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Universal property of the Kaplansky ideal transform and affineness of open subsets

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Abstract

Let R be an integral domain, I an ideal of R and $\Omega_R(I)$ the Kaplansky transform of R with respect to I. A ring homomorphism $\alpha : R \to A$ is called an I-morphism if $\alpha^{-1}(Q) \not\supseteq I$ for each prime ideal Q of A. We denote by $K_R(I,A)$ the set of all the I-morphisms from R to A. It is easy to see that $K_R(I,-)$ defines a covariant functor from *Ring* to *Set*. We prove that the following statements are equivalent: (i) $K_R(I,-) : \mathit{Ring} \to \mathit{Set}$ is a representable functor; (ii) the natural embedding $R \to \Omega_R(I)$ is an I-morphism; (iii) $I\Omega_R(I) = \Omega_R(I)$; (iv) $D(I) = \{P \in \mathrm{Spec}(R) \mid P \not\supseteq I\}$ is an open affine subscheme of Spec(R).

MSC: 13G05; 13A15; 13B10; 13C99

1. Introduction and preliminary results

Let A be a commutative unitary ring. We denote by Spec(A) the prime spectrum of A, i.e. the set of all the prime ideals of A endowed with the Zariski topology. If I is an ideal of A, we set:

$$D(I) := \{P \in \mathrm{Spec}(A) \mid P \not\supseteq I\}$$

the *open subset of* Spec(A) *associated to the ideal* I. If $f : A \to B$ is a ring homomorphism, we denote by f^* : Spec(B) $\to$ Spec(A) the continuous map defined by $f^*(Q) := f^{-1}(Q)$ for each $Q \in \mathrm{Spec}(B)$.

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A family of ideals $\mathscr{F}$ of a ring A is a *localizing system for* A if the following properties hold:

(LS1) $I \in \mathscr{F}$, J ideal of A and $I \subseteq J \Rightarrow J \in \mathscr{F}$;

(LS2) $I \in \mathscr{F}$, J ideal of A and $(J :_A iA) \in \mathscr{F}$ for each $i \in I \Rightarrow J \in \mathscr{F}$.

If $\mathscr{F}$ is a localizing system, it is easy to see that if $I, J \in \mathscr{F}$ then $IJ \in \mathscr{F}$ and so, in particular, $I \cap J \in \mathscr{F}$.

If $\mathscr{F}$ is a localizing system of a ring A, for each pair of ideals I, $I \in \mathscr{F}$ with $I \subseteq J$, we have a canonical map of A-modules:

$$h_{J,I} : Hom_A(J,A) \to \mathrm{Hom}_A(I,A), \quad f \mapsto f|_I.$$

It is easy to see that the $h_{J,I}$'s form a direct system of homomorphisms of A-modules. Set

$$A_{(\mathscr{F})} := \varinjlim \{\mathrm{Hom}_A(I,A) \mid I \in \mathscr{F}\}$$

and

$$h = h_{A,\mathscr{F}} := \varinjlim \{h_{A,I} \mid I \in \mathscr{F}\} : A \to A_{(\mathscr{F})}.$$

It is not difficult to show that $A_{(\mathscr{F})}$ has a natural ring structure and $h : A \to A_{(\mathscr{F})}$ is a ring homomorphism, cf. also [3, Chapitre 2, §2, Exercices 16–25; 9, 20 p. 196].

Set

$$t_{\mathscr{F}}(A) := \{a \in A \mid aI = 0 \text{ for some } I \in \mathscr{F}\}.$$

It is obvious that $t_{\mathscr{F}}(A)$ is an ideal of A, called the *torsion radical of* A *associated to the localizing system* $\mathscr{F}$. Note that $t_{\mathscr{F}}(A) = A$ if and only if $(0) \in \mathscr{F}$. In order to avoid the trivial case, we will assume from now on that all the localizing systems that we will consider are such that $t_{\mathscr{F}}(A) \subsetneq A$.

We denote simply by $\bar{A}$ the quotient ring $A/t_{\mathscr{F}}(A)$ and by $\tau = \tau_{A,\mathscr{F}} : A \to \bar{A}$ the canonical surjection. We can consider

$$\bar{\mathscr{F}} := \tau(\mathscr{F}) = \tau_{A,\mathscr{F}}(\mathscr{F}) := \{J = \tau_{A,\mathscr{F}}(I) \mid I \in \mathscr{F}\},$$

then it is easy to see that $\bar{\mathscr{F}}$ is a localizing system of $\bar{A}$. Then, we define the *generalized ring of fractions* $A_{\mathscr{F}}$ *of the ring* A *with respect to the localizing system* $\mathscr{F}$ as follows:

$$A_{\mathscr{F}} := \bar{A}_{(\bar{\mathscr{F}})},$$

cf. also [3, Chapitre 2, §2, Exercice 19; 20, Chapter IX, Lemma 1.6]. We set $\varphi = \varphi_{A,\mathscr{F}} : A \to A_{\mathscr{F}}$ to be the canonical homomorphism obtained by composition as follows:

$$A \xrightarrow{\varphi} A_{\mathscr{F}} := A \xrightarrow{\tau} \bar{A} \xrightarrow{h} \bar{A}_{(\bar{\mathscr{F}})},$$

then it is not difficult to see that $\mathrm{Ker}(\varphi) = t_{\mathscr{F}}(A)$.

It is obvious that if $t_{\mathscr{F}}(A)=0$, then $\bar{A}=A$, $\bar{\mathscr{F}}=\mathscr{F}$ and $A_{\mathscr{F}}=A_{(\mathscr{F})}$.

For instance, let A be any ring and $\mathrm{Tot}(A)$ be the total ring of fractions of A. Assume that $\mathscr{F}$ is a localizing system of A with the property that each ideal I of $\mathscr{F}$ contains a regular element of A, then $t_{\mathscr{F}}(A)=0$. Moreover, in this situation, for each $I\in\mathscr{F}$, the canonical map:

$$\mu=\mu_{I,A}:(A:_{\mathrm{Tot}(A)}I)\to \mathrm{Hom}_A(I,A),\quad z\mapsto \mu(z):=z\cdot -,$$

i.e. $\mu(z)(i):=zi$, for each $i\in I$, is an isomorphism. (In fact, $\mu(z)$ is obviously injective, since each ideal I of $\mathscr{F}$ contains a regular element of A; moreover, if $f\in\mathrm{Hom}_A(I,A)$, and if r is a regular element belonging to I, then $f=\mu(f(r)/r)$, with $f(r)/r\in(A:_{\mathrm{Tot}(A)}I)$.)

For the sake of simplicity, when each ideal I of $\mathscr{F}$ contains a regular element of A (e.g. when A is an integral domain), we set:

$$A_{\mathscr{F}}=\{z\in\mathrm{Tot}(A)\mid zI\subseteq A \text{ for some } I\in\mathscr{F}\}.$$

Remark 1.1. (a) In general, if A is a commutative ring with zero-divisors, the condition $t_{\mathscr{F}}(A)=0$ does not imply that $A_{\mathscr{F}}$ is embedded in $\mathrm{Tot}(A)$.

Let k be a field and let $\{X_n\mid n\geqslant 1\}$ and Y be indeterminates over k. Let $D:=k[Y;X_n\mid n\geqslant 1]$ denote the polynomial ring and let H be the ideal of D generated by the set $\{Y^n\cdot X_n\mid n\geqslant 1\}$. Set $A:=D/H$, $x_n:=X_n+H$ and $y:=Y+H$. Each nonzero element $a\in A$ can be written (uniquely) as follows:

$$a=a_0+a_1y+a_2y^2+\cdots+a_ry^r,\quad \text{with } r\geqslant 0, \text{ and } a_i\in k[x_n\mid n\geqslant 1]. \tag{1.1.1}$$

Note that $ax_1=a_0x_1$, since $y^nx_1=y^{n-1}(yx_1)$ and $y^nx_n=0$, for each $n\geqslant 1$. Therefore, if a is a regular element of A, then $a_0\in k$ and $a_0\neq 0$. By using (1.1.1), it is not difficult to verify that the converse is also true.

Let I be the ideal of A generated by the set $\{x_n\mid n\geqslant 1\}$. Note that:

$$a\in A \text{ and } aI^m=0,\quad \text{for some } m\geqslant 1\Rightarrow a=0 \tag{1.1.2}$$

(in fact, if a is written as in (1.1.1) and if $h\gg 0$, then $0=ax_h^m=a_0x_h^m+a_1yx_h^m+a_2y^2x_h^m+\cdots+a_ry^rx_h^m$ implies that $a=0$). Let $\mathscr{N}(I)$ denote the localizing system of A generated by I (i.e. the smallest localizing system of A containing I and its powers) [8, Proposition 2.1]. Since the localizing system $\mathscr{N}(I)$ contains the multiplicative system of ideals $\mathscr{N}_0(I):=\{I^m\mid m\geqslant 1\}$, it is not hard to show (by transfinite induction, based on the constructive inductive definition of $\mathscr{N}(I)$) that $t_{\mathscr{N}(I)}(A)=0$.

We consider the map $f\in\mathrm{Hom}_A(I,A)$ defined by A-linearity by setting $f(x_n):=y^{n-1}x_n$, on the set $\{x_n\mid n\geqslant 1\}$ of generators of the ideal $I\subset A$. Since

$$A_{\mathscr{N}(I)}=A_{(\mathscr{N}(I))}=\varinjlim\{\mathrm{Hom}_A(J,A)\mid J\in\mathscr{N}(I)\},$$

then $f \in \mathrm{Hom}_A(I, A)$ defines uniquely an element $z \in A_{\mathcal{N}(I)}$ and the following diagram commutes:

$$\begin{array}{ccc} I & \hookrightarrow & A \\ \Big\downarrow f & & \Big\downarrow z\cdot - \\ A & \xrightarrow{\ h\ } & A_{\mathcal{N}(\mathrm{I})} \end{array} \qquad (1.1.3)$$

i.e. $h(f(x_n)) = z \cdot x_n$, where h is the canonical homomorphism, that is

$$h : A \xrightarrow{\sim} \mathrm{Hom}_A(A, A) \to \varinjlim \{\mathrm{Hom}_A(J, A) \mid J \in \mathcal{N}(I)\}$$

[20, Chapter IX, Lemma 1.4].

Note that, if $A_{\mathcal{N}(I)} \subseteq \mathrm{Tot}(A)$, then $z = c/d$, with $c, d \in A$, d nonzero-divisor in A and $f(x_n) = y^{n-1}x_n = z \cdot x_n$, so $(y^{n-1} - z)x_n = 0$ and hence $dy^{n-1}x_n = cx_n$, for each $n \geqslant 1$. As in (1.1.1), let $c = c_0 + c_1 y + c_2 y^2 + \cdots + c_s y^s$, $d = d_0 + d_1 y + d_2 y^2 + \cdots + d_t y^t$, with $s, t \geqslant 0$, and $c_i, d_j \in k[x_n \mid n \geqslant 1]$, and set $c_{i,0}$ (respectively, $d_{j,0}$) equal to the constant term of the polynomial c_i (respectively, d_j). The equality $dy^{n-1}x_n = cx_n$, for $n = 1$, gives $d_0 x_1 = dx_1 = cx_1 = c_0 x_1$. By the regularity of d, we know that $d_0 = d_{0,0}$ is a nonzero element of k, and so $c_0 = c_{0,0}$ is a nonzero element of k.

On the other hand, the condition $dy^{n-1}x_n = cx_n$, implies that $0 = cyx_n$, for each $n \geqslant 1$. Henceforth, $cyI = 0$ and thus, by (1.1.2), $cy = 0$. It is clear that, when $h \geqslant 0$ $(c_i - c_{i,0})y^{h+i} = 0$, for each i, $0 \leqslant i \leqslant s$. Therefore, $0 = cy = cy^h = c_0 y^h + c_1 y^{h+1} + c_2 y^{h+2} + \cdots + c_s y^{h+s}$, hence $c_{0,0} = c_{1,0} = \cdots = c_{s,0} = 0$. This is a contradiction, thus we conclude that $A_{\mathcal{N}(I)} \nsubseteq \mathrm{Tot}(A)$.

(b) Let A be a commutative ring with zero-divisors and let $\mathrm{Tot}(A)$ be the total ring of fractions of A. Set

$$\mathscr{R} := \mathscr{R}(A) := \{I \text{ ideal of } A \mid \text{ there exists a regular element } r \text{ of } A, \text{ with } r \in I\}.$$

It is straightforward to verify that $\mathscr{R}$ is a localizing system of A and $A_{\mathscr{R}} = \mathrm{Tot}(A)$. Let ρ_A or, simply, ρ denote the canonical injective ring homomorphism from A to $\mathrm{Tot}(A)$. For the sake of simplicity, we identify $\rho(A)$ with A inside $\mathrm{Tot}(A)$.

Let $\mathscr{F}$ is a localizing system of A and let $\varphi = \varphi_{A,\mathscr{F}} : A \to A_{\mathscr{F}}$ be the canonical ring homomorphism.

Claim 1. *Assume that $\mathscr{F}$ is such that there exists a ring homomorphism $\psi : A_{\mathscr{F}} \to \mathrm{Tot}(A)$ such that $A \xrightarrow{\varphi} A_{\mathscr{F}} \xrightarrow{\psi} \mathrm{Tot}(A) = A \xrightarrow{\rho} \mathrm{Tot}(A)$. Then*:

(1) *$\varphi : A \to A_{\mathscr{F}}$ is injective (for the sake of simplicity, we identify $\varphi(A)$ with A in $A_{\mathscr{F}}$).*

(2) $t_{\mathscr{F}}(A) = 0$.

(3) *The set of all localizing systems $\tilde{\mathscr{F}}$ of A such that $A_{\tilde{\mathscr{F}}}$ is canonically A-isomorphic to $A_{\mathscr{F}}$ has a smallest element, denoted by $\mathscr{F}_0$.*

(4) $\mathscr{F}_0 \subseteq \mathscr{R}$.

The statement (1) follows immediately from the fact that ρ is injective and (2) is a consequence of (1) and of the fact that $\mathrm{Ker}(\varphi) = t_{\mathscr{F}}(A)$.

(3) Let $\mathscr{F}_0$ be the localizing system of A generated by the set of ideals:

$$\{I \text{ ideal of } A \mid I \supseteq (A :_A zA), \text{ for some } z \in A_{\mathscr{F}}\}$$

(cf. [6, pp. 140–141] for the transfinite inductive construction of the localizing system $\mathscr{F}_0$ of A, and for the proof that $A_{\mathscr{F}_0} \subseteq A_{\mathscr{F}}$). Note that, if $z \in A_{\mathscr{F}}$ then the ideal $(A :_A zA)$ belongs to $\mathscr{F}$, since in the present situation $t_{\mathscr{F}}(A) = 0$ and there exists an ideal $I \in \mathscr{F}$ and a homomorphism $f \in \mathrm{Hom}_A(I, A)$ such that a diagram (1.1.3) commutes. From these considerations it follows that (in $\mathrm{Tot}(A)$) $A_{\mathscr{F}} = A_{\mathscr{F}_0}$ and that $\mathscr{F}_0$ is the smallest localizing system of A with such a property.

(4) If $z \in A_{\mathscr{F}}$ then $\psi(z) \in \mathrm{Tot}(A)$, therefore there exists a regular element $r \in A$ such that $r\psi(z) = \psi(rz) \in A \subseteq \mathrm{Tot}(A)$, i.e. $rz \in A$, so $r \in (A :_A zA)$, hence $(A :_A zA) \in \mathscr{R}$. This fact implies that $\mathscr{F}_0 \subseteq \mathscr{R}$.

Claim 2. *With the notation introduced above, the following statements are equivalent*:

(i) *There exists a ring homomorphism* $\psi : A_{\mathscr{F}} \to \mathrm{Tot}(A)$ *such that* $A \xrightarrow{\varphi} A_{\mathscr{F}} \xrightarrow{\psi} \mathrm{Tot}(A) = A \overset{\rho}{\hookrightarrow} \mathrm{Tot}(A)$.

(ii) $\mathscr{F}_0 \subseteq \mathscr{R}$.

(iii) $A_{\mathscr{F}} = A_{\mathscr{F} \cap \mathscr{R}}$.

(i) $\Rightarrow$ (ii) by Claim (1, 4). (ii) $\Rightarrow$ (i) is obvious, since $\mathscr{F}_0 \subseteq \mathscr{R}$ implies that $A_{\mathscr{F}} = A_{\mathscr{F}_0} \subseteq A_{\mathscr{R}} = \mathrm{Tot}(A)$. Similarly, it is easy to see that (iii) $\Rightarrow$ (i). (ii) $\Rightarrow$ (iii). From the assumption we deduce that $\mathscr{F}_0 \subseteq \mathscr{F} \cap \mathscr{R} \subseteq \mathscr{F}$, thus (in $\mathrm{Tot}(A)$) we have $A_{\mathscr{F}_0} = A_{\mathscr{F} \cap \mathscr{R}} = A_{\mathscr{F}}$.

Note that Claim 2 can be generalized as follows. Let S be a multiplicatively closed set of elements of A and let $\rho : A \to S^{-1}A$ be the canonical ring homomorphism. Set

$$\mathscr{F}_S := \{I \text{ ideal of } A \mid I \cap S \neq \emptyset\}.$$

Assume that $\mathscr{F}$ is a localizing system of A with the property that there exists a ring homomorphism $\psi : A_{\mathscr{F}} \to S^{-1}A$ such that $A \xrightarrow{\varphi} A_{\mathscr{F}} \xrightarrow{\psi} S^{-1}A = A \xrightarrow{\rho} S^{-1}A$. Then $\mathscr{F}_0 \subseteq \mathscr{F}_S$.

A ring homomorphism $f : A \to B$ is called an *epimorphism* if, for any pair of ring homomorphisms $g_1, g_2 : B \to C$, the condition $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. We say that $f : A \to B$ is a *flat epimorphism* if f is a epimorphism and B is a A-module flat via f (i.e. B is $f(A)$-flat), cf. [3, Chapitre 1, §2, 3,15].

The following characterization of a flat epimorphism, due to Popescu [18] and Popescu–Groza [11], will be used later.

Theorem 1.2. *Let* $\alpha : R \to A$ *be a ring homomorphism.*

Then:

(1) $\mathscr{F}(A) := \{I \mid I \text{ ideal of } R,\ \alpha(I)A = A\}$ *is a localizing system of* R.

(2) *Assume that for each* $I \in \mathscr{F}(A)$ *and for each* $f \in \mathrm{Hom}_R(I,A)$, *there exists a unique* $\bar{f} \in \mathrm{Hom}_R(R,A)$ *with* $\bar{f}|_I = f$. *Then, there exists a unique ring homomorphism* $\bar{\alpha}: R_{\mathscr{F}(A)} \to A$ *such that* $\bar{\alpha} \circ \varphi = \alpha$, *where* $\varphi = \varphi_{R,\mathscr{F}(A)}: R \to R_{\mathscr{F}(A)}$ *is the canonical homomorphism.*

(3) *The following statements are equivalent*:

(i) α *is a flat epimorphism*;

(ii) *there exists a ring isomorphism* $\bar{\alpha}: R_{\mathscr{F}(A)} \to A$ *such that* $\bar{\alpha} \circ \varphi = \alpha$;

(iii) (a) *for each* $a \in A$, *there exists* $I \in \mathscr{F}(A)$ *such that* $\alpha(I)a \subseteq \alpha(R)$;
(b) *if* $x \in R$ *and* $\alpha(x) = 0$, *then there exists* $I \in \mathscr{F}(A)$ *such that* $xI = 0$.

(iv) (a) $\alpha^*: \mathrm{Spec}(A) \to \mathrm{Spec}(R)$ *is injective*;
(b) α *is a GD-homomorphism*;
(c) *if* $\beta: R \to B$ *is a ring homomorphism such that* $\beta^*(\mathrm{Spec}(B)) \subseteq \alpha^*(\mathrm{Spec}(A))$, *then there exists a unique ring homomorphism* $\bar{\beta}: A \to B$ *such that* $\bar{\beta} \circ \alpha = \beta$.

Proof. (1) (LS1) holds trivially. If J is an ideal of R and $(J :_R iR) \in \mathscr{F}(A)$ for each $i \in I$ and $I \in \mathscr{F}(A)$, then

$$A = \alpha((J :_R iR))A \subseteq (\alpha(J)A :_A \alpha(i)A)$$

hence $\alpha(i)A \subseteq \alpha(J)A$, for each $i \in I$. Therefore, $\alpha(I)A \subseteq \alpha(J)A$ and so $\alpha(J)A = A$, because $I \in \mathscr{F}(A)$.

(2) follows from [3, Chapitre 2, §2, Exercice 19(d), (e); 17, Lemma 1.3].

(3) For the equivalence (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) cf. [18, Theorem 16.6, p. 261].

The equivalence (i) $\Leftrightarrow$ (iv) is proved in [11, Theorem 2.2]. □

Remark 1.3. From a local-global point of view, if $\alpha: R \to A$ is an arbitrary ring homomorphism, it is well known that the following statements are equivalent:
(i) $\alpha: R \to A$ is a flat epimorphism;
(ii) $\alpha \otimes_R A_P: R_P \to A_P$ is a flat epimorphism, for each $P \in \mathrm{Spec}(R)$;
(iii) for each $P \in \mathrm{Spec}(R)$, either $\alpha(P)A = A$ or $\alpha \otimes_R A_P: R_P \to A_P$ is a isomorphism; cf. [15, Proposition 2.4].

Let R be an integral domain with quotient field K and let I be an ideal of R. The following overring of R is called the *Nagata transform of I with respect to R*:

$$T(I) = T_R(I) := \{z \in K \mid (R :_R zR) \supseteq I^n, \text{ for some } n \geqslant 1\}.$$

When considering the non-Noetherian case, it seems preferable to replace the Nagata transform with a more general notion of ideal transform, introduced by Kaplansky [14] (see also [13]), which we call the *Kaplansky ideal transform of I with respect to R*:

$$\Omega(I) = \Omega_R(I) := \{z \in K \mid \mathrm{rad}((R :_R zR)) \supseteq I\};$$

cf. also [5, (3.2)].

It is straightforward to check that $T(I) \subseteq \Omega(I)$ and $T(I) = \Omega(I)$, if I is finitely generated.

Proposition 1.4. *Let I be a ideal of an integral domain R. Then*:

(1) $\mathscr{K}(I) = \mathscr{K}_R(I) := \{J \mid J$ *is an ideal of* R, $\mathrm{rad}(J) \supseteq \mathrm{rad}(I)\}$ *is a localizing system of* R.
(2) $R_{\mathscr{K}(I)} = \Omega_R(I) = \bigcap\{R_P \mid P \not\supseteq I\}$.

Proof. See [5, Lemma 3.1, 3.2 and 4.3]. □

Since R is an integral domain, $\Omega_R(I)$ (respectively, $T_R(I)$) is an overring of R. We will let $\omega = \omega_{R,I} : R \hookrightarrow \Omega_R(I)$ (respectively, $\theta = \theta_{R,I} : R \hookrightarrow T_R(I)$) denote the canonical embedding, hence $R \overset{\omega}{\hookrightarrow} \Omega_R(I) = R \overset{\theta}{\hookrightarrow} T_R(I) \subseteq \Omega_R(I)$, and we will identify $\omega(R)$ and $\theta(R)$ with R inside $\Omega_R(I)$.

As in the classical theory developed by Nagata and Kaplansky, we will consider ideal transforms only with respect to integral domains. We note that some type of construction of ideal transform, including the Nagata ideal transforms, have been defined and studied also for rings with zero-divisors, cf. for instance [4,19]. Kaplansky ideal transforms were considered until now only for integral domains. For this reason we shall limit our investigation in the present paper to this case. More precisely, in this work, we pursue the study of the Kaplansky ideal transform, looking for a "universal property" of the canonical embedding $\omega : R \to \Omega_R(I)$. In this investigation we come across a "geometric" aspect of ideal transforms. From classical results by Chevalley [10, I.6.7.1], Nagata [16] and Hartshorne [12] (cf. also [1]), it can be shown that if R is a Noetherian integral domain and I an ideal of R, then the Nagata transform $T_R(I)$ is the ring of global sections over the open subspace $D(I)$ of $\mathrm{Spec}(R)$, and $D(I)$ is affine if and only if $IT_R(I) = T_R(I)$. If R is not Noetherian, $D(I)$ may be affine with $IT_R(I) \neq T_R(I)$ (for an example cf. [5, Section 4]).

When considering the non-Noetherian case, the Kaplansky ideal transform seems preferable to the notion of ideal transform previously considered by Nagata (cf. for instance [5,7,2]). One of the purposes of this paper is to provide further evidence to this aspect of the theory of the Kaplansky transforms. More precisely:

Definition 1.5. Let I be an ideal of a given arbitrary ring R and let $\alpha : R \to A$ be a ring homomorphism. We say that α is an *I-morphism* if $\alpha^*(\mathrm{Spec}(A)) \subseteq D(I)$.

We denote by $K_R(I, A)$ the set of all I-morphisms from R to A. It is easy to see that $K_R(I, -)$ defines a covariant functor from the category of rings, *Ring*, to the category of sets, *Set*. Then, we will prove in Section 2 that the following statements are equivalent:
(i) $K_R(I, -) : Ring \to Set$ is a representable functor;
(ii) $\omega : R \hookrightarrow \Omega_R(I)$ is a I-morphism;
(iii) $I\Omega_R(I) = \Omega_R(I)$;
(iv) $D(I)$ is an affine open subscheme of $\mathrm{Spec}(R)$.

2. Main results

Proposition 2.1. *In the situation of Definition* 1.5, $\alpha : R \to A$ *is an I-morphism if and only if* $\alpha(I)A = A$ (*i.e.* $I \in \mathscr{F}(A)$, *where* $\mathscr{F}(A)$ *is the localizing system of* R *introduced* *in Theorem* 1.2(1)).

Proof. If α is an I-morphism and $\alpha(I)A \neq A$, then there exists $Q \in \mathrm{Spec}(A)$ such that $\alpha(I)A \subseteq Q$. Henceforth, $I \subseteq \alpha^{-1}(Q)$ and this is a contradiction.

Conversely, if $P := \alpha^{-1}(Q)$ for some $Q \in \mathrm{Spec}(A)$ and if $P \supseteq I$ then $\alpha(I)A \subseteq \alpha(\alpha^{-1}(Q))A \subseteq Q \neq A$ and this is a contradiction. □

Proposition 2.2. *Let I be an ideal of a given arbitrary ring R, the assignment*:

$$A \rightsquigarrow K_R(I,A) := \{\alpha : R \to A \mid \alpha \text{ is an } I\text{-morphism}\}$$

defines a (*covariant*) *functor $K_R(I,-)$ from the category of rings and ring homomorphisms*, Ring, *to the category of sets and functions*, Set.

Proof. Straightforward. □

Proposition 2.3. *Let R be an integral domain, I an ideal of R and $\alpha : R \to A$ a ring homomorphism. We denote by $\Omega_R(I)$ the Kaplansky transform of I with respect to R and by $\omega : R \hookrightarrow \Omega_R(I)$ the canonical embedding. If $\alpha \in K_R(I,A)$, then there exists a unique ring homomorphism $\bar{\alpha} : \Omega_R(I) \to A$ such that $\bar{\alpha} \circ \omega = \alpha$.*

Proof. Since $\alpha \in K_R(I,A)$, then $\alpha(I)A = A$ (Proposition 2.1) and hence $\alpha(J)A = A$, for each $J \in \mathscr{K}(I)$ (Proposition 1.4(1)). This means that

there exists $n \geqslant 1,\ j_1, j_2, \ldots, j_n \in J,\ a_1, a_2, \ldots, a_n \in A$ such that :

$$\sum_{k=1}^{n} \alpha(j_k)a_k = 1. \tag{2.3.1}$$

If $z \in \Omega_R(I)$, then there exists $J \in \mathscr{K}(I)$ such that $zJ \subseteq R$ (Proposition 1.4(2)). Set:

$$r_k := zj_k \in R, \quad \text{for each } 1 \leqslant k \leqslant n. \tag{2.3.2}$$

We can define $\bar{\alpha} : \Omega_R(I) \to A$ by setting:

$$\bar{\alpha}(z) := \sum_{k=1}^{n} \alpha(r_k)a_k, \quad \text{for each } z \in \Omega_R(I). \tag{2.3.3}$$

□

Claim 1. *$\bar{\alpha} : \Omega_R(I) \to A$ is well defined* (*i.e. $\bar{\alpha}(z)$ is independent of the choice of J and the choice of the elements j_k and a_k satisfying condition* (2.3.1)).

If $J' \in \mathscr{K}(I)$ with $zJ' \subseteq R$ and if $m \geqslant 1,\ j'_1, j'_2, \ldots, j'_m \in J'$ and $a'_1, a'_2, \ldots, a'_m \in A$ are such that:

$$\sum_{h=1}^{m} \alpha(j'_h)a'_h = 1,$$

then we can consider $r'_h := zj'_h \in R$, for each $1 \leqslant h \leqslant m$, and the element:

$$z' := \sum_{h=1}^{m} \alpha(r'_h)a'_h.$$

We want to show that $z' = \bar{\alpha}(z)$. As a matter of fact, for each h,

$$\alpha(r'_h) = r'_h\alpha(1) = \sum_{k=1}^{m} r'_h\alpha(j_k)a_k$$

and thus:

$$\begin{aligned}\sum_h \alpha(r'_h)a'_h &= \sum_h\sum_k r'_h\alpha(j_k)a_k a'_h\\ &= \sum_h\sum_k \alpha(r'_h j_k)a_k a'_h\\ &= \sum_h\sum_k \alpha(zj'_h j_k)a_k a'_h\\ &= \sum_h\sum_k \alpha(j'_h r_k)a_k a'_h\\ &= \left(\sum_h \alpha(j'_h)a'_h\right)\left(\sum_k \alpha(r_k)a_k\right) = 1\,\bar{\alpha}(z) = \bar{\alpha}(z).\end{aligned}$$

Claim 2. $\bar{\alpha}\circ\omega = \alpha$.

If $z \in R$, then $zR \subseteq R$ with $R \in \mathscr{K}(I)$, hence for $n = 1$, $J = R$, $j = 1$, $a = 1$ we have $\alpha(1)\cdot 1 = 1$ (2.3.1). Therefore, $r := z \cdot 1$ (2.3.2), and so, by (2.3.3), $\bar{\alpha}(\omega(z)) = \alpha(r)\cdot 1 = \alpha(z)$.

Claim 3. *The map* $\bar{\alpha}: \Omega_R(I) \to A$ *is a ring homomorphism.*

Note that, from Claim 2, it follows that $\bar{\alpha}(1) = 1$. We start by showing that, if $z_1, z_2 \in \Omega_R(I)$, $\bar{\alpha}(z_1z_2) = \bar{\alpha}(z_1)\bar{\alpha}(z_2)$. Let $J_1, J_2 \in \mathscr{K}(I)$ be such that $z_1J_1 \subseteq R$ and $z_2J_2 \subseteq R$. Set $J := J_1J_2 \in \mathscr{K}(I)$. Then $z_1z_2J \subseteq R$, $z_1J \subseteq R$ and $z_2J \subseteq R$. Let $n, m \geqslant 1$, $j_{1,k} \in J_1$, $j_{2,h} \in J_2$, $a_{1,k}, a_{2,h} \in A$, with $1 \leqslant k \leqslant n$ and $1 \leqslant h \leqslant m$ be such that:

$$\sum_k \alpha(j_{1,k})a_{1,k} = 1 = \sum_h \alpha(j_{2,h})a_{2,h}.$$

Set $r_{1,k} := z_1 j_{1,k}$ and $r_{2,h} := z_2 j_{2,h}$ for all k and h, with $1 \leqslant k \leqslant n$ and $1 \leqslant h \leqslant m$. Then:

$$\bar{\alpha}(z_1) = \sum_k \alpha(r_{1,k})a_{1,k}, \quad \bar{\alpha}(z_2) = \sum_h \alpha(r_{2,h})a_{2,h}.$$

For all k and h, set:

$$j_{k;h} := j_{1,k}j_{2,h} \in J, \quad a_{k;h} := a_{1,k}a_{2,h} \in A, \quad r_{k;h} := z_1z_2 j_{k;h}.$$

It is easy to see that:

$$\sum_{k,h} \alpha(j_{k;h})a_{k;h} = 1.$$

Therefore, we have:

$$\bar{\alpha}(z_1 z_2) = \sum_{k,h} \alpha(r_{k;h})a_{k;h}.$$

Since,

$$\sum_{k,h} \alpha(r_{k;h})a_{k;h} = \left(\sum_{k} \alpha(r_{1,k})a_{1,k}\right)\left(\sum_{h} \alpha(r_{2,h})a_{2,h}\right)$$

then we conclude that $\bar{\alpha}(z_1 z_2) = \bar{\alpha}(z_1)\bar{\alpha}(z_2)$.

The argument for showing that $\bar{\alpha}(z_1 + z_2) = \bar{\alpha}(z_1) + \bar{\alpha}(z_2)$ is similar and we omit the details.

Claim 4. *Assume that $\bar{\alpha}' : \Omega_R(I) \to A$ is a ring homomorphism such that $\bar{\alpha}' \circ \omega = \alpha$, then $\bar{\alpha}' = \bar{\alpha}$.*

Let $z \in \Omega_R(I)$ and let $J \in \mathscr{K}(I)$ be such that $zJ \subseteq R$, then $\bar{\alpha}'(zJ) = \alpha(zJ) = \bar{\alpha}(zJ)$. On the other hand $\bar{\alpha}'(J) = \alpha(J) = \bar{\alpha}(J)$, hence $(\bar{\alpha}'(z) - \bar{\alpha}(z))\alpha(J) = 0$. Moreover $\alpha(J)A = A$, since $\alpha \in K_R(I, A)$ and $J \in \mathscr{K}(I)$, hence:

$$(\bar{\alpha}'(z) - \bar{\alpha}(z))A = (\bar{\alpha}'(z) - \bar{\alpha}(z))\alpha(J)A = 0,$$

thus $\bar{\alpha}'(z) - \bar{\alpha}(z) = 0$, for each $z \in \Omega_R(I)$. □

Theorem 2.4. *Let R be an integral domain and I an ideal of R. The following statements are equivalent*:

(i) *$\omega : R \hookrightarrow \Omega_R(I)$ is an I-morphism.*
(ii) *$\omega : R \hookrightarrow \Omega_R(I)$ is a flat epimorphism.*
(iii) *$D(I)$ is an affine open subspace of* Spec(R).
(iv) *The functor $K_R(I, -) : Ring \to Set$ is representable. In particular, if $K_R(I, -)$ is represented by a ring W, then W is canonically isomorphic to $\Omega_R(I)$.*
(v) *There exists a finitely generated ideal $J \subseteq I$ such that $D(J) = D(I)$ and $D(J)$ is an affine open subspace of* Spec(R).
(vi) *There exists a finitely generated ideal $J \subseteq I$ such that $T_R(J) = \Omega_R(J) = \Omega_R(I)$ and $JT_R(J) = T_R(J)$.*

Proof. (i) ⇔ (ii) ⇔ (iii). We already know that (iii) is equivalent to each of the following properties:

(j) $I\Omega_R(I) = \Omega_R(I)$;
(jj) $\omega : R \hookrightarrow \Omega_R(I)$ is flat, and if $P \in \text{Spec}(R)$ and $P \supseteq I$ then $P\Omega_R(I) = \Omega_R(I)$;

cf. [5, Theorem 4.4(i) ⇔ (iii) ⇔ (v)]. Moreover (j) is equivalent to (i) by Proposition 2.1, and (jj) is equivalent to (ii) as a consequence of Remark 1.3 and Proposition 1.4(2).

(i)$\Rightarrow$(iv). This implication follows from Proposition 2.3 since, for each ring A, the map

$$K_R(I,A) \to \mathrm{Hom}(\Omega_R(I),A), \quad \alpha \mapsto \bar{\alpha},$$

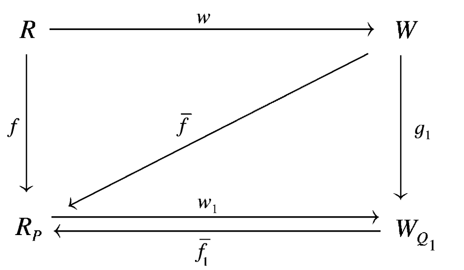
is a natural bijection.

(iv) $\Rightarrow$ (i) and the proof of the second part of the statement. Let W be a ring representing the functor $K_R(I,-): Ring \to Set$. Let $w: R \to W$ be the ring homomorphism in $K_R(I,W)$ corresponding to the identity $1_W \in \mathrm{Hom}(W,W)$, under the natural bijection $K_R(I,W) \xrightarrow{\sim} \mathrm{Hom}(W,W)$. Our aim is to show that there exists a ring isomorphism $\sigma: W \to \Omega_R(I)$ such that $\sigma \circ w = \omega$. This fact will imply in particular that $\omega \in K_R(I,\Omega_R(I))$, since $w \in K_R(I,W)$.

Claim 1. $w^*: \mathrm{Spec}(W) \to \mathrm{Spec}(R)$ *is injective*.

Assume that $Q_1, Q_2 \in \mathrm{Spec}(W)$ are such that $w^{-1}(Q_1) = w^{-1}(Q_2) =: P$. Consider the canonical ring homomorphism $f: R \to R_P$, then f is an I-morphism because $P \in w^*(\mathrm{Spec}(W)) \subseteq D(I)$. Therefore, since $K_R(I,-)$ is represented by W, by the universal property of $w: R \to W$ we can find a unique ring homomorphism $\bar{f}: W \to R_P$ such that $\bar{f} \circ w = f$.

Consider the following diagram of ring homomorphisms:



where $g_1: W \to W_{Q_1}$ is the canonical homomorphism, $w_1: R_P \to W_{Q_1}$ (respectively, $\bar{f}_1: W_{Q_1} \to R_P$) is the canonical extension of w (respectively, $\bar{f}$) to the ring of fractions. (Note that $\bar{f}_1$ is uniquely determined from $\bar{f}$, since if $y \in W \setminus Q_1$ then $\bar{f}(y) \notin PR_P$, otherwise $g_1(y) = w_1(\bar{f}(y))$ would be in $Q_1 W_{Q_1}$ which is a contradiction). Since $\bar{f}_1 \circ g_1 = \bar{f}$ then $\bar{f}_1 \circ g_1 \circ w = \bar{f} \circ w = f$, and thus $w_1 \circ \bar{f}_1 \circ g_1 \circ w = w_1 \circ f = g_1 \circ w$. By the universal property of w, we deduce that $w_1 \circ \bar{f}_1 \circ g_1 = g_1$ and so $w_1 \circ \bar{f}_1 = 1_{W_{Q_1}}$, since $g_1: W \to W_{Q_1}$ is an epimorphism of rings.

If we show that $\mathrm{Ker}(w_1) = 0$, then we obtain that w_1 (and $\bar{f}_1$) is a ring isomorphism. Let $x = f(r)/f(s) \in R_P$ with $r \in R$ and $s \in R \setminus P$. Assume that $w_1(x) = 0$. Then $0 = w_1(f(r)) = g_1(w(r))$ hence, for some $t \in W \setminus Q_1$, $tw(r) = 0$. Therefore $0 = \bar{f}(tw(r)) = \bar{f}(t)\bar{f}(w(r)) = \bar{f}(t)f(r)$ and thus $f(r) = 0$ and so $x = 0$.

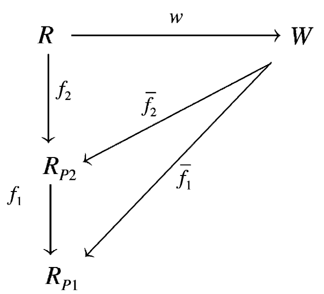
With a similar argument, by interchanging Q_1 with Q_2, we can define $g_2: W \to W_{Q_2}$, $w_2: R_P \to W_{Q_2}$ and $\bar{f}_2: W_{Q_2} \to R_P$ and we can prove that w_2 (and $\bar{f}_2$) is a ring isomorphism and, moreover, that

$$\bar{f}_1 \circ g_1 \circ w = f = \bar{f}_2 \circ g_2 \circ w.$$

Again, by the universality of w, we deduce that $\bar{f}_1 \circ g_1 = \bar{f}_2 \circ g_2$. This fact implies immediately that $Q_1 = Q_2$.

Claim 2. *$w: R \to W$ is a GD-homomorphism.*

Let $P_1 \subset P_2 := w^{-1}(Q_2)$ with $P_1 \in \mathrm{Spec}(R)$ and $Q_2 \in \mathrm{Spec}(W)$. Since the prime ideal $P_2 \in w^*(\mathrm{Spec}(W)) \subseteq D(I)$, then also $P_1 \in D(I)$. Let $f_i : R \to R_{P_i}$ be the canonical homomorphism, for $i = 1, 2$. Then, clearly, $f_i \in K_R(I, R_{P_i})$ and thus, by the universality of w, there exists a unique ring homomorphism $\bar{f}_i : W \to R_{P_i}$ such that $\bar{f}_i \circ w = f_i$, for $i = 1, 2$. In particular, using Claim 1, we deduce that $\bar{f}_2^{-1}(P_2 R_{P_2}) = Q_2$. Set $Q_1 := \bar{f}_1^{-1}(P_1 R_{P_1}) \in \mathrm{Spec}(W)$. From the commutativity of the following diagram of canonical ring homorphisms:



we deduce immediately that $Q_1 \subset Q_2$ and $w^{-1}(Q_1) = P_1$.

Claim 3. *If $\alpha : R \to A$ is a ring homomorphism such that $\alpha^*(\mathrm{Spec}(A)) \subseteq w^*(\mathrm{Spec}(W))$, then there exists a unique ring homomorphism $\bar{\alpha} : W \to A$ such that $\bar{\alpha} \circ w = \alpha$.*

Since w is an I-morphism then $\alpha^*(\mathrm{Spec}(A)) \subseteq D(I)$ and so α is also an I-morphism. The conclusion follows from the universality property of $w: R \to W$.

From Claims 1, 2 and 3 and from Theorem 1.2(3) ((iv) $\Rightarrow$ (i)), we deduce that $w: R \to W$ is a flat epimorphism and, thus, there exists a canonical isomorphism $\bar{w} : R_{\mathscr{F}(W)} \xrightarrow{\sim} W$ such that $\bar{w} \circ \varphi = w$, where $\mathscr{F}(W) := \{J \mid J \text{ ideal of } R,\ w(J)W = W\}$ and $\varphi : R \to R_{\mathscr{F}(W)}$ is the canonical embedding (Theorem 1.2(3) ((i) $\Rightarrow$ (ii))).

Claim 4. $w^*(\mathrm{Spec}(W)) = D(I)$.

Since $w^*(\mathrm{Spec}(W)) \subseteq D(I)$, we need only to prove the reverse inclusion. Let $P \in D(I)$ and let $f : R \to R_P$ be the canonical homomorphism, which is obviously an I-morphism. By the universal property of w, there exists a unique ring homomorphism $\bar{f} : W \to R_P$ such that $\bar{f} \circ w = f$. Then, clearly, $Q := \bar{f}^{-1}(PR_P)$ is a prime ideal of W such that $w^{-1}(Q) = P$.

Claim 5. $\mathscr{F}(W) = \mathscr{K}(I) := \{J \mid J$ *ideal of* $R, \mathrm{rad}(J) \supseteq \mathrm{rad}(I)\}$.

If $J \in \mathscr{K}(I)$, then $w(J)W = W$. As a matter of fact, if $w(J)W \subseteq Q$, for some $Q \in \mathrm{Spec}(W)$, then $J \subseteq w^{-1}(Q)$ and so $\mathrm{rad}(I) \subseteq \mathrm{rad}(J) \subseteq w^{-1}(Q)$. This fact contradicts the inclusion $w^*(\mathrm{Spec}(W)) \subseteq D(I)$.

Conversely, if $J \in \mathscr{F}(W)$, then $w(J)W = W$. Since $w^*(\mathrm{Spec}(W)) = D(I)$ (Claim 4), if $Q \in \mathrm{Spec}(W)$ then $P := w^{-1}(Q) \in D(I)$, thus necessarily $P \in D(J)$ (otherwise $w(J)W \subseteq Q$). From the inclusion $D(I) \subseteq D(J)$ we deduce that $\mathrm{rad}(I) \subseteq \mathrm{rad}(J)$, i.e. $J \in \mathscr{K}(I)$.

Since $R_{\mathscr{K}(I)} = \Omega_R(I)$ (Proposition 1.4(2)), from Claim 5 and from the remark following Claim 3, we deduce that there exists a canonical isomorphism

$$\sigma : \Omega_R(I) = R_{\mathscr{K}(I)} = R_{\mathscr{F}(W)} \xrightarrow[\sim]{\bar{w}} W$$

such that $\sigma \circ \omega = w$.

(iii) $\Rightarrow$ (v) and (vi). Since (iii) is equivalent to $I\Omega_R(I) = \Omega_R(I)$ [5, Theorem 4.4(i) $\Leftrightarrow$ (iii)], then there exist $n \geqslant 1$, $a_1, a_2, \ldots, a_n \in I$, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Omega_R(I)$ such that $\sum_{i=1}^{n} a_i \alpha_i = 1$. Set $J := (a_1, a_2, \ldots, a_n)R$, then obviously, $J \subseteq I$. Moreover:

Claim 6. $\mathrm{rad}_R(J) = \mathrm{rad}_R(I)$, *thus* $\mathscr{K}(J) = \mathscr{K}(I)$ *and* $\Omega_R(J) = \Omega_R(I)$.

As a matter of fact, if P is a prime ideal of R and $J \subseteq P$, then obviously $\Omega_R(I) = J\Omega_R(I) \subseteq P\Omega_R(I) \subseteq \Omega_R(I)$ and thus, by Proposition 1.4(2), $P \notin D(I)$, i.e. $I \subseteq P$. Therefore $\mathrm{rad}_R(J) = \mathrm{rad}_R(I)$. From Proposition 1.4, we deduce immediately that $\mathscr{K}(I) = \mathscr{K}(J)$ and $\Omega_R(I) = \Omega_R(J)$.

Since $D(I) = D(J)$ if and only if $\mathrm{rad}_R(I) = \mathrm{rad}_R(J)$, from Claim 6, we deduce immediately that (iii) $\Rightarrow$ (v). Moreover, when J is finitely generated, then $T_R(J) = \Omega_R(J)$ and $D(J)$ is an affine open subspace of $\mathrm{Spec}(R)$ if and only if $JT_R(J) = J\Omega_R(J) = \Omega_R(J) = T_R(J)$ [5, Theorem 4.4(i) $\Leftrightarrow$ (iii)]. Henceforth, using again Claim 6, we deduce immediately that (iii) $\Rightarrow$ (vi).

The implication (v) $\Rightarrow$ (iii) is trivial.

(vi) $\Rightarrow$ (iii). It is enough to show that $I\Omega_R(I) = \Omega_R(I)$ [5, Theorem 4.4(iii) $\Rightarrow$ (i)]. From the assumption, we have $\Omega_R(I) = \Omega_R(J) = T_R(J) = JT_R(J) \subseteq IT_R(J) \subseteq I\Omega_R(J) = I\Omega_R(I)$. $\square$

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