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On the star class group of a pullback

Marco Fontana [a,*], Mi Hee Park [b]

[a] *Dipartimento di Matematica, Università degli Studi Roma Tre, Largo San Leonardo Murialdo, 1, 00146 Roma, Italy*

[b] *Department of Mathematics, Chung-Ang University, Seoul 156-756, Republic of Korea*

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Abstract

For the domain R arising from the construction T, M, D, we relate the star class groups of R to those of T and D. More precisely, let T be an integral domain, M a nonzero maximal ideal of T, D a proper subring of $k := T/M$, $\varphi : T \to k$ the natural projection, and let $R = \varphi^{-1}(D)$. For each star operation $*$ on R, we define the star operation $*_\varphi$ on D, i.e., the "projection" of $*$ under φ, and the star operation $(*)_T$ on T, i.e., the "extension" of $*$ to T. Then we show that, under a mild hypothesis on the group of units of T, if $*$ is a star operation of finite type, then the sequence of canonical homomorphisms $0 \to \mathrm{Cl}^{*_\varphi}(D) \to \mathrm{Cl}^{*}(R) \to \mathrm{Cl}^{(*)_T}(T) \to 0$ is split exact. In particular, when $* = t_R$, we deduce that the sequence $0 \to \mathrm{Cl}^{t_D}(D) \to \mathrm{Cl}^{t_R}(R) \to \mathrm{Cl}^{(t_R)_T}(T) \to 0$ is split exact. The relation between $(t_R)_T$ and t_T (and between $\mathrm{Cl}^{(t_R)_T}(T)$ and $\mathrm{Cl}^{t_T}(T)$) is also investigated.

Keywords: Class group; Picard group; Star operation; Pullback; t-ideal; Prüfer multiplication domain

1. Introduction and background results

The interest for constructing a general theory of the class group, extending the theory of the divisor class group of a Krull domain, was implicitly present already in the work

* Corresponding author.
E-mail addresses: fontana@mat.uniroma3.it (M. Fontana), mhpark@cau.ac.kr (M.H. Park).

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by Claborn and Fossum (cf. Fossum's book [24]). One of the main objectives for this type of extension was to establish a general functorial theory by exploiting class-group-type techniques in a more general setting than that of Krull domains. An approach to this problem, using star operations, was initiated by D.F. Anderson in 1988 [4], where he studied in a systematic way the star class group $\mathrm{Cl}^\star(R)$ of an integral domain R, equipped with a star operation $\star$. The key point of this construction is that, when $\star$ is the identity operation d, $\mathrm{Cl}^d(R)$ coincides with the Picard group $\mathrm{Pic}(R)$ (which is, in fact, the "classical" class group of the nonzero fractional ideals when R is a Dedekind domain); when $\star$ is the v-operation on a Krull domain, $\mathrm{Cl}^v(R)$ coincides with the "usual" divisor class group of R; when $\star$ is the t-operation, $\mathrm{Cl}^t(R)$, which is defined on arbitrary domain R, is commonly considered the best generalization of the "usual" divisor class group to the general setting (cf. the pioneering work in this area by Bouvier and Zafrullah [12,13,42] and the recent excellent survey paper by D.F. Anderson [5]).

Since various divisibility properties are often reflected in group-theoretic properties of the class groups, a particular interest was given in recent years to the computation of the t-class group where the functorial properties can be applied in a very effective way (for instance, cf. [2,26,36]).

In case of the rings arising from pullback construction of various type (cf. [14,16]), the t-class group was extensively studied by several authors (cf. for instance [6,7,9,10,19,32]).

It is well known that, even in the case of an embedding $A \subset B$ of Krull domains, it is not possible in general to define a canonical homomorphism between the divisor class groups $\mathrm{Cl}(A) \to \mathrm{Cl}(B)$ (the condition (PDE), i.e., "pas d'éclatement", was introduced in 1964 by Samuel [40] in order to characterize the existence of this canonical homomorphism). In case of star class groups, the technical difficulties for establishing functorial properties were surmounted by D.F. Anderson by introducing the notion of compatibility between star operations. More precisely, let A be a subdomain of an integral domain B and let $\star_A$ (respectively, $\star_B$) be a star operation on A (respectively, on B), then $\star_A$ and $\star_B$ are compatible if $(IB)^{\star_B} = (I^{\star_A}B)^{\star_B}$ for each nonzero fractional ideal I of A. In this situation, the extension map $I \mapsto IB$ induces a natural group homomorphism $\mathrm{Cl}^{\star_A}(A) \to \mathrm{Cl}^{\star_B}(B)$. Unfortunately, the compatibility condition is a sufficient but not a necessary condition for the existence of the natural homomorphism $\mathrm{Cl}^{\star_A}(A) \to \mathrm{Cl}^{\star_B}(B)$ [4, page 823]. Moreover, the identity operation d_A on A is compatible with any star operation on B while it is very common that the t-operation t_A (respectively, the v-operation v_A) on A is not compatible with the t-operation t_B (respectively, the v-operation v_B) on B.

In the present paper we mainly consider the following situation:

($\square$) *T represents an integral domain, M a nonzero maximal ideal of T, k the residue field T/M, D a proper subring of k and $\varphi : T \to k$ the canonical projection. Let $R := \varphi^{-1}(D) =: T \times_k D$ be the integral domain arising from the following pullback of canonical homomorphisms*:

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ T & \xrightarrow{\varphi} & k = T/M. \end{array}$$

It is easy to see that $M = (R : T)$ is the conductor of the embedding $\iota : R \hookrightarrow T$. In this situation, we will say that we are dealing with *a pullback of type* ($\square$) and we will still denote by φ the restriction $\varphi|_R$, giving rise to a canonical surjective homomorphism from $R = \varphi^{-1}(D)$ onto D.

Let L denote the field of quotients of D (and hence, $L \subseteq k$). *If we assume, moreover, that* $L = k$, then we will say that we are dealing with *a pullback of type* ($\square^+$).

The main goal of this work is to establish functorial relations among the star class groups of R, D, and T, by using the theory that we have recently developed in [22] concerning the "lifting" and the "projection" of a star operation under a surjective homomorphism of integral domains, the "extension" of a star operation to its overrings and the "glueing" of star operations in pullback diagrams of a rather general type. One of the principal results proven in this paper is that, given a pullback diagram of type ($\square^+$) and a star operation $*$ of finite type on R, if $*_\varphi$ denotes the "projection" of $*$ onto D (respectively, $(*)_T$ denotes the "extension" of $*$ to T), under a mild hypothesis on the group of units of T, the sequence of canonical homomorphisms

$$0 \longrightarrow \mathrm{Cl}^{*_\varphi}(D) \xrightarrow{\bar{\boldsymbol{\alpha}}} \mathrm{Cl}^{*}(R) \xrightarrow{\bar{\boldsymbol{\beta}}} \mathrm{Cl}^{(*)_T}(T) \longrightarrow 0$$

is split exact (Theorem 2.17). In particular, when $* = t_R$, we deduce that the sequence

$$0 \longrightarrow \mathrm{Cl}^{t_D}(D) \xrightarrow{\bar{\boldsymbol{\alpha}}} \mathrm{Cl}^{t_R}(R) \xrightarrow{\bar{\boldsymbol{\beta}}} \mathrm{Cl}^{(t_R)_T}(T) \longrightarrow 0$$

is split exact. The relation between $(t_R)_T$ and t_T (and between $\mathrm{Cl}^{(t_R)_T}(T)$ and $\mathrm{Cl}^{t_T}(T)$) is also investigated. Among the applications of the main results of this paper, a characterization of when R is a Prüfer $*$-multiplication domain is given.

Let D be an integral domain with quotient field L. Let $\overline{\boldsymbol{F}}(D)$ denote the set of all nonzero D-submodules of L and let $\boldsymbol{F}(D)$ be the set of all nonzero fractional ideals of D, i.e., all $E \in \overline{\boldsymbol{F}}(D)$ such that there exists a nonzero $d \in D$ with $dE \subseteq D$. Let $\boldsymbol{f}(D)$ be the set of all nonzero finitely generated D-submodules of L. Then, obviously $\boldsymbol{f}(D) \subseteq \boldsymbol{F}(D) \subseteq \overline{\boldsymbol{F}}(D)$.

For each pair of fractional ideals E, F of D, we denote as usual by $(E :_L F)$ the fractional ideal of D given by $\{y \in L \mid yF \subseteq E\}$; in particular, for each fractional ideal I of D, we set $I^{-1} := (D :_L I)$.

We recall that a mapping $\star : \overline{\boldsymbol{F}}(D) \to \overline{\boldsymbol{F}}(D)$, $E \mapsto E^\star$, is called a *semistar operation on* D if the following properties hold for all $0 \neq x \in L$, and $E, F \in \overline{\boldsymbol{F}}(D)$:

($\star_1$) $(xE)^\star = xE^\star$;
($\star_2$) $E \subseteq F \Rightarrow E^\star \subseteq F^\star$;
($\star_3$) $E \subseteq E^\star$ and $E^\star = (E^\star)^\star =: E^{\star\star}$

(cf. for instance [17,33,34,37,38]).

Example 1.1. (a) If $\star$ is a semistar operation on D such that $D^\star = D$, then the map (still denoted by) $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $E \mapsto E^\star$, is called a *star operation on* D. Recall [27, (32.1)] that a star operation $\star$ satisfies the properties $(\star_2)$, $(\star_3)$ for all $E, F \in \boldsymbol{F}(D)$; moreover, for each $0 \neq x \in L$ and for each $E \in \boldsymbol{F}(D)$, a star operation $\star$ satisfies the following "stronger" version of $(\star_1)$ (when restricted to $\boldsymbol{F}(D)$):

$(\star\star_1)$ $(xD)^\star = xD$, $(xE)^\star = xE^\star$.

Conversely, if $\star : \boldsymbol{F}(D) \to \boldsymbol{F}(D)$, $E \mapsto E^\star$, is a star operation on D (i.e., if $\star$ satisfies the properties $(\star\star_1)$, $(\star_2)$ and $(\star_3)$), then $\star$ can be extended trivially to a semistar operation on D, denoted by $\star_e$ (or, sometimes, just by $\star$), by setting $E^{\star_e} := L$, when $E \in \overline{\boldsymbol{F}}(D) \setminus \boldsymbol{F}(D)$, and $E^{\star_e} := E^\star$, when $E \in \boldsymbol{F}(D)$.

A semistar operation $\star$ on D such that $D \subsetneq D^\star$ is called a *proper semistar operation on* D.

(b) The *identity semistar operation* d_D *on* D (simply denoted by d) is a trivial semistar (in fact, star) operation on D defined by $E^{d_D} := E$ for each $E \in \overline{\boldsymbol{F}}(D)$ (d_D, when restricted to $\boldsymbol{F}(D)$, is a star operation on D).

(c) For each $E \in \overline{\boldsymbol{F}}(D)$, set $E^{\star_f} := \bigcup\{F^\star \mid F \subseteq E,\ F \in \boldsymbol{f}(D)\}$. Then $\star_f$ is also a semistar operation on D, which is called *the semistar operation of finite type associated to* $\star$. Obviously, $F^\star = F^{\star_f}$ for each $F \in \boldsymbol{f}(D)$; moreover, if $\star$ is a star operation, then $\star_f$ is also a star operation. If $\star = \star_f$, then the semistar (respectively, the star) operation $\star$ is called a *semistar* (respectively, *star*) *operation of finite type*.

Note that $\star_f \leqslant \star$, i.e., $E^{\star_f} \subseteq E^\star$ for each $E \in \overline{\boldsymbol{F}}(D)$. Thus, in particular, if $E = E^\star$, then $E = E^{\star_f}$. Note also that $\star_f = (\star_f)_f$.

More generally, if $\star_1$ and $\star_2$ are two semistar operations on D, we say that $\star_1 \leqslant \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$ for each $E \in \overline{\boldsymbol{F}}(D)$. In this situation, it is easy to see that $(E^{\star_1})^{\star_2} = E^{\star_2} = (E^{\star_2})^{\star_1}$.

There are several examples of nontrivial semistar or star operations of finite type; the best known is probably the t-operation. Indeed, we start from the v_D *star operation* on an integral domain D (simply denoted by v), which is defined by

$$E^{v_D} := \left(E^{-1}\right)^{-1} = \left(D :_L (D :_L E)\right)$$

for any $E \in \boldsymbol{F}(D)$, and we set $t_D := (v_D)_f$ (or, simply, $t = v_f$).

(d) Let $\iota : R \hookrightarrow T$ be an embedding of integral domains with the same field of quotients K and let $*$ be a semistar operation on R. Define $*_\iota : \overline{\boldsymbol{F}}(T) \to \overline{\boldsymbol{F}}(T)$ by setting

$$E^{*_\iota} := E^* \quad \text{for each } E \in \overline{\boldsymbol{F}}(T)\ \left(\subseteq \overline{\boldsymbol{F}}(R)\right).$$

Then, it is easy to verify (cf. also [20, Proposition 2.8]) that:

(d1) *If ι is not the identity map, then $*_\iota$ is a semistar, possibly nonstar, operation on T, even if $*$ is a star operation on R* (*obviously, if ι is the identity map, then $*_\iota = *$ and thus this phenomenon does not occur*).

Note that when $*$ is a star operation on R and $(R :_K T) = (0)$, a fractional ideal E of T is not a fractional ideal of R, hence $*_\iota$ is not necessarily defined as a star operation on T.

(d2) *If $*$ is of finite type on R, then $*_\iota$ is also of finite type on T.*
(d3) *If $T = R^*$, then $*_\iota$ defines a star operation on T.*

(e) Let $\star$ be a semistar operation on the overring T of R. Define $\star^\iota : \overline{\boldsymbol{F}}(R) \to \overline{\boldsymbol{F}}(R)$ by setting

$$E^{\star^\iota} := (ET)^\star \quad \text{for each } E \in \overline{\boldsymbol{F}}(R).$$

Then, we know [20, Proposition 2.9, Corollary 2.10]:

(e1) *$\star^\iota$ is a semistar operation on R.*
(e2) *If $\star = d_T$, then $(d_T)^\iota$ is a semistar operation of finite type on R, which is also denoted by $\star_{\{T\}}$ (i.e., it is the semistar operation on R defined by $E^{\star_{\{T\}}} := ET$ for each $E \in \overline{\boldsymbol{F}}(R)$).*
(e3) *For each semistar operation $\star$ on T, $(\star^\iota)_\iota = \star$.*

(f) Let Δ be a set of prime ideals of an integral domain D with quotient field L. The mapping $E \mapsto E^{\star_\Delta}$, where $E^{\star_\Delta} := \bigcap\{ED_P \mid P \in \Delta\}$ for each $E \in \overline{\boldsymbol{F}}(D)$, defines a semistar operation on D. Note that $\star_\Delta$ (restricted to the nonzero fractional ideals of D) is a star operation on D if and only if $D = \bigcap\{D_P \mid P \in \Delta\}$. Moreover ([17, Lemma 4.1] or [1, Theorem 1]):

(f1) *For each $E \in \overline{\boldsymbol{F}}(D)$ and for each $P \in \Delta$, $ED_P = E^{\star_\Delta}D_P$.*
(f2) *The semistar operation $\star_\Delta$ is stable (with respect to the finite intersections), i.e., for all $E, F \in \overline{\boldsymbol{F}}(D)$, we have $(E \cap F)^{\star_\Delta} = E^{\star_\Delta} \cap F^{\star_\Delta}$.*

A semistar operation $\star$ on D is called *spectral* if there exists a subset Δ of $\mathrm{Spec}(D)$ such that $\star = \star_\Delta$; in this case we say that $\star$ is *the spectral semistar operation associated with Δ.*

(g) Let $\star$ be a star operation on D. If $E \in \boldsymbol{F}(D)$, we say that E is *a $\star$-ideal* if $E = E^\star$. We denote by $\boldsymbol{F}^\star(D)$ (respectively, $\boldsymbol{f}^\star(D)$) the set $\{E \in \boldsymbol{F}(D) \mid E = E^\star\}$ (respectively, $\{E \in \boldsymbol{F}(D) \mid E = F^\star$ where $F \in \boldsymbol{f}(D)\}$. Obviously, $\boldsymbol{F}^d(D) = \boldsymbol{F}(D)$ (respectively, $\boldsymbol{f}^d(D) = \boldsymbol{f}(D)$) and the set $\boldsymbol{F}^v(D)$ is called *the set of divisorial ideals of D.*

Set $\mathcal{P}(\star) := \mathrm{Spec}^\star(D) := \{P \in \mathrm{Spec}(D) \mid P = P^\star\}$ and $\mathcal{M}(\star) := \mathrm{Max}^\star(D)$ which is the (possibly empty) set of all the maximal elements of the set $\{I$ proper ideal of $D \mid I = I^\star\}$. Assume that each proper $\star$-ideal of D is contained in some prime ideal of $\mathrm{Spec}^\star(D)$, then it is known that $\star_{\mathcal{P}(\star)}$ is a star operation on D [1, Theorem 3]. In particular, for each star operation $\star$ on D which is not a field, $\mathcal{M}(\star_f)$ is a nonempty subset of $\mathcal{P}(\star_f)$ and it satisfies the property that each proper $\star_f$-ideal of D is contained in some prime ideal of $\mathcal{M}(\star_f)$. Then $\widetilde{\star} := \star_{\mathcal{M}(\star_f)}$ is a star operation of finite type and stable on D, which is called *the* *stable operation of finite type associated to* $\star$. It is easy to see that $\widetilde{\star_f} = \widetilde{\star} = (\widetilde{\star})_f$ and $\widetilde{\star} = \star_{\mathcal{P}(\star_f)}$. Note that [17, Corollary 3.9]

$$\star = \widetilde{\star} \quad \Leftrightarrow \quad \star \textit{ is a stable star operation of finite type.}$$

Particularly interesting is the case in which $\star = v$. Using the notation introduced by Wang Fanggui and R.L. McCasland [41], we will denote by w_D (or, simply, w) the star operation $\widetilde{v_D} = \widetilde{t_D}$ (simply, $w := \widetilde{v} = \widetilde{t}$; cf. also [3,29]).

Note that if $\star_1$ and $\star_2$ are two star operations on D, then

$$\star_1 \leqslant \star_2 \quad \Leftrightarrow \quad \boldsymbol{F}^{\star_2}(D) \subseteq \boldsymbol{F}^{\star_1}(D).$$

It is well known that for each star operation $\star$, we have $\widetilde{\star} \leqslant \star_f \leqslant \star$ [3, Theorem 2.3]. Thus, in particular, if $E = E^\star$, then $E = E^{\widetilde{\star}} = E^{\star_f}$. Moreover, note that

$$\boldsymbol{f}^\star(D) = \boldsymbol{f}^{\star_f}(D) \subseteq \boldsymbol{F}^\star(D) \subseteq \boldsymbol{F}^{\star_f}(D).$$

It is also known that if $\star_1$ and $\star_2$ are two star operations on D and $\star_1 \leqslant \star_2$, then $(\star_1)_f \leqslant (\star_2)_f$ and $\widetilde{\star_1} \leqslant \widetilde{\star_2}$. In particular, for each star operation $\star$, we have $\star \leqslant v$ [27, Theorem 34.1(4)] and so $\star_f \leqslant t$ and $\widetilde{\star} \leqslant w$. Thus we get

$$\boldsymbol{F}^v(D) \subseteq \boldsymbol{F}^t(D) \subseteq \boldsymbol{F}^w(D) \subseteq \boldsymbol{F}(D),$$
$$\boldsymbol{F}^v(D) \subseteq \boldsymbol{F}^\star(D), \qquad \boldsymbol{F}^t(D) \subseteq \boldsymbol{F}^{\star_f}(D), \qquad \boldsymbol{F}^w(D) \subseteq \boldsymbol{F}^{\widetilde{\star}}(D).$$

(h) Let $\iota : R \hookrightarrow T$ be an embedding of integral domains with the same field of quotients K and let $*$ be a semistar operation on R. It is not difficult to prove:

$$* \textit{ is stable on } R \quad \Rightarrow \quad *_\iota \textit{ is stable on } T.$$

(k) If $\{\star_\lambda \mid \lambda \in \Lambda\}$ is a family of semistar (respectively, star) operations on D, then $\bigwedge\{\star_\lambda \mid \lambda \in \Lambda\}$ (simply denoted by $\wedge\star_\lambda$), defined by

$$E^{\wedge\star_\lambda} := \bigcap\left\{E^{\star_\lambda} \mid \lambda \in \Lambda\right\}, \quad \text{for each } E \in \overline{\boldsymbol{F}}(D) \text{ (respectively, } E \in \boldsymbol{F}(D)),$$

is a semistar (respectively, star) operation on D. Note that if at least one of the semistar operations in the family $\{\star_\lambda \mid \lambda \in \Lambda\}$ is a star operation on D, then $\wedge\star_\lambda$ is still a star operation on D.

Let $\star$ be a star operation on an integral domain D and let $F \in \boldsymbol{F}(D)$. We say that F is $\star$-*invertible* if $(FF^{-1})^\star = D$. In particular, when $\star = d$ (respectively, v, t, w) is the identity star operation (respectively, the v-operation, the t-operation, the w-operation), we reobtain the classical notion of *invertibility* (respectively, v-*invertibility*, t-*invertibility*, w-*invertibility*) of a fractional ideal. Recall that:

Lemma 1.2. *Let $\star$, $\star_1$, $\star_2$ be star operations on an integral domain D. Let $\mathrm{Inv}(D,\star)$ be the set of all $\star$-invertible fractional ideals of D and $\mathrm{Inv}(D)$ (instead of $\mathrm{Inv}(D,d)$) the set of all invertible fractional ideals of D. Then*

(1) $D \in \mathrm{Inv}(D,\star)$.
(2) *If $\star_1 \leqslant \star_2$, then $\mathrm{Inv}(D,\star_1) \subseteq \mathrm{Inv}(D,\star_2)$. In particular, $\mathrm{Inv}(D) \subseteq \mathrm{Inv}(D,\tilde{\star}) \subseteq \mathrm{Inv}(D,\star_f) \subseteq \mathrm{Inv}(D,\star)$ and so $\mathrm{Inv}(D) \subseteq \mathrm{Inv}(D,w) \subseteq \mathrm{Inv}(D,t) \subseteq \mathrm{Inv}(D,v)$.*
(3) *$I, J \in \mathrm{Inv}(D,\star)$ if and only if $IJ \in \mathrm{Inv}(D,\star)$.*
(4) *If $I \in \mathrm{Inv}(D,\star)$, then $I^{-1} \in \mathrm{Inv}(D,\star)$.*
(5) *If $I \in \mathrm{Inv}(D,\star)$, then $I^v \in \mathrm{Inv}(D,\star)$.*

Let $\star$ be a star operation on D. Then $\boldsymbol{F}^\star(D)$ is a commutative monoid under the $\star$-multiplication defined by $(I,J) \mapsto (IJ)^\star$ for each $I,J \in \boldsymbol{F}^\star(D)$. If $\star_1$ and $\star_2$ are two star operations on D with $\star_1 \leqslant \star_2$, then while $\boldsymbol{F}^{\star_2}(D) \subseteq \boldsymbol{F}^{\star_1}(D)$, $\boldsymbol{F}^{\star_2}(D)$ is not a submonoid of $\boldsymbol{F}^{\star_1}(D)$ in general (see [4, page 811]). However, there is a special submonoid of $\boldsymbol{F}^\star(D)$ which reverses the inclusion:

Lemma 1.3 (D.F. Anderson [4, Proposition 3.3]). *Let $\star$, $\star_1$, $\star_2$ be star operations on an integral domain D and suppose that $\star_1 \leqslant \star_2$. Let $\mathrm{Inv}^\star(D) := \{I \in \mathrm{Inv}(D,\star) \mid I = I^\star\}$ be the set of all $\star$-invertible $\star$-ideals of D and let $\mathrm{Inv}(D)$ (instead of $\mathrm{Inv}^d(D)$) be the set of all invertible fractional ideals of D. Then*

(1) *$\mathrm{Inv}^\star(D)$ is a submonoid of $\boldsymbol{F}^\star(D)$; moreover, it is an abelian group.*
(2) *$\mathrm{Inv}^{\star_1}(D)$ is a subgroup of $\mathrm{Inv}^{\star_2}(D)$ (in symbol, $\mathrm{Inv}^{\star_1}(D) \leqslant \mathrm{Inv}^{\star_2}(D)$). In particular, for each star operation $\star$ on D, $\mathrm{Inv}(D) \leqslant \mathrm{Inv}^\star(D) \leqslant \mathrm{Inv}^v(D)$, $\mathrm{Inv}(D) \leqslant \mathrm{Inv}^{\star_f}(D) \leqslant \mathrm{Inv}^t(D)$ and $\mathrm{Inv}(D) \leqslant \mathrm{Inv}^{\tilde{\star}}(D) \leqslant \mathrm{Inv}^{\star_f}(D) \leqslant \mathrm{Inv}^\star(D)$.*

In [22] we considered the problem of “lifting a star operation” with respect to a surjective ring homomorphism between two integral domains. More precisely:

Lemma 1.4 [22, Corollary 2.4]. *Let R be an integral domain with field of quotients K, M a prime ideal of R. Let D be the quotient-domain R/M and let $\varphi : R \to D$ be the canonical projection. Assume that $\star$ is a star operation on D. For each nonzero fractional ideal E of R, we set*

$$E^{\star^\varphi} := \bigcap \left\{ x^{-1}\varphi^{-1}\left(\left(\frac{xE+M}{M}\right)^\star\right) \;\middle|\; x \in E^{-1},\ x \neq 0 \right\}$$
$$= \bigcap \left\{ x\varphi^{-1}\left(\left(\frac{x^{-1}E+M}{M}\right)^\star\right) \;\middle|\; x \in K,\ E \subseteq xR \right\},$$

where, if $\frac{zE+M}{M}$ is the zero ideal of D, then we set $\varphi^{-1}((\frac{zE+M}{M})^\star) = M$. Then $\star^\varphi$ is a star operation on R.

In [22] we also considered the problem of "projecting a star operation" with respect to a surjective homomorphism of integral domains, with particular emphasis on pullback constructions of a "special" kind. More precisely:

Lemma 1.5 [22, Propositions 2.6, 2.7, 2.9 and Theorem 2.12]. *Let $\varphi : R \to D$ be a surjective homomorphism of integral domains, let $*$ be a star operation on R and let L be the quotient field of D. For each nonzero fractional ideal F of D, we set*

$$F^{*_\varphi} := \bigcap \bigl\{ y\varphi\bigl(\bigl(\varphi^{-1}\bigl(y^{-1}F\bigr)\bigr)^*\bigr) \bigm| y \in L,\ F \subseteq yD \bigr\}.$$

(1) *$*_\varphi$ is a star operation on D.*

Assume, now, that we are dealing with a pullback diagram of type $(\square)$. Then

(2) *$F^{*_\varphi} = \varphi((\varphi^{-1}(F))^*) = (\varphi^{-1}(F))^*/M$ for each $F \in \boldsymbol{F}(D)$.*
(3) *$(\star^\varphi)_\varphi = \star$ for each star operation $\star$ on D.*
(4) *$* \leqslant (*_\varphi)^\varphi$ for each star operation $*$ on R.*

2. Main results

Lemma 2.1. *Assume that we are dealing with a pullback diagram of type $(\square^+)$. Let $*$ be a star operation on R and let $*_\varphi$ be the star operation on D defined in Lemma 1.5. Then the map $\boldsymbol{\alpha}(\varphi, *)$ (or, simply, $\boldsymbol{\alpha}$) : $\mathrm{Inv}(D, *_\varphi) \to \mathrm{Inv}(R, *)$, defined by $J \mapsto \varphi^{-1}(J)$, is injective with $\mathrm{Im}(\boldsymbol{\alpha}) = \{I \in \mathrm{Inv}(R, *) \mid M \subsetneq I \subseteq I^{v_R} \subsetneq T\}$. Moreover, if we use the same notation $\boldsymbol{\alpha} = \boldsymbol{\alpha}(\varphi, *)$ for the restriction of the map $\boldsymbol{\alpha}$ to the subset $\mathrm{Inv}^{*_\varphi}(D)$, then $\boldsymbol{\alpha} : \mathrm{Inv}^{*_\varphi}(D) \to \mathrm{Inv}^*(R)$ is still injective with $\mathrm{Im}(\boldsymbol{\alpha}) = \{I \in \mathrm{Inv}^*(R) \mid M \subsetneq I \subseteq I^{v_R} \subsetneq T\}$.*

Proof. Recall first that the map $J \mapsto \varphi^{-1}(J)$ establishes a 1–1 correspondence between $\boldsymbol{F}(D)$ and the set $\{H \in \boldsymbol{F}(R) \mid M \subsetneq H \subseteq H^{v_R} \subsetneq T\}$ [19, Corollary 1.9]. Let $J \in \boldsymbol{F}(D)$. Then by applying Lemma 1.5(2), we have $J^{*_\varphi} = (\varphi^{-1}(J))^*/M$. Therefore,

$$J = J^{*_\varphi} \quad \Leftrightarrow \quad \varphi^{-1}(J) = \bigl(\varphi^{-1}(J)\bigr)^*,$$
$$\bigl(JJ^{-1}\bigr)^{*_\varphi} = D \quad \Leftrightarrow \quad \bigl(\varphi^{-1}\bigl(JJ^{-1}\bigr)\bigr)^* = R.$$

By [19, Propositions 1.6 and 1.8(a)], $\varphi^{-1}(JJ^{-1}) = \varphi^{-1}(J)\varphi^{-1}(J^{-1}) = \varphi^{-1}(J)(\varphi^{-1}(J))^{-1}$. Therefore,

$$\bigl(JJ^{-1}\bigr)^{*_\varphi} = D \quad \Leftrightarrow \quad \bigl(\varphi^{-1}(J)\bigl(\varphi^{-1}(J)\bigr)^{-1}\bigr)^* = R. \qquad \square$$

Let $\mathrm{Prin}(D)$ be the subgroup of $\mathrm{Inv}^\star(D)$ of all *the nonzero fractional principal ideals of D*. We recall that the quotient group

$$\mathrm{Cl}^\star(D) := \frac{\mathrm{Inv}^\star(D)}{\mathrm{Prin}(D)}$$

is called *the class group of an integral domain D with respect to a star operation $\star$ on D*.

If $\star = d$ is the identity star operation on D, then $\mathrm{Cl}^d(D)$ is denoted by $\mathrm{Pic}(D)$ and it is called *the Picard group of an integral domain* D.

Lemma 2.2. *Let* $\star$, $\star_1$, $\star_2$ *be star operations on an integral domain* D *and suppose that* $\star_1 \leqslant \star_2$. *Then* $\mathrm{Cl}^{\star_1}(D)$ *is a subgroup of* $\mathrm{Cl}^{\star_2}(D)$. *In particular, for each star operation* $\star$ *on* D, $\mathrm{Pic}(D) \leqslant \mathrm{Cl}^{\star}(D) \leqslant \mathrm{Cl}^{v}(D)$, $\mathrm{Pic}(D) \leqslant \mathrm{Cl}^{\star_f}(D) \leqslant \mathrm{Cl}^{t}(D)$ *and* $\mathrm{Pic}(D) \leqslant \mathrm{Cl}^{\tilde{\star}}(D) \leqslant \mathrm{Cl}^{\star_f}(D) \leqslant \mathrm{Cl}^{\star}(D)$.

Proof. Easy consequence of Lemma 1.3. $\square$

Remark 2.3. Note that the previous statement can be strengthened, since Anderson–Cook (in [3, Theorem 2.18]) proved that *for any star operation* $\star$ *on an integral domain* D, $\mathrm{Inv}^{\tilde{\star}}(D) = \mathrm{Inv}^{\star_f}(D)$, *and thus* $\mathrm{Cl}^{\tilde{\star}}(D) = \mathrm{Cl}^{\star_f}(D)$.

Lemma 2.4. *Assume that we are dealing with a pullback diagram of type* $(\square^+)$. *Then the following statements are equivalent*:

(1) *the canonical map* $\tilde{\varphi}: \mathcal{U}(T) \to k^\bullet/\mathcal{U}(D)$, $u \mapsto \varphi(u)\mathcal{U}(D)$, *is a surjective group homomorphism, where* $k^\bullet$ *is the multiplicative group of the nonzero elements of the field* k *and* $\mathcal{U}(T)$ (*respectively*, $\mathcal{U}(D)$) *is the group of units of* T (*respectively*, D);
(2) *for each nonzero element* $x \in k$, $\varphi^{-1}(xD)$ *is a fractional principal ideal of* R;
(3) *the map* $\bar{\boldsymbol{\alpha}}(\varphi, *)$ (*or, simply*, $\bar{\boldsymbol{\alpha}}$) : $\mathrm{Cl}^{*_\varphi}(D) \to \mathrm{Cl}^{*}(R)$, $[J] \mapsto [\varphi^{-1}(J)]$ $(= [\boldsymbol{\alpha}(J)]$, *where* $\boldsymbol{\alpha}$ *is defined in Lemma* 2.1), *is a well-defined group homomorphism for any star operation* $*$ *on* R.

Proof. $(1) \Leftrightarrow (2) \Leftarrow (3)$. See [19, Theorem 2.3 (i) $\Leftrightarrow$ (ii) $\Leftarrow$ (iv)]. The direction $(2) \Rightarrow (3)$ is a consequence of Lemma 2.1. $\square$

Remark 2.5. General examples for which the map $\tilde{\varphi}: \mathcal{U}(T) \to k^\bullet/\mathcal{U}(D)$ is surjective are provided in [19, Proposition 2.9].

The next theorem presents a generalization of the result by D.F. Anderson [4, Proposition 5.5]:

Theorem 2.6. *Assume that we are dealing with a pullback diagram of type* $(\square)$. *If, moreover,* T *is quasilocal, then the canonical map* $\bar{\boldsymbol{\alpha}}(= \bar{\boldsymbol{\alpha}}(\varphi, *)): \mathrm{Cl}^{*_\varphi}(D) \to \mathrm{Cl}^{*}(R)$ *is an isomorphism for any star operation* $*$ *on* R.

Proof. We adapt the argument used in the proof of [4, Proposition 5.5]. We first show that $\mathrm{Cl}^{*}(R) = 0$ when D is a proper subfield of k. In this case, R is quasilocal, since R and T have the same prime spectrum [8]. Let $I \in \mathrm{Inv}^{*}(R)$. As $M = (R : T)$ is a divisorial ideal of R, if $II^{-1} \subseteq M$, then $(II^{-1})^{*} \subseteq M^{*} = M$, a contradiction. Then, necessarily, $II^{-1} = R$; thus I is invertible in the quasilocal domain R, and hence I is principal. Thus $\mathrm{Cl}^{*}(R) = 0$.

Without loss of generality, we may assume that D is a proper subring of k with quotient field k, i.e., that we are dealing with a pullback diagram of type $(\square^+)$. In this situation, the map $\bar{\boldsymbol{\alpha}}:\mathrm{Cl}^{*_\varphi}(D) \to \mathrm{Cl}^*(R)$ is a homomorphism, because when T is quasilocal, the condition (1) of Lemma 2.4 holds [19, Proposition 2.9].

Let $J \in \mathrm{Inv}^{*_\varphi}(D)$ such that $\varphi^{-1}(J)$ is principal in R, say $\varphi^{-1}(J) = xR$ for some nonzero $x \in T$. Then $J = xR/M = \varphi(x)D$ is principal in D. Therefore $\bar{\boldsymbol{\alpha}}$ is injective.

Conversely, let $I \in \mathrm{Inv}^*(R)$. Then, necessarily, $II^{-1} \nsubseteq M$, and hence $II^{-1}T = T$, i.e., IT is invertible in T. Since $T = R_M$ is quasilocal [19, Corollary 0.5], $IT = IR_M$ is principal, say $IT = iR_M$ for some $i \in I$. Set $I_1 := i^{-1}I$. Then, obviously, $I_1 \in \mathrm{Inv}^*(R)$ and $R \subseteq I_1 \subseteq T = I_1T$. To prove that $\varphi(I_1) = I_1/M$ belongs to $\mathrm{Inv}^{*_\varphi}(D)$, it suffices to show that $(I_1)^v \subsetneq T$ by Lemma 2.1, because $\varphi^{-1}(\varphi(I_1)) = I_1$. Suppose that $(I_1)^v = T$, then $I_1^{-1} = (R:T) = M$. So $R = (I_1I_1^{-1})^* = (I_1M)^* \subseteq (TM)^* = M^* = M$, a contradiction. Thus, necessarily, we have $(I_1)^v \subsetneq T$. Therefore $[I] = [i^{-1}I] = [I_1] = [\varphi^{-1}(I_1/M)] = \bar{\boldsymbol{\alpha}}([I_1/M])$. Hence $\bar{\boldsymbol{\alpha}}$ is also surjective and thus we conclude that $\bar{\boldsymbol{\alpha}}$ is an isomorphism. □

Corollary 2.7. *Assume that we are dealing with a pullback diagram of type* $(\square)$. *If, moreover, T is quasilocal, then we have the following canonical isomorphisms*:

$$\mathrm{Pic}(D) \cong \mathrm{Pic}(R), \qquad \mathrm{Cl}^t(D) \cong \mathrm{Cl}^t(R), \qquad \mathrm{Cl}^w(D) \cong \mathrm{Cl}^w(R), \qquad \mathrm{Cl}^v(D) \cong \mathrm{Cl}^v(R).$$

Proof. Since $(d_R)_\varphi = d_D$, $(t_R)_\varphi = t_D$, $(w_R)_\varphi = w_D$ and $(v_R)_\varphi = v_D$ [22, Propositions 3.3, 3.7, Corollaries 3.10 and 2.13], the conclusion follows from the above theorem. The third isomorphism also follows from the second one by Remark 2.3. □

Corollary 2.8. *Assume that we are dealing with a pullback diagram of type* $(\square)$. *Let T be quasilocal. Then*

(1) *The canonical homomorphism* $\bar{\boldsymbol{\alpha}}(\varphi, \star^\varphi):\mathrm{Cl}^\star(D) \to \mathrm{Cl}^{\star^\varphi}(R)$ *is an isomorphism for any star operation $\star$ on D.*
(2) $\mathrm{Cl}^*(R) = \mathrm{Cl}^{(*_\varphi)^\varphi}(R)$ *for any star operation $*$ on R.*

Proof. (1) Set $* := \star^\varphi$. Then $*_\varphi = (\star^\varphi)_\varphi = \star$ by Lemma 1.5(3). The conclusion follows immediately from Theorem 2.6.

(2) Recall that $* \leqslant (*_\varphi)^\varphi$ and $((*_\varphi)^\varphi)_\varphi = *_\varphi$ by Lemma 1.5(3) and (4). Then, if we apply Theorem 2.6 to both the star operations $(*_\varphi)^\varphi$ and $*$ on R, we have the following chain of canonical isomorphisms:

$$\mathrm{Cl}^{(*_\varphi)^\varphi}(R) \cong \mathrm{Cl}^{((*_\varphi)^\varphi)_\varphi}(D) = \mathrm{Cl}^{*_\varphi}(D) \cong \mathrm{Cl}^*(R).$$

Since these isomorphisms are canonical and $\mathrm{Cl}^*(R)$ is a subgroup of $\mathrm{Cl}^{(*_\varphi)^\varphi}(R)$ (Lemma 2.2), we easily conclude that $\mathrm{Cl}^{(*_\varphi)^\varphi}(R) = \mathrm{Cl}^*(R)$. □

Remark 2.9. (1) We present an example of a pullback diagram of type $(\square^+)$ in which T is quasilocal and $* \lneq (*_\varphi)^\varphi$ (with $\mathrm{Cl}^*(R) = \mathrm{Cl}^{(*_\varphi)^\varphi}(R)$ by Corollary 2.8(2)). Let D be an integral domain in which each nonzero ideal is divisorial (e.g., a Dedekind domain) and let

k be the quotient field of D. Set $T := k[X^2, X^3]_Q$, where $Q := X^2k[X]$, and $M := QT$. Let φ and R be as in $(\square^+)$. Then $((d_R)_\varphi)^\varphi = (d_D)^\varphi = (v_D)^\varphi = v_R$ [22, Proposition 3.3 and Corollary 2.13]. Meanwhile, since $T^{v_R} = (R : (R : T)) = (R : M) \supseteq k[X]$ but $T \not\supseteq k[X]$, $d_R \neq v_R = ((d_R)_\varphi)^\varphi$.

(2) We give an example to show that the quasilocal hypothesis is essential in Corollary 2.8(2). Let D be an integral domain in which each nonzero ideal is divisorial and let k be the quotient field of D. Let B be the polynomial ring $k[\{X_i\}_{i=1}^\infty]$ and let T be the subring of B generated over k by the products X_iX_j for all pairs $i, j \geqslant 1$. Then it is known that T is a Krull domain [24, Example 1.10]. Let $N := (1 + X_1, X_2, X_3, \ldots)B$ and let $M := N \cap T$. Since $k \subseteq T/M \subseteq B/N \cong k$, $T/M \cong k$ and $T = k + M$. Let φ and R be as in $(\square^+)$. Then $((d_R)_\varphi)^\varphi = (d_D)^\varphi = (v_D)^\varphi = v_R$ [22, Proposition 3.3 and Corollary 2.13]. Let $Q := X_1B \cap T$ and note that $X_1B (\not\subseteq N)$ is a prime ideal of height one in the Krull domain B. Since B is integral over the integrally closed domain T, Q is a prime ideal of height one in T. Note that $Q \not\subseteq M$, because $X_1^2 \in Q \setminus N$. Since $R = D + M$, $T = R_{D\setminus\{0\}}$, thus $Q = \boldsymbol{q}T$, where $\boldsymbol{q} := Q \cap R$ and $\boldsymbol{q} \not\subseteq M$. Since Q is a prime ideal of height one in the Krull domain, Q is a t_T-invertible t_T-ideal of T, thus $\boldsymbol{q}$ is a t_R-invertible t_R-ideal of R by [7, Lemma 3.1 and Theorem 2.2(6)]. Moreover, since Q is not finitely generated as an ideal of T [24, Example 1.10], $\boldsymbol{q}$ is not finitely generated as an ideal of R and hence it is not invertible. Therefore $\mathrm{Pic}(R) = \mathrm{Cl}^{d_R}(R) \subsetneq \mathrm{Cl}^{t_R}(R) \subseteq \mathrm{Cl}^{v_R}(R)$, thus $\mathrm{Cl}^{d_R}(R) \neq \mathrm{Cl}^{v_R}(R) = \mathrm{Cl}^{((d_R)_\varphi)^\varphi}(R)$.

This example also shows that the quasilocal hypothesis is essential in Corollary 2.8(1): Choose D to be a PID. Then $\mathrm{Cl}^{d_D}(D) = \mathrm{Pic}(D) = 0$, but since $\mathrm{Cl}^{d_R}(R) \subsetneq \mathrm{Cl}^{v_R}(R) = \mathrm{Cl}^{(d_D)^\varphi}(R)$, we have $\mathrm{Cl}^{(d_D)^\varphi}(R) \neq 0$.

The next goal is to give a complete description of $\mathrm{Cl}^*(R)$ by means of $\mathrm{Cl}^{*_\varphi}(D)$ and of an "appropriate star class group" of T. For this purpose, recall that, in [22], we also considered the problem of "extending a star operation" defined on an integral domain R to some overring T of R.

We need the following notation. Let $*$ be a star operation on an integral domain R and let T be an overring of R such that $(R : T) \neq 0$. Then, for each $E \in \boldsymbol{F}(T) (\subseteq \boldsymbol{F}(R))$, we set

$$E^{(*)_T} := E^* \cap \big(T : (T : E)\big) = E^* \cap E^{v_T}.$$

Lemma 2.10. *Assume that we are dealing with a pullback diagram of type $(\square^+)$. Let $\iota : R \hookrightarrow T$ be the canonical embedding and let $*$ be a star operation on R.*

(1) *$(*)_T$ is a star operation on T with $(*)_T = *_\iota \wedge v_T$.*
(2) *If $*$ is a star operation of finite type on R, then $(*)_T$ coincides with $*_\iota$ (restricted to the fractional ideals of T) and it is a star operation of finite type on T.*
(3) *If $*_1, *_2$ are two star operations on R, then*

$$*_1 \leqslant *_2 \quad \Rightarrow \quad (*_1)_T \leqslant (*_2)_T.$$

(4) *$(*_f)_T \leqslant ((*)_T)_f$.*
(5) *$(\widetilde{*})_T = \widetilde{(*)_T}$.*

Proof. (1) follows from [22, Examples 1.2 and 1.5(a)] and the observation that $T^{(*)_T} = T^* \cap T^{v_T} = T^* \cap T = T$.

For (2), we need the following:

Claim 1. *T is a t_R-ideal of R.*

Choose a nonzero $r \in M$, then obviously rT is an integral t_T-ideal of T and $rT \subseteq M \subset R$. Since T is R-flat, $rT = rT \cap R$ is a t_R-ideal of R by [19, Proposition 0.7(a)]. Therefore, $T = r^{-1} \cdot rT$ is a t_R-ideal of R.

By using Claim 1, we can complete the proof of (2). As a matter of fact, if $*$ is a star operation of finite type on R, then $* \leqslant t_R$, thus the map $E \mapsto E^{*_\iota} := E^*$, for each $E \in \boldsymbol{F}(T)(\subseteq \boldsymbol{F}(R))$, defines a star operation on T (since $T \subseteq T^* \subseteq T^{t_R} = T$). In particular, $*_\iota \leqslant v_T$, and so $(*)_T = *_\iota$ (being $*_\iota$ restricted to the fractional ideals of T). Finally, it is straightforward that if $*$ is a star operation of finite type on R, then $*_\iota$ $(= (*)_T)$ is of finite type on T (cf. also for instance [22, Example 1.2(b)]).

(3) is a straightforward consequence of the definition.

(4) follows from (3) and (2) since $(*_f)_T$ is a star operation of finite type on T.

(5) Note that $(\widetilde{*})_T$ is a star operation of finite type and $(\widetilde{*})_T = (\widetilde{*})_\iota$ (by (2)). Moreover, $(\widetilde{*})_\iota$ is stable, since $\widetilde{*}$ is stable. Therefore $(\widetilde{*})_T = \widetilde{(\widetilde{*})_T}$, and hence we conclude by (3) that $(\widetilde{*})_T \leqslant \widetilde{(*)_T}$.

Claim 2. *For each star operation $\star$ on R, $M = M^{\star_f} = M^\star$.*

It follows from the fact that $M = (R : T)$ is a divisorial ideal of R.

Claim 3. $\mathrm{Max}^{(*_f)_T}(T)$ *coincides with the set of maximal elements of* $\{PT \mid P \in \mathrm{Spec}^{*_f}(R),\ PT \neq T\}$.

Since T is R-flat [19, Lemma 0.3], each ideal of T is extended from R. In particular, each prime ideal Q of T is equal to $(Q \cap R)T$. Note that $\mathrm{Max}^{(*_f)_T}(T) \subseteq \{PT \mid P \in \mathrm{Spec}^{*_f}(R),\ PT \neq T\}$. Indeed, let $Q \in \mathrm{Max}^{(*_f)_T}(T)$ and let $P := Q \cap R$. Then $P \subseteq P^{*_f} \subseteq Q^{*_f} = Q^{(*_f)_T} = Q$, hence $P \subseteq P^{*_f} \subseteq Q \cap R = P$.

Now let PT be a maximal element in the set $\{PT \mid P \in \mathrm{Spec}^{*_f}(R),\ PT \neq T\}$. Suppose $(PT)^{(*_f)_T} = T$. Then $1 \in (PT)^{(*_f)_T} = (PT)^{*_f}$, i.e., $1 \in F^*$ for some $F \in \boldsymbol{f}(R)$ such that $F \subseteq PT$. Let $m \in M \setminus \{0\}$. Then $m \in mF^* = (mF)^* \subseteq (mPT)^{*_f} \subseteq (PR)^{*_f} = P^{*_f} = P$. Thus we have $M \subseteq P$. Since $PT \neq T$, $M \not\subset P$, and hence $M = P$. Then $T = (PT)^{(*_f)_T} = M^{(*_f)_T} = M^{*_f} = M$ (Claim 2), a contradiction. Therefore, $(PT)^{(*_f)_T} \neq T$.

Let $Q' \in \mathrm{Max}^{(*_f)_T}(T)$ such that $(PT)^{(*_f)_T} \subseteq Q'$. Then by the above argument, $Q' \cap R \in \mathrm{Spec}^{*_f}(R)$. Since $PT \subseteq Q' = (Q' \cap R)T$, $PT = Q'$ by the maximality of PT. Thus we have $PT \subseteq (PT)^{(*_f)_T} \subseteq Q' = PT$ and so $PT \in \mathrm{Max}^{(*_f)_T}(T)$.

Claim 4. $\mathrm{Max}^{(*_f)_T}(T) = \mathrm{Max}^{((*)_T)_f}(T)$.

Let $Q \in \mathrm{Max}^{((*)_T)_f}(T)$ and let $P := Q \cap R$. Then $P \subseteq P^{*_f} \subseteq Q^{*_f} = Q^{(*_f)_T} \subseteq Q^{((*)_T)_f} = Q$ (by (4)), and hence $P \subseteq P^{*_f} \subseteq Q \cap R = P$. Thus we have $\mathrm{Max}^{((*)_T)_f}(T) \subseteq \{PT \mid P \in \mathrm{Spec}^{*_f}(R),\, PT \neq T\}$.

Now let PT be a maximal element in the set $\{PT \mid P \in \mathrm{Spec}^{*_f}(R),\, PT \neq T\}$. Suppose $(PT)^{((*)_T)_f} = T$. Then $1 \in (PT)^{((*)_T)_f}$, i.e., $1 \in G^{(*)_T}$ for some $G \in \boldsymbol{f}(T)$ such that $G \subseteq PT$. We may assume that $G = JT$ for some $J \in \boldsymbol{f}(R)$ such that $J \subseteq P$. Let $m \in M \setminus \{0\}$. Then $m \in mG^{(*)_T} = (mG)^{(*)_T} = (mJT)^{(*)_T} \subseteq (mJT)^{*_\iota} = (mJT)^* \subseteq (JR)^* = J^* \subseteq P^{*_f} = P$. Thus we have $M \subseteq P$. Since $PT \neq T$, $M \not\subset P$, and hence $M = P$. Then $T = (PT)^{((*)_T)_f} = M^{((*)_T)_f} \subseteq M^{(*)_T} \subseteq M^{*_\iota} = M^* = M$ (Claim 2), a contradiction. Therefore, $(PT)^{((*)_T)_f} \neq T$.

Let $Q' \in \mathrm{Max}^{((*)_T)_f}(T)$ such that $(PT)^{((*)_T)_f} \subseteq Q'$. Then since $PT \subseteq Q' = (Q' \cap R)T$ and since we have already proved that $Q' \cap R \in \mathrm{Spec}^{*_f}(R)$, we conclude that $PT = Q'$ by the maximality of PT. Thus $PT \subseteq (PT)^{((*)_T)_f} \subseteq Q' = PT$ and so $PT \in \mathrm{Max}^{((*)_T)_f}(T)$.

Claim 5.

(a) *For each prime ideal P of R such that $P \not\supseteq M$, $R_P = TR_P = T_{PT}$.*
(b) *For each prime ideal P of R such that $P \supseteq M$, $R_P \subseteq R_M = T_M$, and moreover, $TR_P = T_M$.*

The statement (a) and the first part of (b) are well known [16, Theorem 1.4 and its proof]. Since $TR_P \subseteq T_M$ for each $P \in \mathrm{Spec}(R)$ with $P \supseteq M$, to prove the equality, it suffices to show that if a prime ideal Q' of T is such that $Q' \cap R \subseteq P$, then Q' is contained in M. Suppose not, i.e., $Q' \not\subseteq M$, then $Q' \cap R \not\subseteq M$. Choose $a \in (Q' \cap R) \setminus M$. Then $M + aT = T$, so $1 = m + at$ for some $m \in M$, $t \in T$. Then $1 - m = at \in aT \cap R \subseteq Q' \cap R \subseteq P$. Since $m \in M \subseteq P$, $1 \in P$, a contradiction.

Claim 6. $\mathrm{Max}^{(*_f)_T}(T) = \{PT \mid P \in \mathrm{Max}^{*_f}(R),\, P \not\supseteq M\} \cup \{M\}$.

Note that, the condition $PT \neq T$ (or, equivalently, $PT \in \mathrm{Spec}(T)$) implies that $P \not\supset M$, since M is a maximal ideal in T. Moreover, by Claim 2, M belongs to $\mathrm{Spec}^{*_f}(R)$, thus $MT = M$ belongs, in any case, to $\mathrm{Max}^{(*_f)_T}(T)$ by Claim 3.

Recall that, by the properties of the prime ideals in a pullback of type $(\square^+)$, it follows that the canonical map $\mathrm{Spec}(T) \to \mathrm{Spec}(R)$ is an order preserving embedding, and if $Q \in \mathrm{Spec}(T)$ and $Q \cap R \subseteq P$ for some $P \in \mathrm{Spec}(R)$ with $P \supseteq M$, then $Q \subseteq M$ (see also the proof of Claim 5). By the previous ordering properties and Claim 3, we easily conclude that $\{PT \mid P \in \mathrm{Max}^{*_f}(R),\, P \not\supseteq M\} \cup \{M\} = \mathrm{Max}^{(*_f)_T}(T)$.

Claim 7. $\widetilde{(*_f)_T} = (\tilde{*})_T$.

Note that, by Claim 4,

$$\widetilde{(*_f)_T} = \widetilde{\big((*)_T\big)_f} = \widetilde{(*)_T}.$$

Now we want to show that $\widetilde{(*_f)_T} = (\tilde{*})_T$.

Set $\mathcal{P}_1^{*_f} := \{P \in \operatorname{Spec}^{*_f}(R) \mid P \not\supseteq M\}$ and $\mathcal{P}_2^{*_f} := \{P \in \operatorname{Spec}^{*_f}(R) \mid P \supseteq M\}$. If we let $\mathcal{P}_0^{*_f}$ be the set of maximal elements in the set $\mathcal{P}_1^{*_f}$, then $\{PT \mid P \in \mathcal{P}_0^{*_f}\} = \{Q \in \operatorname{Max}^{(*_f)_T}(T) \mid Q \neq M\}$ by Claim 6.

Let $E \in \boldsymbol{F}(T)$, then by using Claims 5 and 6, we have

$$\begin{aligned}
E^{(\widetilde{*})_T} = E^{(\widetilde{*})_\iota} = E^{\widetilde{*}} = (ET)^{\widetilde{*}} &= \bigcap\big\{ETR_P \mid P \in \operatorname{Spec}^{*_f}(R)\big\} \\
&= \Big(\bigcap\big\{ETR_P \mid P \in \mathcal{P}_1^{*_f}\big\}\Big) \cap \Big(\bigcap\big\{ETR_P \mid P \in \mathcal{P}_2^{*_f}\big\}\Big) \\
&= \Big(\bigcap\big\{ETR_P \mid P \in \mathcal{P}_0^{*_f}\big\}\Big) \cap ET_M \\
&= \bigcap\big\{ET_{PT} \mid P \in \operatorname{Max}^{*_f}(R),\ P \not\supseteq M\big\} \cap ET_M \\
&= \bigcap\big\{ET_Q \mid Q \in \operatorname{Max}^{(*_f)_T}(T)\big\} \\
&= E^{\widetilde{(*_f)_T}}. \qquad \square
\end{aligned}$$

Remark 2.11. (1) We were not able to prove or disprove the equality in the statement (4) of Lemma 2.10. However $(*_f)_T = ((*)_T)_f$ for the case $* = v_R$, which is the most important star operation of nonfinite type. More precisely, *in the situation of Lemma* 2.10, *we have*

$$(t_R)_T = \big((v_R)_f\big)_T = \big((v_R)_T\big)_f.$$

Since $(t_R)_T \leqslant ((v_R)_T)_f$ and both terms are star operations of finite type (Lemma 2.10(2)), it suffices to show that $H^{(t_R)_T} \supseteq H^{(v_R)_T}$ for all nonzero finitely generated integral ideals H of T. Let H be a nonzero finitely generated integral ideal of T. Then $H = IT$ for some finitely generated ideal I of R.

If IT_M is not principal, then $I^{v_R} = I^{v_R}T$ by [25, Proposition 2.7(1b)]. Therefore, $H^{(v_R)_T} \subseteq H^{(v_R)_\iota} = (IT)^{v_R} = (I^{v_R}T)^{v_R} = I^{v_R} = I^{t_R} \subseteq H^{t_R} = H^{(t_R)_\iota} = H^{(t_R)_T}$.

Now assume that IT_M is principal. Then $H^{v_T} \subseteq (HT_M)^{v_{T_M}} = (IT_M)^{v_{T_M}} = IT_M$. Let $R(M)$ be the CPI-extension of R with respect to M, i.e., $R(M)$ is defined by the following pullback diagram [11]:

$$\begin{tikzcd}
R(M) := \varphi^{-1}(D) \arrow[r] \arrow[d] & D \arrow[d] \\
T_M \arrow[r, "\varphi"] & k = T_M/MT_M.
\end{tikzcd}$$

Then by [19, Lemma 1.3], $R = R(M) \cap T$. Note first that $TR(M) = T_M$, because $TR(M) = \bigcap\{TR(M)_{\bar{N}} \mid \bar{N} \in \operatorname{Max}(R(M))\} = \bigcap\{TR_N \mid N \in \operatorname{Max}(R)$ such that $N \supseteq M\} = T_M$ by Claim 5(b) in the proof of Lemma 2.10. Now by [1, Theorem 2(4)], $H^{(t_R)_T} = H^{t_R} \supseteq (HR(M))^{t_{R(M)}} \cap (HT)^{t_T} = (ITR(M))^{t_{R(M)}} \cap H^{v_T} = (IT_M)^{t_{R(M)}} \cap H^{v_T} \supseteq IT_M \cap H^{v_T} = H^{v_T} \supseteq H^{(v_R)_T}$.

(2) As another special case, we have the following positive result.

Consider a pullback diagram of type $(\square^+)$, *let* $\star'$ *be a star operation on* D *and* $\star''$ *a star operation on* T. *Set* $\diamond := \star'^{\varphi} \wedge \star''^{\iota}$. *We know that* $\diamond$ *is a star operation on* R [22, *Corollary* 2.5]. *If* $((\star'^{\varphi})_T)_f = ((\star'^{\varphi})_f)_T$ (*e.g., this hypothesis is satisfied in each one of the following cases*: (a) $\star' = v_D$, (b) $(\star'_f)^{\varphi}$ *is a star operation of finite type on* R, (c) T *is a Prüfer domain*), *then* $(\diamond_f)_T = ((\diamond)_T)_f$.

Claim 1. *If* $*_1$ *and* $*_2$ *are two semistar operations on an integral domain* R, *then*

$$(*_1 \wedge *_2)_f = (*_1)_f \wedge (*_2)_f.$$

This is an easy consequence of the fact that "$\bigcup_\alpha$ distributes over $\cap$".

Claim 2. *Let* $\iota : R \hookrightarrow T$ *be an embedding of an integral domain* R *in one of its overrings* T *and let* $\star$ *be a semistar operation on* T. *Then, in* R, $(\star^\iota)_f = (\star_f)^\iota$, *and in* T, $\star = (\star^\iota)_\iota$ (*Example* 1.1(e3)).

Let $E \in \overline{\boldsymbol{F}}(R)$ and let $G \in \boldsymbol{f}(T)$ be contained in ET. Then $G := (x_1t_1, x_2t_2, \dots, x_nt_n)T$ for some $n \geqslant 1$, $\{x_1, x_2, \dots, x_n\} \subseteq E$, and $\{t_1, t_2, \dots, t_n\} \subseteq T$. Thus $G \subseteq HT$, where $H := (x_1, x_2, \dots, x_n)R \in \boldsymbol{f}(R)$ (and $H \subseteq E$). Therefore

$$\begin{aligned}
E^{(\star^\iota)_f} &= \bigcup\bigl\{F^{\star^\iota} \mid F \in \boldsymbol{f}(R),\ F \subseteq E\bigr\} \\
&= \bigcup\bigl\{(FT)^{\star} \mid F \in \boldsymbol{f}(R),\ F \subseteq E\bigr\} \\
&= \bigcup\bigl\{G^{\star} \mid G \in \boldsymbol{f}(T),\ G \subseteq ET\bigr\} \\
&= (ET)^{\star_f} = E^{(\star_f)^\iota}.
\end{aligned}$$

Claim 3. *Let* $\iota : R \hookrightarrow T$ *be an embedding of an integral domain* R *in one of its overrings* T *and let* $*_1$ *and* $*_2$ *be two semistar operations on* R. *Then* $(*_1 \wedge *_2)_\iota = (*_1)_\iota \wedge (*_2)_\iota$.

This is an obvious consequence of the definitions.

Claim 4. *Let* $\iota : R \hookrightarrow T$ *be an embedding of an integral domain* R *in one of its overrings* T *and let* $*$ *be a semistar operation on* R. *Then* $(*_f)_\iota$ *is a semistar operation of finite type on* T.

For each $E \in \overline{\boldsymbol{F}}(T)$, we have

$$\begin{aligned}
E^{(*_f)_\iota} &= E^{*_f} = \bigcup\bigl\{F^{*} \mid F \in \boldsymbol{f}(R),\ F \subseteq E\bigr\} \\
&= \bigcup\Bigl\{\bigcup\bigl\{F^{*} \mid F \in \boldsymbol{f}(R),\ F \subseteq G\bigr\} \mid G \in \boldsymbol{f}(T),\ G \subseteq E\Bigr\} \\
&= \bigcup\bigl\{G^{*_f} \mid G \in \boldsymbol{f}(T),\ G \subseteq E\bigr\} \\
&= \bigcup\bigl\{G^{(*_f)_\iota} \mid G \in \boldsymbol{f}(T),\ G \subseteq E\bigr\} \\
&= E^{((*_f)_\iota)_f}.
\end{aligned}$$

Claim 5. *In a pullback diagram of type* ($\Box$), *let* $\star$ *be a star operation on* D. *Then* $(\star^{\varphi})_{\iota} = (v_R)_{\iota}$ (*when restricted to* $\boldsymbol{F}(T)$), *and hence* $(\star^{\varphi})_T = (v_R)_T$. *Moreover, in a pullback diagram of type* ($\Box^+$), $((\star^{\varphi})_T)_f = (t_R)_T$ *by* (1).

Let I be a nonzero integral ideal of T. Note that

$$x \in (R:I) \quad \Rightarrow \quad xIT = xI \subseteq R \quad \Rightarrow \quad xI \subseteq (R:T) = M \quad \big(\Leftrightarrow \quad I \subseteq x^{-1}M\big).$$

Therefore we have

$$\begin{aligned} I^{(\star^{\varphi})_{\iota}} = I^{\star^{\varphi}} &= \bigcap\left\{x^{-1}\varphi^{-1}\left(\left(\frac{xI+M}{M}\right)^{\star}\right) \;\Big|\; x \in (R:I),\ x \neq 0\right\} \\ &= \bigcap\big\{x^{-1}M \mid x \in (R:I),\ x \neq 0\big\} = I^{v_R} = I^{(v_R)_{\iota}}. \end{aligned}$$

Note that $T^{(\star^{\varphi})_{\iota}} = T^{(v_R)_{\iota}} = T^{v_R}$, thus $(\star^{\varphi})_{\iota}$ (when restricted to $\boldsymbol{F}(T)$) is a star operation on T if and only if $T = T^{v_R}$.

Now we use the previous claims to prove the statement. By applying Claims 2, 3, and 5, we have

$$\begin{aligned} (\diamond)_T &= \diamond_{\iota} \wedge v_T = \big(\star'^{\varphi} \wedge \star''^{\iota}\big)_{\iota} \wedge v_T \\ &= \big(\star'^{\varphi}\big)_{\iota} \wedge \big(\star''^{\iota}\big)_{\iota} \wedge v_T = \big(\star'^{\varphi}\big)_{\iota} \wedge \star'' \wedge v_T \\ &= \big(\star'^{\varphi}\big)_{\iota} \wedge \star'' = (v_R)_{\iota} \wedge \star'' \quad \text{or equivalently} \\ &= \big(\star'^{\varphi}\big)_T \wedge \star'' = (v_R)_T \wedge \star''. \end{aligned}$$

Therefore, by Claim 1 and (1), we have

$$\big((\diamond)_T\big)_f = \big(\big(\star'^{\varphi}\big)_{\iota}\big)_f \wedge \star''_f = \big(\big(\star'^{\varphi}\big)_T\big)_f \wedge \star''_f = \big((v_R)_T\big)_f \wedge \star''_f = (t_R)_T \wedge \star''_f.$$

On the other hand, by Lemma 2.10(2), Claims 1, 2 and 3, we have

$$\begin{aligned} (\diamond_f)_T = (\diamond_f)_{\iota} &= \big(\big(\star'^{\varphi}\big)_f\big)_{\iota} \wedge \big(\big(\star''^{\iota}\big)_f\big)_{\iota} = \big(\big(\star'^{\varphi}\big)_f\big)_{\iota} \wedge \big(\big(\star''_f\big)^{\iota}\big)_{\iota} \\ &= \big(\big(\star'^{\varphi}\big)_f\big)_{\iota} \wedge \star''_f = \big(\big(\star'^{\varphi}\big)_f\big)_T \wedge \star''_f. \end{aligned}$$

It is obvious now that, if $((\star'^{\varphi})_T)_f = ((\star'^{\varphi})_f)_T$, then $(\diamond_f)_T = ((\diamond)_T)_f$.

Finally, we check the parenthetical statement.

Assume that $\star' = v_D$, then we know that $(v_D)^{\varphi} = v_R$ [22, Corollary 2.13]. Therefore $((\star'^{\varphi})_f)_T = (t_R)_T$ and so $((\star'^{\varphi})_f)_T$ coincides with $((\star'^{\varphi})_T)_f = ((v_R)_T)_f$ by (1).

Assume that $(\star'_f)^{\varphi}$ is a star operation of finite type. Note that, from the fact that $(\star'_f)^{\varphi} \leqslant \star'^{\varphi}$ and from the assumption, it follows that $(\star'_f)^{\varphi} \leqslant (\star'^{\varphi})_f$. Therefore, by [22, Proposition 2.9, Theorem 2.12 and Proposition 3.6(b)], we have

$$\big(\star'^{\varphi}\big)_f \leqslant \big(\big(\big(\star'^{\varphi}\big)_f\big)_{\varphi}\big)^{\varphi} = \big(\big(\big(\star'^{\varphi}\big)_{\varphi}\big)_f\big)^{\varphi} = \big(\star'_f\big)^{\varphi},$$

thus $({\star'}_f)^{\varphi} = ({\star'}^{\varphi})_f$. In this situation, by Claim 5, we have

$$(t_R)_\iota \leqslant (v_R)_\iota = \big(({\star'}_f)^{\varphi}\big)_\iota = \big(({\star'}^{\varphi})_f\big)_\iota \leqslant (t_R)_\iota.$$

Therefore,

$$(t_R)_\iota = \big(({\star'}_f)^{\varphi}\big)_\iota = \big(({\star'}^{\varphi})_f\big)_\iota = (v_R)_\iota = \big((v_R)_\iota\big)_f$$

and so, in particular, $(t_R)_T = (({\star'}_f)^{\varphi})_T = (({\star'}^{\varphi})_f)_T = (v_R)_T$. On the other hand, by Claim 5, we know that $(({\star'}^{\varphi})_T)_f = ((v_R)_T)_f = (t_R)_T$.

Assume that T a Prüfer domain, then clearly T has a unique star operation of finite type, since $d_T = t_T$. In this situation, obviously $d_T = (({\star'}^{\varphi})_f)_T = (t_R)_T = t_T$, and from Claim 5, we have $(({\star'}^{\varphi})_T)_f = (t_R)_T$.

(3) Under the assumptions of Lemma 2.10, as a consequence of Claims 3 and 6 in its proof, we have that $\mathrm{Max}^{(t_R)_T}(T)$ coincides with the set of the maximal elements of $\{PT \in \mathrm{Spec}(T) \mid P \in \mathrm{Spec}^{t_R}(R)\}$ (which is equal to the set $\{PT \mid P \in \mathrm{Max}^{t_R}(R), P \not\supseteq M\} \cup \{M\}$).

We can give a little different proof of this result under the additional assumption that the map $\tilde{\varphi} : \mathcal{U}(T) \to k^{\bullet}/\mathcal{U}(D)$ is surjective. Let $Q \in \mathrm{Max}^{(t_R)_T}(T)$ and let $P := Q \cap R$. Then $Q = PT$ and $Q = Q^{(t_R)_T} = Q^{t_R}$. Therefore $P \subseteq P^{t_R} \subseteq Q^{t_R} \cap R = Q \cap R = P$ and so $\mathrm{Max}^{(t_R)_T}(T) \subseteq \{PT \in \mathrm{Spec}(T) \mid P \in \mathrm{Spec}^{t_R}(R)\}$.

Conversely, let $Q := PT$ be a maximal element of the set $\{PT \in \mathrm{Spec}(T) \mid P \in \mathrm{Spec}^{t_R}(R)\}$. Assume that $P = M$, then since $M = MT$ is a maximal ideal of T and $M = M^{t_R}$, M is also a $(t_R)_T$-ideal of T, thus $M = MT \in \mathrm{Max}^{(t_R)_T}(T)$. Assume that $P \neq M$. Then $P \not\subseteq M$ by the maximality of $Q = PT$. Now, if $\boldsymbol{S} := \mathcal{U}(T) \cap R$, then by [7, Theorem 2.2(5) and Lemma 3.1] we have $(PT)^{t_T} = (PR_{\boldsymbol{S}})^{t_T} = P^{t_R}R_{\boldsymbol{S}} = PR_{\boldsymbol{S}} = PT$. Since $Q = PT \in \mathrm{Spec}^{t_T}(T)$, $Q \in \mathrm{Spec}^{(t_R)_T}(T)$.

Lemma 2.12. *Assume that we are dealing with a pullback diagram of type* $(\Box^+)$. *Let* $*$ *be a star operation of finite type on* R *and let* $(*)_T$ *be the star operation on* T *defined just before Lemma* 2.10.

(1) *If* $H \in \mathrm{Inv}^*(R)$, *then* $HT \in \mathrm{Inv}^{(*)_T}(T)$.
(2) *The canonical map* $\boldsymbol{\beta}(\varphi, *)$ (*or, simply,* $\boldsymbol{\beta}$) : $\mathrm{Inv}^*(R) \to \mathrm{Inv}^{(*)_T}(T)$, $H \mapsto HT$, *is a group-homomorphism.*
(3) *The map* $\boldsymbol{\beta}$, *defined in* (2), *induces a group-homomorphism* $\overline{\boldsymbol{\beta}}(\varphi, *)$ (*or, simply,* $\overline{\boldsymbol{\beta}}$) : $\mathrm{Cl}^*(R) \to \mathrm{Cl}^{(*)_T}(T)$, $[H] \mapsto [HT]$.

Proof. (1) Note that if H is a $*$-invertible $*$-ideal of R and $* = *_f$, then H is a t_R-invertible t_R-ideal of R (Lemma 1.3(2)). Moreover, T is a flat overring of R [19, Lemma 0.3], and hence HT is a t_T-invertible t_T-ideal of T [19, Proposition 0.7(b)]. We know by Lemma 2.10(2) that $(*)_T$ is a star operation of finite-type on T, so $(*)_T \leqslant t_T$, and hence HT is a $(*)_T$-ideal of T. Now, we show that HT is also $(*)_T$-invertible:

$$\begin{aligned}\big(HT(HT)^{-1}\big)^{(*)_T} &= \big(HT(HT)^{-1}\big)^{*} \cap \big(HT(HT)^{-1}\big)^{v_T} = \big(HT(HT)^{-1}\big)^{*} \cap T\\ &\supseteq \big(HH^{-1}T\big)^{*} \cap T = \big(\big(HH^{-1}\big)^{*}T\big)^{*} \cap T\\ &= (RT)^{*} \cap T = T^{*} \cap T = T,\end{aligned}$$

thus $1 \in (HT(HT)^{-1})^{(*)_T}$ and so $T = (HT(HT)^{-1})^{(*)_T}$.

(2) is an obvious consequence of (1) and (3) follows from (2). $\square$

Theorem 2.13. *Assume that we are dealing with a pullback diagram of type* $(\square^+)$. *Suppose that the map* $\tilde{\varphi}:\mathcal{U}(T) \to k^\bullet/\mathcal{U}(D)$ *is surjective and that* $*$ *is a star operation of finite type on* R. *Then* $\overline{\boldsymbol{\beta}} := \overline{\boldsymbol{\beta}}(\varphi, *) : \mathrm{Cl}^*(R) \to \mathrm{Cl}^{(*)_T}(T)$ *is surjective.*

Proof. Let J be an integral $(*)_T$-invertible $(*)_T$-ideal of T. Then $J = (IT)^{(*)_T} = (IT)^{t_T}$ for some finitely generated integral ideal I of R ([4, Propositions 3.1 and 3.2] and [19, Lemma 0.3]).

Claim 1. *Without loss of generality, we may assume that* $I \nsubseteq M$.

Suppose that $II^{-1} \subseteq M$. Then

$$\begin{aligned}(JJ^{-1})^{(*)_T} &= \big((IT)^{(*)_T}\big((IT)^{(*)_T}\big)^{-1}\big)^{(*)_T} = \big((IT)(IT)^{-1}\big)^{(*)_T} = \big(II^{-1}T\big)^{(*)_T}\\ &\subseteq (MT)^{(*)_T} = M^{(*)_T} = M,\end{aligned}$$

which contradicts that J is $(*)_T$-invertible. Thus, $II^{-1} \nsubseteq M$ and so we can choose $x \in I^{-1}$ such that $xI \nsubseteq M$. Set $I' := xI$ and $J' := xJ$. Then $I' \nsubseteq M$ and $J' = (I'T)^{(*)_T}$. Since the classes $[J]$ and $[J']$ in $\mathrm{Cl}^{(*)_T}(T)$ are the same, we can replace J by J' and I by I'.

Set $\boldsymbol{S} := \mathcal{U}(T) \cap R$ (as in Remark 2.11) and $\boldsymbol{N} := \{x \in R \mid \varphi(x) \in \mathcal{U}(D)\}$. Then $T = R_{\boldsymbol{S}}$ and $\boldsymbol{S} \cdot \boldsymbol{N} = R \setminus M$ [7, Lemma 3.1]. Since we may assume that $I \nsubseteq M$, by [7, Theorem 2.2(2)] we have $I^{t_R} = ((S_1)(N_1))^{t_R}$ for some nonempty finite subsets S_1 of $\boldsymbol{S}$ and N_1 of $\boldsymbol{N}$. Again by [7, Theorem 2.2], $J = (IT)^{t_T} = I^{t_R}T = ((S_1)(N_1))^{t_R}T = ((S_1)(N_1)T)^{t_T} = ((N_1)T)^{t_T} = (N_1)^{t_R}T$, and hence $JJ^{-1} = ((N_1)^{t_R}T)(((N_1)T)^{t_T})^{-1} = ((N_1)^{t_R}T)((N_1)T)^{-1} = (N_1)^{t_R}(N_1)^{-1}T$.

Claim 2. *If* $* = \widetilde{*}$, *then* $\overline{\boldsymbol{\beta}}$ *is surjective.*

Let $P' \in \mathrm{Spec}^*(R)$ such that $M \nsubseteq P'$. Then there exists a unique prime ideal Q' of T such that $Q' \cap R = P'$ and $R_{P'} = T_{Q'}$ [16, Theorem 1.4, point (c) of the proof]. Since $T = (JJ^{-1})^{(*)_T} = (JJ^{-1})^{*_\iota} = \bigcap\{JJ^{-1}R_P \mid P \in \mathrm{Max}^*(R)\} = \bigcap\{JJ^{-1}R_P \mid P \in \mathrm{Spec}^*(R)\} \subseteq JJ^{-1}R_{P'} = JJ^{-1}T_{Q'}$, $JJ^{-1} \nsubseteq Q'$, and hence $(N_1)^{t_R}(N_1)^{-1} \nsubseteq P'$.

Now let $P'' \in \mathrm{Spec}^*(R)$ such that $M \subseteq P''$. Then $P'' \cap \boldsymbol{N} = \emptyset$, because if $x \in P'' \cap \boldsymbol{N}$, then $\varphi(x) \in P''/M \in \mathrm{Spec}(D)$, which contradicts that $\varphi(x) \in \mathcal{U}(D)$. Therefore $(N_1)^{t_R}(N_1)^{-1} \nsubseteq P''$.

Thus since $(N_1)^{t_R}(N_1)^{-1} \nsubseteq P$ for all $P \in \mathrm{Spec}^*(R)$, $((N_1)^{t_R}(N_1)^{-1})^* = R$, i.e., $(N_1)^{t_R}$ is a $*$-invertible $*$-ideal of R. Therefore, passing to the classes, $[J] = [(N_1)^{t_R}T] = \overline{\boldsymbol{\beta}}([(N_1)^{t_R}])$.

Claim 3. $\mathrm{Cl}^{(*)_T}(T) = \mathrm{Cl}^{(\widetilde{*})_T}(T)$ (it does hold without the condition $\tilde{\varphi}:\mathcal{U}(T) \to k^\bullet/\mathcal{U}(D)$ is surjective).

By [3, Theorem 2.18] and Lemma 2.10(5), $\mathrm{Cl}^{(*)_T}(T) = \mathrm{Cl}^{\widetilde{(*)_T}}(T) = \mathrm{Cl}^{(\widetilde{*})_T}(T)$.

Finally, since $\mathrm{Cl}^*(R) = \mathrm{Cl}^{\widetilde{*}}(R)$ by [3, Theorem 2.18], $\overline{\boldsymbol{\beta}}(\varphi, *) = \overline{\boldsymbol{\beta}}(\varphi, \widetilde{*})$ and hence the conclusion follows. $\square$

From Claim 3 in the proof of Theorem 2.13 we deduce immediately:

Corollary 2.14. *Assume that we are dealing with a pullback diagram of type* $(\square^+)$. *Then* $\mathrm{Cl}^{(t_R)_T}(T) = \mathrm{Cl}^{(w_R)_T}(T)$.

In order to give a description of $\mathrm{Cl}^*(R)$ by means of $\mathrm{Cl}^{*_\varphi}(D)$ and $\mathrm{Cl}^{(*)_T}(T)$, we need the following result from [19]:

Lemma 2.15 [19, Lemma 2.2 and the subsequent considerations]. *Assume that we are dealing with a pullback diagram of type* $(\square^+)$.

(1) *For each* $H \in \mathrm{Inv}^t(R)$ *there exist a nonzero element* z *in the quotient field of* R *and* $H' \in \mathrm{Inv}^t(R)$, *with* $H' \not\subseteq M$, $H' \subseteq R$, *and* $H = zH'$.
(2) *The map* $\overline{\boldsymbol{\gamma}}:\mathrm{Cl}^{t_R}(R) \to \mathrm{Cl}^{t_D}(D)$, $[H] \mapsto [(\varphi(H'))^{v_D}]$, *is a well-defined group-homomorphism* (*where* H' *is chosen as in* (1)).

Corollary 2.16. *Assume that we are dealing with a pullback diagram of type* $(\square^+)$. *Let* $\overline{\boldsymbol{\gamma}} : \mathrm{Cl}^{t_R}(R) \to \mathrm{Cl}^{t_D}(D)$ *be as in Lemma* 2.15 *and let* $*$ *be a star operation of finite type on* R. *Then, by restriction to* $\mathrm{Cl}^*(R)(\subseteq \mathrm{Cl}^{t_R}(R))$, $\overline{\boldsymbol{\gamma}}$ *defines a group-homomorphism* $\overline{\boldsymbol{\gamma}} =: \overline{\boldsymbol{\gamma}}(\varphi, *):\mathrm{Cl}^*(R) \to \mathrm{Cl}^{*_\varphi}(D)$.

Proof. We want to show that $\overline{\boldsymbol{\gamma}}(\mathrm{Cl}^*(R)) \subseteq \mathrm{Cl}^{*_\varphi}(D) \subseteq \mathrm{Cl}^{t_D}(D)$. First, recalling that $*_\varphi \leqslant (t_R)_\varphi = t_D$ [22, Proposition 3.7], we have $\mathrm{Cl}^{*_\varphi}(D) \subseteq \mathrm{Cl}^{t_D}(D)$. Now let H be a $*$-invertible $*$-ideal of R such that $H \subseteq R$ and $H \not\subseteq M$. Choose $r \in H \setminus M$. Then $rH^{-1} \subseteq R$ and $rH^{-1} \not\subseteq M$. By using the fact that $\varphi(r)D$ is a divisorial ideal of D and [22, Proposition 2.7], we have

$$\begin{aligned}
\varphi(r)D &= \big(\varphi(r)D\big)^{*_\varphi} = \big(\varphi(rR)\big)^{*_\varphi} = \big(\varphi\big(r\big(HH^{-1}\big)^*\big)\big)^{*_\varphi} \\
&= \left(\frac{r(HH^{-1})^* + M}{M}\right)^{*_\varphi} = \frac{(r(HH^{-1})^* + M)^*}{M} \\
&= \frac{(rHH^{-1} + M)^*}{M} = \left(\frac{rHH^{-1} + M}{M}\right)^{*_\varphi} \\
&= \left(\frac{H + M}{M}\frac{rH^{-1} + M}{M}\right)^{*_\varphi} = \big(\varphi(H)\varphi\big(rH^{-1}\big)\big)^{*_\varphi}.
\end{aligned}$$

Hence $\varphi(H)$ is $*_\varphi$-invertible, and so $(\varphi(H))^{v_D}$ is a $*_\varphi$-invertible $*_\varphi$-ideal of D (Lemma 1.2(5)). Therefore $\overline{\boldsymbol{\gamma}}$ induces a homomorphism $\overline{\boldsymbol{\gamma}}(\varphi, *) : \mathrm{Cl}^*(R) \to \mathrm{Cl}^{*_\varphi}(D)$. $\square$

Theorem 2.17. *Assume that we are dealing with a pullback diagram of type* $(\square^+)$. *Suppose that the map* $\tilde{\varphi} : \mathcal{U}(T) \to k^\bullet/\mathcal{U}(D)$ *is surjective and that* $*$ *is a star operation of finite type on* R. *Then the sequence*

$$0 \longrightarrow \mathrm{Cl}^{*_\varphi}(D) \xrightarrow{\overline{\boldsymbol{\alpha}}} \mathrm{Cl}^*(R) \xrightarrow{\overline{\boldsymbol{\beta}}} \mathrm{Cl}^{(*)_T}(T) \longrightarrow 0$$

is split exact.

Proof. It is obvious that $\overline{\boldsymbol{\alpha}}$ is injective, because if $J \in \mathrm{Inv}^{*_\varphi}(D)$ such that $\boldsymbol{\alpha}(L) = \varphi^{-1}(J)$ is pricipal in R, say $\varphi^{-1}(J) = xR$ for some $x \in T$, then $J = \varphi(x)D$ is also principal in D. The surjectivity of $\overline{\boldsymbol{\beta}}$ follows from Theorem 2.13. To see that $\mathrm{Im}(\overline{\boldsymbol{\alpha}}) = \mathrm{Ker}(\overline{\boldsymbol{\beta}})$, let $[H] \in \mathrm{Im}(\overline{\boldsymbol{\alpha}})$. We can assume that $H = \varphi^{-1}(J)$ for some $J \in \mathrm{Inv}^{*_\varphi}(D)$ and so $M \subsetneq H \subseteq T$. Hence, in particular, $HT = T$, because M is a maximal ideal of T, and thus $\overline{\boldsymbol{\beta}}([H]) = [HT] = [T]$. Conversely, let $[H] \in \mathrm{Ker}(\overline{\boldsymbol{\beta}})$. Without loss of generality, we can assume that $H \in \mathrm{Inv}^*(R)$ and $HT = T$. Then by [19, Proposition 1.1] and [4, Proposition 3.1(a)], $M \subsetneq H = H^{v_R} \subseteq T$. Moreover, since T is not a $*$-invertible ($*$-)ideal of R, $H^{v_R} \subsetneq T$. By Lemma 2.1, $H = \varphi^{-1}(J)$ for some $*_\varphi$-invertible $*_\varphi$-ideal J of D, hence $H \in \mathrm{Im}(\overline{\boldsymbol{\alpha}})$. Thus the sequence is exact.

Lastly, by the definitions of $\overline{\boldsymbol{\alpha}} = \overline{\boldsymbol{\alpha}}(\varphi, *)$ and $\overline{\boldsymbol{\gamma}} = \overline{\boldsymbol{\gamma}}(\varphi, *)$ (Lemma 2.4 and Corollary 2.16), we immediately obtain that $\overline{\boldsymbol{\gamma}} \circ \overline{\boldsymbol{\alpha}} : \mathrm{Cl}^{*_\varphi}(D) \to \mathrm{Cl}^*(R) \to \mathrm{Cl}^{*_\varphi}(D)$ is such that $[J] \mapsto \overline{\boldsymbol{\gamma}}([\varphi^{-1}(J)]) = [(\varphi(\varphi^{-1}(J)))^{v_D}] = [J^{v_D}] = [J]$, i.e., it is the identity map. Therefore the above exact sequence splits. $\square$

Corollary 2.18. *Assume that we are dealing with a pullback diagram of type* $(\square^+)$ *and that the map* $\tilde{\varphi} : \mathcal{U}(T) \to k^\bullet/\mathcal{U}(D)$ *is surjective. Then the sequence*

$$0 \longrightarrow \mathrm{Cl}^{t_D}(D) \xrightarrow{\overline{\boldsymbol{\alpha}}} \mathrm{Cl}^{t_R}(R) \xrightarrow{\overline{\boldsymbol{\beta}}} \mathrm{Cl}^{(t_R)_T}(T) \longrightarrow 0$$

is split exact.

Proof. Recall that $(t_R)_\varphi = t_D$ [22, Proposition 3.7]. Then apply Theorem 2.17. $\square$

Note that, when we are dealing with a pullback diagram of type $(\square^+)$, $(t_R)_T \leqslant t_T$ (Lemma 2.10(2)) and so $\mathrm{Cl}^{(t_R)_T}(T)$ is a subgroup of $\mathrm{Cl}^{t_T}(T)$. In general, it can happen that $(t_R)_T \lneq t_T$ (for instance, when M is not a t_T-ideal). We will show, moreover, that $\mathrm{Cl}^{(t_R)_T}(T)$ can be a proper subgroup of $\mathrm{Cl}^{t_T}(T)$ (Remark 2.20).

Corollary 2.19. *Under the same notation and hypotheses of Corollary* 2.18, *if we assume that* T *is quasilocal, then* $\mathrm{Cl}^{(t_R)_T}(T) = 0$. (*In particular, we reobtain that* $\mathrm{Cl}^{t_D}(D) \cong \mathrm{Cl}^{t_R}(R)$, *see Corollary* 2.7.)

Proof. Let J be a $(t_R)_T$-invertible $(t_R)_T$-ideal of T. Then $J = (IT)^{(t_R)_T}$ for some nonzero finitely generated fractional ideal I of R [23, Proposition 2.6]. By the same argument as in Claim 1 of the proof of Theorem 2.13, we have $II^{-1} \nsubseteq M$. Therefore $JJ^{-1} \supseteq (IT)(IT)^{-1} = II^{-1}T = II^{-1}R_M = R_M = T$, and so J is invertible in T. Since T is quasilocal, we conclude that J is principal. Therefore $\mathrm{Cl}^{(t_R)_T}(T) = 0$. $\square$

Remark 2.20. Note that for a pullback diagram of type $(\square^+)$ with T quasilocal, it is quite common that $\mathrm{Cl}^{t_T}(T)$ is nonzero, but $\mathrm{Im}(\overline{\boldsymbol{\beta}}) = 0$ (Corollaries 2.18 and 2.19). An explicit example can be obtained as follows. Let $T := \mathbb{Q}[X^2, XY, Y^2]_{(X^2,XY,Y^2)}$, $M := (X^2, XY, Y^2)T$, thus $T = \mathbb{Q} + M$, and set $R := \mathbb{Z} + M$. Then, clearly $T = R_M$ and M is a t_R-prime of R. In this situation, the map $\overline{\boldsymbol{\beta}} : \mathrm{Cl}^{t_R}(R) \to \mathrm{Cl}^{t_T}(T) = \mathrm{Cl}^{t_{R_M}}(R_M)$ is the zero map, while $\mathrm{Cl}^{t_T}(T)$ is nonzero [9, Proposition 2.3 and Example 3.4]. Therefore in this case, by Corollary 2.19, $\mathrm{Cl}^{(t_R)_T}(T) \neq \mathrm{Cl}^{t_T}(T)$.

From Theorem 2.17 applied to $* = d_R$, we reobtain [19, Theorem 2.5(c)], since $(d_R)_\varphi = d_D$ [22, Proposition 3.3] and $(d_R)_T = d_T$. More precisely,

Corollary 2.21. *Assume that we are dealing with a pullback diagram of type* $(\square^+)$. *Suppose that the map* $\tilde{\varphi} : \mathcal{U}(T) \to k^\bullet/\mathcal{U}(D)$ *is surjective. Then* $\mathrm{Pic}(R) \cong \mathrm{Pic}(D) \oplus \mathrm{Pic}(T)$.

Remark 2.22. Note that, in [19, Remark 2.7], it was proved more generally that: *Assume that we are dealing with a pullback diagram of type* $(\square)$. *The map* $\tilde{\varphi} : \mathcal{U}(T) \to k^\bullet/\mathcal{U}(D)$ *is surjective if and only if* $\mathrm{Pic}(R) \cong \mathrm{Pic}(D) \oplus \mathrm{Pic}(T)$. A similar result was reobtained in [7, Theorem 3.9].

The next goal is to study the behavior of the property of being a Prüfer star multiplication domain in a pullback diagram of type $(\square)$. Recall that, given a star operation $*$ on an integral domain R, we say that R is a P$*$MD if for each nonzero finitely generated fractional ideal I of R, $(II^{-1})^{*_f} = R$ (cf. for instance [18,28,30,31,35]).

Theorem 2.23. *Consider a pullback diagram of type* $(\square)$ *and let* $*$ *be a star operation on* R. *Then* R *is a P$*$MD if and only if* k *is the quotient field of* D, D *is a P$*_\varphi$MD,* T *is a P$(*)_T$MD, and* T_M *is a valuation domain.*

Proof. If R is a P$*$MD, then R is a PvMD, and hence k is the quotient field of D and T_M is a valuation domain by [19, Theorem 4.1]. It is easy to see that if R is a P$*$MD, then D is a P$*_\varphi$MD and T is a P$(*)_T$MD. Actually, to prove that T is a P$(*)_T$MD, let J be a nonzero finitely generated ideal of T. Since T is R-flat, $J = IT$ for some finitely generated ideal I of R. Then by Lemma 2.10(4), $(JJ^{-1})^{((*)_T)_f} \supseteq (JJ^{-1})^{(*_f)_T} = (II^{-1}T)^{(*_f)_T} = (II^{-1}T)^{*_f} = ((II^{-1})^{*_f}T)^{*_f} = T^{*_f} = T$.

Conversely, assume that k is the quotient field of D, D is a P$*_\varphi$MD, T is a P$(*)_T$MD, and T_M is a valuation domain. Since D and T are PvMDs, R is a PvMD by [19, Theorem 4.1]. Let I be a nonzero finitely generated fractional ideal of R. Then $(II^{-1})^{t_R} = R$, and hence $II^{-1} \nsubseteq M$. To show that I is $*_f$-invertible, we may assume that I is a nonzero finitely generated integral ideal of R such that $I \nsubseteq M$.

Since D is a P$*_\varphi$MD, $(\varphi(I)\varphi(I)^{-1})^{(*_\varphi)_f} = D$. Since $(*_\varphi)_f = (*_f)_\varphi$ [22, Proposition 3.6], $(\varphi(I)\varphi(I)^{-1})^{(*_f)_\varphi} = D$, i.e., $((I+M)(I+M)^{-1})^{*_f} = R$, which implies that $(II^{-1}+M)^{*_f} = R$. Now suppose $II^{-1} \subseteq P$ for some $P \in \mathrm{Max}^{*_f}(R)$. Then $M \not\subseteq P$, because otherwise $R = (II^{-1}+M)^{*_f} \subseteq P^{*_f} = P$. Note that $PT \in \mathrm{Max}^{((*)_T)_f}(T)$ (by Claims 4 and 6 in the proof of Lemma 2.10). But since T is R-flat and T is a P$(*)_T$MD, $(IT(IT)^{-1})^{((*)_T)_f} = (II^{-1}T)^{((*)_T)_f} = T$, which contradicts that $II^{-1}T \subseteq PT$. Therefore $II^{-1} \not\subseteq P$ for all $P \in \mathrm{Max}^{*_f}(R)$, i.e., $(II^{-1})^{*_f} = R$. Thus R is a P$*$MD. $\square$

Corollary 2.24. *Consider a pullback diagram of type* ($\Box$). *R is a Pv_RMD ($= Pt_R$MD $= Pw_R$MD) if and only if k is the quotient field of D, D is a Pv_DMD ($= Pt_D$MD $= Pw_D$MD), T is a $P(v_R)_T$MD ($= P(t_R)_T$MD $= P(w_R)_T$MD), and T_M is a valuation domain.*

Proof. We can use Theorem 2.23 and the following facts:

(1) for any star operation $*$ on an integral domain A, A is a P$*$MD if and only if A is a P$\tilde{*}$MD [18, Theorem 3.1];
(2) $(v_R)_\varphi = v_D$ [22, Corollary 2.13];
(3) when k is the quotient field of D, $\widetilde{(v_R)_T} = (\widetilde{v_R})_T = (w_R)_T \leqslant (t_R)_T = ((v_R)_f)_T \leqslant ((v_R)_T)_f \leqslant (v_R)_T$ (Lemma 2.10). $\square$

Remark 2.25. Given a star operation $*$ on an integral domain R, recall that R is a P$*$MD if and only if R is a Pv_RMD and $\tilde{*} = t_R$ (or, equivalently, $*_f = t_R$) [18, Proposition 3.4]. Therefore (using Lemma 2.10(5) and [22, Proposition 3.9]) the previous theorem can be restated as follows: *Consider a pullback diagram of type* ($\Box$) *and let $*$ be a star operation on R. Then $\tilde{*} = t_R$ and R is a Pv_RMD if and only if k is the quotient field of D, $\tilde{*}_\varphi = t_D$, $(\tilde{*})_T = t_T$, D is a Pv_DMD, T is a Pv_TMD, and T_M is a valuation domain.*

Lemma 2.26. *Let R be a Pv_RMD and let T be a flat overring of R such that $(R : T) \neq 0$. Then $(w_R)_T = (t_R)_T = t_T = w_T$.*

Proof. Since T is a flat overring of R, T is a subintersection of R and hence T is a Pv_T*MD* [31, Theorem 3.11]. Recalling the fact that $w_A = t_A$ on a Pv_AMD A ([39, Theorem 2.4] or [18, Proposition 3.4]), it suffices to show that $(t_R)_T = t_T$.

Note first that T is a w_R-ideal of R and hence a t_R-ideal of R. Let $x \in T^{w_R}$. Then $xI \subseteq T$ for some finitely generated ideal I of R such that $I^{v_R} = R$ [21, Remark 2.8]. By flatness, $(IT)^{v_T} = (I^{v_R}T)^{v_T} = T$, and thus $x \in T$.

Then $(t_R)_T \leqslant t_T$ and both are star operations on T of finite type. Let J be a nonzero finitely generated integral ideal of T. Then $J = IT$ for some finitely generated ideal I of R. By [15, Proposition 2.17], $I^{v_R}T$ is a v_T-ideal of T, and hence $J^{t_T} = (I^{t_R}T)^{t_T} = (I^{v_R}T)^{t_T} = I^{v_R}T \subseteq (I^{v_R}T)^{t_R} = (I^{t_R}T)^{t_R} = (IT)^{t_R} = J^{t_R} = J^{(t_R)_T}$. Thus we have $(t_R)_T = t_T$. $\square$

Corollary 2.27. *Consider a pullback diagram of type* (□). *Then R is a Pv_RMD if and only if k is the quotient field of D, D is a Pv_DMD, T is a Pv_TMD, and T_M is a valuation domain. Moreover, in this situation,* $(w_R)_T = (t_R)_T = t_T = w_T$.

Proof. The first statement is [19, Theorem 4.1] and the "moreover" statement follows from Lemma 2.26. □

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References

[1] D.D. Anderson, Star-operations induced by overrings, Comm. Algebra 16 (1988) 2535–2553.
[2] D.D. Anderson, D.F. Anderson, M. Zafrullah, Splitting the t-class group, J. Pure Appl. Algebra 74 (1991) 17–37.
[3] D.D. Anderson, S.J. Cook, Two star-operations and their induced lattices, Comm. Algebra 28 (2000) 2461–2475.
[4] D.F. Anderson, A general theory of class groups, Comm. Algebra 16 (1988) 805–847.
[5] D.F. Anderson, The class group and the local class group of an integral domain, in: S.T. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, Kluwer Academic, 2000, pp. 33–55.
[6] D.F. Anderson, S. El Baghdadi, S. Kabbaj, On the class group of $A + XB[X]$ domains, in: D.E. Dobbs, M. Fontana, S. Kabbaj (Eds.), Advances in Commutative Ring Theory, in: Lecture Notes in Pure and Appl. Math., vol. 205, Dekker, New York, 1999, pp. 73–85.
[7] D.F. Anderson, G.W. Chang, The class group of integral domains, J. Algebra 264 (2003) 535–552.
[8] D.F. Anderson, D.E. Dobbs, Pairs of rings with the same prime spectrum, Canad. J. Math. 32 (1980) 362–384.
[9] D.F. Anderson, A. Ryckaert, The class group of $D + M$, J. Pure Appl. Algebra 52 (1988) 199–212.
[10] S. El Baghdadi, On the class group of a pullback, J. Pure Appl. Algebra 169 (2002) 159–173.
[11] M.P. Boisen, P.B. Sheldon, CPI-extensions: overrings of integral domains with special prime spectrum, Canad. J. Math. 29 (1977) 722–737.
[12] A. Bouvier, Le groupe des classes d'un anneau intègre, in: 107ème Congrès National des Sociétés Savantes, Brest, IV, 1982, pp. 85–92.
[13] A. Bouvier, M. Zafrullah, On some class groups of integral domains, Bull. Sc. Math. Grèce 29 (1988) 45–59.
[14] P.-J. Cahen, Couples d'anneaux partageant un idéal, Arch. Math. 51 (1988) 505–514.
[15] D.E. Dobbs, E.G. Houston, T.G. Lucas, M. Zafrullah, t-linked overrings and Prüfer v-multiplication domains, Comm. Algebra 17 (11) (1989) 2835–2852.
[16] M. Fontana, Topologically defined classes of commutative rings, Ann. Mat. Pura Appl. 129 (1980) 331–355.
[17] M. Fontana, J.A. Huckaba, Localizing systems and semistar operations, in: S.T. Chapman, S. Glaz (Eds.), Non-Noetherian Commutative Ring Theory, Kluwer Academic, 2000, pp. 169–198.
[18] M. Fontana, P. Jara, E. Santos, Prüfer $\star$-multiplication domains and semistar operations, J. Algebra Appl. 2 (2003) 21–50.
[19] M. Fontana, S. Gabelli, On the class group and the local class group of a pullback, J. Algebra 181 (1996) 803–835.
[20] M. Fontana, K.A. Loper, Kronecker function rings: a general approach, in: D.D. Anderson, I.J. Papick (Eds.), Ideal Theoretic Methods in Commutative Algebra, in: Lecture Notes in Pure and Appl. Math., vol. 220, Dekker, 2001, pp. 189–205.

[21] M. Fontana, K.A. Loper, Nagata rings, Kronecker function rings, and related semistar operations, Comm. Algebra 31 (2003) 4775–4805.
[22] M. Fontana, M.H. Park, Star operations and pullbacks, J. Algebra 274 (2004) 387–421.
[23] M. Fontana, G. Picozza, Semistar invertibility of integral domains, Algebra Colloq. 12 (2005) 645–664.
[24] R.M. Fossum, The Divisor Class Group of a Krull Domain, Springer, New York, 1973.
[25] S. Gabelli, E. Houston, Coherentlike conditions in pullbacks, Michigan Math. J. 44 (1997) 99–123.
[26] S. Gabelli, M. Roitman, On Nagata's theorem for the class group, J. Pure Appl. Algebra 66 (1990) 31–42.
[27] R. Gilmer, Multiplicative Ideal Theory, Dekker, New York, 1972.
[28] M. Griffin, Some results on v-multiplication rings, Canad. J. Math. 19 (1967) 710–721.
[29] J.R. Hedstrom, E.G. Houston, Some remarks on star-operations, J. Pure Appl. Algebra 18 (1980) 37–44.
[30] E.G. Houston, S.B. Malik, J.L. Mott, Characterization of $\star$-multiplication domains, Canad. Math. Bull. 27 (1984) 48–52.
[31] B.G. Kang, Prüfer v-multiplication domains and the ring $R[X]_{N_v}$, J. Algebra 123 (1989) 151–170.
[32] M. Khalis, D. Nour El Abidine, On the class group of a pullback, in: P.-J. Cahen, M. Fontana, E.G. Houston, S. Kabbaj (Eds.), Commutative Ring Theory, in: Lecture Notes in Pure and Appl. Math., vol. 185, Dekker, New York, 1997, pp. 377–386.
[33] R. Matsuda, I. Sato, Note on star operations and semistar operations, Bull. Fac. Sci. Ibaraki Univ. Ser. A 28 (1996) 5–22.
[34] R. Matsuda, T. Sugatani, Semistar operations on integral domains, II, Math. J. Toyama Univ. 18 (1995) 155–161.
[35] J.L. Mott, M. Zafrullah, On Prüfer v-multiplication domains, Manuscripta Math. 35 (1981) 1–26.
[36] D. Nour El Abidine, Sur le groupe des classes d'un anneau intègre, Ann. Univ. Ferrara 36 (1990) 175–183.
[37] A. Okabe, R. Matsuda, Star operations and generalized integral closures, Bull. Fac. Sci. Ibaraki Univ. Ser. A 24 (1992) 7–13.
[38] A. Okabe, R. Matsuda, Semistar operations on integral domains, Math. J. Toyama Univ. 17 (1994) 1–21.
[39] M.H. Park, Group rings and semigroup rings over strong Mori domains, J. Pure Appl. Algebra 163 (3) (2001) 301–318.
[40] P. Samuel, Lectures on Unique Factorization Domains, Notes by P. Murty, Tata Institute for Fundamental Research, No. 30, Bombay, 1964.
[41] F. Wang, R.L. McCasland, On w-modules over strong Mori domains, Comm. Algebra 25 (1997) 1285–1306.
[42] M. Zafrullah, A general theory of almost factoriality, Manuscripta Math. 51 (1985) 29–62.