of commutative cancellative monoids or commutative rings with nonzero zerodivisors.

C.5 Dedekind and Prüfer domains

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All rings in this Section are assumed to be commutative.

Dedekind Domains

C.5.1 Definition An integral domain R is called a ***Dedekind domain*** if each nonzero proper ideal of R can be represented uniquely (apart from order) as a finite product of prime ideals of R (cf. Section C.1 and Section C.4).

Any principal ideal domain (e.g., $\mathbb{Z}$ or $K[X]$, where K is a field) is a Dedekind domain, and since the integral closure of a Dedekind domain in a finite field extension of its quotient field is also a Dedekind domain, it follows that the ring of algebraic integers in an algebraic number field and the ring of integral functions in a field of algebraic functions of one variable are also Dedekind domains (Zariski and Samuel 1958, Vol. 1, Section 6, pp. 270–271).

The theory of Dedekind domains was established by Emmy Noether, in the mid twenties, as a generalization of results concerning factorization properties of algebraic integers obtained primarily by Richard Dedekind in 1871. E. Noether provided a fundamental characterization for Dedekind domains which prominently exhibited the role of chain conditions for ideals. Namely,

C.5.2 Theorem *An integral domain R is a Dedekind domain if and only if R satisfies the following properties (called* ***Noether Axioms****):*

(1) *R is Noetherian (ascending chain condition for ideals);*
(2) *R is* ***integrally closed*** *(R coincides with the set of the roots of its monic polynomials);*
(3) *R/I is Artinian (descending chain condition for ideals), for each nonzero ideal I of R.*

Note that condition **(3)** can be replaced by

(3') each nonzero prime ideal of R is maximal.

As a consequence, it follows that *Dedekind domains are Noetherian integral domains such that each localization at maximal ideals is a discrete valuation domain.* In particular, as we have mentioned already, each principal ideal domain is a Dedekind domain, but not conversely (e.g., $\mathbb{Z}[i\sqrt{5}]$).

The structural diversity of Dedekind domains can be observed by the following sampling of some classical characterizations:

C.5.3 Theorem *An integral domain R is a Dedekind domain if and only if one of the following statement holds:*

(a) (W. Krull, 1935) *the nonzero fractional ideals of R form a group under multiplication;*

(b) (I. Kaplansky, 1952 (the necessity) and S. U. Chase, 1960 (the sufficiency) *each extension of a torsion module of bounded rank by a torsion-free module splits;*

(c) (C. U. Jensen, 1963) *R/I is a principal ideal ring, for each nonzero ideal I of R.*

The proofs or appropriate references for Theorem C.5.2 and Theorem C.5.3 can be found in (Kaplansky 1970, Gilmer 1972a, Narkiewicz 1990).

Dedekind domains have played a crucial role in the development of Algebraic Number Theory and Algebraic Geometry. In fact, it was observed by R. Dedekind and H. Weber in 1882 that several properties concerning rings of algebraic integers also apply to rings of integral elements in function fields. This realization contained the germ of the fact that the coordinate ring of a nonsingular irreducible curve is a Dedekind domain.

Prüfer Domains

In view of Krull's characterization of Dedekind domains (see Theorem C.5.3 (a)), it is natural to consider the following:

C.5.4 Definition An integral domain R having its set of nonzero finitely generated fractional ideals form a group under multiplication is called ***Prüfer domain***.

Dedekind domains are Prüfer domains, but not conversely (see below). Papers of H. Prüfer in 1932 and of W. Krull in 1936 introduced the study of Prüfer domains, although the first place the name *Prüfer ring* appears in the literature seems to be in the classic book by Cartan and Eilenberg (1956). Prüfer and Krull were interested in the ideal and overring theory of Prüfer domains, whereas Cartan and Eilenberg examined the homological nature of these domains. Many researchers during the second half of the 20th century investigated the structure of Prüfer domains and a reasonable sampling of this work has been chronicled in (Kaplansky 1970, Gilmer 1972a, Fontana, Huckaba, and Papick 1997).

A brief overview of some fundamental results concerning Prüfer domains will help to illustrate important research topics from the late 30's to the early 70's. The previously mentioned sources contain precise acknowledgement of the listed results.

C.5.5 Theorem *Let R be an integral domain. The following statements are equivalent to R being a Prüfer domain:*

(i) *Each valuation overring of R is a ring of fractions.*
(ii) *Each finitely generated torsion-free R-module is projective.*
(iii) *Each overring of R is integrally closed.*
(iv) *Each overring S of R is R-flat (i.e., for each monomorphism of R-modules $N \to M$, $N \otimes_R S \to M \otimes_R S$ is also a monomorphism).*
(v) *R is integrally closed and there exists a positive integer $n \geq 2$ such that $(a,b)^n = (a^n, b^n)$, for each a, b in R.*
(vi) *For all ideals I, J, L of R, $I + (J \cap L) = (I + J) \cap (I + L)$ (or, equivalently, the Chinese Remainder Theorem for ideals holds in R).*

From Theorem C.5.5 and Theorem C.5.3, it follows that: **(a)** *the integral closure of a valuation domain (and, more generally, of a Prüfer domain) in any algebraic extension field of its field of quotients is a Prüfer domain;* **(b)** *Prüfer domains coincide with the integral domains such that each localization at maximal ideals is a valuation domain;* **(c)** *Noetherian Prüfer domains coincide with Dedekind domains.*

Note that the ring of all algebraic integers (i.e., the integral closure of $\mathbb{Z}$ in the field of complex numbers) is a Prüfer *non* Dedekind domain, since it is not Noetherian. Further interesting examples of Prüfer domains include: **(1)** Bezout domains (i.e., integral domains in which every finitely generated ideal is principal) ***Bezout domain*** and hence, in particular, Kronecker function rings (W. Krull, 1936); **(2)** the ring of entire functions (O. Helmer, 1940); **(3)** the intersection of any finite family of valuation domains on a given field (M. Nagata, 1953); **(4)** the ring $\mathrm{Int}(R)$ of all integer valued polynomials, where R is a Dedekind domain with finite residue fields (D. Brizolis, 1979; D. L. McQuillan, 1985; J.-L. Chabert, 1987) hence, in particular, the classical ring of all integer-valued rational polynomials $\mathrm{Int}(\mathbb{Z}) := \{f(X) \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}$ is a Prüfer (but *not* a Dedekind) domain.

Extensions and ramifications of these ideas have been pursued in great detail since the 70's, and more recent work focuses on specific properties of ideals and modules over Prüfer domains. For example, an ideal in a Dedekind domain requires no more than two generators and early evidence suggested that finitely generated ideals in a Prüfer domain also might require only two generators. In fact, H. Prüfer in his 1932 paper showed that if each two-generated ideal of a domain R is invertible, then R is a Prüfer domain. However, in 1979 H. Schülting gave an example of a Prüfer domain with a finitely generated ideal requiring three generators, which in turn, led to several new studies investigating the number of generators needed for finitely generated ideals in Prüfer domains.

Another notable example of recent directions in the theory of Prüfer domains is the Y. Lequain and A. Simis extension of the D. Quillen and A. Suslin solutions to the Serre Conjecture. They proved in 1980 that if R is a Prüfer domain then, for all positive integers n, all finitely generated projective modules over $R[X_1, X_2, \ldots, X_n]$ are extended from R (recall that, if S is an R-algebra, an S-module N is ***extended*** from R if there exists an R-module M such that N is isomorphic to $M \otimes_R S$).

Research activity involving Prüfer domains has remained strong since the initial work of H. Prüfer and W. Krull, and will continue to flourish for many years to come (Chapman and Glaz 2000). The rich and interesting structure of these rings, as well as their natural presence, make them important and useful objects in the context of commutative algebra.

C.6 Local Rings

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Examples, History, and Conventions

Consider the ring $A = \{\, a/b \mid a, b \in \mathbb{Z},\ b \text{ odd} \,\}$. In A, all odd primes are units, and $2A$ is the only maximal ideal. The ring $B = k[[x, y]]$ of formal power series in x, y with coefficients in a field k has a similar feature: any power series outside of $xB + yB$ can be formally inverted, so $xB + yB$ is the only maximal ideal of B. Yet another "similar" ring is $C = \mathbb{Z}_{p^n}$, for any prime p, except that C is no longer a domain when $n > 1$.

A commutative ring R is called a ***local ring*** if it has exactly one maximal ideal, or equivalently, $R \neq 0$ and the nonunits of R form an