when $\star$ is a semistar operation.

Let $I \subseteq D$ be a nonzero ideal of D and let $\star$ be a semistar operation on D. We say that I is *a quasi–$\star$–ideal of* D if $I^\star \cap D = I$. Similarly, we designate by *quasi–$\star$–prime* [respectively, *$\star$–prime*] of D a quasi–$\star$–ideal [respectively, an integral $\star$–ideal] of D which is also a prime ideal. We designate by *quasi–$\star$–maximal* [respectively, *$\star$–maximal*] of D a maximal element in the set of all proper quasi–$\star$–ideals [respectively, integral $\star$–ideals] of D.

Note that if $I \subseteq D$ is a $\star$–ideal, it is also a quasi–$\star$–ideal and, when $D = D^\star$ the notions of quasi–$\star$–ideal and integral $\star$–ideal coincide. When $D \subsetneq D^\star \subsetneq K$, we can "restrict" the semistar operation $\star$ on D to the nontrivial (semi)star operation on $D^\star$, denoted by $\dot{\star}^{D^\star}$, or simply by $\dot{\star}$, and defined in Remark 2.1 (c), and we have a strict relation between the quasi–$\star$–ideals of D and the $\dot{\star}$–ideals of $D^\star$.

Lemma 2.2. *Assume the notation of the preceding paragraph. Then:*
I is a quasi–$\star$–ideal of D $\Leftrightarrow$ $I = L \cap D$, where $L \subseteq D^\star$ is a $\dot{\star}$–ideal of $D^\star$.

Proof. The proof follows from $I^\star \cap D \subseteq (I^\star \cap D)^\star \cap D \subseteq I^{\star\star} \cap D^\star \cap D = I^\star \cap D$. □

Note that this also gives a means of constructing quasi–$\star$–ideals and, in particular, quasi–$\star$–ideals containing a given ideal. If $I \subseteq D$ is a nonzero ideal of D, then $I^\star \cap D$ is a quasi–$\star$–ideal of D which contains I.
We denote by $\mathrm{Spec}^\star(D)$ [respectively, $\mathrm{Max}^\star(D)$; $\mathrm{QSpec}^\star(D)$; $\mathrm{QMax}^\star(D)$] the set of all $\star$–primes [respectively, $\star$–maximals; quasi–$\star$–primes; quasi–$\star$–maximals] of D.

As in the classical star-operation setting, we associate to a semistar operation $\star$ of D a new semistar operation $\star_f$ as follows. Let $\star$ be a semistar operation of a domain D. If $E \in \overline{\boldsymbol{F}}(D)$ we set $E^{\star_f} := \cup\{F^\star \mid F \subseteq E,\, F \in \boldsymbol{f}(D)\}$.
We call $\star_f$ *the semistar operation of finite type of* D associated to $\star$. If $\star = \star_f$, we say that $\star$ is *a semistar operation of finite type of* D. Note that $\star_f \leq \star$ and $(\star_f)_f = \star_f$, so $\star_f$ is a semistar operation of finite type of D. For instance, if v is *the v–(semi)star operation on* D defined by $E^v := (E^{-1})^{-1}$, for each $E \in \overline{\boldsymbol{F}}(D)$, with $E^{-1} := (D :_K E) := \{z \in K \mid zE \subseteq D\}$ [16, Example 1.3 (c) and Proposition 1.6 (5)], then the semistar operation of finite type v_f associated to v is called *the t–(semi)star operation on* D (in this case $D^v = D^t = D$).

Both the Kronecker function rings and the Nagata rings considered in the present paper are defined in a natural way for a general semistar operation. A principal theme of the paper is that both of these classes of rings can be recast as Kronceker function rings and Nagata rings of certain natural semistar operations of finite type. So the entire theory could be stated in terms of semistar operations of finite type. It seems worthwhile to us to keep the more general setting so that, for example, we can talk about the Kronecker function ring and the Nagata ring associated to the classical v operation (which is rarely of finite type).

Lemma 2.3. *Let $\star$ be a semistar operation of an integral domain D. Assume that $\star$ is not trivial and that $\star = \star_f$. Then*

- **(1)** *Each proper quasi–$\star$–ideal is contained in a quasi-$\star$-maximal.*
- **(2)** *Each quasi–$\star$–maximal is a quasi–$\star$-prime.*
- **(3)** *If Q is a quasi–$\star$–maximal ideal of D then $Q = M \cap D$, for some $\dot{\star}$–maximal ideal M of $D^\star$.*
- **(4)** *If $L \subseteq D^\star$ is a $\dot{\star}$–prime ideal of $D^\star$, then $L \cap D$ is a quasi–$\star$–prime ideal of D.*
- **(5)** *Set $\Pi^\star := \{P \in \mathrm{Spec}(D) \mid P \neq 0$ and $P^\star \cap D \neq D\}$.*
  Then $\mathrm{QSpec}^\star(D) \subseteq \Pi^\star$ and the set of maximal elements of $\Pi^\star$, denoted by $\Pi^\star_{\max}$, is nonempty and coincides with $\mathrm{QMax}^\star(D)$.

Proof. The proof is straightforward. □

Note that, in general, the restriction to D of a $\dot{\star}$–maximal ideal of $D^\star$ is a quasi–$\star$–prime ideal of D, but not necessarily a quasi–$\star$–maximal ideal of D, and if L is an ideal of $D^\star$ and $L \cap D$ is a quasi–$\star$–prime ideal of D, then L is not necessarily a $\dot{\star}$–prime ideal of $D^\star$, (cf. the Remark 3.6).

For the sake of simplicity, when $\star = \star_f$, we will denote simply by $\mathcal{M}(\star)$, the nonempty set $\Pi^\star_{\max} = \mathrm{QMax}^\star(D)$.

If Δ is a nonempty set of prime ideals of an integral domain D, then the semistar operation $\star_\Delta$ defined on D as follows

$$E^{\star_\Delta} := \cap\{ED_P \mid P \in \Delta\}\,, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D)\,,$$

is called *the spectral semistar operation associated to* Δ. If $\Delta = \emptyset$, then we can extend the previous defintion by setting $E^{\star_\emptyset} := K$, for each $E \in \overline{\boldsymbol{F}}(D)$, i.e. $\star_\emptyset$ is the trivial semistar operation on D (constant onto K; cf. Remark 2.1 (b)).

Lemma 2.4. *Let D be an integral domain and let $\emptyset \neq \Delta \subseteq \mathrm{Spec}(D)$. Then:*

(1) *$E^{\star_\Delta} D_P = ED_P$, for each $E \in \overline{\boldsymbol{F}}(D)$ and for each $P \in \Delta$.*
(2) *$(E \cap F)^{\star_\Delta} = E^{\star_\Delta} \cap F^{\star_\Delta}$, for all $E, F \in \overline{\boldsymbol{F}}(D)$.*
(3) *$P^{\star_\Delta} \cap D = P$, for each $P \in \Delta$.*
(4) *If I is a nonzero integral ideal of D and $I^{\star_\Delta} \cap D \neq D$ then there exists $P \in \Delta$ such that $I \subseteq P$.*
(5) *Assume that the set of maximal elements $\Delta_{\max}$ of Δ is also nonempty and that each $P \in \Delta$ is contained in some $Q \in \Delta_{\max}$. Then:*

$$\star_\Delta = \star_{\Delta_{\max}}\,.$$

Proof: [16, Lemma 4.1] and, for (5), [16, Remark 4.5]. □

A semistar operation $\star$ of an integral domain D is called *a spectral semistar operation* if there exists $\Delta \subseteq \mathrm{Spec}(D)$ such that $\star = \star_\Delta$. We say that $\star$ *posesses enough primes* or that $\star$ is *a quasi-spectral semistar operation of* D if, for each nonzero ideal I of D such that $I^\star \cap D \neq D$, there exists a quasi–$\star$–prime P of D such that $I \subseteq P$. Finally, we say that $\star$ is *a stable semistar operation on* D if $(E \cap F)^\star = E^\star \cap F^\star$, for all $E, F \in \overline{\boldsymbol{F}}(D)$.

Lemma 2.5. *Let $\star$ be a nontrivial semistar operation of an integral domain D.*

(1) *$\star$ is spectral if and only if $\star$ is quasi-spectral and stable.*
(2) *Assume that $\star = \star_f$. Then $\star$ is quasi-spectral and $\mathcal{M}(\star) \neq \emptyset$.*

Proof. (1) The "only if" part is a consequence of Lemma 2.4 (2) and (4). The "if" part is proved in [16, Theorem 4.12 (3)]. (2) is a restatement of Lemma 2.3. □

If $\star$ is a semistar operation of an integral domain D and if $\Pi^\star \neq \emptyset$, the nontrivial semistar operation

$$\star_{sp} := \star_{\Pi^\star}$$

is called *the spectral semistar operation associated to* $\star$.

Lemma 2.6. *Let $\star$ be a nontrivial semistar operation.*

(1) *$\star$ is spectral if and only if $\star = \star_{sp}$.*
(2) *Assume that $\Pi^\star \neq \emptyset$. Then the following statements are equivalent:*

(i) *$\star_{sp} \leq \star$;*
(ii) *$\star$ is quasi-spectral;*
(iii) *$E^\star = \cap\{E^\star D_P \mid P \in \Pi^\star\}$, for each $E \in \overline{\boldsymbol{F}}(D)$.*

Proof: (1) [16, Corollary 4.10]; (2) [16, Proposition 4.8]. □

Corollary 2.7. *Let $\star$ be a nontrivial semistar operation. Set*

$$\tilde{\star} := (\star_f)_{sp}\,.$$

(1) *$\tilde{\star} = \star_{\mathcal{M}(\star_f)} \leq \star_f$ (in particular, $\tilde{\star}$ is not trivial).*
(2) *For each $E \in \overline{\boldsymbol{F}}(D)$,*
  (a) *$E^{\star_f} = \cap\{E^{\star_f} D_Q \mid Q \in \mathcal{M}(\star_f)\}$;*
  (b) *$E^{\tilde{\star}} = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$.*

Proof. (1) is a consequence of Lemma 2.5 (1), Lemma 2.4 (5), Lemma 2.6 (2) and Lemma 2.3 (3). (2). The first equality follows from Lemma 2.6 (2) ((ii) ⇒ (iii)), Lemma 2.5 (1) and Lemma 2.3 (3). The second equality follows from (1) and from the definition of spectral semistar operation. □

Remark 2.8. **(a)** Note that, when $\star$ is the (semi)star v–operation, then the (semi)star operation $\tilde{v}$ coincides with *the (semi)star operation* w defined as follows:

$$E^{w} := \cup\{(E : H) \mid H \in \boldsymbol{f}(D) \text{ and } H^{v} = D\}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D).$$

This (semi)star operation was first considered by J. Hedstrom and E. Houston in 1980 [27, Section 3] under the name of F_{∞}–operation. Later, starting in 1997, this operation was intensively studied by W. Fanggui and R. McCasland (cf. [12], [13], [14], [15],) under the name of w–operation. Note also that the notion of w–ideal coincides with the notion of semi-divisorial ideal considered by S. Glaz and W. Vasconcelos in 1977 [22]. Finally, in 2000, for each (semi)star operation $\star$, D.D. Anderson and S.J. Cook [4] considered the $\star_w$–operation which can be defined as follows:

$$E^{\star_w} := \cup\{(E : H) \mid H \in \boldsymbol{f}(D) \text{ and } H^{\star} = D\}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D).$$

From their theory it follows that $\star_w = \tilde{\star}$ [4, Corollary 2.10]. The relation between $\tilde{\star}$ and the localizing systems of ideals was established in [16].

(b) If Δ is a nonempty quasi-compact subset of $\mathrm{Spec}(D)$, then $\star_{\Delta} = (\star_{\Delta})_f$ and $\mathcal{M}(\star_{\Delta}) = \Delta_{\max}$, [16, Proposition 4.3 (B)].

The collection of all quasi–$\star$–ideals of a domain D, associated to a given semistar operation $\star$, can be an unwieldy object. We now turn to the use of ultrafilters to gain some control over this collection. A similar course was followed in [9] for the special case of the t–operation. We generalize the results given there. We begin with some notation/terminology/definitions.

- Let D be a domain and let $\mathcal{J} = \mathcal{J}(\Lambda) := \{J_{\lambda} \mid \lambda \in \Lambda\}$ be a collection of ideals of D.
- Let $\mathcal{U} = \mathcal{U}(\Lambda)$ be an ultrafilter on the index set Λ given above.
- For $I \subseteq D$ let $B(I) := \{\lambda \in \Lambda \mid I \subseteq J_{\lambda}\}$.
- Let $J_{\mathcal{U}} := \{d \in D \mid B(d) \in \mathcal{U}\}$, more explicitly $J_{\mathcal{U}} = \cup\{\ \cap\{J_{\lambda} \mid \lambda \in B\} \mid B \in \mathcal{U}\ \}$. We call $J_{\mathcal{U}}$ *the* $\mathcal{U}$*–ultrafilter limit of the collection* $\mathcal{J}$.

Proposition 2.9. *Assume the notation/terminology/definitions given above. Assume also that $\star$ is a star operation on D and that each $J_{\lambda} \in \mathcal{J}$ is a $\star_f$–ideal [respectively, $\star_f$–prime] of D. If $J_{\mathcal{U}}$ is nonzero, then it is also a $\star_f$–ideal [respectively, $\star_f$–prime] of D.*

Proof. The proof is the same as that given in [9, Proposition 2.5] with the t–operation replaced by $\star$ an arbitrary star operation of finite type. The "prime ideal part" of the statement follows from [9, Lemma 2.4]. □

Corollary 2.10. *Generalize the setting of Proposition 2.9 to the case where $\star$ is a semistar operation and each $J_{\lambda} \in \mathcal{J}$ is a proper quasi–$\star_f$–ideal [respectively, quasi–$\star_f$–prime] of D. If $J_{\mathcal{U}}$ is nonzero, then it is also a proper quasi–$\star_f$–ideal [respectively, quasi–$\star_f$–prime] of D.*

Proof. For ease of notation we set $* := \star_f$. As noted in Lemma 2.2, the quasi–$*$–ideals of D are precisely the contractions to D of the $\dot{*}$–ideals of D^* (where $\dot{*}$ is a (semi)star operation on the domain $D^{\star_f}$ defined in Remark 2.1 (c)). The result follows easily by using Proposition 2.9 and Lemma 2.3. □

3. Semistar Nagata Rings

If R is a ring and X an indeterminate over R, then the ring:

$$R(X) := \{f/g \mid f, g \in R[X] \text{ and } \boldsymbol{c}(g) = R\}$$

is called *the Nagata ring of* R [20, Proposition 33.1]. Some results proved in [31, Proposition 2.1] are generalized in the following:

Proposition 3.1. *Let $\star$ be a nontrivial semistar operation of an integral domain D. Set $N(\star) := N_D(\star) := \{h \in D[X] \mid h \neq 0$ and $\boldsymbol{c}(h)^\star = D^\star\}$.*

(1) *$N(\star) = N(\star_f)$ is a saturated multiplicatively closed subset of $D[X]$.*
(2) *$N(\star) = D[X] \setminus \cup\{Q[X] \mid Q \in \mathcal{M}(\star_f)\}$.*
(3) *$\mathrm{Max}(D[X]_{N(\star)}) = \{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$.*
(4) *$D[X]_{N(\star)} = \cap\{D_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.*
(5) *$\mathcal{M}(\star_f)$ coincides with the canonical image in $\mathrm{Spec}(D)$ of the maximal spectrum of $D[X]_{N(\star)}$; i.e. $\mathcal{M}(\star_f) = \{M \cap D \mid M \in \mathrm{Max}(D[X]_{N(\star)})\}$.*

Proof. (1) It is obvious that $N(\star) = N(\star_f)$; the remaining part is a standard consequence of (2) [32, Theorem 2].

(2) We start by proving the following:

Claim: *Let $h \in D[X], h \neq 0$. Then:*

$$\boldsymbol{c}(h)^\star = D^\star \;\Leftrightarrow\; \boldsymbol{c}(h) \not\subseteq Q\,, \text{ for each } Q \in \mathcal{M}(\star_f)\,.$$

($\Leftarrow$) If $\boldsymbol{c}(h)^\star \neq D^\star$, then $0 \neq \boldsymbol{c}(h) \subseteq \boldsymbol{c}(h)^\star \cap D \subsetneq D$. Since $\boldsymbol{c}(h)^{\star_f} \cap D = \boldsymbol{c}(h)^\star \cap D$ is a proper quasi–$\star$–ideal of D, we can find $Q \in \mathcal{M}(\star_f)$ such that $\boldsymbol{c}(h) \subseteq \boldsymbol{c}(h)^\star \cap D \subseteq Q$ (Lemma 2.3 (1)). This fact contradicts the assumption.

($\Rightarrow$) is trivial.

Using the claim, we have:

$$\begin{aligned} h \in N(\star) \;\Leftrightarrow\; \boldsymbol{c}(h)^\star = D^\star \;&\Leftrightarrow\; \boldsymbol{c}(h) \not\subseteq Q\,, \text{ for each } Q \in \mathcal{M}(\star_f) \;\Leftrightarrow \\ &\Leftrightarrow\; h \not\subseteq Q[X]\,, \text{ for each } Q \in \mathcal{M}(\star_f)\,. \end{aligned}$$

(3) By using [20, (4.7) and Proposition 4.8], it is sufficient to show that each prime ideal H of $D[X]$ contained inside $\cup\{Q[X] \mid Q \in \mathcal{M}(\star_f)\}$ is contained in $Q[X]$, for some $Q \in \mathcal{M}(\star_f)$. Let $\boldsymbol{c}(H)$ be the ideal generated by $\{\boldsymbol{c}(h) \mid h \in H\}$. It is easy to see that $\boldsymbol{c}(H)$ is an ideal of D and that:

$$H \subseteq \cup\{Q[X] \mid Q \in \mathcal{M}(\star_f)\} \;\Rightarrow\; \boldsymbol{c}(H)^{\star_f} \neq D^{\star_f} = D^\star\,.$$

As a matter of fact, if $\boldsymbol{c}(H)^{\star_f} = D^\star$, then we can find a polynomial $\ell \in \boldsymbol{c}(H)[X]$ such that $\boldsymbol{c}(\ell)^\star = \boldsymbol{c}(\ell)^{\star_f} = D^\star$. Now,

$$\ell \in \boldsymbol{c}(h_1)[X] + \boldsymbol{c}(h_2)[X] + \ldots + \boldsymbol{c}(h_r)[X] = (\boldsymbol{c}(h_1) + \boldsymbol{c}(h_2) + \ldots + \boldsymbol{c}(h_r))[X]$$

with $(h_1, h_2, \ldots, h_r) \subseteq H$. Since $\boldsymbol{c}(h_1) + \boldsymbol{c}(h_2) + \ldots + \boldsymbol{c}(h_r) \subseteq \boldsymbol{c}(H)$ and $\boldsymbol{c}(H)$ is an ideal of D, then $\boldsymbol{c}(h_1) + \boldsymbol{c}(h_2) + \ldots + \boldsymbol{c}(h_r) = \boldsymbol{c}(h)$, for some $h \in H$. Therefore $\boldsymbol{c}(\ell) \subseteq \boldsymbol{c}(h)$ and thus $\boldsymbol{c}(\ell)^\star = \boldsymbol{c}(h)^\star = D^\star$. This is a contradiction, since $h \in H$ and thus $\boldsymbol{c}(h)^\star = \boldsymbol{c}(h)^{\star_f} \subseteq Q$, for some $Q \in \mathcal{M}(\star_f)$. By the fact that $\boldsymbol{c}(H)^{\star_f} \neq D^{\star_f}$ we deduce that $\boldsymbol{c}(H) \subseteq Q$, for some $Q \in \mathcal{M}(\star_f)$. This implies that $H \subseteq Q[X]$, for some $Q \in \mathcal{M}(\star_f)$.

(4) and (5) are easy consequences of (3), since:

$$(D[X]_{N(\star)})_{Q[X]_{N(\star)}} = D[X]_{Q[X]} = D_Q(X)\,,$$

(cf. also [20, Corollary 5.3 and Proposition 33.1]). □

We set:

$$\mathrm{Na}(D, \star) := D[X]_{N_D(\star)}$$

and we call it *the Nagata ring of D with respect to the semistar operation $\star$*. Obviously, $\mathrm{Na}(D, \star) = \mathrm{Na}(D, \star_f)$. If $\star = d$ is the identity (semi)star operation of D, then $\mathrm{Na}(D, d) = D(X)$.

Corollary 3.2. *Let D be an integral domain, then:*

Q is a maximal t–ideal of D $\Leftrightarrow$ $Q = M \cap D$, for some $M \in \mathrm{Max}(\mathrm{Na}(D, v))$.

Proof. It is a straightforward consequence of Proposition 3.1 (5). □

Example 3.3. (1) Let P be a nonzero prime ideal of an integral domain D and let $\star := \star_{\{D_P\}}$ be the semistar operation of D defined as follows:

$$E^{\star_{\{D_P\}}} := ED_P\,, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D)\,.$$

Then, *it is easy to verify that:*

(a) $\mathcal{M}(\star_f) = \{P\}$;

(b) $\mathrm{Na}(D, \star) = D_P(X)$;

(c) $\star = \star_f = \star_{sp} = \tilde{\star}$.

(2) The previous example can be generalized as follows. *Let D be an integral domain, let Δ be a nonempty subset of* $\mathrm{Spec}(D)$ *and set $\star := \star_\Delta$. Let $\Delta_{\max}$ be the set of all the maximal elements of Δ and let*

$$\Delta^{\downarrow} := \{H \in \mathrm{Spec}(D) \mid H \subseteq P\,, \ \textit{for some } \ P \in \Delta\}\,.$$

Assume that each $P \in \Delta$ is contained in some $Q \in \Delta_{\max}$. Then, under the previous assumptions, $\star = \star_{\Delta_{\max}}$ (Lemma 2.4 (5)) and moreover:

(a) $\Delta \subseteq \mathrm{QSpec}^\star(D) \subseteq \Delta^{\downarrow}$, *thus* $\mathrm{QMax}^\star(D) = \Delta_{\max}$.

Assume also that $\Delta_{\max}$ is a quasi-compact subspace of $\mathrm{Spec}(D)$. *Then:*

(b) $\mathrm{Na}(D, \star_\Delta) = \cap\{D_Q(X) \mid Q \in \Delta_{\max}\} = \cap\{D_P(X) \mid P \in \Delta\}$.

(c) $\widetilde{(\star_\Delta)} = \star_\Delta$.

Proof. (a) If $P \in \Delta$, then $P^\star = P^{\star_\Delta} = PD_P \cap (\,\cap\{D_{P'} \mid P' \in \Delta\,,\ P' \not\subseteq P\}) = PD_P \cap D^{\star_\Delta}$, and so $P \subseteq P^\star \cap D \subseteq PD_P \cap D^{\star_\Delta} \cap D = PD_P \cap D = P$. This shows that $\Delta \subseteq \mathrm{QSpec}^\star(D)$. Let $H \in \mathrm{QSpec}^\star(D)$. Then $H^\star \neq D^\star$ and so, for some $P \in \Delta$, $HD_P = H^\star D_P \subseteq PD_P \neq D_P$, (cf. also Lemma 2.3 (1)). Henceforth, $H \subseteq HD_P \cap D \subseteq PD_P \cap D = P$.

(b) If $\Delta_{\max}$ is quasi-compact, then $\star = \star_\Delta = \star_{\Delta_{\max}} = (\star_{\Delta_{\max}})_f = \star_f$, [16, Corollary 4.6 (2)] and so, by (a) and Lemma 2.3 (3), $\Delta_{\max} = \mathcal{M}((\star_\Delta)_f) = \mathrm{QMax}^{\star_f}(D) = \mathrm{QMax}^\star(D)$. The conclusion follows from Proposition 3.1 (4).

(c) Since $\star_\Delta = (\star_\Delta)_f$ and $\Delta_{\max} = \mathcal{M}((\star_\Delta)_f)$ (Remark 2.8 (b)), then we conclude by Corollary 2.7 and Lemma 2.6 (cf. also Lemma 2.4 (1)). □

Proposition 3.4. *Let $\star$ be a nontrivial semistar operation of an integral domain D with quotient field K. Let $\tilde{\star} := (\star_f)_{sp}$ be the spectral semistar operation considered in Corollary 2.7. For each $E \in \overline{\boldsymbol{F}}(D)$, we have:*

(1) $E\mathrm{Na}(D, \star) = \cap\{ED_Q(X) \mid Q \in \mathcal{M}(\star_f)\}$.

(2) $E\mathrm{Na}(D, \star) \cap K = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$.

(3) $E^{\tilde{\star}} = E\mathrm{Na}(D, \star) \cap K$, *hence if $E = E^\star$, then $E = E\mathrm{Na}(D, \star) \cap K$.*

(4) *If $\star = \star_f$, then $\tilde{\star} = \star_{sp}$, hence $D^{\star_{sp}} = \cap\{D_Q \mid Q \in \mathcal{M}(\star)\}$, and $\star_{sp}$ is a semistar operation of finite type.*

Proof. (1) By Proposition 3.1 (3) and [8, Chapitre 2, Corollaire 3, p. 112] (see also the proof of Proposition 3.1 (4)), we have:

$$\begin{aligned} E\mathrm{Na}(D,\star) \ &= \ \cap\{(ED[X]_{N(\star)})_M \mid M \in \mathrm{Max}(D[X]_{N(\star)}\} = \\ &= \ \cap\{(ED[X]_{N(\star)})_{Q[X]_{N(\star)}} \mid Q \in \mathcal{M}(\star_f)\} = \\ &= \cap\,\{ED[X]_{Q[X]} \mid Q \in \mathcal{M}(\star_f)\} = \cap\{ED_Q(X) \mid Q \in \mathcal{M}(\star_f)\}\,. \end{aligned}$$

(2) By using (1) and [20, Proposition 33.1 (4)], we have:

$$\begin{aligned} ED[X]_{N(\star)} \cap K &= \cap\{ED_Q(X) \mid Q \in \mathcal{M}(\star_f)\} \ \cap \ K = \\ &= \cap\{ED_Q(X) \cap K \mid Q \in \mathcal{M}(\star_f)\} = \\ &= \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}. \end{aligned}$$

(3) From Corollary 2.7 (2) we know that $E^{\tilde{\star}} = \cap\{ED_Q \mid Q \in \mathcal{M}(\star_f)\}$, thus the first statement of (3) is a straightforward consequence of (2). Since $\tilde{\star} \leq \star_f \leq \star$ (Corollary 2.7 (1)), then obviously if $E = E^\star$, then $E = E^{\tilde{\star}}$ and thus $E = E\mathrm{Na}(D,\star) \cap K$.

For a direct proof of the second statement of (3), it is enough to show that, if $z \in E\mathrm{Na}(D,\star) \cap K$, then $z \in E$. Let $g, h \in D[X]$ with $h \neq 0$, $\boldsymbol{c}(h)^\star = D^\star$ and $zh = g$. Then $\boldsymbol{c}(g)^\star = \boldsymbol{c}(zh)^\star = z\boldsymbol{c}(h)^\star = zD^\star = (zD)^\star \subseteq E^\star = E$.

(4) follows directely from the definitions, [16, Proposition 3.2] and from (2) and (3). □

Corollary 3.5. *Let $\star$ be a nontrivial semistar operation of an integral domain D, let $\tilde{\star}$ be the semistar operation of D considered in Corollary 2.7, and let $\dot{\tilde{\star}}$ $(:= \dot{\tilde{\star}}^{D^{\tilde{\star}}})$ be the (semi)star operation of $D^{\tilde{\star}}$ associated to $\tilde{\star}$ (Remark 2.1).*

- **(1)** $(\tilde{\star})_f = \tilde{\star} = (\tilde{\star})_{sp} = \tilde{\tilde{\star}}$.
- **(2)** $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.
- **(3)** $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D^{\tilde{\star}}, \dot{\tilde{\star}})$.

Proof. (1) follows easily from Proposition 3.4 (4) and Lemma 2.6.

(2) Let $Q \in \mathcal{M}(\star_f)$. Then $Q^{\star_f} \cap D = Q$. Since $\tilde{\star} \leq \star_f$ (Corollary 2.7 (1)), then necessarily $Q^{\tilde{\star}} \cap D = Q$. By (1) we know that $\tilde{\star}$ is a semistar operation of finite type. Hence we know that the quasi–$\tilde{\star}$–ideal Q of D is contained in some $H \in \mathcal{M}(\tilde{\star})$ (Lemma 2.3 (1)).

Conversely, let $H \in \mathcal{M}(\tilde{\star})$. Then $H = H^{\tilde{\star}} \cap D = \cap\{HD_Q \mid Q \in \mathcal{M}(\star_f)\} \cap D$ (Corollary 2.7 (2)). In particular, we have $HD_Q \neq D_Q$ for some $Q \in \mathcal{M}(\star_f)$, since otherwise $H^{\tilde{\star}} \cap D$ would be equal to D. Therefore, $H = H^{\tilde{\star}} \cap D \subseteq HD_Q \cap D \subseteq QD_Q \cap D = Q$, for some $Q \in \mathcal{M}(\star_f)$.
By the previous properties, we deduce immediately that $\mathcal{M}(\star_f) = \mathcal{M}(\tilde{\star})$.

(3) Since, from (1), we know that $\tilde{\star}$ is a semistar operation of finite type, then, by Proposition 3.1 (4), $\mathrm{Na}(D,\tilde{\star}) = \cap\{D_H(X) \mid H \in \mathcal{M}(\tilde{\star})\}$. Since, by (2), we know that $\mathcal{M}(\tilde{\star}) = \mathcal{M}(\star_f)$, then we have $\mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D,\star_f) = \mathrm{Na}(D,\star)$.

Claim. $\mathcal{M}(\dot{\tilde{\star}}) = \{QD_Q \cap D^{\tilde{\star}} \mid Q \in \mathcal{M}(\star_f)\}$.

If $M \in \mathcal{M}(\dot{\tilde{\star}})$, then $M = M^{\dot{\tilde{\star}}} = M^{\tilde{\star}} = \cap\{MD_Q \mid Q \in \mathcal{M}(\tilde{\star}) = \mathcal{M}(\star_f)\}$, hence $MD_Q \neq D_Q$, thus $M \subseteq QD_Q \cap D^{\tilde{\star}}$, for some $Q \in \mathcal{M}(\star_f)$. On the other hand, it is easy to verify that $QD_Q \cap D^{\tilde{\star}}$ is a $\dot{\tilde{\star}}$–ideal of $D^{\tilde{\star}}$ (Lemma 2.2), hence the claim is proved.

The last equality in (3) is a straightforward consequence of Proposition 3.1 (4), of the Claim and of the fact that $D^{\tilde{\star}}_{QD_Q \cap D^{\tilde{\star}}} = D_Q$, for each $Q \in \mathcal{M}(\star_f)$. □

Remark 3.6. Let $\star$ be a nontrivial semistar operation of an integral domain D, let $\tilde{\star}$ be the semistar operation considered in Corollary 2.7 and let $\dot{\star} := \dot{\star}^{D^\star}$ [respectively, $\tilde{\dot{\star}} := \tilde{\dot{\star}}^{D^\star}$] be the (semi)star operation of $D^\star$ associated to $\star$ [respectively, $\dot{\star}$] and defined in Remark 2.1 (c) [respectively, Corollary 2.7]. Then, in general, *the semistar Nagata ring* $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star})$ *is different from the star Nagata ring* $\mathrm{Na}(D^\star, \dot{\star}) = \mathrm{Na}(D^\star, \tilde{\dot{\star}})$.

Let L be a field and X, Y, U, and Z indeterminates over L. Set:
$V := L(X)[Y]_{(Y)} = L(X) + M$, $M := Y L(X)[Y]_{(Y)}$, $R := L + M$.

It is well known that R and V are 1-dimensional local domains with the same field of quotients $F := L(X,Y)$, with the same set of prime ideals $\{(0), M\}$, and moreover V is a discrete valuation domain, $\dim(V[U]) = 2$ and $\dim(R[U]) = 3$. The last property follows from the fact that in $R[U]$ we have the following inclusions of prime ideals:

$$\begin{aligned}(0) \subset Q_1 &:= (U-X)F[U] \cap R[U] = (YU - YX)F[U] \cap R[U] \subset \\ &\subset Q_2 := M[U] \subset Q_3 := (M,U)R[U]\,,\end{aligned}$$

but, a similar inclusion does not hold in $V[U]$:

$$(0) \subset P_1 := (U-X)F[U] \cap V[U] = (U-X)V[U] \not\subseteq M[U]\,;$$

in fact, more generally, no height 1 prime ideal P of $V[U]$, with $P \cap V = (0)$, is contained in $M[U]$ [32, Theorem 39, Theorem 68; Exercise 18, page 42].

Set $D := R(U)$, $T := V(U)$, and let $\star := \star_{\{T\}}$ be the semistar operation of D considered in Remark 2.1 (d). Since all the prime ideals of D [respectively, of T], different from (0) and $M[U]$, are of the type $fF[U] \cap D$ [respectively, $fF[U] \cap T$], where $f \in F[U]$ is irreducible [32, Theorem 36], then it follows that the canonical map $\mathrm{Spec}(T) \to \mathrm{Spec}(D)$ is a bijection. From this fact we deduce immediately that $\mathcal{M}(\star) = \mathrm{Max}(D)$, and, hence, that $\tilde{\star} = d_D$, where d_D is the identity (semi)star operation of D.

Note that, in the present situation, $D^\star = T$ and it is obvious that $\dot{\star}$ coincides with d_T, where d_T is the identity (semi)star operation of T, and hence $\dot{\tilde{\star}} = \dot{d_D} = d_T$. We deduce that $\mathcal{M}(\dot{\star}) = \mathcal{M}(\dot{\tilde{\star}}) = \mathrm{Max}(T)$ and, obviously, that $\widetilde{d_T} = d_T$.

Furthermore, note that P_1T is a maximal ideal of T (because $P_1 \not\subseteq M[U] \subset V[U]$), but $P_1T \cap D = Q_1D$ is not maximal in D (because $Q_1 \subset M[U] \subset R[U]$). Therefore the statement in Lemma 2.3 (3) is not reversible. Note also that this example shows that $(Q_1D)^\star = Q_1T \subsetneq P_1T$ (because $Q_1V[U] = (YU - YX)V[U] \subsetneq P_1 = (U-X)V[U]$) and so $(Q_1D)^\star$ is not a $\dot{\star}$–prime of $D^\star = T$, even though $(Q_1D)^\star \cap D = Q_1D$ is a quasi–$\star$–prime of D; so also the statement in Lemma 2.3 (4) is not reversible.

Finally note that:

$$\begin{aligned}\mathrm{Na}(D,\star) &= \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D^{\tilde{\star}}, \dot{\tilde{\star}}) = \mathrm{Na}(D, d_D) = D(Z) \subsetneq \\ &\subsetneq \mathrm{Na}(D^\star, \dot{\star}) = \mathrm{Na}(D^\star, \tilde{\dot{\star}}) = \mathrm{Na}(T, d_T) = T(Z)\,.\end{aligned}$$

Corollary 3.7. *Let $\star$ be a semistar operation of an integral domain D. Assume that $\Pi^\star \neq \emptyset$ and that $\star$ is quasi-spectral. Then:*

$$\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star_{sp}) = \mathrm{Na}(D,\tilde{\star})\,.$$

Proof. Under the present assumptions, we can define the nontrivial semistar operation $\star_{sp}$ and we have that $\tilde{\star} = (\star_f)_{sp} \leq \star_{sp} \leq \star$, (Lemma 2.6 (2) and Corollary 2.7). Since it is easy to see that $\star_1 \leq \star_2$ implies that $\mathrm{Na}(D,\star_1) \subseteq \mathrm{Na}(D,\star_2)$, then the conclusion follows immediately from Corollary 3.5 (3). □

The content of Proposition 3.1 (5) is that, when the maximal ideals of $\mathrm{Na}(D,\star)$ are contracted to D, the result is exactly the prime ideals of D in $\mathcal{M}(\star_f)$. We now prove that this result can be reversed: the maximal ideals of $\mathrm{Na}(D,\star)$ can be obtained by extending to $\mathrm{Na}(D,\star)$ the prime ideals of D in $\mathcal{M}(\star_f)$. In particular:

Theorem 3.8. *Let $\star$ be a semistar operation of an integral domain D. Then* $\mathrm{Max}(\mathrm{Na}(D,\star)) = \{QD_Q(X) \cap \mathrm{Na}(D,\star) \mid Q \in \mathcal{M}(\star_f)\}$.

Proof. Proposition 3.1 (3) indicates that the maximal ideals of $\mathrm{Na}(D,\star)$ are exactly the ideals of the set $\{Q[X]_{N(\star)} \mid Q \in \mathcal{M}(\star_f)\}$. The result follows easily since these ideals are maximal in $\mathrm{Na}(D,\star)$ and are each contained in an ideal of the form $QD_Q(X)$, where $Q \in \mathcal{M}(\star_f)$. □

Note that the previous result indicates a strong similarity between the Nagata rings and the Kronecker function rings associated to a given semistar operation. In particular, the maximal spectrum of each ring consists of restrictions of the maximal ideals of local overrings of the form $R(X)$ where R is a local overring of D (cf. Theorem 3.8 and [17, Theorem 3.5]). The difference is that, in the Kronecker case, the overrings R are valuation overrings of D and, in the Nagata case, they are localizations of D at certain prime ideals. This is perhaps an indication that the Nagata and Kronecker constructions are actually each special cases of a more general construction involving more general classes of overrings.

We now turn our attention to the question of valuation overrings. The notion that we recall next is due to P. Jaffard [29] (cf. also [23], [25], [18]). For a domain D and a semistar operation $\star$ on D, we say that a valuation overring V of D is *a $\star$–valuation overring of D* provided $F^\star \subseteq FV$, for each $F \in \boldsymbol{f}(D)$. Note that, by definition the $\star$–valuation overrings coincide with the $\star_f$–valuation overrings; by [18, Proposition 3.3] the $\star$–valuation overrings also coincide with the $\star_a$–valuation overrings.

Theorem 3.9. *Let D be a domain and let $\star$ be a semistar operation on D. A valuation overring V of D is a $\tilde{\star}$–valuation overring of D if and only if V is an overring of D_P, for some $P \in \mathcal{M}(\star_f)$.*

Proof. To avoid the trivial case, we assume that $V \neq K$. First suppose that V is a valuation overring of D_P, for some $P \in \mathcal{M}(\star_f)$. It is clear from the definition of $\tilde{\star}$ that V is a $\tilde{\star}$–valuation overring of D.

Now assume that V is a $\tilde{\star}$–valuation overring of D. Let M be the maximal ideal of V and let $P := M \cap D$. We need to show that P is contained in a prime $Q \in \mathcal{M}(\star_f)$. We consider two cases.

Case 1. Suppose that there is a finitely generated ideal J of D contained in P such that $J \not\subseteq Q$, for each $Q \in \mathcal{M}(\star_f)$. Then $J^{\tilde{\star}} = \cap\{JD_Q \mid Q \in \mathcal{M}(\star_f)\} = \cap\{D_Q \mid Q \in \mathcal{M}(\star_f)\} = D^{\tilde{\star}}$. However, $JV \subseteq PV$ is a proper ideal of V and so cannot contain $J^{\tilde{\star}}$. This contradicts our assumption that V was a $\tilde{\star}$–valuation overring of D. We conclude that no such ideal J can exist.

Case 2. Suppose that every finitely generated ideal J which is contained in P is also contained in some ideal $Q \in \mathcal{M}(\star_f)$. Let $\mathcal{M}(\star_f) = \{Q_\lambda \mid \lambda \in \Lambda\}$. Note that, by assumption, for any finitely generated ideal $J \subseteq P$, the set $B(J) := \{\lambda \in \Lambda \mid J \subseteq Q_\lambda\}$ is not empty. Let $\mathcal{U}$ be an ultrafilter on Λ which contains each set $B(J)$, where J runs through the finitely generated ideals contained in P. Such an ultrafilter exists because the intersection of any finite collection of sets $\{B(J_1), B(J_2), \ldots, B(J_n)\}$ is simply $B(J_1 + J_2 + \ldots + J_n)$ and so must be nonempty. Then the ultrafilter limit ideal $Q_{\mathcal{U}}$ of the collection of prime ideals $\mathcal{M}(\star_f)$ must be a $\star_f$–prime (Proposition 2.9) and it also clearly contains P. This completes the proof. □

4. Semistar Kronecker function rings

Let $\star$ be a semistar operation on an integral domain D. We say that $\star$ is *an e.a.b. (endlich arithmetisch brauchbar) semistar operation of* D if, for all $E, F, G \in \boldsymbol{f}(D)$, $(EF)^\star \subseteq (EG)^\star$ implies that $F^\star \subseteq G^\star$, [17, Definition 2.3 and Lemma 2.7].

It is possible to associate to any semistar operation $\star$ of D an e.a.b. semistar operation of finite type $\star_a$ of D, defined as follows:

$$\begin{aligned} F^{\star_a} &:= \cup\{((FH)^\star : H^\star) \mid H \in \boldsymbol{f}(D)\}, \text{ for each } F \in \boldsymbol{f}(D); \\ E^{\star_a} &:= \cup\{F^{\star_a} \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}, \text{ for each } E \in \overline{\boldsymbol{F}}(D). \end{aligned}$$

The semistar operation $\star_a$ is called *the e.a.b. semistar operation associated to* $\star$ [16, Definition 4.4]. Note the previous construction is essentially due to P. Jaffard [29] (cf. also F. Halter-Koch [24]). Note that $D^{\star_a}$ is integrally closed and contains the integral closure of D in K [17, Proposition 4.5] (cf. also [23] [25], [41] and [18]). When $\star = v$, then D^{v_a} coincides with the pseudo-integral closure of D introduced by D.F. Anderson, Houston and Zafrullah [7].

If $\star$ is a semistar operation of an integral domain D, then we call *the Kronecker function ring of* D *with respect to* $\star$ the following domain:

$$\mathrm{Kr}(D,\star) := \{f/g \mid \begin{array}{l} f, g \in D[X]\setminus\{0\} \text{ and there exists } h \in D[X]\setminus\{0\} \\ \text{such that } (\boldsymbol{c}(f)\boldsymbol{c}(h))^\star \subseteq (\boldsymbol{c}(g)\boldsymbol{c}(h))^\star \} \cup \{0\}\}, \end{array}$$

[17, Theorem 5.1] (cf. also [25], [37] and [43]).

In the following statement we collect some of the main properties related to the Kronecker function ring of an integral domain with respect to a semistar operation (cf. [17, Proposition 3.3, Theorem 3.11, Proposition 4.5, Theorem 5.1 and the proof of Corollary 5.2]).

Proposition 4.1. *Let* $\star$ *be a semistar operation of an integral domain* D *with quotient field* K, *let* $\star_a$ *be the e.a.b. semistar operation of* D *associated to* $\star$ *and let* $\dot{\star}_a$ $(= \dot{\star}_a^{D^{\star_a}})$ *be the (semi)star operation of* $D^{\star_a}$ *associated to* $\star_a$ *and defined in Remark 2.1 (c). Then:*

- **(1)** $\star_f \leq \star_a$.
- **(2)** $\mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_f) = \mathrm{Kr}(D,\star_a) = \mathrm{Kr}(D^{\star_a},\dot{\star}_a)$.
- **(3)** $\mathrm{Kr}(D,\star)$ *is a Bézout domain with quotient field* $K(X)$.
- **(4)** $\mathrm{Na}(D,\star) \subseteq \mathrm{Kr}(D,\star)$.
- **(5)** $E^{\star_a} = E\mathrm{Kr}(D,\star) \cap K$, *for each* $E \in \overline{\boldsymbol{F}}(D)$. □

Remark 4.2. Note that if $\star$ is not the trivial semistar operation, then $\star_a$ is also different from the trivial semistar operation. As a matter of fact, if $D^\star \neq K$, then $D^{\star_a} = \cup\{(H^\star : H^\star) \mid H \in \boldsymbol{f}(D)\} \neq K$. Otherwise $(H^\star : H^\star)$ would be equal to K, for some $H \in \boldsymbol{f}(D)$ and $H \subseteq D$. This implies easily that $H^\star = K$ and this contradicts the assumption that $D^\star \neq K$.

Theorem 4.3. *Let* $\star$ *be a nontrivial semistar operation of an integral domain* D. *Assume that* $\star = \star_f$. *Let* $\star_a$ *be the e.a.b. semistar operation of finite type canonically associated to* $\star$.

- **(1)** *Let* (W, N) *be a nontrivial valuation overring of* $\mathrm{Kr}(D,\star)$. *Set* $N_0 := N \cap D$ *and let* $N_1 := N \cap D[X]$. *Then:*
  - **(a)** $N_1 = N_0[X]$, $N \cap \mathrm{Na}(D,\star) = N_0\mathrm{Na}(D,\star) = N_1\mathrm{Na}(D,\star)$ *and* $N \cap \mathrm{Na}(D,\star_a) = N_0\mathrm{Na}(D,\star_a) = N_1\mathrm{Na}(D,\star_a)$

(b) *N_0 is a quasi–$\star_a$–prime ideal (in particular, a quasi–$\star$–prime ideal) of D.*

(2) *If P is a quasi–$\star_a$–prime ideal of D, then there exists a quasi–$\star_a$–maximal ideal Q of D and a valuation overring (W, N) of $\mathrm{Kr}(D, \star)$ such that $P \subseteq Q = N \cap D$.*

(3) *$\mathcal{M}(\star_a)$ is contained in the canonical image in D of $\mathrm{Max}(\mathrm{Kr}(D, \star))$.*

(4) *For each $Q \in \mathcal{M}(\star_a)$, there exists a $\star$–valuation overring (V, M) of D dominating D_Q.*

Proof. As usual, we denote by K the field of fractions of D. It is obvious that: (1, a) $N_0[X] \subseteq N_1 = N \cap D[X]$, and if $f := f_0 + f_1X + \ldots + f_rX^r \in N \cap D[X]$, then $\boldsymbol{c}(f)\mathrm{Kr}(D, \star) = f\mathrm{Kr}(D, \star) \subseteq N$ [17, Theorem 3.11 (2) and Theorem 5.1 (2)]. Therefore, $f_i \in N \cap D = N_0$, for each i with $0 \leq i \leq r$. This fact implies that $f \in N_0[X]$.
Since $\mathrm{Na}(D, \star)$ (and $\mathrm{Na}(D, \star_a)$) is a ring of fractions of $D[X]$ and $N_0[X] = N_1 = N \cap D[X]$, we have immediately that $N_0\mathrm{Na}(D, \star) = N_1\mathrm{Na}(D, \star) = N \cap \mathrm{Na}(D, \star)$ and $N_0\mathrm{Na}(D, \star_a) = N_1\mathrm{Na}(D, \star_a) = N \cap \mathrm{Na}(D, \star_a)$.

(1, b) Recall that $\mathrm{Na}(D, \star_a) \subseteq \mathrm{Kr}(D, \star_a) = \mathrm{Kr}(D, \star)$ and $N_0^{\star_a} = N_0\mathrm{Kr}(D, \star) \cap K$, (Proposition 4.1 (2), (4) and (5)). Since $N_0\mathrm{Kr}(D, \star) \subseteq N \cap \mathrm{Kr}(D, \star)$, then $N_0^{\star_a} \subseteq N \cap \mathrm{Kr}(D, \star) \cap K = N \cap D^{\star_a}$, (Proposition 4.1 (5)). Thus $N_0^{\star_a} \cap D \subseteq N \cap D^{\star_a} \cap D = N \cap D = N_0$, i.e. N_0 is a quasi–$\star_a$–prime ideal of D. Since $\star = \star_f \leq \star_a$ (Proposition 4.1 (1)) then, in particular, N_0 is also a quasi–$\star$–prime ideal of D.

(2) Each quasi–$\star_a$–ideal of D, like P, is contained in a quasi–$\star_a$–maximal ideal Q of D (Lemma 2.3 (1)). In particular, we have $P \subseteq Q = Q^{\star_a} \cap D = Q\mathrm{Kr}(D, \star) \cap K \cap D = Q\mathrm{Kr}(D, \star) \cap D$. Since $Q^{\star_a} \cap D \neq D$, then necessarily $Q\mathrm{Kr}(D, \star) \neq \mathrm{Kr}(D, \star)$. Therefore there exists a maximal ideal of $\mathrm{Kr}(D, \star)$ containing $Q\mathrm{Kr}(D, \star)$ or, equivalently, a valuation overring (W, N) of $\mathrm{Kr}(D, \star)$ with center in $\mathrm{Kr}(D, \star)$ containing $Q\mathrm{Kr}(D, \star)$. By (1, b), $N \cap D$ is a quasi–$\star_a$–prime ideal of D. Since it contains Q, by the maximality of Q, we deduce that $Q = N \cap D$.

(3) Since $\mathrm{Kr}(D, \star)$ is a Bézout domain (Proposition 4.1 (3)) then the maximal spectrum of $\mathrm{Kr}(D, \star)$ is described by the centers in $\mathrm{Kr}(D, \star)$ of the minimal valuation overrings of $\mathrm{Kr}(D, \star)$. The conclusion follows immediately from (2).

(4) Recall that if V is a $\star$–valuation overring of D, the map $V \mapsto V(X)$ (where $V(X)$ is the trivial extension of V into $K(X)$ [20, page 218]) defines an order preserving bijection between the set of all the $\star$–valuation overrings of D and the set of all the valuation overrings of $\mathrm{Kr}(D, \star)$ [18, Theorem 3.5]. Therefore, by (3), if $Q \in \mathcal{M}(\star_a)$, we can find a (minimal) valuation overring (W, N) of $\mathrm{Kr}(D, \star)$ such that $N \cap D = Q$. Then, we can consider $V := W \cap K$ and $M := N \cap K = N \cap V$. By the previous remark, V is a $\star$–valuation overring of D and its maximal ideal M is such that $M \cap D = Q$. Hence, (V, M) dominates D_Q. □

Corollary 4.4. *Let $\star$ be a nontrivial semistar operation of an integral domain D. Assume that $\star$ is an e.a.b. semistar operation of finite type with $D = D^\star$. Then each $\star$–maximal ideal of D is the center in D of a minimal $\star$–valuation overring of D.*

Proof. In the present situation, we have $\mathcal{M}(\star_a) = \mathrm{Max}^\star(D)$. The conclusion follows easily from Theorem 4.3 (3) and (4). □

Corollary 4.5. *Let $\star$ be a nontrivial semistar operation of an integral domain D. Assume that $\star = \star_f$. Then:*

(1) $\tilde{\star} \leq \widetilde{(\star_a)} = (\star_a)_{sp} \leq \star_a$ *and* $\tilde{\star} \leq (\tilde{\star})_a \leq \star_a$.
(2) $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star}) \subseteq \mathrm{Na}(D,\widetilde{(\star_a)}) = \mathrm{Na}(D,\star_a) \subseteq \mathrm{Kr}(D,\star_a) = \mathrm{Kr}(D,\star)$.
(3) $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\tilde{\star}) \subseteq \mathrm{Na}(D,(\tilde{\star})_a) \subseteq \mathrm{Kr}(D,(\tilde{\star})_a) = \mathrm{Kr}(D,\tilde{\star}) \subseteq \mathrm{Kr}(D,\star)$.
(4) *For each* $E \in \overline{\boldsymbol{F}}(D)$,
  (a) $E^{\widetilde{(\star_a)}} = E\mathrm{Na}(D,\star_a) \cap K$ ($\supseteq E\mathrm{Na}(D,\star) \cap K = E^{\tilde{\star}}$) ;
  (b) $E^{(\tilde{\star})_a} = E\mathrm{Kr}(D,\tilde{\star}) \cap K$ ($\subseteq E\mathrm{Kr}(D,\star) \cap K = E^{\star_a}$) .

Proof. (1) follows trivially from the fact that if $\star_1$ and $\star_2$ are two semistar operations of finite type, then $\star_1 \leq \star_2$ implies that $(\star_1)_{sp} = \widetilde{\star_1} \leq \widetilde{\star_2} = (\star_2)_{sp}$, and from Corollary 2.7 (1) and Proposition 4.1 (1).

(2) and (3) are consequences of Corollary 3.5 (3) and of Proposition 4.1 (2).

(4) follows from Proposition 3.4 (3) and Proposition 4.1 (5). □

5. The semistar operations $\tilde{\star}$ and $\star_a$

In this section we consider more closely the two operations which are naturally associated with the semistar Nagata rings and the semistar Kronecker function rings:

$\tilde{\star}$ associated with $\mathrm{Na}(D,\star)$ and $\star_a$ associated with $\mathrm{Kr}(D,\star)$.

An elementary first question to ask is whether the two semistar operations are actually the same - or usually the same - or rarely the same. Theorem 3.9 indicates that for a semistar operation $\star$ on a domain D, the $\tilde{\star}$–valuation overrings of D are all the valuation overrings of the localizations of D at the primes in $\mathcal{M}(\star_f)$. On the other hand, [18, Proposition 3.3 and Theorem 3.5] indicates that the $\star_a$–valuation overrings (or, equivalently, the $\star$–valuation overrings) of D correspond exactly to the valuation overrings of the Kronecker function ring $\mathrm{Kr}(D,\star)$. It is easy to imagine that these two collections of valuation domains can frequently be different. We consider several different examples.

Example 5.1. *A (semi)star operation $\star$ of an integral domain D such that $\tilde{\star} \neq \star_a$, but the $\tilde{\star}$–valuation overrings coincide with the $\star_a$–valuation overrings (and so* $\mathrm{Kr}(D,\tilde{\star}) = \mathrm{Kr}(D,\star_a) = \mathrm{Kr}(D,\star)$, [17, Corollary 3.8] *and* [18, Theorem 3.5]*).*

Let L be a field and let D be the localization $L[X,Y]_M$ of the polynomial ring $L[X,Y]$ at the maximal ideal $M := (X,Y)$. Let $\star := d$ be the identity (semi)star operation on D (defined by $E^d := E$, for all $E \in \overline{F}(D)$). Clearly, $\star_f = d = \star$ and every prime ideal of D is a $\star_f$–prime. It follows that $\tilde{\star} = \star = d$ (Corollary 2.7 (2, b)). On the other hand, note that in general:

Claim. *For each integral domain D, the e.a.b. semistar operation d_a associated to the identity (semi)star operation d of D coincides with the (semi)star operation b defined, for each $E \in \overline{F}(D)$, by $E^b := \cap\{EV \mid V$ is a valuation overring of $D\}$.*

The claim follows from Proposition 4.1 (5) and [18, Theorem 3.5], since:

$$\begin{aligned} E^{d_a} &= E\mathrm{Kr}(D,d) \cap K = \cap\{EV(X) \mid V \text{ is a valuation overring of } D\} \cap K = \\ &= \cap\{EV \mid V \text{ is a valuation overring of } D\} = E^b . \end{aligned}$$

Note that $d \neq b = d_a$ in D (otherwise D would be a Prüfer domain by [20, Theorem 24.7]), hence $\star = d$ is not e.a.b. [17, Proposition 4.5 (5)] and so $\tilde{\star}$ ($= \star = d$) $\neq \star_a$ ($= d_a = b$) . Moreover, every valuation overring of D is (obviously) a $\tilde{\star}$–valuation overring and also (by the claim) every valuation overring of D is a $\star_a$–valuation overring of D . Therefore, $\tilde{\star}$ and $\star_a$ are different, but have the same collection of "associated" valuation overrings. Finally, observe that, for the particular $\star$ we are considering here, we have (using a "new" indeterminate Z):

$$\begin{aligned}\mathrm{Na}(D,\star) &= \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D,d) = D(Z) \subsetneq \\ &\subsetneq \mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\tilde{\star}) = \mathrm{Kr}(D,\star_a) = \mathrm{Kr}(D,d) = \\ &= \cap\{V(Z) \mid V \text{ is an overring of } D\}\,.\end{aligned}$$ □

We noted in the preceding example that although $\tilde{\star}$ and $\star_a$ were different, nevertheless, the collection of the $\tilde{\star}$–valuation overrings of D coincides with the collection of the $\star_a$–valuation overrings of D. The next example displays wider differences between the two operations.

Example 5.2. *A (semi)star operation $\star$ of an integral domain D such that $\tilde{\star} \neq \star_a$, the $\star_a$–valuation overrings form a proper subset of the set of $\tilde{\star}$–valuation overrings, but $\tilde{\star} = \widetilde{(\star_a)}$.*

Let L and D be as in Example 5.1. Let $N := MD$ denote the maximal ideal of D. For each irreducible polynomial $f \in M$, let $W_f := L[X,Y]_{(f)} = D_{(f)}$. Then W_f is a DVR overring of D. Let V_X be the two dimensional valuation overring of D with maximal ideal generated by Y and with $W_X = L[X,Y]_{(X)}$ as a one dimensional (valuation) overring, i.e.

$$V_X := L[Y]_{(Y)} + XL[X,Y]_{(X)} \ (\subsetneq W_X)\,.$$

We consider the following family of valuation overrings of D:

$$\mathcal{W} := \{W_f \mid f \in M\,,\ f \neq X\,,\ f \text{ irreducible in } L[X,Y]\} \cup \{V_X\}$$

and we define a semistar operation $\star := \star_{\mathcal{W}}$ of D by setting $E^\star := \cap\{EW \mid W \in \mathcal{W}\}$ for all $E \in \overline{F}(D)$. It is well known that $\star_{\mathcal{W}}$ is an e.a.b. (in fact, a.b.) semistar operation of D and the Kronecker function ring associated with $\star$ (in the "new" variable Z) is then $\mathrm{Kr}(D,\star) = \cap\{W(Z) \mid W \in \mathcal{W}\}$ [18, Corollary 3.8]. We claim that:

Claim. *The maximal ideals of $\mathrm{Kr}(D,\star)$ are exactly the centers of the maximal ideals of the valuation domains $W(Z)$, when $W \in \mathcal{W}$.*

To prove the claim note that, from the fact that $\mathrm{Kr}(D,\star)$ is a Prüfer (in fact, Bézout) domain (Proposition 4.1 (3)) and from [17, Theorem 3.5], there exists a canonical bijection between the maximal ideals of $\mathrm{Kr}(D,\star)$ and the valuation overrings of $\mathrm{Kr}(D,\star)$ of the type $V(Z)$, where V is a minimal $\star$–valuation overring of D (cf. also [10]). Moreover observe that, by definition, each $W \in \mathcal{W}$ is a $\star$–valuation overring and that the intersection $\cap\{W(Z) \mid W \in \mathcal{W}\}$ is irredundant, i.e., if any one of the valuation domains W was omitted, the intersection would be different (in fact, it is easy to see that the first intersection in the following formula

$$\begin{aligned}D &= \cap\{W_f \mid f \in M\,,\ f \text{ irreducible in } L[X,Y]\}) = \\ &= \cap\{W_f \mid f \in M\,,\ f \neq X\,,\ f \text{ irreducible in } L[X,Y]\} \cap \{V_X\} = \\ &=: \cap\{W \mid W \in \mathcal{W}\}\end{aligned}$$

is irredundant, because D is a Krull domain, so it is the same for the last intersection; this property implies easily the irredudancy of the $\cap\{W(Z) \mid W \in \mathcal{W}\}$).

Note that the family of valuation overrings $\mathcal{W}(Z) := \{W(Z) \mid W \in \mathcal{W}\}$ of the Prüfer domain $\mathrm{Kr}(D,\star)$ has finite character (in the sense that each nonzero element of $\mathrm{Kr}(D,\star)$ is a nonunit in at most finitely many valuation overrings of $\mathcal{W}(Z)$).

As a matter of fact, in this case $\star = \star_a$ hence $(a_0, a_1, \ldots, a_n)^\star = f\mathrm{Kr}(D,\star) \cap K$, for each $0 \neq f := \sum_{k=0}^{n} a_k Z^k \in D[Z]$, (cf. [18, Theorem 3.11 (1), (2) and Theorem 5.1 (2)]) and $(a_0, a_1, \ldots, a_n)^\star \in \boldsymbol{f}(D)$, because D is a Noetherian ring;

moreover, each nonzero finitely generated ideal of D is contained in at most finitely many height 1 prime ideals of D, because D is a Krull domain.

The claim then follows immediately from [21, Corollary 1.11]. In other words, each maximal ideal H of $\mathrm{Kr}(D,\star)$ contracts onto a prime ideal of D and thus contains a polynomial $f \in M$, f irreducible in the polynomial ring $L[X,Y]$; henceforth, if $f \neq X$, then $H = fW_f(Z) \cap \mathrm{Kr}(D,\star)$, if $f = X$, then $H = YV_X(Z) \cap \mathrm{Kr}(D,\star)$.

The import of this claim is that the collection $\{W_f \mid f \in M\,, f \neq X\,, f$ irreducible in $L[X,Y]\} \cup \{V_X,\ W_X\} = \mathcal{W} \cup \{W_X\}$, constitutes the collection of all nontrivial $\star_a$–valuation (or, equivalently, $\star$–valuation) overrings of D.

On the other hand, note that D is local and the maximal ideal of the valuation overring $V_X \in \mathcal{W}$ is centered on the maximal ideal N of D. Moreover, D is Noetherian, so $\star$ is a semistar operation of finite type on D. It follows that the maximal ideal N of D belongs to $\mathcal{M}(\star_f)$. Since (obviously) $D_N = D$ this leads to the conclusion that $\mathcal{M}(\star_f) = \{N\} = \mathrm{Max}(D)$, and so $\tilde{\star} = d$ (the identity (semi)star operation) of D. As noted in the previous example, this implies that every valuation overring of D is a $\tilde{\star}$–valuation overring of D. Therefore, $\star_a$ and $\tilde{\star}\ (= d\ )$ are not only different as semistar operations, but they are also associated with different sets of valuation overrings (e.g. for each $f \in M\,, f \neq X\,, f$ irreducible in $L[X,Y]$, the two dimensional valuation overring V_f of D, having $W_f = L[X,Y]_{(f)}$ as a one dimensional (valuation) overring and dominating D, is a valuation overring of D, but is not a $\star_a$–valuation overring of D).

In the present situation, observe that we have that $d = \tilde{\star} = \widetilde{(\star_a)}$. As a matter of fact, from Theorem 4.3 (3) and from the fact that $\mathrm{Kr}(D,\star)$ is a Prüfer domain, we have that each member of $\mathcal{M}(\star_a)$ is the center in D of a minimal $\star$–valuation overring of D, thus $\mathcal{M}(\star_a) = \{N\}$, hence $\mathcal{M}(\star_a) = \mathcal{M}(\star_f)$. Finally, we have:

$$\begin{aligned}\mathrm{Na}(D,\star) &= \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D,\widetilde{(\star_a)}) = \mathrm{Na}(D,\star_a) = \mathrm{Na}(D,d) = D(Z) \subsetneq \\ &\subsetneq \mathrm{Kr}(D,d) = \mathrm{Kr}(D,b) = \cap\{V(Z) \mid V \text{ is an overring of } D\} \subsetneq \\ &\subsetneq \mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_a) = \cap\{W(Z) \mid W \in \mathcal{W}\}\,. \qquad \square\end{aligned}$$

In Example 5.2 $\star_a \neq \tilde{\star}$, however $\widetilde{(\star_a)} = \tilde{\star}$. It seems plausible that something of this type holds in general. The next example demonstrates that it does not and illustrates why.

Example 5.3. *A (semi)star operation $\star$ of an integral domain D such that $\tilde{\star} \neq \star_a$, the $\star_a$–valuation overrings form a proper subset of the set of $\tilde{\star}$–valuation overrings and $\tilde{\star} \neq \widetilde{(\star_a)}$.*

Let D and N be as in the two previous examples. We construct a (semi)star operation $\star$ on D as follows:

1. If dD is any nonzero principal ideal of D, then $(dD)^\star := dD$.
2. If $J \subseteq D$ is a nonzero ideal of D which is not contained in any proper principal ideal of D, then $J^\star := N$.
3. If $J \subseteq D$ is a nonzero ideal of D which is not principal, but is contained in a principal ideal, then we factor J as $J = fI$, where f is a GCD of a set of generators of J and $I := (J :_D fD)$ is not contained in any proper principal ideal of D by the choice of f. Then $J^\star := fN$.
4. If J is a nonzero fractionary ideal of D which is not contained in D, choose a nonzero element $d \in D$ such that $dJ \subseteq D$. Then define $J^\star := (1/d)(dJ)^\star$.
5. If $J \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$ we define $J^\star := L(X,Y)$.

Since D is Noetherian, then $\star$ is of finite type. Henceforth, it is clear that $\mathcal{M}(\star_f) = \{N\}$. Thus, as in the previous example, $\tilde{\star} = d$.
However, since D is integrally closed and Noetherian, it is easy to see from the definition of $\star_a$ that $D^{\star_a} = D$, $N^{\star_a} = D$ and $(fD)^{\star_a} = fD$, for each $f \in M$ and f irreducible in $L[X,Y]$. Hence, $\mathcal{M}(\star_a) = \{fD \mid f \in M \text{ and } f \text{ irreducible in } L[X,Y]\ \}$ coincides with the set of all the height 1 primes of D. Moreover, d_a is the classical b operation (Claim in Example 5.2) and thus $(\tilde{\star})_a = d_a = b$.

On the other hand, $\star_a$ coincides with the t (semi)star operation of D. As a matter of fact, we observed already that $\mathcal{M}(\star_a)$ coincides with the set of all the height 1 primes of D, and this implies that $\widetilde{(\star_a)} = t$ because D is a Krull domain [20, Proposition 44.13 or Theorem 44.2]. Since for (semi)star operations of finite type we have always the inequalities $\widetilde{(\star_a)} \leq \star_a \leq t$ (Corollary 4.5 (1) and [20, Theorem 34.1 (4)]), we deduce immediately that $\widetilde{(\star_a)} = \star_a = t$.

Observe that $\mathcal{M}(b) = \mathrm{Max}(D)$, since every valuation overring is a b–valuation overring and every prime ideal of D is a b–prime of D. We conclude, for the particular $\star$ we are considering here, that:

$$b = (\tilde{\star})_a \neq \widetilde{(\star_a)} = t\,, \quad \text{and} \quad d = \tilde{b} = \widetilde{(\tilde{\star})_a} \neq \widetilde{\widetilde{(\star_a)}} = \widetilde{(\star_a)} = t\,.$$

So it is hopeless to try to attain an equality by applying $\widetilde{(\text{-})}$ and $(\text{-})_a$ in different orders.
Finally, observe that, for the particular $\star$ we are considering here, we have (using a "new" indeterminate Z):

$$\begin{aligned}
\mathrm{Na}(D,\star) &= \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D,d) = D(Z) \subsetneq \\
&\subsetneq \mathrm{Na}(D,\widetilde{(\star_a)}) = \mathrm{Na}(D,\star_a) = \mathrm{Na}(D,t) = \mathrm{Na}(D,v) = \\
&= \cap\{W_f(Z) \mid f \in M\,,\ f \text{ irreducible in } L[X,Y]\ \} = \\
&= \mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\widetilde{(\star_a)}) = \mathrm{Kr}(D,\star_a) = \\
&= \mathrm{Kr}(D,t) = \mathrm{Kr}(D,v) = \cap\{W(Z) \mid W \in \mathcal{W}\}\,. \\
\mathrm{Na}(D,\star) &= \mathrm{Na}(D,\tilde{\star}) = \mathrm{Na}(D,d) = D(Z) \subsetneq \\
&\subsetneq \mathrm{Kr}(D,d) = \mathrm{Kr}(D,\tilde{b}) = \mathrm{Kr}(D,b) = \\
&= \cap\{V(Z) \mid V \text{ is an overring of } D\} \subsetneq \\
&\subsetneq \mathrm{Kr}(D,\star) = \mathrm{Kr}(D,\star_a) = \cap\{W(Z) \mid W \in \mathcal{W}\}\,.
\end{aligned}$$

□

It is possible to make a positive statement about the relationship between $\widetilde{(\text{-})}$ and $(\text{-})_a$ under conditions made clear in the preceding example.

Proposition 5.4. *Let $\star$ be a semistar operation of an integral domain D. Then, the following conditions are equivalent*

- **(i)** $\tilde{\star} = \widetilde{(\star_a)}$;
- **(ii)** $\mathcal{M}(\star_f) = \mathcal{M}(\star_a)$;
- **(iii)** $\mathrm{Na}(D,\star) = \mathrm{Na}(D,\star_a)$.

Proof. (ii) ⇒ (i). If $\mathcal{M}(\star_f) = \mathcal{M}(\star_a)$, then $\tilde{\star} = \widetilde{(\star_a)}$ follows immediately from the definition of the $\widetilde{(\text{-})}$ operator and the fact that $\star_a$ is always (by definition) of finite type.

(i) ⇒ (ii). If $\mathcal{M}(\star_f) \neq \mathcal{M}(\star_a)$, then $\tilde{\star} \neq \widetilde{(\star_a)}$ by Proposition 3.4 (2) and (3), again taking into account the fact that $(\star_a)_f = \star_a$.

(iii) ⇒ (i) and (i) ⇒ (iii) follow from Proposition 3.1 (5) and Corollary 3.5. □

Remark 5.5. Let $\star$ be a semistar operation of an integral domain D. For semistar Kronecker function rings, we can easily state a result "analogous" to Proposition 5.4 and concerning $\star$ and $\tilde{\star}$. More precisely, from Proposition 4.1 (5), [17, Theorem 3.1] and [18, Corollary 3.8, Theorem 5.1 (3)], we have that *the following conditions are equivalent:*

(i) $\star_a = (\tilde{\star})_a$;
(ii) *the set of $\tilde{\star}$–valuation overrings D coincides with the set of $\star$–valuation overrings of D ;*
(iii) $\mathrm{Kr}(D, \tilde{\star}) = \mathrm{Kr}(D, \star)$.
Moreover, each of the previous conditions implies
(iv) $\mathcal{M}(\star_a) = \mathcal{M}((\tilde{\star})_a)$.
On the other hand, observe that, from Example 5.2 (for $\star = \star_{\mathcal{W}}$), we have that (iv) $\not\Rightarrow$ (iii), since $\mathcal{M}(\star_a) = \mathcal{M}(\star_f) = \{N\} = \mathcal{M}(b) = \mathcal{M}(d_a) = \mathcal{M}((\tilde{\star})_a)$ and $\mathrm{Kr}(D, \tilde{\star}) = \mathrm{Kr}(D, b) \subsetneq \mathrm{Kr}(D, \star)$.

This line of thinking motivates our final result, tying our investigation of differently constructed semistar operations back to the topic of Nagata rings.

Proposition 5.6. *Suppose $\star_1$ and $\star_2$ are semistar operations on a domain D. Then,* $\mathrm{Na}(D, \star_1) = \mathrm{Na}(D, \star_2)$ *if and only if* $\mathcal{M}((\star_1)_f) = \mathcal{M}((\star_2)_f)$.

Proof. First, suppose $\mathrm{Na}(D, \star_1) = \mathrm{Na}(D, \star_2)$. Then $\mathcal{M}((\star_1)_f) = \mathcal{M}((\star_2)_f)$ follows from Proposition 3.1 (5).

Now suppose that $\mathcal{M}((\star_1)_f) = \mathcal{M}((\star_2)_f)$. Then $\mathrm{Na}(D, \star_1) = \mathrm{Na}(D, \star_2)$ follows from Proposition 3.1 (4). □

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