MAT.632 - Elective subjects Mathematics Topological Methods in Commutative Ring Theory

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To the rose in the desert

CONTENTS

NOTATION AND CONVENTIONS.

Let X be a set and $\mathcal{F}$ be a collection of subsets of X. Then the union (resp., the intersection) of the members of X will be often denoted simply by $\bigcup \mathcal{F}$ (resp., $\bigcap \mathcal{F}$).

Any ring is assumed to be commutative with $1 \neq 0$. Any ring homomorphism $f : A \longrightarrow B$ sends, by definition, the multiplicative identity of A to that of B. If A is a ring and $\mathfrak{a}$ is an ideal of A, we will denote by $\sqrt{\mathfrak{a}} := \{x \in A : x^r \in \mathfrak{a}$, for some positive integer $r\}$ the radical of $\mathfrak{a}$. An ideal $\mathfrak{a}$ is radical if $\mathfrak{a} = \sqrt{\mathfrak{a}}$. If $a_1, \ldots, a_n \in A$, we will denote by $(a_1, \ldots, a_n)A$ the ideal generated by $a_1, \ldots, a_n$. Set $\mathrm{Spec}(A) := \{\text{prime ideals of } A\}$, $\mathrm{Max}(A) := \{\text{maximal ideals of } A\}$. We assume the convention that any prime ideal is a proper ideal. Of course, we have $\mathrm{Max}(A) \subseteq \mathrm{Spec}(A)$. It is well known that, for any ideal $\mathfrak{a}$ of A, then

$$\sqrt{\mathfrak{a}} = \bigcap \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\}$$

Let $A \subseteq B$ be a ring extension. Recall that B is finite over A if B is finitely generated as an A-module. We say that B is of finite type over A if $B = A[b_1, \ldots, b_r]$, for some $b_1, \ldots, b_r \in B$. If B is of finite type over A, say $B = A[b_1, \ldots, b_r]$ and $b_1, \ldots, b_r$ are integral over A, then B is integral over A.

1. Hilbert's Nullstellensatz.

In the following, K is any field, $T_1, \ldots, T_n$ are indeterminates over K and $\mathbb{A}^n_K$ is the n-dimensional affine space over K.

(1.1) Definition. *If S is any subset of the polynomial ring $K[T_1, \ldots, T_n]$, the* zero set *of S is the following subset*

$$Z(S) := \{p \in \mathbb{A}^n_K : f(p) = 0, \textit{for any } f \in S\}$$

of $\mathbb{A}^n_K$. We say that a subset X of $\mathbb{A}^n_K$ is algebraic *if $X = Z(S)$, for some $S \subseteq K[T_1, \ldots, T_n]$.*

(1.2) Remark. Take subsets S, S' of $K[T_1, \ldots, T_n]$.

(a) If $S \subseteq S'$, then $Z(S) \supseteq Z(S')$.

(b) If $\mathfrak{a}$ is the ideal of $K[T_1, \ldots, T_n]$ generated by S, then $Z(S) = Z(\mathfrak{a})$.

(c) For any ideal $\mathfrak{a}$ of $K[T_1, \ldots, T_n]$ we have $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$.

Part (a) is trivial. (b): since $S \subseteq \mathfrak{a}$ we have, by part (a), $Z(S) \supseteq Z(\mathfrak{a})$. Conversely, take a point $p \in Z(S)$ and a polynomial $f \in \mathfrak{a}$. By definition, there are polynomials $f_1, \ldots, f_m \in K[T_1, \ldots, T_n]$, $s_1, \ldots, s_n \in S$ such that $f = \sum_{i=1}^m s_i f_i$. Since $p \in Z(S)$ we have $s_i(p) = 0$, for $1 \leq i \leq m$, and thus $f(p) = 0$. (c): by part (a), the inclusion $\mathfrak{a} \subseteq \sqrt{\mathfrak{a}}$ implies $Z(\mathfrak{a}) \supseteq Z(\sqrt{\mathfrak{a}})$. Take now a point $p \in Z(\mathfrak{a})$ and a polynomial $f \in \sqrt{\mathfrak{a}}$, and let r be a positive integer such that $f^r \in \mathfrak{a}$. Since $p \in Z(\mathfrak{a})$ we have $(f^r)(p) := (f(p))^r = 0$, that is, $f(p) = 0$.

We observe now that any algebraic set is the zero set of a *finite* collection of polynomials.

(1.3) Remark. Let X be an algebraic subset of $\mathbb{A}^n_K$. Then there are polynomials $f_1, \ldots, f_m \in K[T_1, \ldots, T_n]$ such that $X = Z(\{f_1, \ldots, f_m\})$.

Indeed, by definition $X = Z(S)$ for some subset S of $K[T_1, \ldots, T_n]$. On the other hand, $X = Z(\mathfrak{a})$, where $\mathfrak{a}$ is the ideal of $K[T_1, \ldots, T_n]$ generated by S, by (1.2b). Since $K[T_1, \ldots, T_n]$ is a Noetherian ring, the ideal $\mathfrak{a}$ is finitely generated, say by $f_1, \ldots, f_m$. Thus $X = Z(\{f_1, \ldots, f_m\})$, again by (1.2b).

(1.4) Definition. *Let X be any subset of $\mathbb{A}^n_K$. Then the following subset*

$$I(X) := \{f \in K[T_1, \ldots, T_n] : f(p) = 0, \ \textit{for any } p \in X\}$$

is an ideal of $K[T_1, \ldots, T_n]$, called the ideal of X.

(1.5) Remark. Take a subset $X \subseteq \mathbb{A}^n_K$.

(a) $I(X)$ is a radical ideal of $K[T_1, \ldots, T_n]$.

(b) If X is algebraic, then $X = Z(I(X))$.

(a): take a polynomial $f \in \sqrt{I(X)}$ and a positive integer r such that $f^r \in I(X)$. For any point $p \in X$ we have $(f(p))^r = 0$, that is, $f(p) = 0$. In other words, $f \in I(X)$.

(b): Let $\mathfrak{a}$ be an ideal of $K[T_1, \ldots, T_n]$ such that $X = Z(\mathfrak{a})$. The obvious inclusion $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$ and (1.2a) imply $Z(\mathfrak{a}) \supseteq Z(I(Z(\mathfrak{a})))$. Conversely, take a point $p \in Z(\mathfrak{a})$ and a polynomial $f \in I(Z(\mathfrak{a}))$. By definition, any point of $Z(\mathfrak{a})$ is a zero of f and, in particular, $f(p) = 0$.

(1.6) Remark. For any ideal $\mathfrak{a}$ of $K[T_1, \ldots, T_n]$, then $\sqrt{\mathfrak{a}} \subseteq I(Z(\mathfrak{a}))$. The trivial proof is left to the reader. Note that the inclusion $\sqrt{\mathfrak{a}} \subseteq I(Z(\mathfrak{a}))$ may be strict. Indeed, if $n := 1$, $K := \mathbb{R}$ and $\mathfrak{a} := (T^2 + 1)\mathbb{R}[T]$, then $\mathfrak{a}$ is a maximal ideal of $\mathbb{R}[T]$ and, in particular, it is a radical ideal. Thus $\mathfrak{a} = \sqrt{\mathfrak{a}} \subsetneq I(Z(\mathfrak{a})) = I(\emptyset) = \mathbb{R}[T]$.

The statement of (1.6) is much more precise when K is algebraically closed.

(1.7) Theorem (Hilbert's Nullstellensatz). *If K is an algebraically closed field and $\mathfrak{a}$ is an ideal of $K[T_1, \ldots, T_n]$, then $\sqrt{\mathfrak{a}} = I(Z(\mathfrak{a}))$.*

First, we give an immediate corollary of Hilbert's Nullstellensatz.

(1.8) Corollary. *Let K be an algebraically closed field and define the map*

$$\psi : \{\text{algebraic subsets of } \mathbb{A}^n_K\} \longrightarrow \{\text{radical ideals of } K[T_1, \ldots, T_n]\}$$

by setting $\psi(X) := I(X)$. Then ψ is a order reversing bijection.

Proof. Apply (1.5b) and (1.7). □

Before giving a proof of Hilbert's Nullstellensatz, we now note that this very famous and powerful statement admits several equivalent forms.

(1.9) Proposition. *The following conditions are equivalent.*

(i) *Hilbert's Nullstellensatz holds.*
(ii) *If K is an algebraically closed field, then*

$$\operatorname{Max}(K[T_1, \ldots, T_n]) = \{(T_1 - a_1, \ldots, T_n - a_n)K[T_1, \ldots, T_n] : a_1, \ldots, a_n \in K\}.$$

(iii) *If K is an algebraically closed field and $\mathfrak{a}$ is a proper ideal of $K[T_1, \ldots, T_n]$, then $Z(\mathfrak{a}) \neq \emptyset$.*

Proof. (i) $\Longrightarrow$ (ii). The inclusion $\supseteq$ is an exercise. Let $\mathfrak{m}$ be a maximal ideal of $K[T_1, \ldots, T_n]$. Being $\mathfrak{m}$, in particular, a radical ideal, by statement (i) we have $\mathfrak{m} = I(Z(\mathfrak{m}))$ and thus $Z(\mathfrak{m})$ is nonempty (otherwise $\mathfrak{m} = I(\emptyset) = K[T_1, \ldots, T_n]$). Take a point $(a_1, \ldots, a_n) \in Z(\mathfrak{m})$. We claim that $T_i - a_i \in \mathfrak{m}$, for $1 \leq i \leq n$. If not, there is an index i such that $T_i - a_i \notin \mathfrak{m}$ and thus $\mathfrak{m} + (T_i - a_i)K[T_1, \ldots, T_n] = K[T_1, \ldots, T_n]$, since $\mathfrak{m}$ is maximal. It would follow that there are polynomials $m \in \mathfrak{m}, f \in K[T_1, \ldots, T_n]$ such that $1 = m + (T_i - a_i)f$. Since $(a_1, \ldots, a_n) \in Z(\mathfrak{m})$, we have $1 = m(a_1, \ldots, a_n) + (a_i - a_i)f(a_1, \ldots, a_n) = 0$, a contradiction. This proves that $T_i - a_i \in \mathfrak{m}$, for $1 \leq i \leq n$, that is, $\mathfrak{n} := (T_1 - a_1, \ldots, T_n - a_n) \subseteq \mathfrak{m}$. Since $\mathfrak{n}$ is a maximal ideal, it follows $\mathfrak{m} = \mathfrak{n}$.

(ii) $\Longrightarrow$ (iii). Take a proper ideal $\mathfrak{a}$ of $K[T_1, \ldots, T_n]$ and let $\mathfrak{m}$ be a maximal ideal of $K[T_1, \ldots, T_n]$ containing $\mathfrak{a}$. By (ii), there are elements $a_1, \ldots, a_n \in K$ such that $\mathfrak{m} = (T_1 - a_1, \ldots, T_n - a_n)K[T_1, \ldots, T_n]$ and clearly, by (1.2a), we have $(a_1, \ldots, a_n) \in Z(\mathfrak{m}) \subseteq Z(\mathfrak{a})$.

(iii) $\Longrightarrow$ (i). Take any ideal $\mathfrak{a}$ of $K[T_1, \ldots, T_n]$. By (1.6), we have to show that $I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$. Take a nonzero polynomial $f \in I(Z(\mathfrak{a})))$ and consider the ring inclusion $K[T_1, \ldots, T_n] \subseteq B := K[T_1, \ldots, T_n, U]$, where U is a new indeterminate over K. Let $\mathfrak{b}$ be the ideal of B generated by $\mathfrak{a}$ and the polynomial $Uf - 1$. We claim that the subset $Z(\mathfrak{b})$ of $\mathbb{A}^{n+1}_K$ is empty. We argue by contradiction, and pick a point $p := (a_1, \ldots, a_n, \alpha) \in Z(\mathfrak{b})$. Since $Uf - 1 \in \mathfrak{b}$, we have $\alpha f(a_1, \ldots, a_n) - 1 = 0$. Moreover, since $\mathfrak{a} \subseteq \mathfrak{b}$, we have $g(a_1, \ldots, a_n) = 0$, for any $g \in \mathfrak{a}$, that is, $(a_1, \ldots, a_n) \in Z(\mathfrak{a})$. Since $f \in I(Z(\mathfrak{a}))$ it follows $f(a_1, \ldots, a_n) = 0$ and thus the equality $\alpha f(a_1, \ldots, a_n) - 1 = 0$ implies $-1 = 0$, a contradiction. This argument proves that $Z(\mathfrak{b}) = \emptyset$. Thus, applying condition (iii) to the algebraically closed field K and to the ideal $\mathfrak{b}$ of the polynomial ring B we have $\mathfrak{b} = B$. Pick polynomials $r_1, \ldots, r_h, s \in B$, $f_1, \ldots, f_h \in \mathfrak{a}$ such that $1 = \sum_{i=1}^h r_i f_i + s(Uf - 1)$, and consider the ring homomorphism $\phi : B \longrightarrow K(T_1, \ldots, T_n)$ such that $T_i \mapsto T_i$, for $1 \leq i \leq n$,

$U \mapsto f^{-1}$ and $k \mapsto k$, for any $k \in K$. Thus, in particular, ϕ is the identity on $K[T_1,\ldots,T_n]$. Then we have

$$1 = \sum_{i=1}^{h} \phi(r_i)\phi(f_i)+\phi(s)(\phi(U)\phi(f)-1) = \sum_{i=1}^{h} \phi(r_i)f_i+\phi(s)(f^{-1}f-1) = \sum_{i=1}^{h} \phi(r_i)f_i \quad (\star).$$

Moreover we have

$$\phi(r_i) := \phi(r_i(T_1,\ldots,T_n,U)) = r_i(T_1,\ldots,T_n,f^{-1}) = \frac{\rho_i}{f^{m_i}}$$

for some $\rho_1,\ldots,\rho_h \in K[T_1,\ldots,T_n]$ and some $m_i \in \mathbb{N}$. Thus $(\star)$ is equivalent to $1 = \sum_{i=1}^{h} \frac{\rho_i}{f^{m_i}} f_i$ and, if $m = \max\{m_1,\ldots,m_h\}$, we get, by multiplying both sides for f^m, the equality $f^m = \sum_{i=1}^{h} \widetilde{\rho}_i f_i$, for some $\widetilde{\rho}_1,\ldots,\widetilde{\rho}_h \in K[T_1,\ldots,T_n]$, that is $f \in \sqrt{\mathfrak{a}}$. $\square$

1.1. **G-ideals and Hilbert rings: toward a proof of Hilbert's Nullstellesatz.** We will present a proof of Hilbert's Nullstellensatz due to Goldman and Krull. It is based on the notions of independent interest, that of G-ideal and Hilbert ring. Thus, these notions will be central in the next step of our investigation.

(1.10) Definition. *Let A be a ring and let $\mathfrak{p} \in \mathrm{Spec}(A)$. We say that $\mathfrak{p}$ is a G-ideal of A if $\mathfrak{p}$ is not the intersection of a family of prime ideals which are strictly bigger than $\mathfrak{p}$.*

For example, any maximal ideal of a ring is a G-ideal.

(1.11) Lemma. *Let K be a field. Then, the polynomial ring $K[T]$ has infinitely many monic and irredicible polynomials. In particular, $\mathrm{Spec}(K[T])$ is infinite.*

Proof. We argue by contradiction. Assume that the set

$$\Sigma := \{\text{monic and irredicible polynomials of } K[T]\}$$

is finite, say $\Sigma = \{f_0 := T, f_1,\ldots,f_n\}$, and set $f := 1 + Tf_1f_2\ldots f_n$. Then f is monic and $\deg(f) > \deg(f_i)$, for $0 \le i \le n$. It follows that $f \notin \Sigma$, that is, f is reducible. Since $K[T]$ is a UFD, f has at least an irreducible factor g, and we can clearly assume that g is monic. Then $g = f_i$, for some $0 \le i \le n$. Since g divides both f and $Tf_1\ldots f_n$, it follows that g divides 1, a contradiction.

To prove the last statement note that, for an $f \in \Sigma$, $fK[T]$ is a maximal ideal of $K[T]$. Moreover, if $f, g \in \Sigma$ and $fK[T] = gK[T]$, then $f = g$. The conclusion follows since Σ is infinite. $\square$

(1.12) Proposition. *For any integral domain D, (0) is not a G-ideal of the polynomial ring $D[T]$.*

Proof. First, we prove the following fact.

Claim. *If K is a field, then $(0) = \bigcap\{\mathfrak{m} : \mathfrak{m} \in \mathrm{Max}(K[T])\}$.* Indeed, by (1.11), if $f \in \mathfrak{m}$, for any $\mathfrak{m}\mathrm{Max}(K[T])$, then f would admit infinitely many irreducible factors, against the fact that $K[T]$ is a UFD.

Now we can prove the proposition. Let K be the quotient field of D, and let $\mathfrak{m}$ be a maximal ideal of $K[T]$. Then $\mathfrak{m}$ is generated by an irreducible polynomial f. Since

K is the quotient field of D, there is a nonzero element $d \in D$ such that $df \in D[T]$ (and clearly $df \in \mathfrak{m}$). It follows that $\{\mathfrak{m} \cap D[T] : \mathfrak{m} \in \mathrm{Max}(K[T])\}$ is a collection of nonzero prime ideals of $D[T]$. Then, by the Claim,

$$\bigcap_{\mathfrak{m} \in \mathrm{Max}(K[T])} (\mathfrak{m} \cap D[T]) = (\bigcap_{\mathfrak{m} \in \mathrm{Max}(K[T])} \mathfrak{m}) \cap D[T] = (0).$$

The proof is now complete. □

(1.13) Definition. *An integral domain D is called* a G-domain *if* (0) *is a G-ideal of D.*

(1.14) Example. (1) Any field is a G-domain (because (0) is a maximal ideal).

(2) By (1.12), for any integral domain D, the polynomial ring $D[T]$ is not a G-domain.

(1.15) Remark. Let A be a ring and $\mathfrak{p} \in \mathrm{Spec}(A)$. For any $\mathfrak{q} \in \mathrm{Spec}(A)$, with $\mathfrak{q} \supseteq \mathfrak{p}$, let $\overline{\mathfrak{q}}$ denote the prime ideal of the integral domain $A/\mathfrak{p}$ corresponding to $\mathfrak{q}$. Then

$$\mathfrak{p} \text{ is a G-ideal of } A$$
$$\Updownarrow$$
$$\bigcap\{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \supsetneq \mathfrak{p}\} \supsetneq \mathfrak{p}$$
$$\Updownarrow$$
$$\bigcap\{\overline{\mathfrak{q}} \in \mathrm{Spec}(A/\mathfrak{p})) : \overline{\mathfrak{q}} \supsetneq \overline{\mathfrak{p}}\} \supsetneq (0)$$
$$\Updownarrow$$
$$A/\mathfrak{p} \text{ is a G-domain}$$

(1.16) Proposition. *Let D be an integral domain and K be the quotient field of D. Then, the following conditions are equivalent.*

(i) *D is a G-domain.*
(ii) *There exists a nonzero element $\alpha \in D$ such that $K = D[\alpha^{-1}]$.*
(iii) *There exists a nonzero element $x \in K$ such that $K = D[x]$.*
(iv) *K is of finite type over D.*

Proof. (i) $\Longrightarrow$ (ii). By definition, there is a nonzero element $\alpha \in D$ such that $\alpha \in \mathfrak{p}$, for any nonzero prime ideal $\mathfrak{p}$ of D. Fix any element $x \in K$, say $x = \dfrac{a}{b}$, for some $a, b \in D, b \neq 0$. Then, clearly

$$\alpha \in \bigcap\{\mathfrak{p} \in \mathrm{Spec}(D) : \mathfrak{p} \neq (0)\} \subseteq \bigcap\{\mathfrak{p} \in \mathrm{Spec}(D) : b \in \mathfrak{p}\} = \sqrt{bD},$$

and thus there is a positive integer r and an element $d \in D$ such that $\alpha^r = bd$, that is, $b^{-1} = \dfrac{d}{\alpha^r}$. Then, $x = \dfrac{a}{b} = \dfrac{ad}{\alpha^r} \in D[\alpha^{-1}]$.

(iii) $\Longrightarrow$ (i). Suppose there is an element $x := \dfrac{a}{b}$ $(a, b \in D, b \neq 0)$ such that $K = D[x]$, let $\mathfrak{p}$ be any nonzero prime ideal of D and let $y \in \mathfrak{p} - (0)$. By definition, there are suitable $d_0, \ldots, d_n \in D$ such that $y^{-1} = \displaystyle\sum_{i=0}^{n} d_i \frac{a^i}{b^i}$. The last equality is equivalent to the following:

$$y^{-1}b^n = d_0 b^n + d_1 a b^{n-1} + \ldots + d_n a^n =: z$$

and finally $b^n = yz \in \mathfrak{p}$. Since $\mathfrak{p}$ is a prime ideal, it follows $b \in \mathfrak{p}$. Thus $b \in \bigcap\{\mathfrak{p} \in \operatorname{Spec}(D) : \mathfrak{p} \neq (0)\}$.

The equivalence of (i) and (iv) is left to the reader as an exercise. □

We recall now the following well known fact.

(1.17) Proposition. *Let A be a ring, S be a multiplicative subset of A and let $\mathfrak{a}$ be an ideal of A such that $\mathfrak{a} \cap S = \emptyset$. Then the following statements hold.*

(a) *The collection of ideals $\Sigma := \{\mathfrak{b} : \mathfrak{b}$ is an ideal of $A, \mathfrak{b} \supseteq \mathfrak{a}, \mathfrak{b} \cap S = \emptyset\}$ has maximal elements, under inclusion.*
(b) *Any maximal element of Σ is a prime ideal of A.*

(1.18) Lemma. *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. Then*

$$\sqrt{\mathfrak{a}} = \bigcap\{\mathfrak{p} : \mathfrak{p} \text{ is a G-ideal of } A, \mathfrak{p} \supseteq \mathfrak{a}\}$$

Proof. The inclusion $\subseteq$ is trivial. Then, it suffices to show that, if $u \in A - \sqrt{\mathfrak{a}}$, then there is a G-ideal $\mathfrak{p}$ of A such that $\mathfrak{p} \supseteq \mathfrak{a}$ and $u \notin \mathfrak{p}$. Take an element $u \in A - \sqrt{\mathfrak{a}}$ and consider the multiplicative subset $S := \{1, u^n : n \geq 1\}$ of A. Then $S \cap \mathfrak{a} = \emptyset$. By (1.18), there is a prime ideal $\mathfrak{p}$ of A which is maximal in the family of ideals $\Sigma := \{\mathfrak{b} : \mathfrak{b}$ is an ideal of $A, \mathfrak{b} \supseteq \mathfrak{a}, \mathfrak{b} \cap S = \emptyset\}$, partially ordered by inclusion. In particular, $\mathfrak{p} \supseteq \mathfrak{a}$, $\mathfrak{p} \cap S = \emptyset$ and thus $u \notin \mathfrak{p}$. Now, let $\mathfrak{q}$ any prime ideal of A such that $\mathfrak{q} \supsetneq \mathfrak{p}$. By maximality of $\mathfrak{p}$, we have $\mathfrak{q} \notin \Sigma$, and since $\mathfrak{q}$ clearly contains $\mathfrak{a}$, we infer that $S \cap \mathfrak{q} \neq \emptyset$. By primality of $\mathfrak{q}$ it follows $u \in \mathfrak{q}$. This proves that $u \in \bigcap\{\mathfrak{q} \in \operatorname{Spec}(A) : \mathfrak{q} \supsetneq \mathfrak{p}\} - \mathfrak{p}$, that is, $\mathfrak{p}$ is a G-ideal. The proof is now complete. □

(1.19) Definition. *We say that a ring A is* a Hilbert ring *if any G-ideal of A is maximal.*

(1.20) Example. For any field K, the polynomial ring $K[T]$ is a Hilbert ring. Indeed, the unique non maximal prime ideal of $K[T]$ is (0), and it is not a G-ideal, by (1.12).

(1.21) Proposition. *Let A be a ring. Then, the following conditions are equivalent.*

(i) *A is a Hilbert ring.*
(ii) *For any ideal $\mathfrak{a}$ of A, $\sqrt{\mathfrak{a}}$ is an intersection of maximal ideals of A.*

Proof. (i)$\Longrightarrow$(ii). Apply (1.18).

(ii)$\Longrightarrow$(i). Let $\mathfrak{p}$ be a non maximal prime ideal of A. By (ii), there is a set Y of maximal ideals of A such that $\mathfrak{p} = \bigcap\{\mathfrak{m} : \mathfrak{m} \in Y\}$. Since $\mathfrak{p}$ is not a maximal ideal, we must have $\mathfrak{p} \subsetneq \mathfrak{m}$, for any $\mathfrak{m} \in Y$, and thus $\mathfrak{p}$ is not a G-ideal. □

We recall now the following well-known fact concerning integral dependence.

(1.22) Proposition. *Let $A \subseteq B$ an integral extension of integral domains. Then A is a field if and only if B is a field.*

(1.23) Lemma. *Let $A \subseteq B$ be an extension of integral domains such that $B = A[t]$, for some element $t \in B$ which is algebraic over A. Then A is a G-domain if and only if B is a G-domain.*

Proof. Let K (resp., L) be the quotient field of A (resp., B). Then it is straightforward that $L = K[t]$ and clearly L is algebraic over K.

($\Longleftarrow$). Suppose that B is a G-domain. By (1.16), there is an element $c \in B$ such that $L = B[c^{-1}] = A[t, c^{-1}]$. Clearly, c^{-1} is algebraic over A, and thus we can pick nonzero polynomials $f, g \in A[T]$ such that $f(c^{-1}) = 0$ and $g(t) = 0$. Let a (resp., b) be the leading coefficient of f (resp., g), and consider the ring $\widetilde{A} := A[a^{-1}, b^{-1}]$. Then, the elements c^{-1}, t are integral over $\widetilde{A}$: indeed, the polynomials $f' := a^{-1}f, g' := b^{-1}g \in \widetilde{A}[T]$ are monic and clearly $f'(c^{-1}) = g'(t) = 0$. It follows that the field $L = \widetilde{A}[c^{-1}, t]$ is integral over $\widetilde{A}$. Thus, by (1.22), $\widetilde{A}$ is a field and, since clearly $\widetilde{A}$ is a field between A and the quotient field K of A, we have $\widetilde{A} = K$. This proves that K is of finite type over A, and thus, in view of (1.16), A is a G-domain.

($\Longrightarrow$). Suppose now that A is a G-domain and, by (1.16), take a nonzero element $a \in A$ such that $K = A[a^{-1}]$. Then $L = K[t] = A[a^{-1}, t] = B[a^{-1}]$. Again by (1.16), B is a G-domain. □

(1.24) Lemma. *Let A be a ring and let $\mathfrak{q}$ be a G-ideal of the polynomial ring $A[T]$. Then $\mathfrak{q} \cap A$ is a G-ideal of A.*

Proof. If $\mathfrak{p} := A \cap \mathfrak{q}$, then clearly $\mathfrak{p}[T] \subseteq \mathfrak{q}$. By (1.15) and (1.12), $A[T]/\mathfrak{q}$ is a G-domain and $A[T]/\mathfrak{p}[T] \cong (A/\mathfrak{p})[T]$ is not a G-domain. It follows that $\mathfrak{p}[T] \subsetneq \mathfrak{q}$. Consider now the extension of integral domains $D := A/\mathfrak{p} \subseteq E := A[T]/\mathfrak{q}$ (we can identify D as a subring of E via the ring embedding $a + \mathfrak{p} \mapsto a + \mathfrak{q}$, for any $a \in A$). If $t := T + \mathfrak{q}$ is the class of T modulo the ideal $\mathfrak{q}$, then it is immediately seen that $E = A[t]$. In view of (1.23) and (1.15), it is enough to show that t is algebraic over A. Since $\mathfrak{p}[T] \subsetneq \mathfrak{q}$, we can pick a polynomial $f \in \mathfrak{q} - \mathfrak{p}[T]$. Then, the canonical image $\overline{f} \in D[T] \cong A[T]/\mathfrak{p}[T]$ of f is nonzero. We immediately get $\overline{f}(t) = 0$. The proof is now complete. □

(1.25) Lemma. *Let A be a ring and let $\mathfrak{p}$ be a G-ideal of A. Then, there exists a maximal ideal $\mathfrak{m}$ of the polynomial ring $A[T]$ such that $\mathfrak{m} \cap A = \mathfrak{p}$.*

Proof. By definition, the integral domain $D := A/\mathfrak{p}$ is a G-domain and thus, by (1.16), the quotient field of of D is $D[d^{-1}]$, for some $d \in D$, $d \neq 0$. It follows that the kernel $\mathfrak{n}$ of the canonical surjective ring homomorphism $\pi : D[T] \longrightarrow D[d^{-1}]$, $f(T) \mapsto f(d^{-1})$ is a maximal ideal of $D[T]$. Since the canonical ring homomorphism $\varphi : A[T] \longrightarrow D[T]$ is surjective, we infer that $\mathfrak{m} := \varphi^{-1}(\mathfrak{n})$ is a maximal ideal of $A[T]$ and clearly $\mathfrak{m} \supseteq \mathfrak{p}[T]$. It follows $\mathfrak{m} \cap A \supseteq \mathfrak{p}[T] \cap A = \mathfrak{p}$. Take now an element $x \in \mathfrak{m} \cap A$. By definition, $\varphi(x) := x + \mathfrak{p} \in D \cap \mathfrak{n} = (0)$, that is, $x \in \mathfrak{p}$. □

(1.26) Theorem. *Let A be a ring and let $\mathfrak{p} \in \operatorname{Spec}(A)$. Then, the following conditions are equivalent.*

(i) *$\mathfrak{p}$ is a G-ideal of A.*
(ii) *There exists a maximal ideal of the polynomial ring $A[T]$ such that $\mathfrak{m} \cap A = \mathfrak{p}$.*
(iii) *There exists a G-ideal $\mathfrak{q}$ of the polynomial ring $A[T]$ such that $\mathfrak{q} \cap A = \mathfrak{p}$.*

Proof. Apply (1.24) and (1.25). □

Exercise. Let $f : A \longrightarrow B$ be a surjective ring homomorphism.

(1) If A is a Hilbert ring, then B is a Hilbert ring.
(2) If $\mathfrak{q}$ is a prime ideal of B such that $f^{-1}(\mathfrak{q})$ is a G-ideal of A, then $\mathfrak{q}$ is a G-ideal of B.

(1.27) Proposition. *Let A be a ring and T be an indeterminate over A. Then A is a Hilbert ring if and only if $A[T]$ is a Hilbert ring. Furthermore, if A is a Hilbert* *ring and* $\mathfrak{q}$ *is any maximal ideal of* $A[T]$ *and* $\mathfrak{m} := A \cap \mathfrak{q}$*, then* $\mathfrak{q}$ *is generated by* $\mathfrak{m}$ *and by a polynomial* $f \in A[T]$ *whose canonical image in* $(A/\mathfrak{m})[T]$ *is irreducible.*

Proof. By the first part of the previous exercise it follows that if $A[T]$ is a Hilbert ring, then A is a Hilbert ring. Thus, assume that A is a Hilbert ring and take any G-ideal $\mathfrak{q}$ of $A[T]$. We have to show that $\mathfrak{q}$ is a maximal ideal of $A[T]$. By (1.24) and the fact that A is a Hilbert ring, $\mathfrak{m} := \mathfrak{q} \cap A$ is a maximal ideal of A. Consider the canonical ring homomorphism $\pi : A[T] \longrightarrow K[T]$, where $K := A/\mathfrak{m}$, and note that $\mathfrak{q} \supseteq \mathfrak{m}[T] = \mathrm{Ker}(\pi)$. Then, there exists a unique prime ideal $\overline{\mathfrak{q}}$ of $K[T]$ such that $\pi^{-1}(\overline{\mathfrak{q}}) = \mathfrak{q}$. By the second part of the previous exercise and the fact that $K[T]$ is a Hilbert ring (see (1.20)), $\overline{\mathfrak{q}}$ is a maximal ideal of $K[T]$, that is, $\overline{\mathfrak{q}} = \overline{f}K[T]$, where $f \in A[T]$ and the canonical image $\overline{f}$ in $K[T]$ is irreducible. It follows that $\mathfrak{q}$ is a maximal ideal of $A[T]$ (being it the inverse image of a maximal ideal, namely $\overline{\mathfrak{q}}$, under a surjective ring homomorphism) and moreover $\mathfrak{q}$ is generated by $\mathfrak{m}$ and f. The proof is now complete. □

(1.28) Corollary. *Let* K *be a field and* $T_1, \ldots, T_n$ *be indeterminates over* K*. Then* $K[T_1, \ldots, T_n]$ *is a Hilbert ring.*

Proof. It is enough to apply (1.27) and induction, keeping in mind that any field is a Hilbert ring. □

We can now prove Hilbert's Nullstellensatz (precisely, the equivalent form (ii) of it given in (1.9)).

(1.29) Theorem. *Let* K *be an algebraically closed field. Then*

$$\mathrm{Max}(K[T_1, \ldots, T_n]) = \{(T_1 - a_1, \ldots, T_n - a_n)K[T_1, \ldots, T_n] : a_1, \ldots, a_n \in K\}.$$

Proof. The inclusion $\supseteq$ is an easy exercise. We prove the converse inclusion by induction of the number n of the indeterminates over K. If $n = 1$, the maximal ideals of the PID $K[T]$ are principal generated by the monic irreducible polynomials over K and, since K is algebraically closed, such polynomials are linear. Suppose $n \geq 1$ and assume that

$$\mathrm{Max}(K[T_1, \ldots, T_n]) = \{(T_1 - a_1, \ldots, T_n - a_n)K[T_1, \ldots, T_n] : a_1, \ldots, a_n \in K\}.$$

Let $\mathfrak{m}$ be a maximal ideal of the polynomial ring $K[T_1, \ldots, T_n, T]$. Since $A := K[T_1, \ldots, T_n]$ is a Hilbert domain, then $\mathfrak{m} \cap A$ is a maximal ideal of A, by (1.24). By inductive assumption, $\mathfrak{m} \cap A = (T_1 - a_1, \ldots, T_n - a_n)A$, for some $a_1, \ldots, a_n \in K$. Furthermore, by (1.27), $\mathfrak{m}$ is generated by $\mathfrak{m} \cap A$ and by a polynomial $f \in A[T]$ whose canonical image in $(A/(\mathfrak{m} \cap A))[T]$ is irreducible. Since clearly $A/(\mathfrak{m} \cap A) \cong K$ is algebraically closed, a suitable polynomial f can be choosen of the type $f := T - a$, for some $a \in K$. The proof is now complete. □

(1.30) Remark. Let K be an algebraically closed field. From the previous version of Hilbert's Nullstellensatz it immediately follows that the map

$$\varphi : \mathbb{A}^n_K \longrightarrow \mathrm{Max}(K[T_1, \ldots, T_n])$$

$$(a_1, \ldots, a_n) \mapsto (T_1 - a_1, \ldots, T_n - a_n)K[T_1, \ldots, T_n]$$

is a bijection. We will see soon that this map is a homeomorphism of topological spaces too. To do this we need to define natural topologies on $\mathbb{A}^n_K$ and on $\mathrm{Max}(K[T_1, \ldots, T_n])$. This motivates the next section.

2. The Zariski topology on the affine space $\mathbb{A}^n_K$

As in the previous section, K is a field and $T_1, \ldots, T_n$ are indeterminates over K. Through these notes, we assume the usual convention that any topological space is nonempty. The closure of a subset Y of a topological space X will be denoted, as usual, by $\overline{Y}$.

(2.1) Proposition. *The algebraic subsets of* $\mathbb{A}^n_K$ *are the closed sets for a topology on* $\mathbb{A}^n_K$. *This topology is called* the Zariski topology.

Proof. Keeping in mind (1.2b), is sufficient to note that

(1) $Z(\{0\}) = \mathbb{A}^n_K$, $Z(\{1\}) = \emptyset$;
(2) $Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$, for any pair of ideals $\mathfrak{a}, \mathfrak{b}$ of $K[T_1, \ldots, T_n]$;
(3) $Z(\bigcup_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} Z(\mathfrak{a}_i)$, for any collection of ideals $\{\mathfrak{a}_i : i \in I\}$ of $K[T_1, \ldots, T_n]$.

Statement (1) is trivial. In view of (1.2a), it is enough to show that $Z(\mathfrak{a} \cap b) \subseteq Z(\mathfrak{a}) \cap Z(\mathfrak{b})$ and that $\bigcap_{i \in I} Z(\mathfrak{a}_i) \subseteq Z(\bigcup_{i \in I} \mathfrak{a}_i)$.

Take an element $p \in Z(\mathfrak{a} \cap \mathfrak{b})$. If $p \notin Z(\mathfrak{a}) \cup Z(\mathfrak{b})$, there are polynomials $f \in \mathfrak{a}, g \in \mathfrak{b}$ such that $f(p), g(p) \neq 0$. Clearly, the polynomial $fg \in \mathfrak{a} \cap \mathfrak{b}$ and $fg(p) \neq 0$, agaist the fact that $p \in Z(\mathfrak{a} \cap \mathfrak{b})$.

Take an element $p \in \bigcap_{i \in I} Z(\mathfrak{a}_i)$, and let $f \in \bigcup_{i \in I} \mathfrak{a}_i$. Then, $f \in \mathfrak{a}_j$, for some $j \in I$ and since, in particular, $p \in Z(\mathfrak{a}_j)$, we have $f(p) = 0$. □

(2.2) Remark. Let K be a field.

(a) $\mathbb{A}^n_K$ is a T_1 space (that is, any singleton is a closed set). As a matter of fact, for any $p = (a_1, \ldots, a_n) \in \mathbb{A}^n_K$, we have $\{p\} = Z(\{T_1 - a_1, \ldots, T_n - a_n\})$.
(b) $\mathbb{A}^1_K$ has the cofinite topology. Indeed, let C be a proper closed subset of $\mathbb{A}^1_K$. Then, keeping in mind that $K[T]$ is a PID, there is a nonzero polynomial $f \in K[T]$ such that $C = Z(fK[T]) = Z(\{f\})$. Thus C is finite since any nonzero polynomial in one indeterminate over a field has finitely many roots.
(c) The collection of the open sets of $\mathbb{A}^n_K$ of the form $D_f := \mathbb{A}^n_K - Z(\{f\})$, for any $f \in K[T_1, \ldots, T_n]$, is a basis of the Zariski topology. As a matter of fact, let $\Omega \subseteq \mathbb{A}^n_K$ be an open set and let $p \in \Omega$. Since $\Omega = \mathbb{A}^n_K - V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of $K[T_1, \ldots, T_n]$, there is a polynomial $f \in \mathfrak{a}$ such that $f(p) \neq 0$. It follows $p \in D_f \subseteq \Omega$.

(2.3) Definition. *A topological space is called to be* Noetherian *if any countable ascending chain of open sets stabilizes (or, equivalently, if any countable descending chain of closed sets stabilizes).*

(2.4) Remark. Let X be a topological space.

(a) Then is Noetherian if and only if any nonempty collection of closed subsets of X has a minimal element under inclusion.
(b) If X is Noetherian, then any subspace of X is Noetherian.

The easy proof is left as an exercise.

(2.5) Proposition. *Let K be a field. Then $\mathbb{A}^n_K$ is Noetherian, endowed with the Zariski topology.*

Proof. Let

$$C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$$

be a descending chain of closed subsets of $\mathbb{A}^n_K$. Clearly we get the ascending chain of radical ideals

$$I(C_1) \subseteq I(C_2) \subseteq \ldots \subseteq I(C_n) \subseteq \ldots$$

of the Noetherian ring $K[T_1, \ldots, T_n]$. Thus this chain stabilizes and thus there is a positive integer i such that $I(C_m) = I(C_i)$, for any $m \geq i$. Finally, by (1.5b) we have $C_m = C_i$, for any $m \geq i$. □

(2.6) Proposition. *Any Noetherian space is compact.*

Proof. Assume that X is a Noetherian space and let $\mathcal{F}$ be a nonempty collection of closed subsets of X with the finite intersection property. Let Σ denote the collection of all the intersections of finite subfamilies of $\mathcal{F}$. By assumption, Σ consists of nonempty sets. By (2.4a), Σ has a minimal element, say C_0, and, since Σ is closed under finite intersections, C_0 is the minimum of Σ. It follows immediately that $\bigcap\{C : C \in \mathcal{F}\} = C_0 \neq \emptyset$. □

(2.7) Definition. *A topological space X is called to be* irreducible *if it not a finite union of closed proper subspaces of X (in other words, two nonempty open subsets of X have nonempty intersection, that is, any nonempty open subset of X is dense in X).*

(2.8) Example. If K is a finite field, then $\mathbb{A}^n_K$ is reducible. Indeed it can be realized as finite union of singletons and each of them is closed, by (2.2).

We give now a criterion to decide when an algebraic set is irreducible (with the subspace Zariski topology).

(2.9) Proposition. *Let X be a closed subset of $\mathbb{A}^n_K$. Then X is irreducible if and only if the ideal $I(X)$ of X is a prime ideal of $K[T_1, \ldots, T_n]$.*

Proof. Assume that X is reducible and take closed proper subsets X_1, X_2 of X such that $X_1 \cup X_2 = X$. Keeping in mind (1.5b), for each $i = 1, 2$, we have $I(X) \subsetneq I(X_i)$, and thus there is an element $f_i \in I(X_i) - I(X)$. It immediately follows that the polynomial $f := f_1 f_2 \in I(X)$, that is, $I(X)$ is not a prime ideal.

Conversely, suppose that $I(X)$ is not a prime ideal, and take polynomials $f, g \in K[T_1, \ldots, T_n] - I(X)$ such that $fg \in I(X)$, and take points $p, q \in X$ such that $f(p), g(q) \neq 0$. Consider the closed subspaces $C := Z(\{f\}), D := Z(\{g\})$ of $\mathbb{A}^n_K$. It immediately follows that $X = (C \cap X) \cup (D \cap X)$ and that $C \cap X, D \cap X$ are closed proper subsets of X (since $p \in X - C, q \in X - D$). Thus X is reducible. □

(2.10) Corollary. *Let K be an algebraically closed field. Then, the map ψ defined in (1.8) restricts to a order reversing bijection*

$$\{\textit{Irreducible closed subsets of } \mathbb{A}^n_K\} \longrightarrow \operatorname{Spec}(K[T_1, \ldots, T_n])$$

Proof. Apply (1.8) and (2.9). □

(2.11) Lemma. *Let K be an infinite field and let $f \in K[T_1, \ldots, T_n]$ be a nonzero polynomial. Then the set $\mathbb{A}^n_K - Z(\{f\})$ is infinite.*

Proof. We can assume that f is not a constant and show the statement by induction on n. The case $n = 1$ is trivial, because a nonconstant polynomial in one variable over a field has only finitely many roots. Suppose now that $n \geq 1$ and that $\mathbb{A}^n_K - Z(\{f\})$ is infinite, for any nonconstant polynomial $f \in K[T_1, \ldots, T_n]$. Take a nonconstant polynomial $g \in K[T_1, \ldots, T_n, T]$. We can clearly assume that T appears in the expression of g, and thus we can write $g = \phi_0 + \phi_1 T + \ldots + \phi_r T^r$, where r is a positive integer, $\phi_i \in K[T_1, \ldots, T_n]$, for $1 \leq i \leq r$ and $\phi_r \neq 0$. By the inductive assumption, there are infinitely many points $p \in \mathbb{A}^n_K$ such that $\phi_r(p) \neq 0$, thus let $(a_1, \ldots, a_n)$ be one of them. Then

$$g'(T) := \phi_0(a_1, \ldots, a_n) + \phi_1(a_1, \ldots, a_n)T + \ldots + \phi_r(a_1, \ldots, a_n)T^r$$

is a nonzero polynomial in one variable. Since K is infinite, there are infinitely many elements $\alpha \in K$ such that $g'(\alpha) \neq 0$, and then $\mathbb{A}^{n+1}_K - Z(\{g\})$ is infinite, containing it $\{(a_1, \ldots, a_n, \alpha) : g'(\alpha) \neq 0\}$. □

(2.12) Example. Let K be a infinite field. By (2.11), we have $I(\mathbb{A}^n_K) = \{0\}$. Thus, by (2.9), $\mathbb{A}^n_K$ is irreducible.

(2.13) Definition. *Let X be a topological space. An irreducible subset of X which is maximal under inclusion is called* an irreducible component of X.

(2.14) Proposition. *Let X be a topological space. Then, X is union of its irreducible components. In particular, X has irreducible components.*

Proof. Take a point $x \in X$. It suffices to show that there exists an irreducible component of X containing x. Consider the collection

$$\Sigma := \{Y \subseteq X : x \in Y, Y \text{ is irreducible}\}$$

of subsets of X, partially orederd by inclusion, and note that Σ is nonempty, because $\{x\}$ is irreducible. Let $\mathcal{C} \subseteq \Sigma$ be a chain and let C be the union of the sets in the chain $\mathcal{C}$. We claim that C is irreducible. Suppose this is false, and take closed subsets $F, G \subseteq X$ such that $C = (F \cap C) \cup (G \cap C)$ and $F \cap C, G \cap C \subsetneq C$. Take points $x_F, x_G \in C$ such that $x_F \notin F, x_G \notin G$, and let $C_F, C_G \in \mathcal{C}$ be such that $x_F \in C_F, x_G \in C_G$. Since $\mathcal{C}$ is a chain we can assume that $C_F \subseteq C_G$. From the obvious inclusions $C_G \subseteq C \subseteq F \cup G$ it follows that $C_G = (F \cap C_G) \cup (G \cap C_G)$ and, moreover, $F \cap C_G, G \cap C_G$ are closed proper subspaces of C_G, a contradiction, because $C_G \in \mathcal{C} \subseteq \Sigma$ implies that C_G is irreducible. Then Zorn's Lemma implies that Σ has maximal elements and, clearly, any maximal element is an irreducible component of X containing x. □

(2.15) Proposition. *Let X be a topological space.*

(a) *If a subset Y of X is irreducible, then $\overline{Y}$ is irreducible too.*
(b) *The irreducible components of X are closed.*
(c) *If X is irreducible and $f : X \longrightarrow S$ is a continuous surjective function of topological spaces, then S is irreducible.*

Proof. (a). Since $\overline{Y}$ is closed in X, then the closed subspaces of $\overline{Y}$ are closed in X too. Let C_1, C_2 be closed subsets of X such that $\overline{Y} = C_1 \cup C_2$ and note that $Y = (C_1 \cap Y) \cup (C_2 \cap Y)$. The subsets $C_i \cap Y$, $i = 1, 2$, are closed subspaces of Y (in the subspace topology), and, since Y is irreducible, we can assume $Y = C_1 \cap Y$, that is $Y \subseteq C_1$. Since C_1 is closed in X we have $\overline{Y} \subseteq C_1$, i.e., $\overline{Y} = C_1$.

(b). It is a trivial consequence of (a).

(c). The proof is straightforward and it is left to the reader. □

The proof of the following lemma is left to the reader.

(2.16) Lemma. *Let X be a topological space and let $\mathcal{F}$ be a finite collection of closed irreducible subspaces of X such that $\bigcup \mathcal{F} = X$. Then, any irreducible component of X belongs to $\mathcal{F}$. In particular, X has only finitely many irreducible components. Furthermore, if the members of $\mathcal{F}$ are pairwise incomparable, then $\mathcal{F}$ is the family of all irreducible components of X.*

(2.17) Proposition. *Let X be a Noetherian space. Then X has only finitely many irreducible components.*

Proof. Suppose that the statement is false. Then, the collection Σ of all the closed subspaces of X having infinitely many irreducible components is nonempty and, since X is Noetherian, Σ has a minimal element, say T. In particular, T is reducible, and thus there are closed proper subsets of T, say C, D such that $T = C \cup D$ (note that, since T is closed in X, then C, D are closed in X too). By minimality we have $C, D \notin \Sigma$, and thus they have only finitely many irreducible components. Let $E_1, \ldots, E_n$ (resp., $E_{n+1}, \ldots, E_m$) be the irreducible components of C (resp., D). Then, T is the union of the finite collection $\mathcal{F} := \{E_i : 1 \leq i \leq m\}$ of closed irreducible subspaces, and thus (2.16) implies that T has only finitely many irreducible components, a contradiction, because $T \in \Sigma$. The proof is now complete. □

(2.18) Remark. Let K be an algebraically closed field, and let $\mathfrak{p}$ be a prime ideal of $K[T_1, \ldots, T_n]$. Then $Z(\mathfrak{p})$ is irreducible. Indeed, by Hilbert's Nullstellensatz, we have $I(Z(\mathfrak{p})) = \sqrt{\mathfrak{p}} = \mathfrak{p}$, and thus the conclusion follows from (2.9). As a particular case, if $f \in K[T_1, \ldots, T_n]$ is an irreducible polynomial, then the ideal $\mathfrak{p}$ generated by f is prime, and thus $Z(\{f\})$ is irreducible.

The conclusions of the previous remark may fail when K is not algebraically closed, as the following example shows.

(2.19) Example. Let K be a finite field, consider the irreducible polynomial $f := T \in K[T, U]$ and set $C := Z(\{f\})$. Then clerly $C = \{(0, a) : a \in K\}$ is a finite union of singletons, and thus it is reducible.

(2.20) Example. Let K be an algebraically closed field and let $f \in K[T_1, \ldots, T_n]$. Suppose that $f = c f_1^{n_1} \cdots f_r^{n_r}$, where $f_i^{n_i} \in K[T_1, \ldots, T_n]$ is irreducible, for $1 \leq i \leq r$, $c \in K$, and f_i is not associate with f_j, for any $i \neq j$. Then

$$Z(\{f\}) = Z(\{f_1\}) \cup \ldots \cup Z(\{f_r\})$$

and, by (2.18), each set $Z(\{f_i\})$ is irreducible. The fact that f_i is not associate with f_j, for $i \neq j$, easily implies that the sets $Z(\{f_i\})$ are pairwise incomparable. Thus, by (2.16), $\{Z(\{f_j\}) : 1 \leq i \leq r\}$ is the collection of all the irreducible components of $Z(\{f\})$.

In the general case, i.e., when a closed set X is not the zero set of a unique polynomial, we need further properties of Noetherian rings to find the irreducible components of X.

(2.21) Definition. *Let A be a ring and let $\mathfrak{a}$ be a proper ideal of A. We say that $\mathfrak{a}$ is* irreducible *if, given ideals $\mathfrak{b}, \mathfrak{c}$ of A such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$, we have necessarily either $\mathfrak{a} = \mathfrak{b}$ or $\mathfrak{a} = \mathfrak{c}$.*

Clearly, any prime ideal is irreducible.

(2.22) Proposition. *Let A be a Noetherian ring. Then, any proper ideal of A is a finite intersection of irreducible ideals of A.*

Proof. If the statement is false, the set Σ of all the ideals of A which are not a finite intersection of irreducible ideals of A is nonempty. Since A is Noetherian, Σ has a maximal element, say $\mathfrak{a}$. Of course, $\mathfrak{a}$ is reducible, and thus there are ideals $\mathfrak{b}, \mathfrak{c} \supsetneq \mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{b} \cap \mathfrak{c}$. By maximality of $\mathfrak{a}$, we have $\mathfrak{b}, \mathfrak{c} \notin \Sigma$, and thus each of them is a finite intersection of irreducible ideals of A and, a fortiori, so is $\mathfrak{a}$. This is a contradiction. □

(2.23) Definition. *Let A be a ring and let $\mathfrak{a}$ be a proper ideal of A. We say that $\mathfrak{a}$ is* primary *if, given elements $x, y \in A$ such that $xy \in \mathfrak{a}$, then either $x \in \mathfrak{a}$ or $y \in \sqrt{\mathfrak{a}}$.*

Clearly, any prime ideal is primary.
The following remark is easy and we left it to the reader.

(2.24) Remark. Let A be a ring.

(a) If $\mathfrak{a}$ is a primary ideal of A, then $\mathfrak{p} := \sqrt{\mathfrak{a}}$ is a prime ideal of A. We will say that $\mathfrak{a}$ is *a $\mathfrak{p}$-primary ideal.*
(b) If $\mathfrak{a}$ is an ideal of A whose radical is a maximal ideal, then $\mathfrak{a}$ is primary.
(c) If $\mathfrak{a}, \mathfrak{b}$ are $\mathfrak{p}$-primary ideals, then $\mathfrak{a} \cap \mathfrak{b}$ is a $\mathfrak{p}$-primary ideal.
(d) If $\mathfrak{a}$ is a $\mathfrak{p}$-primary ideal and $x \in A - \mathfrak{a}$, then

$$(\mathfrak{a} :_A x) := \{a \in A : ax \in \mathfrak{a}\}$$

is a $\mathfrak{p}$-primary ideal of A.

(2.25) Definition. *Let A be a ring and let $\mathfrak{a}$ be a proper ideal of A.*

(a) A primary decomposition *of $\mathfrak{a}$ is a finite collection of primary ideals whose intersection is $\mathfrak{a}$. We say that $\mathfrak{a}$* is decomposable *if $\mathfrak{a}$ admits a primary decomposition.*
(b) *Let $\mathcal{P} := \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$ be a primary decomposition of $\mathfrak{a}$. We say that $\mathcal{P}$ is irredundant if, for any $1 \leq i \leq n$, we have $\sqrt{\mathfrak{q}_j} \neq \sqrt{\mathfrak{q}_i}$, for any $j \neq i$, and $\mathfrak{q}_i \not\supseteq \bigcap_{j \neq i} \mathfrak{q}_j$.*

Keeping in mind (2.24c) and the fact that we can omit any redundant term, we infer that any primary decomposition of an ideal can be refined to an irredundant primary decomposition.

(2.26) Proposition. *An irreducible ideal of a Noetherian ring is primary. In particular, any proper ideal of a Noetherian ring is decomposable.*

Proof. Let $\mathfrak{a}$ be an irreducible ideal of a Noetherian ring A, and take elements $x, y \in A$ such that $xy \in \mathfrak{a}$. For any positive integer n, consider the ideal

$$\mathfrak{a}_n := \{a \in A : ax^n \in \mathfrak{a}\}$$

of A, and note that $\mathfrak{a}_n \subseteq \mathfrak{a}_m$, for any $n \leq m$. Since A is Noetherian, there exists a positive integer ν such that $\mathfrak{a}_\nu = \mathfrak{a}_m$, for any $m \geq \nu$. Take any element $\lambda \in (x^\nu A + \mathfrak{a}) \cap (yA + \mathfrak{a})$. Then, there are elements $\alpha, \beta \in A, i, j \in \mathfrak{a}$ such that

$$\lambda = x^\nu \alpha + i = y\beta + j.$$

The previous equality implies that $x^{\nu+1}\alpha = -ix + xy\beta + jx$ and, keeping in mind that $xy \in \mathfrak{a}$, we get $x^{\nu+1}\alpha \in \mathfrak{a}$, that is, $\alpha \in \mathfrak{a}_{\nu+1} = \mathfrak{a}_\nu$. It follows that $x^\nu\alpha \in \mathfrak{a}$ and thus $\lambda = x^\nu + i \in \mathfrak{a}$. This proves that

$$\mathfrak{a} = (x^\nu A + \mathfrak{a}) \cap (yA + \mathfrak{a}),$$

and since $\mathfrak{a}$ is irreducible we have either $\mathfrak{a} = x^\nu A + \mathfrak{a}$ or $\mathfrak{a} = yA + \mathfrak{a}$, that is, either $x^\nu \in \mathfrak{a}$ or $y \in A$. This shows that $\mathfrak{a}$ is primary.

The last statement follows from the first one and (2.22). □

(2.27) Example. Let K be an algebraically closed field and let C be a closed subset of $\mathbb{A}^n_K$. We can assume that $C = Z(\mathfrak{a})$ for some ideal $\mathfrak{a}$ of $K[T_1, \ldots, T_n]$. Since $K[T_1, \ldots, T_n]$ is a Noetherian ring, the ideal $\mathfrak{a}$ has a primary decomposition, say $\mathfrak{a} = \bigcap_{i=1}^r \mathfrak{q}_i$. Thus, keeping in mind (1.2c) and the proof of (2.1), we have

$$C = Z(\mathfrak{q}_1) \cup \ldots \cup Z(\mathfrak{q}_r) = Z(\sqrt{\mathfrak{q}_1}) \cup \ldots \cup Z(\sqrt{\mathfrak{q}_r}).$$

By (2.18) and (2.24a), any $Z(\sqrt{\mathfrak{q}_i})$ is a closed irreducible subspace of C. Thus, by (2.16), any irreducible component of C belongs to the family $\{Z(\sqrt{\mathfrak{q}_i}) : 1 \le i \le n\}$.

(2.28) Example. An ideal can have distinct irredundant primary decompositions. For example, let K be a field let T, U be indeterminates over K, and consider the ideal $\mathfrak{a} := (T^2, TU)K[T, U]$. If

$$\mathfrak{p} := TK[T, U], \mathfrak{m} := (T, U)K[T, U], \mathfrak{q} := (T^2, U)K[T, U],$$

then $\{\mathfrak{p}, \mathfrak{m}^2\}, \{\mathfrak{p}, \mathfrak{q}\}$ are distinct irredundant primary decompositions of $\mathfrak{a}$. The proof is left to the reader.

Let A be a ring and let $\mathfrak{a}$ be an ideal of A which admists an irredundant primary decomposition, say $\mathcal{P} := \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$. We show now that the set $\{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_n}\}$ depends only on the ideal $\mathfrak{a}$ and not on $\mathcal{P}$. Recall the following easy and well-known fact.

(2.29) Proposition. *Let A be a ring, let $\mathcal{F}$ be a finite collection of ideals of A and let $\mathfrak{p}$ be a prime ideal of A.*

(a) *If $\mathfrak{p} \supseteq \bigcap \mathcal{F}$, then $\mathfrak{p} \supseteq \mathfrak{a}$, for some $\mathfrak{a} \in \mathcal{F}$*
(b) *If $\mathfrak{p} = \bigcap \mathcal{F}$, then $\mathfrak{p} = \mathfrak{a}$, for some $\mathfrak{a} \in \mathcal{F}$.*

(2.30) Proposition. *Let A be a ring, let $\mathfrak{a}$ be an ideal of A which admits an irredundant primary decomposition $\mathcal{P} := \{\mathfrak{q}_1, \ldots, \mathfrak{q}_n\}$. Then*

$$\{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_n}\} = \mathrm{Spec}(A) \cap \{\sqrt{(\mathfrak{a} :_A x)} : x \in A\}.$$

Thus, the set of prime ideals $\mathrm{Ass}(\mathfrak{a}) := \{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_n}\}$ *is uniquely determined by $\mathfrak{a}$, and it is called the set of* prime ideals associated with $\mathfrak{a}$.

Proof. Set $\mathfrak{p}_i := \sqrt{\mathfrak{q}_i}$, for $i = 1, \ldots, n$, and note that, for any $x \in A$, we have $(\mathfrak{a} :_A x) = \bigcap_{i=1}^{n} (\mathfrak{q}_i :_A x)$, since $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$. Taking radicals and keeping in mind (2.24) and that, if $x \in \mathfrak{q}_i$, for some i, then $(\mathfrak{q}_i :_A x) = A$, we have $\sqrt{(\mathfrak{a} :_A x)} = \bigcap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j$. Then, the inclusion $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \supseteq \mathrm{Spec}(A) \cap \{\sqrt{(\mathfrak{a} :_A x)} : x \in A\}$ follows immediately from the previous proposition. Conversely, take a prime ideal $\mathfrak{p}_i$; since $\mathcal{P}$ is irredundant, then $\bigcap_{j\neq i} \mathfrak{q}_j \not\subseteq \mathfrak{q}_i$, and thus there is an element $x \in \mathfrak{q}_j$, for any $j \neq i$ and $x \notin \mathfrak{q}_i$. It follows $\sqrt{(\mathfrak{a} :_A x)} = \bigcap_{x \notin \mathfrak{q}_j} \mathfrak{p}_j = \mathfrak{p}_i$. □

(2.31) Definition. *Let A be a ring and let $\mathfrak{a}$ be a proper ideal of A. The minimal elements, under inclusion, of the set $\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq \mathfrak{a}\}$ are called* minimal prime ideals over $\mathfrak{a}$.

A straightforward application of Zorn's Lemma implies that there are minimal prime ideals over any proper ideal of a ring. In particular, there are minimal prime ideals (i.e., the minimal prime ideals over (0)).

(2.32) Proposition. *Let A be a ring and let $\mathfrak{a}$ be a decomposable ideal of A. Then the minimal prime ideals of A over $\mathfrak{a}$ are precisely the minimal elements, under inclusion, of* $\mathrm{Ass}(\mathfrak{a})$*. In particular, there are only finitely many prime ideals over $\mathfrak{a}$.*

Proof. Clearly, any element of $\mathrm{Ass}(\mathfrak{a}) =: \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is a prime ideal of A containing $\mathfrak{a}$. Let $\mathfrak{p}$ be a minimal prime ideal of A over $\mathfrak{a}$. Then $\mathfrak{p} \supseteq \sqrt{\mathfrak{a}} = \bigcap_{i=1}^{n} \mathfrak{p}_i$. By (2.29), we have $\mathfrak{p} \supseteq \mathfrak{p}_i$, for some $i \in \{1, \dots, n\}$, and thus $\mathfrak{p} = \mathfrak{p}_i$, by minimality. The conclusion is now clear. □

(2.33) Remark. Preserve the notation of (2.27). By (2.10), the minimal elements of $\mathrm{Ass}(\mathfrak{a}) = \{\sqrt{\mathfrak{q}_i} : 1 \leq i \leq r\}$ (i.e., the minimal prime ideals over $\mathfrak{a}$, by (2.32)) correspond to the maximal elements of $\{Z(\sqrt{\mathfrak{q}_i}) : 1 \leq i \leq r\}$, that is, to the irreducible components of $Z(\mathfrak{a})$.

3. The Zariski topology on the prime spectrum of a ring.

Let A be a ring and let S be a subset of A. Define

$$V_A(S) := V(S) = \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \supseteq S\}.$$

With a small abuse of notation, for any $f \in A$ we will write $V(f)$ to mean $V(\{f\})$.

(3.1) Proposition. *Let A be a ring. Then, the subsets of* $\mathrm{Spec}(A)$ *of the form $V(S)$, for any $S \subseteq A$, form the family of the closed sets for a topology, called* the Zariski topology*. Precisely, we have*

(a) $V(S) = V(SA) = V(\sqrt{SA})$*, for any $S \subseteq A$ (where SA denotes the ideal of A generated by S).*
(b) $V(\{1\}) = \emptyset$ *and* $V(\{0\}) = \mathrm{Spec}(A)$.
(c) $V(\bigcup_{i\in I} S_i) = \bigcap_{i\in I} V(S_i)$*, for any collection $\{S_i : i \in I\}$ of subsets of A.*
(d) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$*, for any ideals $\mathfrak{a}, \mathfrak{b}$ of A.*

The proof of the previous equalities is easy and it is left to the reader.

(3.2) Remark. Let A be a ring.

(a) By (3.1a), any closed subset of $\mathrm{Spec}(A)$ is of the form $V(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of A.
(b) Moreover, again by (3.1a), the canonical map

$$\varphi : \{\text{radical ideals of } A\} \longrightarrow \{\text{closed subsets of } \mathrm{Spec}(A)\}$$

defined by setting $\varphi(\mathfrak{a}) := V(\mathfrak{a})$ is a order reversing bijection.

(c) For any nonempty subset Y of $\mathrm{Spec}(A)$, we have $\overline{Y} = V(\bigcap Y)$. As a matter of fact, for any $\mathfrak{p} \in Y$ we have obviously $\mathfrak{p} \subseteq \bigcap Y$, that is $\mathfrak{p} \in V(\bigcap Y)$. Then, $Y \subseteq V(\bigcap Y)$ and, since the second set is closed, $\overline{Y} \subseteq V(\bigcap Y)$. If C is any closed subset of $\mathrm{Spec}(A)$ containing Y, say $C = V(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of A, then clearly $\mathfrak{p} \supseteq \mathfrak{a}$, for any $\mathfrak{p} \in Y$, that is $\bigcap Y \supseteq \mathfrak{a}$. It follows $C = V(\mathfrak{a}) \supseteq V(\bigcap Y)$.

(d) By part (c) it follows that, for any $\mathfrak{p} \in \mathrm{Spec}(A)$, then $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. In particular, $\mathfrak{p}$ is a maximal ideal if and only if $\{\mathfrak{p}\}$ is closed.

(e) $\mathrm{Spec}(A)$ is a T_0 space, that is, points of $\mathrm{Spec}(A)$ are uniquely determined by their closure. Indeed, suppose $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ and that $\overline{\{\mathfrak{p}\}} = \overline{\{\mathfrak{q}\}}$. By part (d), this equality is equivalent to $V(\mathfrak{p}) = V(\mathfrak{q})$, that is, $\mathfrak{p} = \mathfrak{q}$.

(f) For any $f \in A$, set $D_A(f) := D(f) := \mathrm{Spec}(A) - V(f)$. Then, the collection of sets $\mathcal{B} := \{D(f) : f \in A\}$ is a basis for the Zariski topology, and any set $D(f)$ is called *a principal open subset of* $\mathrm{Spec}(A)$. To see that $\mathcal{B}$ is a basis, take any nonempty open subset Ω of $\mathrm{Spec}(A)$ and take a point $\mathfrak{p} \in \Omega$. By definition, $\Omega = \mathrm{Spec}(A) - V(\mathfrak{a})$, for some ideal $\mathfrak{a}$ of A, and thus there is an element $f \in \mathfrak{a} - \mathfrak{p}$. It follows immediately that $\mathfrak{p} \in D(f) \subseteq \Omega$.

(g) Any irreducible closed subset of $\mathrm{Spec}(A)$ is of the form $V(\mathfrak{p})$, for some prime ideal $\mathfrak{p}$ of A. Indeed, by part (d) and (2.15), $V(\mathfrak{p})$ is irreducible, for any $\mathfrak{p} \in \mathrm{Spec}(A)$. Conversely, let C be an irreducible closed subset of $\mathrm{Spec}(A)$ and let $\mathfrak{a} := \bigcap C$. By part (c) we have $C = V(\mathfrak{a})$. Then, it suffices to prove that $\mathfrak{a}$ is prime. Take elements $a, b \in A$ such that $ab \in \mathfrak{a}$. Then we have immediately $C = (V(a) \cap C) \cup (V(b) \cap C)$. Since the sets $V(a) \cap C, V(b) \cap C$ are closed and C is irreducible, we have either $C = V(a) \cap C$ or $C = V(b) \cap C$, that is, either $C \subseteq V(a)$ or $C \subseteq V(b)$. In other words, either $a \in \mathfrak{a}$ or $b \in \mathfrak{a}$.

Conversely, for any prime ideal $\mathfrak{p}$ the set $V(\mathfrak{p})$ is irreducible, by part (d) and (2.15a).

(h) By part (g) the irreducible components of $\mathrm{Spec}(A)$ are precisely the sets of the form $V(\mathfrak{p})$, where $\mathfrak{p}$ is any minimal prime ideal of A.

(3.3) Proposition. *Let A be a ring. Then,* $\mathrm{Spec}(A)$ *is compact.*

Proof. Let $\mathcal{A}$ be an open cover of $\mathrm{Spec}(A)$. By (3.2f), we can assume that $\mathcal{A}$ consists of principal open subsets of $\mathrm{Spec}(A)$, say $\mathcal{A} = \{D(f_i) : i \in I\}$. Let $\mathfrak{a}$ be the ideal generated by the set $\{f_i : i \in I\}$. Since $\mathrm{Spec}(A) = \bigcup \mathcal{A}$, for any maximal ideal $\mathfrak{m}$ of A there is some element f_i such that $f_i \notin \mathfrak{m}$, and thus $\mathfrak{a} \nsubseteq \mathfrak{m}$. It follows that $\mathfrak{a} = A$. Then, there are indexes $i_1, \ldots, i_n \in I$ and elements $a_1, \ldots, a_n \in A$ such that $1 = \sum_{j=1}^{n} a_j f_{i_j}$. Since any prime ideal $\mathfrak{p}$ of A is, in particular, a proper ideal, it cannot happen that $f_{i_1}, \ldots, f_{i_n} \in \mathfrak{p}$, and thus $\{D(f_{i_j}) : 1 \leq j \leq n\}$ is a finite subcover of $\mathcal{A}$. □

(3.4) Corollary. *Let A be a ring and $\mathfrak{a}$ be an ideal of A. Then $V(\mathfrak{a})$ is compact.*

Proof. It suffices to recall that any closed subspace of a compact space is compact, and apply (3.3). □

Recall that a continuous map $f : X \longrightarrow Y$ of topological space is called to be a topological embedding if f induces a homeomorphism of X with $f(X)$.

(3.5) Proposition. *Let $f : A \longrightarrow B$ be a ring homomorphism, and consider the map $f^\star : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ defined by $f^\star(\mathfrak{q}) := f^{-1}(\mathfrak{q})$, for any $\mathfrak{q} \in \mathrm{Spec}(B)$.*

(a) *For any $a \in A$ we have ${f^\star}^{-1}(D_A(a)) = D_B(f(a))$. In particular, the map $f^\star$ is continuous and it is called* the canonical continuous function induced by the homomorphism f.
(b) *If f is surjective, then $f^\star$ is a closed topological embedding inducing a homeomorphism of* $\mathrm{Spec}(B)$ *and* $V(\mathrm{Ker}(f))$. *In particular, if $\mathfrak{n}$ is a maximal ideal of B, then $f^\star(\mathfrak{n})$ is a maximal ideal of A. In particular, we have* $f^\star(\mathrm{Max}(B)) = \mathrm{Max}(A) \cap V(\mathrm{Ker}(f))$.
(c) *If S is a multiplicative subset of A, $B := A_S$ is the localization of A at S and f is the canonical map ($a \mapsto \dfrac{a}{1}$, for any $a \in A$), then $f^\star$ is a topological embedding and it induces a homeomorphism of* $\mathrm{Spec}(A_S)$ *with the subspace* $\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}$ *of* $\mathrm{Spec}(A)$.

Proof. (a). The equality ${f^\star}^{-1}(D_A(a)) = D_B(f(a))$ is trivial and thus the last statement follows from (3.2f).

(b). The surjectivity of f implies that $f^\star$ is injective and that the equality $f^\star(\mathrm{Spec}(B)) = V(\mathrm{Ker}(f))$ holds. Furthermore, for any ideal $\mathfrak{b}$ of B, we have $f^\star(V_B(\mathfrak{b})) = V_A(f^{-1}(\mathfrak{b}))$, and this proves that $f^\star$ is closed. For any maximal ideal $\mathfrak{n}$ of B, $\{\mathfrak{n}\}$ is closed in $\mathrm{Spec}(B)$, by (3.2d), and thus $f^\star(\{\mathfrak{n}\}) = \{f^\star(\mathfrak{n})\}$ is a closed point of $\mathrm{Spec}(A)$, since $f^\star$ is closed, that is, $f^\star(\mathfrak{n})$ is a maximal ideal of A.

(c). It is well known that $f^\star$ is injective and that

$$f^\star(\mathrm{Spec}(A_S)) = X := \{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \emptyset\}.$$

Thus, it suffices to show that $f^\star$ is a homeomorphism of $\mathrm{Spec}(A_S)$ with X. Recall that any ideal of A_S is the extension of some ideal of A. Thus, any closed set of $\mathrm{Spec}(A_S)$ is of the form $V_{A_S}(\mathfrak{a}A_S)$ for some ideal $\mathfrak{a}$ of A. Thus the conclusion follows from the equality $f^\star(V_{A_S}(\mathfrak{a}A_S)) = V_A(\mathfrak{a}) \cap X$, whose proof is straightforward. □

(3.6) Remark. Let A be a ring.

(a) In view of (3.5a), for any ideal $\mathfrak{a}$ of A, the closed subspace $V(\mathfrak{a})$ of $\mathrm{Spec}(A)$ is canonically homeomorphic to $\mathrm{Spec}(A/\mathfrak{a})$, via the closed embedding $\pi^\star : \mathrm{Spec}(A/\mathfrak{a}) \longrightarrow \mathrm{Spec}(A)$, where $\pi : A \longrightarrow A/\mathfrak{a}$ is the canonical projection. In view of (2.15c) and (3.2h), the irreducible components of $V(\mathfrak{a})$ are precisely the sets of the form $V(\mathfrak{p})$, where $\mathfrak{p}$ is any minimal prime ideal over $\mathfrak{a}$.
(b) For any $f \in A$, the principal open subset $D(f)$ of $\mathrm{Spec}(A)$ is compact. As a matter of fact, consider the multiplicative subset $S := \{1, f^n : n \geq 1\}$ of A. By (3.5c), $\mathrm{Spec}(A_S)$ is canonically homeomorphic to

$$\{\mathfrak{p} \in \mathrm{Spec}(A) : \mathfrak{p} \cap S = \emptyset\} = D(f).$$

Then, it is sufficient to apply (3.3).
(c) Let $\mathfrak{p}$ be a prime ideal of A and, as usual, let $A_\mathfrak{p}$ denote the localization of A at $A - \mathfrak{p}$. By (3.5c), $\mathrm{Spec}(A_\mathfrak{p})$ is canonically homeomorphic to

$$\{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \cap (A - \mathfrak{p}) = \emptyset\} = \{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \subseteq \mathfrak{p}\}.$$

In particular, $\{\mathfrak{q} \in \mathrm{Spec}(A) : \mathfrak{q} \subseteq \mathfrak{p}\}$ is compact.

The following remark will justify the reason of the name we gave to the topology which $\mathrm{Spec}(A)$ is endowed with. It is strictly related to the Zariski topology on an affine space, as we will explain now.

(3.7) Remark. Let K be an algebraically closed field and let X be a closed subset of $\mathbb{A}^n_K$. If $I(X)$ is the ideal of X, consider the factor ring $\Gamma(X) := K[T_1, \ldots, T_n]/I(X)$, which is called *the coordinate ring of* X. We claim that $\mathrm{Max}(\Gamma(X))$, as a subspace of $\mathrm{Spec}(\Gamma(X))$, is canonically homeomorphic to X. By applying (3.5b) to the canonical projection $\pi : K[T_1, \ldots, T_n] \longrightarrow \Gamma(X)$, we infer that $\mathrm{Max}(\Gamma(X))$ is canonically homeomorphic, via $\pi^\star$, to $\widetilde{X} := \mathrm{Max}(K[T_1, \ldots, T_n]) \cap I(X)$. Now, let $p := (a_1, \ldots, a_n) \in \mathbb{A}^n_K$ and let $\mathfrak{m}_p := (T_1 - a_1, \ldots, T_n - a_n)K[T_1, \ldots, T_n]$ be the maximal ideal corresponding to p. Then clearly $p \in X$ if and only if $\mathfrak{m}_p \supseteq I(X)$. By Hilbert's Nullstellensatz, the map $\varphi : X \longrightarrow \widetilde{X}$, $p \mapsto \mathfrak{m}_p$ is a bijection. We claim that φ is a homeomorphism. This follows from (2.2c), (3.2f), from the straightforward equality $\varphi^{-1}(D(f) \cap \widetilde{X}) = X \cap D_f$, for any polynomial $f \in K[T_1, \ldots, T_n]$, and from the fact that φ is bijective.

Note that, in view of (2.12), if $X = \mathbb{A}^n_K$ then φ is precisely the map defined in (1.30).

(3.8) Proposition. *Let A be a ring and let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then, the following conditions are equivalent.*

(i) *$\mathfrak{p}$ is a minimal prime ideal of A.*
(ii) *For any $x \in \mathfrak{p}$ there is an element $s \in A - \mathfrak{p}$ such that xs is nilpotent.*

Proof. (i)$\Longrightarrow$(ii). If $\mathfrak{p}$ is minimal then, by (3.6c), the only prime ideal of the local ring $A_\mathfrak{p}$ is $\mathfrak{p}A_\mathfrak{p}$. If $x \in \mathfrak{p}$, then the element $\dfrac{x}{1} \in \mathfrak{p}A_\mathfrak{p}$ is nilpotent. Take a positive integer r such that $\dfrac{x^r}{1} = 0$ (in $A_\mathfrak{p}$). By definition, there is an element $s \in A - \mathfrak{p}$ such that $sx^r = 0$. In particular, sx is nilpotent.

(ii)$\Longrightarrow$(i). Take a prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$, and let $x \in \mathfrak{p}$. By assumption, there is an element $s \in A - \mathfrak{p}$ such that sx is nilpotent and, in particular, $sx \in \mathfrak{q}$. Since $s \notin \mathfrak{p}$ we have $s \notin \mathfrak{q}$, and then $x \in \mathfrak{q}$. □

(3.9) Proposition. *Let A be a ring. Then* $\mathrm{Spec}(A)$ *is a T_1 space if and only if it is Hausdorff.*

Proof. First, we note that $\mathrm{Spec}(A)$ is T_1 if and only if any prime ideal of A is maximal (i.e., $\dim(A) = 0$). Indeed, let $\mathfrak{p} \in \mathrm{Spec}(A)$. Then, $\{\mathfrak{p}\}$ is closed if and only if $V(\mathfrak{p}) = \{\mathfrak{p}\}$, in view of (3.2d), and the last equality is equivalent to state that $\mathfrak{p}$ is maximal. Suppose $\mathrm{Spec}(A)$ is a T_1 space and take distinct prime ideals $\mathfrak{p}, \mathfrak{q}$ of A and take an element $x \in \mathfrak{p} - \mathfrak{q}$. Since $\dim(A) = 0$, any prime ideal is both maximal and minimal. By (3.8), there is an element $s \in A - \mathfrak{p}$ such that xs is nilpotent. Then $D(x)$ (resp., $D(s)$) is an open neighborhood of $\mathfrak{q}$ (resp., $\mathfrak{p}$) and $D(x) \cap D(s) = \emptyset$, since xs is nilpotent.

The converse is trivial, because any Hausdorff space is T_1. □

(3.10) Proposition. *If A is a Noetherian ring, then* $\mathrm{Spec}(A)$ *is a Noetherian space.*

Proof. Let $C_1 \supseteq C_2 \supseteq \ldots \supseteq C_n \supseteq \ldots$ be a descending chain of closed subspaces of $\mathrm{Spec}(A)$, say $C_i = V(\mathfrak{a}_i)$, where $\mathfrak{a}_i$ is some ideal of A, for any $i \geq 1$. By (3.2b) we can assume that any $\mathfrak{a}_i$ is a radical ideal. It follows that $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \ldots \subseteq \mathfrak{a}_n \subseteq \ldots$ is an ascending chain of ideals of the Noetherian ring A. Thus, there is a positive integer m such that $\mathfrak{a}_n = \mathfrak{a}_m$, for any $n \geq m$. In other words, $C_n = C_m$, for any $n \geq m$. □

It is not true, in general, that a ring A is Noetherian provided that $\mathrm{Spec}(A)$ is Noetherian.

(3.11) Example. Let K be a field and let $\mathcal{T} := \{T_i : i \in \mathbb{N}\}$ be an infinite and coutable collection of indeterminates over K. Consider the polynomial ring $A := K[\mathcal{T}]$ and its ideal $\mathfrak{a} := (\{T_i^2 : i \in \mathbb{N}\})A$. By (3.6a), $\mathrm{Spec}(A/\mathfrak{a})$ is canonically homeomorphic to $V(\mathfrak{a})$ and clearly $V(\mathfrak{a}) = \{\mathfrak{m}\}$, where $\mathfrak{m}$ is the maximal ideal of A generated by $\mathcal{T}$. Thus, $V(\mathfrak{a})$ is Noetherian, being it finite, but $A/\mathfrak{a}$ is non Noetherian. The proof is left to the reader.

Now we will show that, for any ring A, there exist a non Noetherian ring A' such that $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A')$ are homeomorphic. In order to do this, we will present now a new ring construction.

Let A be a ring and let M be a A-module. Let $A(+)M$ denote the set $A \times M$ equipped with the ring structure defined by setting

$$(a,m)+(b,n) := (a+b, m+n), \quad (a,m)(b,n) := (ab, an+bm) \quad \text{for all } a,b \in A, m,n \in M.$$

$A(+)M$ is called *the idealization of M, with respect to A.*

(3.12) Remark. We list in the following some straightforward properties of $A(+)M$, the first one will help the reader to understand the reason for this terminology.

(a) A is isomorphic to a subring of $A(+)M$, via the ring embedding $A \longrightarrow A(+)M$, $a \mapsto (a,0)$, and the module M is canonically identified with the ideal $(0) \times M$ of $A(+)M$.

(b) If $M \neq (0)$, then $A(+)M$ is not reduced, being any (non zero) element of $(0) \times M$ nilpotent (of index 2).

(c) If M is not finitely generated, then $A(+)M$ is not Noetherian. Indeed, if $A(+)M$ is Noetherian, then in particular the ideal $(0) \times M$ is finitely generated, say by $(0,m_1), \ldots, (0,m_h)$, and this easily implies that $m_1, \ldots, m_h$ is a finite set of generators of M as a A-module.

(3.13) Proposition. *Let A be a ring and M be an A-module. Let $\pi : A(+)M \longrightarrow A$ denote the projection onto A ($(a,m) \mapsto a$, for any $(a,m) \in A(+)M$). Then the canonical map $\pi^\star : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(A(+)M)$ is a homeomorphism. In particular,*

$$\mathrm{Spec}(A(+)M) = \{\mathfrak{p} \times M : \mathfrak{p} \in \mathrm{Spec}(A)\}.$$

Proof. Since π is a surjective ring homomorphism, $\pi^\star$ is a closed embedding, by (3.5b), and $\pi^\star(\mathrm{Spec}(A)) = V(\mathrm{Ker}(\pi)) = V((0) \times M)$. Then it is enough to note that $V((0) \times M) = \mathrm{Spec}(A(+)M)$, in view of (3.12b). The last statement is now obvious, because, by definition, $\pi^\star(\mathfrak{p}) = \mathfrak{p} \times M$, for any $\mathfrak{p} \in \mathrm{Spec}(A)$. $\square$

(3.14) Example. Let A be a ring and M be a non finitely generated module (for example, a direct sum of infinitely many copies of A). Then $A(+)M$ is non Noetherian, by (3.12c), and $\mathrm{Spec}(A)$ and $\mathrm{Spec}(A(+)M)$ are homeomorphic, by the previous proposition.

4. Fiber products

That of fiber products is a powerful tool for presenting interesting examples and counterexamples in Commutative Ring Theory. Thus in the following we sketch some relevant properties of rings, and their spectra, arising as fiber products.

(4.1) Definition. *Let $f : A \longrightarrow C, g : B \longrightarrow C$ be ring homomorphisms. Then, the subring*

$$D := f \times_C g := \{(a, b) \in A \times B : f(a) = g(b)\}$$

of $A \times B$ is called the fiber product of f and g. *In the following, we will denote by $p : D \longrightarrow A$ (resp., $q : D \longrightarrow B$) the restriction to D of the projection of $A \times B$ into A (resp., into B).*

(4.2) Example. Let $\pi : B \longrightarrow C$ be a ring homomorphism, and let A be a subring of C. Consider the subring $D := \pi^{-1}(A)$ and let $i : A \longrightarrow C$ be the inclusion. Then D is canonically isomorphic to $i \times_C \pi$. Indeed, it is easy to see that the ring homomorphism $D \longrightarrow i \times_C \pi$, $d \mapsto (\pi(d), d)$ for any $d \in D$, is well defined and bijective. For example, the subring $\mathbb{Z} + T\mathbb{Q}[T]$ of the polynomial ring $\mathbb{Q}[T]$ is of this type. Indeed, in this case, $B := \mathbb{Q}[T]$, $C := \mathbb{Q}$, $\pi : \mathbb{Q}[T] \longrightarrow \mathbb{Q}$ is the ring homomorphism sending a polynomial f in its constant term, and $A := \mathbb{Z}$. Thus $\mathbb{Z} + T\mathbb{Q}[T] = \pi^{-1}(\mathbb{Z})$.

First, we will provide a precise description of the prime spectrum of a fiber product $f \times_C g$, under the assumption that one of the ring homomorphisms is surjective.

(4.3) Proposition. *We preserve the notation given in (4.1), and assume that g is surjective. Then p is surjective and it induces a closed embedding $p^\star : \mathrm{Spec}(A) \longrightarrow \mathrm{Spec}(D)$ whose image is $V_D((0) \times \mathrm{Ker}(g))$.*

Proof. It is straightforward. Indeed, take an element $a \in A$. Since g is surjective, there is an element $b \in B$ such that $g(b) = f(a)$. Thus $(a, b) \in D$ and $p(a, b) = a$. The last statement follows immediately from (3.5b) and from the equality $\mathrm{Ker}(p) = (0) \times \mathrm{Ker}(g)$. $\square$

(4.4) Lemma. *We preserve the notation of (4.1) and fix an element $b \in \mathrm{Ker}(g)$. Then $(0, b) \in \mathrm{Ker}(p) \subseteq D$ and the canonical ring homomorphism $\lambda : D_{(0,b)} \longrightarrow B_b$, $\dfrac{(x, y)}{(0, b)^n} \mapsto \dfrac{y}{b^n}$, is a well defined isomorphism.*

Proof. By definition, the image of an element of the form $(0, b)^n$ via the ring homomorphism $\Lambda : D \longrightarrow B_b$, $(x, y) \mapsto \dfrac{y}{1}$, is invertible in B_b ($\Lambda((0, b)^n) = \dfrac{b^n}{1}$). Then λ is well defined and it is a ring homomorphism, since it is induced by Λ (in view of the universal property of localizations). Take now $(x, y) \in D$, $n \geq 1$ such that $\dfrac{y}{b^n} = 0$ in B_b. Then, there is a natural number m such that $b^m y = 0$. It follows

$$\frac{(x, y)}{(0, b)^n} = \frac{(x, y)(0, b)^m}{(0, b)^{n+m}} = 0 \qquad \text{in } D_{(0,b)}.$$

This proves that λ is injective. Now, take any element $\dfrac{y}{b^n} \in B_b$. Clearly, $by \in \mathrm{Ker}(g)$ and thus $(0, by) \in D$. It follows immediately that $\lambda(\dfrac{(0, by)}{(0, b)^{n+1}}) = \dfrac{y}{b^n}$. The proof is now complete. $\square$

(4.5) Lemma. *We preserve the notation of (4.1) and fix an element $b \in \mathrm{Ker}(g)$. Then, the restriction of $q^\star : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(D)$ to the open set $D_B(b)$ of $\mathrm{Spec}(B)$ is a homeomorphism of $D_B(b)$ with $D_D((0, b))$.*

Proof. Let $\mu : D \longrightarrow D_{(0,b)}, \eta : B \longrightarrow B_b$ be the localization maps, and let $\lambda : D_{(0,b)} \longrightarrow B_b$ be the isomorphism given in (4.4). Keeping in mind (3.6b), the maps $\mu^\star : \mathrm{Spec}(D_{(0,b)}) \longrightarrow D_D((0,b))$, $\eta^\star : \mathrm{Spec}(B_b) \longrightarrow D_B(b)$ are homeomorphisms. Moreover, $\lambda^\star$ is a homeomorphism since λ is an isomorphism. Since $\eta \circ q = \lambda \circ \mu$, it immediately follows that

$$D_D((0,b)) = \mu^\star(\mathrm{Spec}(D_{(0,b)})) = \mu^\star(\lambda^\star(\mathrm{Spec}(B_b))) = q^\star(\eta^\star(\mathrm{Spec}(B_b))) = q^\star(D_B(b)).$$

Finally, for any prime ideal $\mathfrak{q} \in D_B(b)$, we have $q^\star(\mathfrak{q}) = \mu^\star(\lambda^\star(\eta^{\star -1}(\mathfrak{q})))$, and thus $q^\star|_{D_B(b)}$ is a homeomorphism of $D_B(b)$ with $D_D((0,b))$. □

(4.6) Lemma. *Let A be a ring and let $\mathfrak{a}$ be an ideal of A. If $\Omega \subseteq \mathrm{Spec}(A) - V(\mathfrak{a})$ is an open set, then $\Omega = \bigcup_{s \in S} D(s)$, for some $S \subseteq \mathfrak{a}$.*

Proof. Let $\mathfrak{p} \in \Omega$. Since Ω is open, by (3.2f), there is an element $x \in A$ such that $\mathfrak{p} \in D(x) \subseteq \Omega$. Moreover, by assumption there is an element $y \in \mathfrak{a} - \mathfrak{p}$. Thus $xy \in \mathfrak{a}$ and $\mathfrak{p} \in D(xy) \subseteq \Omega$. □

(4.7) Proposition. *We preserve the notation of (4.1). Then the canonical continuous map $q^\star : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(D)$ induces, by restriction, a homeomorphism of $\mathrm{Spec}(B) - V(\mathrm{Ker}(g))$ with $\mathrm{Spec}(D) - V(\mathrm{Ker}(p))$.*

Proof. Clearly, $q^\star(\mathrm{Spec}(B) - V(\mathrm{Ker}(g))) \subseteq \mathrm{Spec}(D) - V(\mathrm{Ker}(p))$. Let $\mathfrak{h}$ be a prime ideal of $\mathrm{Spec}(D) - V(\mathrm{Ker}(p))$. Since $\mathrm{Ker}(p) = (0) \times \mathrm{Ker}(g)$, there is an element $b \in \mathrm{Ker}(g)$ such that $(0,b) \notin \mathfrak{h}$. By definition, the fiber $q^{\star -1}(\{\mathfrak{h}\})$ is contained in the open set $D_B(b)$ and thus it consists of one point, by (4.5). Thus $q^\star|_{\mathrm{Spec}(B) - V(\mathrm{Ker}(g))}$ is bijective and, by (4.5) and (4.6), it is open. The proof is now complete. □

(4.8) Corollary. *We preserve the notation of (4.1) and assume that f is surjective. Then the following properties hold.*

(a) $\mathrm{Spec}(D) = p^\star(\mathrm{Spec}(A)) \cup q^\star(\mathrm{Spec}(B) - V(\mathrm{Ker}(g)))$.
(b) $\mathrm{Max}(D) = p^\star(\mathrm{Max}(A)) \cup q^\star(\mathrm{Max}(B) - V(\mathrm{Ker}(g)))$.
(c) *D is local if and only if A is local and $\mathrm{Ker}(g)$ is contained in the Jacobson radical of B.*

Proof. Exercise. □

Let S, T be topological spaces and let $S \sqcup T := (S \times \{0\}) \cup (T \times \{1\})$ denote the disjoint union of S and T. With a small abuse of notation, we will identify S, T with subsets of $S \sqcup T$. Thus, a natural topology on $S \sqcup T$ is that whose open sets are the subsets Ω of $S \sqcup T$ such that $\Omega \cap S$ (resp., $\Omega \cap T$) is open in S (resp., in T). In particular, S, T are clopen subspaces of $S \sqcup T$. Now, let $\alpha : C \longrightarrow T$ be a continuous function, where C is a closed subspace of S. Let $\mathcal{E}$ be the equivalence relation on $S \sqcup T$ generated by identifying c with $\alpha(c)$, for any $c \in C$. Then we will denote by $S \cup_\alpha T$ the quotient space of $S \sqcup T$, with respect to the equivalence relation $\mathcal{E}$.

We recall now the following basic sufficient condition for a function of topological spaces to be continuous.

(4.9) Proposition. *Let $f : X \longrightarrow Y$ be a function of topological spaces and let $\mathcal{A}$ be an open cover of X. If the restriction $f|_A$ is continuous, for any $A \in \mathcal{A}$, then f is continuous.*

Let $f : X \longrightarrow Y$ be a continuous function of topological spaces. Recall that f is called to be a *quotient map* if the topology of Y is the finest topology which makes f a continuous function. We will use the fundamental property of quotient spaces.

(4.10) Proposition. *Let $f : X \longrightarrow Y$ a continuous surjective function of topological spaces and let $\mathcal{K}_f$ be the equivalence relation induced by f. Then the following conditions are equivalent.*

(i) *The canonical map $f_\star : X/\mathcal{K}_f \longrightarrow Y$ is a homeomorphism.*
(ii) *f is a quotient map.*

(4.11) Theorem. [6, (1.4) Theorem] *We preserve the notation of (4.1) and assume that g is surjective. Thus, we can identify* $\mathrm{Spec}(C)$ *with the closed subset* $V(\mathrm{Ker}(g))$ *of* $\mathrm{Spec}(B)$ *(by (3.5b)). Then,* $\mathrm{Spec}(D)$ *is canonically homeomorphic to* $\mathrm{Spec}(A) \cup_{f^\star} \mathrm{Spec}(B)$.

Proof. Set $\mathfrak{b} := \mathrm{Ker}(g), X := \mathrm{Spec}(A), Y := \mathrm{Spec}(B), Z := \mathrm{Spec}(C)$ and $T := X \sqcup Y$, endowed with its natural topology (see the discussion above). Let $\sigma : T \longrightarrow Z$ be the natural map defined by

$$\sigma(\mathfrak{p}) := \begin{cases} p^\star(\mathfrak{p}) & \text{if } \mathfrak{p} \in X \\ q^\star(\mathfrak{p}) & \text{if } \mathfrak{q} \in Y \end{cases}$$

By considering the open cover $\mathcal{A} := \{X, Y\}$ of T and applying (4.9), it follows that σ is continuous. Moreover, by (4.8a), σ is surjective. By definition, the equivalence relation $\mathcal{K}_\sigma$ on T induced by σ is is that which is generated by identifying the prime ideals of C (they correspond to the points of the closed subspace $V_B(\mathfrak{b})$ of Y) with their images under $f^\star$ (the straighforward verification is left to the reader). By (4.10), the conclusion will follow if we prove that σ is a quotient map. Let $\mathcal{T}$ be a topology on Z making σ a continuous map and let $\Omega \subseteq Z$ be open, with respect to the topology $\mathcal{T}$. It is sufficient to show that Ω is open, with respect to the Zariski topology. By definition, $\sigma^{-1}(\Omega)$ is open in T, that is, $\sigma^{-1}(\Omega) \cap X = p^{\star -1}(\Omega)$ (resp., $\sigma^{-1}(\Omega) \cap Y = q^{\star -1}(\Omega)$) is open in X (resp., in Y). Take a prime ideal $\mathfrak{p} \in \Omega$. First, assume that $\mathfrak{p} \notin V_D((0) \times \mathfrak{b})$. By (4.7), there exists a unique prime ideal $\mathfrak{p}' \in Y - V(\mathfrak{b})$ such that $q^\star(\mathfrak{p}') = \mathfrak{p}$. Since $\mathfrak{p}'$ is an element of the open set $q^{\star -1}(\Omega) \cap (Y - V_B(\mathfrak{b}))$, there is an element $b \in \mathfrak{b}$ such that $\mathfrak{p}' \in D_B(b) \subseteq q^{\star -1}(\Omega) \cap (Y - V_B(\mathfrak{b}))$, by (4.6). Then, in view of (4.5), we have $\mathfrak{p} = q^\star(\mathfrak{p}') \in q^\star(D_B(b)) = D_D((0, b)) \subseteq \Omega$.

Suppose now that $\mathfrak{p} \in V_D((0) \times \mathfrak{b}) \cap \Omega$ and, by (4.3), let $\mathfrak{p}'$ be the unique prime ideal of A such that $p^\star(\mathfrak{p}') = \mathfrak{p}$. Since $p^{\star -1}(\Omega)$ is an open neighborhood of $\mathfrak{p}'$, there is an element $a \in A$ such that $\mathfrak{p}' \in D(a) \subseteq p^{\star -1}(\Omega)$. Since g is surjective, take an element $b \in B$ such that $g(b) = f(a)$. We are going to show the following claim.

Claim: $V(\mathfrak{b}) \cap D(b) \subseteq q^{\star -1}(\Omega)$.

As a matter of fact, take a prime ideal $\mathfrak{h} \in V(\mathfrak{b}) \cap D(b)$. Since $\mathfrak{h} \in V(\mathfrak{b})$ there is a unique prime ideal $\mathfrak{h}'$ of C such that $g^\star(\mathfrak{h}') = \mathfrak{h}$, by (3.5b). Since $\mathfrak{h} \in D(b)$, $f(a) = g(b) \notin \mathfrak{h}'$, that is, $f^\star(\mathfrak{h}') \in D(a) \subseteq p^{\star -1}(\Omega)$. Thus, the equality $g \circ q = f \circ p$ implies $q^\star(\mathfrak{h}) = p^\star(f^\star(\mathfrak{h}')) \in \Omega$. This proves the claim.

Let $\mathfrak{c}$ be an ideal of B such that $V(\mathfrak{c}) = Y - q^{\star -1}(\Omega)$. By the claim, we have $V(\mathfrak{b}) \cap D(b) \cap V(\mathfrak{c}) = \emptyset$. Clearly, this implies that the ideals $\mathfrak{b}B_b, \mathfrak{c}B_b$ of B_b are comaximal. Thus, by the Chinese remainder Theorem, the canonical ring homomorphism $\psi : B_b \longrightarrow (B_b/\mathfrak{b}B_b) \times (B_b/\mathfrak{c}B_b)$ is surjective. Choose an element $\frac{x}{b^n} \in \psi^{-1}(\{(1, 0)\})$, that is, $\frac{x}{b^n} - 1 \in \mathfrak{b}B_b$ and $\frac{x}{b^n} \in \mathfrak{c}$. Take an element $c \in \mathfrak{c}$ and a natural number m such that $\frac{x}{b^n} = \frac{c}{b^m}$. It follows that $\frac{c - b^m}{b^m} = \frac{\beta}{b^r}$, for some $\beta \in \mathfrak{b}$ and some $r \in \mathbb{N}$.

The last equality implies that there are natural numbers t, u such that $b^t c - b^u \in \mathfrak{b}$, and we can assume, without loss of generality, that $u \geq 1$. Then

$$0 = g(b^t c) - g(b^u) = g(b^t c) - f(a^u),$$

that is $(a^u, b^t c) \in D$. We claim that $\mathfrak{p} \in D_D((a^u, b^t c)) \subseteq \Omega$. We have $p(a^u, b^t c) = a^u \notin \mathfrak{p}'$, since $\mathfrak{p}' \in D(a)$, and thus $(a^u, b^t c) \notin p^\star(\mathfrak{p}') = \mathfrak{p}$. Finally, take any prime ideal $\mathfrak{q} \in D_D((a^u, b^t c))$. If $\mathfrak{q} \in V_D((0) \times \mathfrak{b})$, take the unique prime ideal $\mathfrak{q}' \in X$ such that $p^\star(\mathfrak{q}') = \mathfrak{q}$. Then $a^u = p(a^u, b^t c) \notin \mathfrak{q}'$, that is $a \notin \mathfrak{q}'$. Thus $\mathfrak{q}' \in D(a) \subseteq p^{\star -1}(\Omega)$ and $\mathfrak{q} = p^\star(\mathfrak{q}') \in \Omega$. If $\mathfrak{q} \in \mathrm{Spec}(D) - V((0) \times \mathfrak{b})$, let $\mathfrak{q}' \in \mathrm{Spec}(B) - V(\mathfrak{b})$ be such that $q^\star(\mathfrak{q}') = \mathfrak{q}$. Thus $b^t c = q(a^u, b^t c) \notin \mathfrak{q}'$, since $\mathfrak{q} \in D_D((a^u, b^t c))$. In particular $c \notin \mathfrak{q}'$ and thus $\mathfrak{q}' \in Y - V(\mathfrak{c}) = q^{\star -1}(\Omega)$. Then, $\mathfrak{q} \in \Omega$. □

5. Eakin-Nagata's Theorem

Let $A \subseteq B$ be a ring extension. As it is well known, if A is Noetherian and B is of finite type over A, then B is Noetherian. Indeed, if $B := A[b_1, \ldots, b_n]$, for some $b_1, \ldots, b_n \in B$, then B is a homomorphic image of the Noetherian polynomial ring $A[T_1, \ldots, T_n]$ (via the ring homomorhism $A[T_1, \ldots, T_n] \longrightarrow B$, $f \mapsto f(b_1, \ldots, b_n)$). If $A \subseteq B$ is a finite extension, it is, in particular, of finite type, and thus B is Noetherian provided that A is Noetherian. In general, if B is a Noetherian A-module, then A is a Noetherian ring, being it a Noetherian A-submodule of B. But, if B is a Noetherian ring it is not necessarily true that A is a Noetherian ring. For example, if $\{T_i : i \in I\}$ is an infinite collection of indeterminates over a field K, then the polynomial ring $A := K[\{T_i : i \in I\}]$ is a non Noetherian domain, but the quotient field of A is Noetherian.

Eakin-Nagata's Theorem states that, under the assumption that the ring extension $A \subseteq B$ is finite, then A is a Noetherian ring provided that B is a Noetherian ring. There are several proofs of this nontrivial result. We will provide that due to Formanek.

First we recall some easy basic facts.

(5.1) Remark. Let A be a ring.

(a) If M is a A-module and N is a submodule of M, then M is Noetherian (resp., Artin) if and only if N and M/N are Noetherian (resp., Artin). Moreover, if N and M/N are finitely generated, then M is finitely generated.

(b) If $M_1, \ldots, M_n$ is a finite collection of Noetherian A-modules, then the direct product $M := M_1 \times \ldots \times M_n$ is Noetherian. As a matter of fact, it suffices to prove the statement for $n := 2$, and then use induction. If $M := M_1 \times M_2$, then M_1 is isomorphic to the submodule $N := M_1 \times (0)$ of M. Moreover, M/N is clearly isomorphic to M_2. Then, the statement follows from part (a).

(c) If A is a Noetherian ring and M is a finitely generated A-module, then M is Noetherian. Indeed, if $\{m_1, \ldots, m_r\}$ is a set of generators of M, there is a unique surjective A-linear map $f : A^r \longrightarrow M$ such that $\mathbf{e}_i \mapsto m_i$, where $\{\mathbf{e}_1, \ldots, \mathbf{e}_r\}$ is the canonical basis of the free module A^r. Then it is enough to note that A^r is a Noetherian A-module, by part (b).

If A is a ring and M is a A-module, then $\mathrm{Ann}_A(M) := \{a \in A : aM = (0)\}$ is an ideal of A, called *the annihilator of* M. The module M is called to be *faithful* if $\mathrm{Ann}_A(M) = (0)$.

(5.2) Remark. Let A be a ring and, let M be an A-module and set $\mathfrak{a} := \mathrm{Ann}_A(M)$. For any ideal $\mathfrak{b} \subseteq \mathfrak{a}$ of A, then M can be endowed with a natural structure of $A/\mathfrak{b}$-module, via the scalar multiplication defined by $(a+\mathfrak{b})\cdot m := am$, for any $a+\mathfrak{b} \in A/\mathfrak{a}$, $m \in M$.

(a) Clearly, for any subset N of M, N is a A-submodule of M if and only if N is a $A/\mathfrak{b}$-submodule of M. This implies that M is Noetherian (resp., Artin) as a A-module if and only if it is Noetherian (resp., Artin) as a $A/\mathfrak{b}$-module.

(b) By definition, M is a faithful $A/\mathfrak{a}$-module.

(5.3) Proposition. *If A is a ring and there is a Noetherian and faithful A-module, then A is a Noetherian ring.*

Proof. Let M be a Noetherian and faithful A-module, and let $\{m_1, \ldots, m_r\}$ be a finite set of generators of M. Consider the A-linear map $f : A \longrightarrow M^r$ defined by $f(a) := (am_1, \ldots, am_r)$, for any $a \in A$. Since, for any $a \in A$, the equalities $am_1 = \ldots = am_r = 0$ imply $aM = 0$, it follows that f is injective, since M is faithful. Thus A can be identified with a A-submodule of the Noetherian module M^r, and thus A is Noetherian as a A-module, that is, A is a Noetherian ring. □

(5.4) Lemma. *Let A be a ring and let X be a finitely generated faithful A-module such that the A-module $X/\mathfrak{a}X$ is Noetherian, for any nonzero ideal $\mathfrak{a}$ of A. Then X is Noetherian.*

Proof. Consider the collection

$$\Sigma := \{N \subseteq X : N \text{ is a submodule of } X \text{ and } X/N \text{ is faithful}\}$$

of submodules of X, partially ordered by inclusion, and note that it is nonempty, since $(0) \in \Sigma$. Let $\mathcal{C} \subseteq \Sigma$ be a chain and note that $N := \bigcup \mathcal{C}$ is a submodule of X. Take an element $a \in \mathrm{Ann}_A(X/N)$ and let $\{x_1, \ldots, x_r\}$ be a finite set of generators of X. This is equivalent to state that $ax_i \in N$, for $1 \leq i \leq r$. Since $\mathcal{C}$ is a chain, there is an element $N' \in \mathcal{C}$ such that $ax_i \in N'$, for $1 \leq i \leq r$. It follows that $a \in \mathrm{Ann}_A(X/N')$ and thus $a = 0$, since $N' \in \Sigma$. This proves that X/N is faithful, that is, $N \in \Sigma$. By Zorn's Lemma, there is a maximal element $M \in \Sigma$. Since X/M is a faithful A-module, if it is Noetherian, then A is a Noetherian ring, by (5.3), and thus X is a Noetherian module, by (5.1c). Thus the conclusion follows if we show that $Y := X/M$ is Noetherian. For any nonzero ideal $\mathfrak{a}$ of A the A-module $Y/\mathfrak{a}Y$ is a quotient of $X/\mathfrak{a}X$, and thus, by assumption, it is Noetherian. Moreover if Z is a nonzero submodule of Y, then $Z = P/M$ for some submodule P of X such that $M \subsetneq P$. Thus $Y/Z = X/P$ is not faithful, since M is maximal in Σ. Thus, keeping in mind the assumptions, Y has the following properties:

(1) Y is finitely generated (it is a quotient of X);

(2) $Y/\mathfrak{a}Y$ is Noetherian, for any nonzero ideal $\mathfrak{a}$ of A;

(3) Y/Z is not faithful, for any nonzero submodule Z of Y.

Fix now any nonzero submodule Z of Y. By (3), the A-module Y/Z is not faithful, and thus there exists a nonzero element $a \in A$ such that $a(Y/Z) = (0)$, that is, $aY \subseteq Z$. Thus the factor module Z/aY is a submodule of the Noetherian module Y/aY (property (2)), and hence Z/aY is finitely generated. Keeping in mind that aY is finitely generated, being it Y (property (1)), it follows that Z is finitely generated, in view of (5.1a). This proves that any submodule of Y is finitely generated, that is, Y is Noetherian. □

The statement of the following remark can be easily verified.

(5.5) Remark. Let $(X, \leq)$ be a partially ordered set. Then the following conditions are equivalent.

(a) $(X, \leq)$ satisfies the ascending chain condition.
(b) Any nonempty subset of X has a maximal element.

(5.6) Theorem ([8]). *Let A be a ring and let M be a finitely generated faithful A-module such that the collection $\{\mathfrak{a}M : \mathfrak{a}$ ideal of $A\}$ of submodules of M satisfies the ascending chain condition. Then, M is Noetherian.*

Proof. By contradiction, suppose that M is non Noetherian. Then, the collection

$$\Sigma := \{\mathfrak{a}M : \mathfrak{a} \text{ ideal of } A \text{ and } M/\mathfrak{a}M \text{ is non Noetherian}\}$$

of submodules of M is nonempty and, by (5.5), it has a maximal element $\mathfrak{a}_0 M$, for some ideal $\mathfrak{a}_0$ of A. Since $\mathfrak{a}_0 M \in \Sigma$, the A-module $X^* := M/\mathfrak{a}_0 M$ is non Noetherian. If $A^* := A/\mathrm{Ann}_A(X^*)$ then, by (5.2), X^* is a faithful and non Noetherian A^*-module. If $\mathfrak{i}$ is any nonzero ideal of A^*, then $\mathfrak{i} = \mathfrak{a}^*/\mathrm{Ann}_A(X^*)$, for some ideal $\mathfrak{a}^*$ of A such that $\mathfrak{a}^* \supsetneq \mathrm{Ann}_A(X^*) \supseteq \mathfrak{a}_0$. It easily follows that $\mathfrak{a}^* M \supsetneq \mathfrak{a}_0 M$. Since $\mathfrak{a}_0 M$ is maximal in Σ, then $M/\mathfrak{a}^* M$ is a Noetherian A-module. Since $\mathrm{Ann}_A(X^*) \subseteq \mathrm{Ann}_A(M/\mathfrak{a}^* M)$, then, in view of (5.2), $M/\mathfrak{a}^* M$ is a Noetherian A^*-module and clearly $X^*/\mathfrak{i}X^* \cong M/\mathfrak{a}^* M$. This contradicts the statement of (5.4). $\square$

(5.7) Theorem (Eakin-Nagata). *Let $A \subseteq B$ be a finite ring extension and assume that B is a Noetherian ring. Then, A is a Noetherian ring.*

Proof. By the conventions stated at the beginning, A and B have the same multiplicative identity, and then B is a faithful A-module. The collection of ideals $\{\mathfrak{a}B : \mathfrak{a}$ ideal of $A\}$ of B satisfies the ascending chain condition, since B is Noetherian. By (5.6), B is a Noetherian A-module, and thus A is a Noetherian ring, by (5.3) (or from the fact that it is a A-submodule of the Noetherian module B). $\square$

Let $f : A \longrightarrow B$ be a ring homomorphism. Clearly, B is a A-module, via the natural scalar multiplication defined by $a \cdot b := f(a)b$, for any $a \in A, b \in B$. We say that f *is finite* (resp., of finite type, integral) if the ring extension $f(A) \subseteq B$ is finite (resp., of finite type, integral). By well known properties on integral dependence, f is finite if and only if it is integral and of finite type.

The proof of the following straightforward lemma is left to the reader.

(5.8) Lemma. *We preserve the notation of (4.1) and assume that g is surjective. If f is finite (resp., of finite type, integral), then q is finite (resp., of finite type, integral).*

(5.9) Proposition. [6, Proposition (1.8)] *We preserve the notation of (4.1) and assume that g is surjective. Then, the following conditions are equivalent.*

(i) *D is a Noetherian ring and q is finite.*
(ii) *A, B are Noetherian rings and f is finite.*

Proof. (i)$\Longrightarrow$(ii). Keeping in mind (4.3), $A = p(D)$ and $q(D)$ are Noetherian rings, being homomorphic images of the Noetherian ring D. Since the ring extension $q(D) \subseteq B$ is finite, then B is a Noetherian ring. By assumption, B is finitely generated as $q(D)$-module, say by $b_1, \ldots, b_n$. We claim that C is generated by $g(b_1), \ldots, g(b_n)$, as a $f(A)$-module. Indeed let $c \in C$ and, since g is surjective, let $b \in B$ such that $g(b) = c$. Take elements $(x_1, y_1), \ldots, (x_n, y_n) \in D$ such that $b = \sum_{i=1}^{n} q(x_i, y_i)b_i = \sum_{i=1}^{n} y_i b_i$. Then $c = \sum_{i=1}^{n} g(y_i)g(b_i) = \sum_{i=1}^{n} f(x_i)g(b_i)$.

(ii)$\Longrightarrow$(ii). Since A is a Noetherian ring and $p : D \longrightarrow A$ is surjective, then A is Noetherian as a D-module, too. Morover, C is a Noetherian ring, being it the homomorphic image of the Noetherian ring B, via g. By (5.8), q is finite and thus, in view of Eakin-Nagata's Theorem, $q(D)$ is a Noetherian ring and, clearly, a Noetherian D-module. The ideal $\mathrm{Ker}(g)$ of $q(D)$ is a D-submodule of $q(D)$, and thus $\mathrm{Ker}(g)$ is a Noetherian D-module and clearly it is isomorphic to $(0) \times \mathrm{Ker}(g) = \mathrm{Ker}(p)$. Since A is isomorphic to $D/\mathrm{Ker}(p)$ and it is a Noetherian D-module (being A a Noetherian ring), the conclusion follows from (5.1a). □

(5.10) Corollary. *Let $\pi : B \longrightarrow C$ be a surjective ring homomorphism, let A be a subring of C and let $D := \pi^{-1}(A)$. The following conditions are equivalent.*

(i) *D is a Noetherian ring and the ring extension $D \subseteq B$ is finite.*
(ii) *A, B are Noetherian rings and the ring extension $A \subseteq C$ is finite.*

Proof. Apply (5.9) to the ring construction (4.2). The easy details are left to the reader. □

(5.11) Corollary. *Let $A \subseteq B$ be a ring extension and let T be an indeterminate over B. Then, the ring $A + TB[T]$ is Noetherian if and only if A is a Noetherian ring and the ring extension $A \subseteq B$ is finite.*

Proof. If $D := A + TB[T]$ is a Noetherian ring, that so is A being it a homomorphic image of D. Now, let $\mathfrak{a}$ be the ideal of D generated by the set $\{bT : b \in B\}$. Since D is Noetherian, then $\mathfrak{a}$ is finitely generated, say by $f_1, \ldots, f_n \in \mathfrak{a}$. Set, for any $1 \leq i \leq n$, $f_i(T) := b_i T + g_i(T)T^2$, for some $b_i \in B$ and $g_i(T) \in B[T]$. Then an easy computation proves that B is generated by $b_1, \ldots, b_n$ as an A-module.

Conversely, assume that A is Noetherian and that $A \subseteq B$ is finite. Then $B[T]$ is Noetherian, by Hilbert's basis Theorem. Now the conclusion follows from (5.10), keeping in mind that $A + TB[T] = \pi^{-1}(A)$, where $\pi : B[T] \longrightarrow B$ is the surjective ring homomorphism defined by $\pi(f) := f(0)$, for any $f \in B[T]$. □

6. The Krull intersection Theorem

Now we state and prove a very famous theorem which will be useful in the following.

(6.1) Theorem (Krull intersection Theorem)**.** *Let A be a Noetherian ring, $\mathfrak{a}$ be an ideal of A and M be a finitely generated A-module. Then*

$$\mathfrak{a}(\bigcap_{n\geq 1} \mathfrak{a}^n M) = \bigcap_{n\geq 1} \mathfrak{a}^n M$$

Proof. By (5.1c), M is a Noetherian A-module. Set $N := \bigcap_{n\geq 1} \mathfrak{a}^n M$. The collection Σ of all the submodules S of M such that $S \cap N = \mathfrak{a}N$ is nonempty, since $\mathfrak{a}N \in \Sigma$, and thus Σ admits a maximal element C, by noetherianity. We are going to show the following claim.

Claim. If $x \in \mathfrak{a}$, there is a positive integer ν such that $x^\nu M \subseteq C$.

For any $i \geq 1$, consider the following A-submodule

$$N_i := \{m \in M : x^i m \in C\}$$

of M. By noetherianity, the ascending chain $N_1 \subseteq N_2 \subseteq \dots$ of submodules of M is eventually constant. Take a positive integer ν such that $N_\nu = N_n$, for any $n \geq \nu$, and note that $(C+x^\nu M)\cap N \supseteq \mathfrak{a}N$. Conversely, take an element $y \in (C+x^\nu M)\cap N$, and let $c \in C, m \in M$ be such that $y = c + x^\nu m$. It follows $xy - xc = x^{\nu+1}m$ and, since $xy \in \mathfrak{a}N \subseteq C$, we have $x^{\nu+1}m \in C$, that is, $m \in N_{\nu+1} = N_\nu$. In other words, $x^\nu m \in C$ and thus $y \in C \cap N = \mathfrak{a}N$. This proves that $(C + x^\nu M) \cap N = \mathfrak{a}N$, i.e., $C + x^\nu M \in \Sigma$ and, since C is maximal in Σ, we infer that $x^\nu M \subseteq C$. The proof of claim is complete.

Keeping in mind that $\mathfrak{a}$ is finitely generated, the claim implies that there exists a positive integer k such that $N \subseteq \mathfrak{a}^k M \subseteq C$, and thus $\mathfrak{a}N = N \cap C = N$. □

The following fact is well-known and we recall it for the reader convenience.

(6.2) Theorem (Nakayama's Lemma). *Let A be a ring, M be a finitely generated A-module and let $\mathfrak{a}$ be an ideal of A which is contained in the Jacobson radical of A. If $\mathfrak{a}M = M$, then $M = (0)$.*

(6.3) Corollary. *Let A be a ring, M be a finitely generated A-module and let $\mathfrak{a}$ be an ideal of A which is contained in the Jacobson radical of A. If N is an A-submodule of M and $M = \mathfrak{a}M + N$, then $M = N$.*

Proof. Apply (6.2) to the finitely generated A-module M/N. □

(6.4) Corollary. *Let A be a Noetherian ring, M be a finitely generated A-module and let $\mathfrak{a}$ be an ideal of A contained in the Jacobson radical of A. Then*

$$\bigcap_{n\geq 1} \mathfrak{a}^n M = (0).$$

Proof. By (5.1c), the submodule $N := \bigcap_{n\geq 1} \mathfrak{a}^n M$ of M is finitely generated and, by the Krull intersection Theorem, $\mathfrak{a}N = N$. By Nakayama's Lemma, $N = (0)$. □

(6.5) Corollary. *Let A be a Noetherian ring and let $\mathfrak{a}$ be an ideal of A which is contained in the Jacobson radical of A. Then $\bigcap_{n\geq 1} \mathfrak{a}^n = (0)$.*

Proof. Apply (6.4) to $M := A$. □

7. The Principal ideal Theorem

The next important result deals with the height of a prime ideal of a Noetherian ring which is minimal over a principal ideal. We will see that such a prime ideal must have height ≤ 1.

We start with some easy remarks.

(7.1) Remark. Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field K, let M be a finitely generated A-module, and let $X := M/\mathfrak{m}M$. By (5.2), X is both an A-module and a $A/\mathfrak{m}$-module, that is a K-vector space. Since M is finitely generated, it follows that X is, in particular, a finitely generated K-vector space, that is, it is both Noetherian and Artin as a K-vector space. Thus, by (5.2a), and furthermore X is both Noetherian and Artin as a A-module.

(7.2) Lemma. *Let A be a local ring with finitely generated maximal ideal $\mathfrak{m}$ and residue field K. Then, for any positive integer r, $\mathfrak{m}^r/\mathfrak{m}^{r+1}$ is both an Noetherian and a Artin A-module.*

Proof. Consider the finitely generated $M := \mathfrak{m}^r$ and note that $M/\mathfrak{m}M = \mathfrak{m}^r/\mathfrak{m}^{r+1}$. Then it suffices to apply (7.1). □

(7.3) Proposition. *Any Noetherian ring with a unique prime ideal is an Artin ring.*

Proof. Let A be a Noetherian ring and $\mathfrak{m}$ be its unique prime ideal (hence, $\mathfrak{m}$ is the nilradical of A). Since $\mathfrak{m}$ is finitely generated, there is a smallest positive integer k such that $\mathfrak{m}^k = (0)$. If $k = 1$ the A is a field and thus we have nothing to prove. Assume that $k \geq 2$. Since, for any positive integer r, $\mathfrak{m}^r$ is a A-submodule of $\mathfrak{m}^{r-1}$ and the factor module $\mathfrak{m}^{r-1}/\mathfrak{m}^r$ is an Artin A-module (by (7.2)), it is easily proved by induction that $\mathfrak{m}, \mathfrak{m}^2, \ldots, \mathfrak{m}^k$ are Artin A-modules, keeping in mind (5.1a). Moreover, the field $A/\mathfrak{m}$ is an Artin A-module (by (7.1)). Then, A is an Artin A-module, that is, an Artin ring, again by (5.1a). □

More generally, it is possible to show that a ring is an Artin ring if and only if it is Noetherian and zero-dimensional.

The proof of the following lemma is an easy exercise.

(7.4) Lemma. *Let D be an integral domain and let $\mathfrak{q}$ be a prime ideal of D. Then, the ideal $\mathfrak{q}^{(n)} := (\mathfrak{q}^n D_\mathfrak{q}) \cap D$ is of D is $\mathfrak{q}$-primary, for any positive integer n.*

(7.5) Theorem (Principal Ideal Theorem)**.** *Let A be a Noetherian ring, $a \in A$ and let $\mathfrak{q} \in \mathrm{Spec}(A)$ be minimal over aA. Then $\mathrm{ht}(\mathfrak{q}) \leq 1$.*

Proof. We argue by contradiction, and assume there are prime ideals $\mathfrak{p}_0, \mathfrak{p}_1$ of A such that $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{q}$. Consider the local domain $D := A_\mathfrak{q}/\mathfrak{p}_0 A_\mathfrak{q}$ and note that, by (3.6a,c), $\mathrm{Spec}(D)$ is homeomorphic to the subspace $\{\mathfrak{h} \in \mathrm{Spec}(A) : \mathfrak{p}_0 \subseteq \mathfrak{h} \subseteq \mathfrak{q}\}$ of $\mathrm{Spec}(A)$. Moreover the maximal ideal $\mathfrak{m}$ of D (which corresponds to $\mathfrak{q}$) is minimal over the principal ideal xD, where x is the canonical image of a in D, and there is a nonzero prime ideal $\mathfrak{p}$ of D (corresponding to $\mathfrak{p}_1$) such that $x \notin \mathfrak{p}$. Since $\mathfrak{m}$ is minimal over x, (3.6a) implies that the Noetherian factor ring D/xD has a unique prime ideal (corresponding to $\mathfrak{m}$), and thus D/xD is an Artin ring, by (7.3). For any positive integer t, let $\overline{\mathfrak{p}^{(t)}}$ denote the canonical image of $\mathfrak{p}^{(t)}$ in D/xD. Since $\overline{\mathfrak{p}^{(1)}} \supseteq \overline{\mathfrak{p}^{(2)}} \supseteq \ldots$ and D/xD is Artin, there is a positive integer ν such that $\overline{\mathfrak{p}^{(t)}} = \overline{\mathfrak{p}^{(\nu)}}$, for any $t \geq \nu$, that is, $\mathfrak{p}^{(t)} + xD = \mathfrak{p}^{(\nu)} + xD$. Take an integer $t \geq \nu$ and an element $v \in \mathfrak{p}^{(\nu)}$. Thus there are elements $w \in \mathfrak{p}^{(t)}, d \in D$ such that $v = w + xd$. It follows $xd \in \mathfrak{p}^{(\nu)} + \mathfrak{p}^{(t)} = \mathfrak{p}^{(\nu)}$. Since $x \notin \mathfrak{p}$ and $\mathfrak{p}^{(\nu)}$ is $\mathfrak{p}$-primary (by (7.4)), it follows $d \in \mathfrak{p}^{(\nu)}$. This proves that $\mathfrak{p}^{(\nu)} = \mathfrak{p}^{(t)} + x\mathfrak{p}^{(\nu)}$ and, by (6.3), we have $\mathfrak{p}^{(\nu)} = \mathfrak{p}^{(t)}$. By applying (6.5) to the Noetherian local domain $D_\mathfrak{p}$ and its maximal ideal $\mathfrak{p}D_\mathfrak{p}$, we infer that $\bigcap_{t\geq 1} \mathfrak{p}^t D_\mathfrak{p} = (0)$ and, a fortiori,

$$(0) = \bigcap_{t\geq 1} \mathfrak{p}^t D_\mathfrak{p} \cap D = \bigcap_{t\geq 1} \mathfrak{p}^{(t)} = \mathfrak{p}^{(\nu)}.$$

Since $\mathfrak{p}^\nu \subseteq \mathfrak{p}^{(\nu)}$, by primality it follows $\mathfrak{p} = (0)$, a contradiction. □

(7.6) Corollary. *Let A be a Noetherian ring and let $\mathfrak{p}, \mathfrak{q} \in \mathrm{Spec}(A)$ be such that $\mathfrak{p} \subsetneq \mathfrak{q}$. If the set $\{\mathfrak{h} \in \mathrm{Spec}(A) : \mathfrak{p} \subsetneq \mathfrak{h} \subsetneq \mathfrak{q}\}$ is nonempty, then it is infinite.*

Proof. Consider the Noetherian domain $D := A/\mathfrak{p}$. By (3.6a), it suffices to prove that if there is a nonzero prime ideal $\mathfrak{h}$ of D such that $\mathfrak{h}_1 \subsetneq \mathfrak{q}' := \mathfrak{q}/\mathfrak{p}$, then there are infinitely many prime ideals of D with the same property. By contradiction, assume that $X := \{\mathfrak{p} \in \mathrm{Spec}(D) : (0) \neq \mathfrak{p} \subsetneq \mathfrak{q}'\} = \{\mathfrak{h}_1, \ldots, \mathfrak{h}_n\}$ and note that, clearly $\bigcup_{i=1}^n \mathfrak{h}_i \subseteq \mathfrak{q}'$. If the equality $\bigcup_{i=1}^n \mathfrak{h}_i = \mathfrak{q}'$ holds, by the prime avoidance theorem, $\mathfrak{q}' \subseteq \mathfrak{h}_k$, for some $1 \leq k \leq n$, against the fact that $\mathfrak{h}_k \in X$. Hence we have $\bigcup_{i=1}^n \mathfrak{h}_i \subsetneq \mathfrak{q}'$. Take an element $x \in \mathfrak{q}' - \bigcup_{i=1}^n \mathfrak{h}_i$ and take a (nonzero) prime ideal $\mathfrak{p} \in \mathrm{Spec}(D)$ minimal over xD and such that $\mathfrak{p} \subseteq \mathfrak{q}'$. By the principal ideal theorem we have $\mathrm{ht}(\mathfrak{p}) \leq 1$ and, since $\mathrm{ht}(\mathfrak{q}') \geq 2$, it follows $\mathfrak{p} \subsetneq \mathfrak{q}'$. By definition, $\mathfrak{p} \in X$, a contradiction, since each prime ideal of X does not contain x. □

(7.7) Remark. It is easy to note that the Principal Ideal Theorem and (7.6) fail in the non Noetherian setting. For example, let V be a two-dimensional valuation domain. As it is well known, the set of all ideals of a valuation domain is totally ordered by inclusion, and thus $\mathrm{Spec}(V) = \{(0), \mathfrak{p}, \mathfrak{m}\}$, with $(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$. Thus, for any element $x \in \mathfrak{m} - \mathfrak{p}$, $\mathfrak{m}$ is minimal over xV and $\mathrm{ht}(V) \geq 2$. Furthermore, there is a unique prime ideal (namely, $\mathfrak{p}$) between (0) and $\mathfrak{m}$.

Furthermore note that $\mathrm{Spec}(V)$ is a Noetherian space, but there are no Noetherian rings A such that $\mathrm{Spec}(A)$ is homeomorphic to $\mathrm{Spec}(V)$, in view of (7.6). Indeed, in general, if $f : \mathrm{Spec}(A_1) \longrightarrow \mathrm{Spec}(A_2)$ is a homeomorphism, then f is order preserving (that is, for any $\mathfrak{h}_1, \mathfrak{h}_2 \in \mathrm{Spec}(A_1)$, then $\mathfrak{h}_1 \subseteq \mathfrak{h}_2$ if and only if $f(\mathfrak{h}_1) \subseteq f(\mathfrak{h}_2)$). The easy proof of this statement is left to the reader.

8. Valuation domains and fiber products

Now we provide a technique for constructing valuation rings, based on fiber products.

We leave the straightforward proof of this easy lemma to the reader.

(8.1) Lemma. *Let $A \subseteq B$ be a ring extension and assume that A and B share a common ideal containing a regular element of B. Then A and B have the same total ring of quotients.*

(8.2) Proposition. [6, Theorem 2.4] *Let V be a local domain with residue field K, $\pi : V \longrightarrow K$ be the canonical projection, and let D be a subring of K whose quotient field is K. Then $\pi^{-1}(D)$ is a valuation domain if and only if V, D are valuation domains.*

Proof. Set $E := \pi^{-1}(D)$. First, note that the maximal ideal $\mathfrak{m} = \mathrm{Ker}(\pi)$ of V is clearly a common ideal of V and E. If $\mathfrak{m} = (0)$, then V is a field and E and D are isomorphic. Thus, in this case, the statement is trivial. Assume now that $\mathfrak{m} \neq (0)$. By (8.1), V and E have the same quotient field, say L. Thus, if E is a valuation domain, then so is V (any overring of a valuation domain is a valuation domain). Moreover, if $x \in K - D$ and $v \in V$ is such that $\pi(v) = x$, then $v \notin E$. Thus, since E is a valuation domain and v is in the quotient field of E, then $v^{-1} \in E$ and $\pi(v^{-1}) \in D$ is the inverse of x in D.

Conversely, assume that D and V are valuation domains and take an element $x \in L - E$. If $x \notin V$, then $x^{-1} \in \mathfrak{m} \subseteq E$, since V is a valuation domain of L. If $x \in V$ then x is a unit of V (otherwise $x \in \mathfrak{m} \subseteq E$, a contradiction). Since $x \in V - E$, then $\pi(x) \in K - D$ and, since D is a valuation domain of K, $(\pi(x))^{-1} \in D$. From $\pi(x)\pi(x^{-1}) = 1$, we infer $\pi(x^{-1}) = (\pi(x))^{-1} \in D$, and finally $x^{-1} \in E$. □

(8.3) Proposition. *Let V be a valuation domain with residue field K, let $\pi : V \longrightarrow K$ be the canonical projection, and let D be any nonzero subring of K. If W is a valuation domain of the quotient field of V such that of $\pi^{-1}(D) \subseteq W \subseteq V$, then $\pi(W)$ is a valuation domain of K containing D and $W = \pi^{-1}(\pi(W))$.*

Proof. Clearly, $\pi(W)$ is a ring containing D. The statement is trivial if V is a field. Thus assume that the maximal ideal $\mathfrak{m}$ of V is nonzero. It follows that $\pi^{-1}(D), W, V$ have the same quotient field, in view of (8.1). Take a nonzero element $k \in K$ and an element $v \in V$ such that $\pi(v) = k$. If $k \notin \pi(W)$, then $v \notin W$. Since V, W have the same quotient field and W is a valuation domain, it follows that $v^{-1} \in W$. Then, $1 = \pi(v^{-1})\pi(v) = \pi(v^{-1})k$, that is, $k^{-1} = \pi(v^{-1}) \in W$. This proves that $\pi(W)$ is a valuation domain of K. Finally, it is obvious that $W \subseteq \pi^{-1}(\pi(W))$. Conversely, take an element $x \in \pi^{-1}(\pi(W))$, and let $w \in W$ be such that $\pi(x) = \pi(w)$. It follows $x - w \in \mathfrak{m} \subseteq \pi^{-1}(D) \subseteq W$, and thus $x \in W$. □

(8.4) Example. Consider the valuation domain $V := \mathbb{Q}[T]_{(T)}$ and let $\pi : V \longrightarrow \mathbb{Q}$ be the canonical projection. Then, by (8.2), the ring $\pi^{-1}(\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)} + T\mathbb{Q}[T]_{(T)}$ is a valuation domain of $\mathbb{Q}(T)$. It is easy to verify that $\dim(\mathbb{Z}_{(p)} + T\mathbb{Q}[T]_{(T)}) = 2$. This can be also seen as a particular case of the following result.

(8.5) Proposition. [6, Proposition 2.1] *Let V be a local ring with maximal ideal $\mathfrak{m}$ and residue field K, and let $\pi : V \longrightarrow K$ be the canonical projection. Let D be a subring of K and set $E := \pi^{-1}(D)$. The following properties hold.*

(a) *Any prime ideal of E is comparable with $\mathfrak{m}$.*
(b) *If V, D are finite dimensional, then E is finite dimensional and we have $\dim(E) = \dim(V) + \dim(D)$.*

Proof. Clearly, $\mathfrak{m} = \mathrm{Ker}(\pi)$ is a common ideal of V and E. Let $i : E \longrightarrow V$ be the inclusion map. Keeping in mind (4.3), (4.7) and (4.2), it follows that $V_E(\mathfrak{m})$ is homeomorphic to $\mathrm{Spec}(D)$, via $(\pi|_E)^\star$, and that $\mathrm{Spec}(E) - V_E(\mathfrak{m})$ is homeomorphic to $\mathrm{Spec}(V) - V_V(\mathfrak{m}) = \{\mathfrak{h} \in \mathrm{Spec}(V) : \mathfrak{h} \subsetneq \mathfrak{m}\}$, via $i^\star$. This proves part (a).

(b). Let $(0) \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_m$ be prime ideals of D and $\mathfrak{h}_0 \subsetneq \mathfrak{h}_1 \subsetneq \ldots \subsetneq \mathfrak{h}_n = \mathfrak{m}$ be prime ideals of V such that $\dim(D) = n, \dim(V) = m$. Note that $\mathfrak{h}_i = \mathfrak{h}_i \cap E$, for any $1 \leq i \leq n$, since $\mathfrak{h}_i \subseteq \mathfrak{m} \subseteq E$. Then,

$$\mathfrak{h}_0 \subsetneq \mathfrak{h}_1 \subsetneq \ldots \subsetneq \mathfrak{h}_n = \mathfrak{m} = \pi^{-1}((0)) \subsetneq \mathfrak{q}_{n+1} := \pi^{-1}(\mathfrak{p}_1) \subsetneq \ldots \subsetneq \mathfrak{q}_{n+m} := \pi^{-1}(\mathfrak{p}_m)$$

is a chain of prime ideals of E of length $n+m$, that is $\dim(E) \leq n+m$. Conversely, let $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \ldots \subsetneq \mathfrak{q}_r$ be a chain of prime ideals of E. By part (a), any prime ideal of this chain is comparable with $\mathfrak{m}$. If $\mathfrak{q}_r \subseteq \mathfrak{m}$, then, by the discussion above, $\mathfrak{q}_0 \subsetneq \mathfrak{q}_1 \subsetneq \ldots \subsetneq \mathfrak{q}_r$ is a chain of prime ideals of V, and thus $r \leq n \leq n+m$. If $\mathfrak{q}_0 \supseteq \mathfrak{m}$, then there are prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \ldots \subsetneq \mathfrak{p}_r$ of D such that $\mathfrak{q}_i = \pi^{-1}(\mathfrak{p}_i)$, for $0 \leq i \leq r$, and thus $r \leq m \leq n+m$. Otherwise, we have $\mathfrak{q}_i \subsetneq \mathfrak{m} \subseteq \mathfrak{q}_{i+1}$, for some index $0 \leq i \leq r$. It follows that $\mathfrak{q}_0 \subsetneq \ldots \subsetneq \mathfrak{q}_i$ is a chain of prime ideals of V, and thus $i \leq n$, and that tha chain $\mathfrak{q}_{i+1} \subsetneq \ldots \subsetneq \mathfrak{q}_r$ corresponds to a chain of prime ideals of D of length $r-i$, that is $r - i \leq m$. Hence, $r \leq n+m$. □

9. Ultrafilters

We will see in the following that there is a very powerful tool for describing the prime spectrum of several classes of rings. This tool is the notion of ultrafilter.

(9.1) Definition. *Let X be a set.*

(a) *A nonempty collection $\mathscr{F}$ of nonempty subsets of X is called to be* a filter on *X if the following properties are verified:*
   - *$F \cap G \in \mathscr{F}$, for any $F, G \in \mathscr{F}$;*
   - *if $F \in \mathscr{F}$ and $F \subseteq Y \subseteq X$, then $Y \in \mathscr{F}$.*

(b) *A maximal element in the collection of all filters on X, partially ordered by $\subseteq$, is called to be* an ultrafilter on X.

(9.2) Example. (a) If X is a set, then $\{X\}$ is a filter on X. Moreover, for any filter $\mathscr{F}$ on X, we have $X \in \mathscr{F}$.

(b) Recall that if X is a topological space and $x \in X$, a subset Y of X is called *a neighborhood of x* if there is an open subset U of X such that $x \in U \subseteq Y$. Then, clearly, the collection $\mathscr{I}(x)$ of the neighborhoods of x is a filter on X.

(c) Now let X be any set and let $x \in X$. Then, the collection

$$\mathscr{U}_x := \{Y \subseteq X : x \in Y\}$$

is an ultrafilter on X, called *the trivial ultrafilter generated by x*. It is obvious that $\mathscr{U}_x$ is a filter on X. If there is a filter $\mathscr{F}$ on X such that $\mathscr{U}_x \subsetneq \mathscr{F}$, then we can pick a set $F \in \mathscr{F} - \mathscr{U}_x$. This implies $x \notin F$, that is, $X - F \in \mathscr{U}_x \subseteq \mathscr{F}$. It follows $\emptyset = F \cap (X - F) \in \mathscr{F}$, a contradiction (by definition, any filter does not contain $\emptyset$).

Recall that a nonempty collection $\mathcal{F}$ of subsets of a set X *has the finite intersection property* if, for any finite subcollection $\mathcal{G}$ of $\mathcal{F}$, we have $\bigcap \mathcal{G} \neq \emptyset$.

(9.3) Remark. Let X be a set.

(a) By definition, any filter on X has the finite intersection property.

(b) Let $\mathcal{F}$ be a collection of subsets of X with the finite intersection property. Then, it is immediately seen that the collection of sets

$$\mathscr{F}(\mathcal{F}) := \{A \subseteq X : A \supseteq \bigcap_{i=1}^{n} F_i, \text{ for some } F_1, \ldots, F_n \in \mathcal{F}\}$$

is a filter on X and $\mathcal{F} \subseteq \mathscr{F}(\mathcal{F})$.

(9.4) Proposition (Ultrafilter Lemma, Tarski, 1930)**.** *Let X be a set and $\mathcal{F}$ be a collection of subsets of X with the finite intersection property. Then there is an ultrafilter $\mathscr{U}$ on X such that $\mathcal{F} \subseteq \mathscr{U}$. In particular, any filter on X can be extended to an ultrafilter on X.*

Proof. By (9.3b), the collection $\Sigma := \{\mathscr{F} : \mathscr{F} \text{ is a filter on } X, \mathscr{F} \supseteq \mathcal{F}\}$, partially ordered by inclusion, is nonempty. Moreover, by definition, the union of a chain of filters is a filter. Thus, any chain in Σ has an upper bound. The conclusion follows by Zorn's Lemma. The last statement follows from (9.3a). □

(9.5) Proposition. *Let X be a set and let $\mathscr{U}$ be a collection of subsets of X. Then, the following conditions are equivalent.*

(i) *$\mathscr{U}$ is an ultrafilter on X.*

(ii) *$\mathscr{U}$ is a filter on X and, if $Y, Z \subseteq X$ satisfy $Y \cup Z \in \mathscr{U}$, then either $Y \in \mathscr{U}$ or $Z \in \mathscr{U}$.*

(iii) *$\mathscr{U}$ is a filter on X and, for any subset Y of X, then either $Y \in \mathscr{U}$ or $X - Y \in \mathscr{U}$.*

Proof. (i)$\Longrightarrow$(ii). Take $Y, Z \subseteq X$ such that $Y \cup Z \in \mathscr{U}$ and $Y \notin \mathscr{U}$. Then, for every $U \in \mathscr{U}$ we have $Z \cap U \neq \emptyset$ (otherwise, $U \subseteq X - Z$ and thus $Y \supseteq U \cap (Y \cup Z) \in \mathscr{U}$; it would follow $Y \in \mathscr{U}$, a contradiction). This proves that the collection $\mathscr{U} \cup \{Z\}$ has the finite intersection property and thus, in view of (9.4), there is an ultrafilter

$\mathscr{V}$ on X such that $\mathscr{U} \cup \{Z\} \subseteq \mathscr{V}$ and, since $\mathscr{U}$ is an ultrafilter, it follows $\mathscr{V} = \mathscr{U} \cup \{X\} = \mathscr{U}$, that is, $Z \in \mathscr{U}$.

(ii)$\Longrightarrow$(iii). Take any subset Y of X and note that, since X belongs to any filter on X, $Y \cup (X - Y) \in \mathscr{U}$. Apply (ii) to get the conclusion.

(iii)$\Longrightarrow$(i). Suppose there exists a filter $\mathscr{F}$ on X such that $\mathscr{U} \subsetneq \mathscr{F}$, and take a set $F \in \mathscr{F} - \mathscr{U}$. By (iii), the set $X - F \in \mathscr{U} \subseteq \mathscr{F}$ and thus $\emptyset = F \cap (X - F) \in \mathscr{F}$, a contradiction. □

(9.6) Corollary. *Any ultrafilter on a finite set is trivial.*

Proof. Set $X := \{x_1, \ldots, x_n\}$ and let $\mathscr{U}$ be an ultrafilter on X. Since

$$X = \{x_1\} \cup \ldots \cup \{x_n\} \in \mathscr{U},$$

condition (ii) of (9.5) implies that $\{x_i\} \in \mathscr{U}$, for some $1 \leq i \leq n$. It immediately follows that $\mathscr{U} \subseteq \mathscr{U}_{x_i}$ and, since $\mathscr{U}$ is an ultrafilter, $\mathscr{U} = \mathscr{U}_{x_i}$. □

(9.7) Proposition. *Any infinite set admits nontrivial ultrafilters.*

Proof. Let X be an infinite set and let $\mathcal{F}$ the collection of all the subsets Y of X such that $X - Y$ is finite. Since X is infinite, $\mathcal{F}$ has the finite intersection property, and thus, in view of (9.4), there is an ultrafilter $\mathscr{U}$ on X such that $\mathcal{F} \subseteq \mathscr{U}$. For any $x \in X$ we have $X - \{x\} \in \mathcal{F} \subseteq \mathscr{U}$, and since $\emptyset \notin \mathscr{U}$, it follows that $\{x\} \notin \mathscr{U}$. This proves that $\mathscr{U}$ is nontrivial. □

9.1. **The Stone-Cech compactification of a discrete space.** Let X be a topological space. Recall that a compactification of X is a compact space Y together with a topological embedding $\iota : X \longrightarrow Y$ such that $\iota(X)$ is dense in Y. Now we will use ultrafilters to give a very important example of compactification of a discrete space. We start by fixing some notation: for any set X let βX be the collection of all the ultrafilters on X and, for any subset Y of X, let $Y^\sharp := \{\mathscr{U} \in \beta X : Y \in \mathscr{U}\}$. Since the collection $\mathcal{B} := \{Y^\sharp : Y \subseteq X\}$ clearly covers the set βX and $(Y \cap Z)^\sharp = Y^\sharp \cap Z^\sharp$, it immediately follows that $\mathcal{B}$ is a basis of open sets for a (unique) topology on βX. We will call it *the Stone-Cech topology.*

(9.8) Proposition. *Let X be a set.*

(a) *For any $Y \subseteq X$ we have $\beta X - Y^\sharp = (X - Y)^\sharp$. In particular, the basic open set $Y^\sharp$ is clopen in βX.*

(b) *βX is a compact and Hausdorff space.*

Proof. (a). The equality $\beta X - Y^\sharp = (X - Y)^\sharp$ holds since exactly one of the sets $Y, X - Y$ belongs to an ultrafilter, in view of (9.5).

(b). First, we show that βX is a Hausdorff space. Take distinct ultrafilters $\mathscr{U}, \mathscr{V}$ on X, and take a set $Y \in \mathscr{U} - \mathscr{V}$. Again by (9.5), $X - Y \in \mathscr{V}$, and thus $Y^\sharp, (X - Y)^\sharp$ are disjoint open neighborhoods of $\mathscr{U}, \mathscr{V}$, respectively.

We show now that βX is compact. Let $\mathcal{A}$ be an open cover of βX. By the definition of the Stone-Cech topology, we can assume, without loss of generality, that $\mathcal{A}$ consists of basic open sets, that is $\mathcal{A} := \{Y^\sharp : Y \in \mathscr{F}\}$, where $\mathscr{F}$ is a collection of subsets of X. By contradiction, assume that $\mathcal{A}$ does not admit any finite subcover, and let $\mathscr{F}' := \{X - Y : Y \in \mathscr{F}\}$. Take finitely many members $Y_1, \ldots, Y_n \in \mathscr{F}$. By assumtion, $\bigcup_{i=1}^n Y_i^\sharp \subsetneq \beta X$, and thus there exists an ultrafilter $\mathscr{U}$ on X such that $Y_i \notin \mathscr{U}$, for any $i = 1, \ldots, n$. By (9.5) and by definition, we have $\bigcap_{i=1}^n (X - Y_i) \in \mathscr{U}$ and, a fortiori, $\bigcap_{i=1}^n (X - Y_i) \neq \emptyset$. This argument shows that $\mathscr{F}'$

has the finite intersection property and hence, in view of (9.4), there is an ultrafilter $\mathscr{V}$ such that $\mathscr{F}' \subseteq \mathscr{V}$. In particular, we have $X - Y \notin \mathscr{V}$, and equivalently $Y \notin \mathscr{V}$, for any $Y \in \mathscr{F}$, against the fact that $\mathcal{A}$ is an open cover of βX. The proof is now complete. □

(9.9) Proposition. *Let X be a set, endowed with the discrete topology, and let $\iota : X \longrightarrow \beta X$ be the function defined by $\iota(x) := \mathscr{U}_x$, for any $x \in X$, where $\mathscr{U}_x$ is the trivial ultrafilter generated by x. Then, βX and ι provide a compactification of X, called* the Stone-Cech compactification.

Proof. It remains to show that ι is a topological embedding and that $\iota(X)$ is dense in βX. The map ι is obviously injective and continuous. Moreover, the straightforward equality $\iota(Y) = Y^\sharp \cap \iota(X)$, which holds for any subset Y of X, shows that ι is a topological embedding. Finally, take any nonempty open subset Ω of βX. In order to prove that $\Omega \cap \iota(X) \neq \emptyset$ it suffices to assume that $\Omega = Y^\sharp$, for some nonempty subset Y of X. Clearly, for any $y \in Y$, the trivial ultrafilter $\mathscr{U}_y$ belongs to $\Omega \cap \iota(X)$. □

We recall here the following easy fact for the reader convenience.

(9.10) Lemma. *Let X be a T_4 space, $U \subseteq X$ be open and $x \in U$. Then, there is an open set V such that $x \in V$ and $\overline{V} \subseteq U$.*

Proof. The sets $\{x\}$ and $X - U$ are closed and disjoint. By assumption, there are disjoint open sets V, Ω of X such that $x \in V$ and $X - U \subseteq \Omega$. It follows that $\overline{V} \subseteq X - \Omega \subseteq U$. □

(9.11) Theorem (Universal property of the Stone-Cech compactification)**.** *Let X be a discrete space and let ι be the topological embedding defined in (9.9). Then, for any compact and Hausdorff space K and any function $f : X \longrightarrow K$, there is a unique continuous function $\widehat{f} : \beta X \longrightarrow K$ such that $f = \widehat{f} \circ \iota$.*

Proof. First, we will show the following claims.

Claim 1. Take an ultrafilter $\mathscr{U}$ on X and set $C(\mathscr{U}) := \bigcap\{\overline{f(Y)} : Y \in \mathscr{U}\}$. If U is an open subset of K and $U \cap C(\mathscr{U}) \neq \emptyset$, then $f^{-1}(U) \in \mathscr{U}$.

As a matter of fact, take an element $k \in U \cap C(\mathscr{U})$. This implies that, for any $Y \in \mathscr{U}$, we have $U \cap f(Y) \neq \emptyset$ and, equivalently, $Y \cap f^{-1}(U) \neq \emptyset$. Thus, keeping in mind (9.5), we easily infer that $f^{-1}(U) \in \mathscr{U}$.

Claim 2. For any ultrafilter $\mathscr{U}$ on X, the set $C(\mathscr{U})$ consists of a unique point.

Indeed, take sets $Y_1, \ldots, Y_n \in \mathscr{U}$ and note that $T := \bigcap_{i=1}^n Y_i \neq \emptyset$, since $T \in \mathscr{U}$. Then $\emptyset \neq \overline{f(T)} \subseteq \bigcap_{i=1}^n \overline{f(Y_i)}$. This proves that the collection $\{\overline{f(Y)} : Y \in \mathscr{U}\}$ is a collection of closed sets of the compact space K, with the finite intersection property. It follows that $C(\mathscr{U}) \neq \emptyset$. Now assume that $x, y \in C(\mathscr{U})$ and that $x \neq y$. Since K is a Hausdorff space, there are disjoint open sets $U, V \subseteq K$ such that $x \in U, y \in V$. By Claim 1 we have $f^{-1}(U), f^{-1}(V) \in \mathscr{U}$, and thus $\emptyset = f^{-1}(U) \cap f^{-1}(V) \in \mathscr{U}$, a contradiction.

Now, let $\widehat{f} : \beta X \longrightarrow K$ be the function such that, for any $\mathscr{U} \in \beta X$, $\widehat{f}(\mathscr{U})$ is the unique element of the set $C(\mathscr{U})$. We show that $\widehat{f}$ is continuous by proving that it is continuous at any point of βX. Take any ultrafilter $\mathscr{U}$ on X and any open neighborhood U of $\widehat{f}(\mathscr{U})$. Since K is normal, being it compact and Hausdorff, we can take an open neighborhood V of $\widehat{f}(\mathscr{U})$ such that $\overline{V} \subseteq U$, in view of (9.10). Since clearly $V \cap C(\mathscr{U}) \neq \emptyset$, we have $f^{-1}(V) \in \mathscr{U}$, by Claim 1, that is, $(f^{-1}(V))^\sharp$

is an open neighborhood of $\mathscr{U}$. Take any ultrafilter $\mathscr{V} \in (f^{-1}(V))^\sharp$. By definition, $\widehat{f}(\mathscr{V}) \in \bigcap\{\overline{f(Y)} : Y \in \mathscr{V}\} \subseteq \overline{f(f^{-1}(V))} \subseteq \overline{V} \subseteq U$. This proves that $\widehat{f}$ is continuous. The fact that $f = \widehat{f} \circ \iota$ follows immediately by Claim 2 and the fact that, for any $x \in X$, then clearly $f(x) \in \bigcap\{\overline{f(Y)} : x \in Y\} =: C(\mathscr{U}_x)$. Finally, if $g : \beta X \longrightarrow K$ is a continuous function such that $g \circ \iota = f$, then $\widehat{f}|_{\iota(X)} = g|_{\iota(X)}$. From the fact that $\iota(X)$ is dense in βX and that K is a Hausdorff space it follows that $\widehat{f} = g$. □

9.2. **The prime spectrum of a product of fields.** Now we will give an interesting application of the Stone-Cech compactification for describing the prime spectrum of any product of fields.

(9.12) Remark. Let $\{K_x : x \in X\}$ be a nonempty collection of fields, where X is any index set, and consider the ring $A := \prod_{x \in X} K_x$. By definition, an element of A is a function $f : X \longrightarrow \bigcup_{x \in X} K_x$ such that $f(x) \in K_x$, for any $x \in X$.

(a) For any $x \in X$, $\mathfrak{m}_x := \{f \in A : f(x) = 0\}$ is a maximal ideal of A, because it is the kernel of the surjective ring homomorphism $p_x : A \longrightarrow K_x$, $f \mapsto f(x)$.

(b) Note that if X is finite, then any prime ideal of A is of the form $\mathfrak{m}_x$, for $x \in X$. As a matter of fact, let $\mathfrak{p}$ any prime ideal of A and, for any $x \in X$, let $f_x \in A$ be defined by

$$f_x(y) := \begin{cases} 1 & \text{if } y \neq x \\ 0 & \text{if } y = x \end{cases}.$$

Since X is finite, the product $\prod_{x \in X} f_x$ is defined and belongs to A and clearly $\prod_{x \in X} f_x = 0 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime, $f_x \in \mathfrak{p}$, for some $x \in X$. Moreover, for any $f \in \mathfrak{m}_x$ we have $f = ff_x \in \mathfrak{p}$. Thus $\mathfrak{m}_x \subseteq \mathfrak{p}$ and, since $\mathfrak{m}_x$ is maximal, the equality holds.

(c) If X is infinite, the prime spectrum of A is much more complicated. This intuition comes from the following easy observation. Let $\mathfrak{a} := \bigoplus_{x \in X} K_x$ be the direct sum of the collection $\{K_x : x \in X\}$, that is,

$$\mathfrak{a} := \{f \in A : \{x \in X : f(x) \neq 0\} \text{ is finite}\}.$$

It is immediately seen that $\mathfrak{a}$ is an ideal of A and, since X is infinite, it is a proper ideal ($1 \notin \mathfrak{a}$). Then, there is a maximal ideal $\mathfrak{m}$ of A such that $\mathfrak{a} \subseteq \mathfrak{m}$. But clearly, for any $x \in X$, it happens that $\mathfrak{m} \neq \mathfrak{m}_x$ since, if f_x is the function defined in part (b), then $1 - f_x \in \mathfrak{a} - \mathfrak{m}_x$.

(9.13) Lemma. *Let A be a ring such that any element of A can be written as the product of an invertible element of A and an idempotent of A. Then A is zero-dimensional.*

Proof. Let $\mathfrak{p}$ be a prime ideal of A. We will show that $A/\mathfrak{p}$ is a field. For any $a \in A$, let $\overline{a} \in A/\mathfrak{p}$ denote the class of a modulo $\mathfrak{p}$. Suppose that $\overline{a} \neq 0$ and take an invertible element $u \in A$ and a idempotent $e \in A$ such that $a = ue$. Thus $\overline{a} = \overline{ue}$ and, since the unique idempotents of an integral domain are $0, 1$, we must have $\overline{e} = 1$ (otherwise $\overline{a} = 0$). It follows $\overline{a} = \overline{u}$, that is, $\overline{a}$ is invertible in $A/\mathfrak{p}$. □

(9.14) Proposition. *Any product of fields is zero-dimensional.*

Proof. Let $\{K_x : x \in X\}$ be a nonempty collection of fields and let $A := \prod_{x\in X} K_x$. For any $f \in A$ define the elements $u, e \in A$ by setting

$$u(x) := \begin{cases} f(x) & \text{if } f(x) \neq 0 \\ 1 & \text{if } f(x) = 0 \end{cases} \qquad\qquad e(x) := \begin{cases} 1 & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0. \end{cases}$$

Clearly, u is an invertible element of A, e is an idempotent and $f = ue$. The conclusion follows from (9.13). □

Let $\{K_x : x \in X\}$ be a nonempty collection of fields and let $A := \prod_{x\in X} K_x$. In view of (9.12a), there is a natural injective map $\lambda : X \longrightarrow \mathrm{Spec}(A)$ defined by $\lambda(x) := \mathfrak{m}_x$, for any $x \in X$. By (3.9) and (9.14), $\mathrm{Spec}(A)$ is a compact and Hausdorff space. Thus, in light of the universal property of the Stone-Cech compactification, there exists a unique continuous function $\widehat{\lambda} : \beta X \longrightarrow \mathrm{Spec}(A)$ such that $\widehat{\lambda} \circ \iota = \lambda$, where $\iota : X \longrightarrow \beta X$ is the canonical topological embedding (now X is endowed with the discrete topology). In the next crucial result we are going to describe the map $\widehat{\lambda}$ and to show that it is a homeomorphism.

(9.15) Theorem. *Let $\{K_x : x \in X\}$ be a nonempty collection of fields and let $A := \prod_{x\in X} K_x$. For any ultrafilter $\mathscr{U}$ on X let*

$$\mathfrak{p}_{\mathscr{U}} := \{f \in A : \{x \in X : f(x) = 0\} \in \mathscr{U}\}.$$

(a) *Then $\mathfrak{p}_{\mathscr{U}}$ is a prime ideal of A.*
(b) *Let $\lambda : X \longrightarrow \mathrm{Spec}(A)$ be the map defined in the above discussion ($\lambda(x) := \mathfrak{m}_x$, for any $x \in X$), and let $\iota : X \longrightarrow \beta X$ be the canonical topological embedding. Then the unique continuous function $\widehat{\lambda} : \beta X \longrightarrow \mathrm{Spec}(A)$ such that $\widehat{\lambda}\circ\iota = \lambda$, induced by the unversal property of the Stone-Cech compactification, is defined by $\widehat{\lambda}(\mathscr{U}) := \mathfrak{p}_{\mathscr{U}}$, for any $\mathscr{U} \in \beta X$, and it is a homeomorphism.*

Proof. For any $f \in A$, set $Z_f := \{x \in X : f(x) = 0\}$.

(a). For any $f, g \in \mathfrak{p}_{\mathscr{U}}$ and any $a \in A$ we clearly have $Z_f \cap Z_g \subseteq Z_{f\pm g}$ and $Z_f \subseteq Z_{af}$. Since $Z_f, Z_g \in \mathscr{U}$, we have $Z_f \cap Z_g \in \mathscr{U}$ and thus $Z_{f\pm g}, Z_{af} \in \mathscr{U}$, that is, $f \pm g, af \in \mathfrak{p}_{\mathscr{U}}$. Moreover $Z_1 = \emptyset \notin \mathscr{U}$. This proves that $\mathfrak{p}_{\mathscr{U}}$ is a proper ideal of A. Take now elements $f, g \in A$ such that $fg \in \mathfrak{p}_{\mathscr{U}}$. Thus $Z_f \cup Z_g = Z_{fg} \in \mathscr{U}$ and, by (9.5), either $Z_f \in \mathscr{U}$ or $Z_g \in \mathscr{U}$. In other words, either $f \in \mathfrak{p}_{\mathscr{U}}$ or $g \in \mathfrak{p}_{\mathscr{U}}$. Thus $\mathfrak{p}_{\mathscr{U}}$ is a prime ideal of A.

(b). Let $\mathscr{U}$ be an ultrafilter on X and, as in the proof of the universal property of the Stone-Cech compactification, let $C(\mathscr{U}) := \bigcap\{\overline{\lambda(U)} : U \in \mathscr{U}\}$. Keeping in mind (3.2c) we have

$$C(\mathscr{U}) = \bigcap_{U\in\mathscr{U}} \overline{\{\mathfrak{m}_x : x \in U\}} = \bigcap_{U\in\mathscr{U}} V(\bigcap_{x\in U} \mathfrak{m}_x) = \bigcap_{U\in\mathscr{U}} V(\{f \in A : f|_U = 0\}).$$

Fix a set $U \in \mathscr{U}$ and note that a function $f \in A$ is such that $f|_U = 0$ if and only if $U \subseteq Z_f$ and thus it follows $Z_f \in \mathscr{U}$, that is $f \in \mathfrak{p}_{\mathscr{U}}$. This proves that $\mathfrak{p}_{\mathscr{U}} \in C(\mathscr{U})$. By Claim 1 of the proof of the universal property of the Stone-Cech compactification, it follows that the unique continuous function $\widehat{\lambda} : \beta X \longrightarrow \mathrm{Spec}(A)$ such that $\widehat{\lambda} \circ \iota = \lambda$ is defined by setting $\widehat{\lambda}(\mathscr{U}) := \mathfrak{p}_{\mathscr{U}}$, for any $\mathscr{U} \in \beta X$. We claim that $\widehat{\lambda}$ is injective. Take distinct ultrafilters $\mathscr{U}, \mathscr{V}$ on X, fix a set $U \in \mathscr{U} - \mathscr{V}$ and consider the function $f \in A$ such that

$$f(x) := \begin{cases} 0 & \text{if } x \in U \\ 1 & \text{if } x \in X - U. \end{cases}$$

Then $U = Z_f \in \mathscr{U} - \mathscr{V}$, that is, $f \in \mathfrak{p}_{\mathscr{U}} - \mathfrak{p}_{\mathscr{V}}$. In order to see that $\widehat{\lambda}$ is surjective, take any prime ideal $\mathfrak{p}$ of A and let $\mathcal{G} := \{Z_f : f \in \mathfrak{p}\}$. Take elements $f_1, \ldots, f_n \in \mathfrak{p}$ and, in light of the proof of (9.14), take invertible elements $u_1, \ldots, u_n \in A$ and idempotents $e_1, \ldots, e_n \in A$ such that $f_i = u_i e_i$, for $1 \leq i \leq n$. Clearly, since $\mathfrak{p}$ is a prime ideal, $e_i \in \mathfrak{p}$, for $1 \leq i \leq n$. If $\bigcap_{i=1}^n Z_{f_i} = \emptyset$, for any $x \in X$ there is an index $i \in \{1, \ldots, n\}$ such that $f_i(x) \neq 0$ and, a fortiori, $e_i(x) \neq 0$. Since $e_i = e_i^2$, we infer that $e_i(x) = 1$. It follows that $\prod_{i=1}^n (1 - e_i) = 0 \in \mathfrak{p}$, and thus $1 - e_i \in \mathfrak{p}$, for some $1 \leq i \leq n$. Thus we have $1 = (1 - e_i) + e_i \in \mathfrak{p}$, a contradiction. This argument shows that $\mathcal{G}$ has the finite intersection property, and hence $\mathcal{G}$ can be extended to an ultrafilter $\mathscr{U}$ on X, by (9.4). By definition, we have $\mathfrak{p} \subseteq \mathfrak{p}_{\mathscr{U}}$ and, by (9.14), $\mathfrak{p} = \mathfrak{p}_{\mathscr{U}}$. Finally, $\widehat{\lambda}$ is closed, being it a continuous function from a compact space into a Hausdorff space (by (9.8) and (3.9)). The conclusion is now clear. □

10. The constructible topology on the prime spectrum of a ring.

Let A be a ring. As we saw in (3.2e), the prime spectrum $\operatorname{Spec}(A)$ of A, endowed with the Zariski topology, is always a T_0 space but it is Hausdorff if and only if A is zero-dimensional, by (3.9). In the following, we are going to define and study a new topology on $\operatorname{Spec}(A)$, introduced by A. Grothendieck, which refines the Zariski topology and makes $\operatorname{Spec}(A)$ a compact and Hausdorff space.

(10.1) Definition. [10, (7.2.11)] *If A is any ring,* the constructible topology *on* $\operatorname{Spec}(A)$ *is the coarsest topology for which the open and compact subspaces of* $\operatorname{Spec}(A)$ *(when equipped with the Zariski topology) are clopen sets. We will denote by* $\operatorname{Spec}(A)^{\text{cons}}$ *the set* $\operatorname{Spec}(A)$*, endowed with the constructible topology.*

(10.2) Remark. Let A be a ring.

(a) In view of (3.2f), the open and compact subspaces of $\operatorname{Spec}(A)$ are precisely the subsets of the form $\bigcup_{i=1}^n D(f_i)$, where $f_1, \ldots, f_n \in A$ and $n \geq 1$, and they form a basis $\mathcal{B}$ of open sets of the Zariski topology. Since, by definition, any member of $\mathcal{B}$ is, in particular, open in $\operatorname{Spec}(A)^{\text{cons}}$, the Zariski topology is coarser than the constructible topology.

(b) By definition, a subbasis of open sets for $\operatorname{Spec}(A)^{\text{cons}}$ is

$$\mathcal{S} := \mathcal{B} \cup \{\operatorname{Spec}(A) - \Omega : \Omega \in \mathcal{B}\},$$

and thus the collection of all finite intersections of members of $\mathcal{S}$ forms a basis of open sets for the constructible topology. Such finite intersections are sets of the form $\Lambda := \bigcap_{i=1}^n (\operatorname{Spec}(A) - V(\mathfrak{a}_i)) \cap \bigcap_{j=1}^m V(\mathfrak{b}_j)$, where the ideals $\mathfrak{a}_i, \mathfrak{b}_j$ are finitely generated. In other words, $\Lambda = (\operatorname{Spec}(A) - V(\mathfrak{a})) \cup V(\mathfrak{b})$, where $\mathfrak{a} = \mathfrak{a}_1 \cdots \mathfrak{a}_n$ and $\mathfrak{b} = \mathfrak{b}_1 + \ldots + \mathfrak{b}_n$ (note that $\mathfrak{a}$ and $\mathfrak{b}$ are finitely generated). If $\mathfrak{a} := (a_1, \ldots, a_r)A$, then $\Lambda = \bigcup_{h=1}^r D(a_i) \cap V(\mathfrak{b})$. Since, by definition, any set of the form $D(a) \cap V(\mathfrak{i})$, where $a \in A$ and $\mathfrak{i}$ is a finitely generated ideal of A, is open in $\operatorname{Spec}(A)^{\text{cons}}$, it finally follows that a basis of open sets for the constructible topology is

$$\mathcal{B}_{\text{cons}} := \{D(a) \cap V(\mathfrak{i}) : a \in A, \mathfrak{i} \text{ finitely generated ideal of } A\}.$$

Note that any set in $\mathcal{B}_{\text{cons}}$ is clopen in $\text{Spec}(A)^{\text{cons}}$.

(c) $\text{Spec}(A)^{\text{cons}}$ is a Hausdorff space. Indeed, if $\mathfrak{p}, \mathfrak{q}$ are distinct prime ideals of A, take an element $f \in \mathfrak{p} - \mathfrak{q}$. By definition, $V(f), D(f)$ are disjoint open neighborhoods of $\mathfrak{p}, \mathfrak{q}$, respectively, in $\text{Spec}(A)^{\text{cons}}$.

(d) $\text{Spec}(A)^{\text{cons}}$ is totally disconnected. Indeed, if C is any subset of $\text{Spec}(A)$ and $\mathfrak{p}, \mathfrak{q} \in C$ are distinct prime ideals, take an element $f \in \mathfrak{p} - \mathfrak{q}$. Then, by definition, $\Omega_1 := D(f) \cap C, \Omega_2 := V(f) \cap C$ are nonempty, disjoint and open subspaces of C (with the subspace topology induced by that of $\text{Spec}(A)^{\text{cons}}$) and $C = \Omega_1 \cap \Omega_2$. Thus C is not connected.

By using just the definition, it is hard to provide an easy description of the closed subsets, with respect to the constructible topology. We will do this by using ultrafilters.

(10.3) Lemma. *Let A be a ring and Y be a nonempty subset of* $\text{Spec}(A)$. *The following properties hold.*

(a) *Let $\mathscr{U}$ be an ultrafilter on Y. Then,*

$$Y_{\mathscr{U}} := \{a \in A : V(a) \cap Y \in \mathscr{U}\}$$

is a prime ideal of A, called the ultrafilter limit point of Y (with respect to the ultrafilter $\mathscr{U}$)*.*

(b) *For any prime ideal $\mathfrak{p} \in Y$, let $\kappa_{\mathfrak{p}}$ denote the residue field of A at $\mathfrak{p}$ (that is, $\kappa_{\mathfrak{p}}$ is the quotient field of $A/\mathfrak{p}$ or, equivalently, the residue field of the local ring $A_{\mathfrak{p}}$). Consider the ring $A_Y := \prod_{\mathfrak{p} \in Y} \kappa_{\mathfrak{p}}$ and let $\lambda : A \longrightarrow A_Y$ be the canonical ring homomorphism (i.e., for any $a \in A$, then $\lambda(a) \in A_Y$ is the function defined by $\lambda(a)(\mathfrak{p}) := a + \mathfrak{p} \in A/\mathfrak{p} \subseteq \kappa_{\mathfrak{p}}$, for any $\mathfrak{p} \in Y$. If $\lambda^\star : \text{Spec}(A_Y) \longrightarrow \text{Spec}(A)$ is the canonical map induced by λ, then*

$$\lambda^\star(\text{Spec}(A_Y)) = \{Y_{\mathscr{U}} : \mathscr{U} \textit{ ultrafilter on } Y\}.$$

Proof. Clearly, it is sufficient to prove part (b). By (9.15), we have

$$\text{Spec}(A_Y) = \{\mathfrak{p}_{\mathscr{U}} : \mathscr{U} \text{ ultrafilter on } Y\},$$

where $\mathfrak{p}_{\mathscr{U}} := \{f \in A_Y : Z_f \in \mathscr{U}\}$ and $Z_f := \{\mathfrak{p} \in Y : f(\mathfrak{p}) = 0\}$. Now, for every $a \in A$,

$$Z_{\lambda(a)} := \{\mathfrak{p} \in Y : \lambda(a)(\mathfrak{p}) = 0\} = \{\mathfrak{p} \in Y : a + \mathfrak{p} = 0 \text{ in } A/\mathfrak{p}\} = V(a) \cap Y.$$

It follows immediately that, for any ultrafilter $\mathscr{U}$ on Y, $\lambda^{-1}(\mathfrak{p}_{\mathscr{U}}) = Y_{\mathscr{U}}$. The conclusion is now clear. □

(10.4) Example. Let A be a ring.

(a) If Y is a subset of $\text{Spec}(A)$, $\mathfrak{p} \in Y$ and $\mathscr{U}_{\mathfrak{p}}$ is the trivial ultrafilter on Y generated by $\mathfrak{p}$, then $\mathfrak{p} = Y_{\mathscr{U}_{\mathfrak{p}}}$.

(b) If $A = \mathbb{Z}$, $Y := \text{Max}(A)$ and $\mathscr{U}$ is any nontrivial ultrafilter on Y, then $Y_{\mathscr{U}} = (0)$. Indeed, if $n \in \mathbb{Z}$ and $V(n) \cap Y \in \mathscr{U}$, then $V(n) \cap Y$ is infinite, since $\mathscr{U}$ is nontrivial, and thus $n = 0$ (any nonzero integer has only finitely many prime factors).

(10.5) Definition. *Let A be a ring and let $Y \subseteq$* $\text{Spec}(A)$. *We say that Y is* ultrafilter closed *if $Y_{\mathscr{U}} \in Y$, for any ultrafilter $\mathscr{U}$ on Y.*

(10.6) Example. Let A be a ring.

(a) For any ideal $\mathfrak{a}$ of A, then $V(\mathfrak{a})$ is ultrafilter closed. Indeed, let $\mathscr{U}$ be an ultrafilter on $Y := V(\mathfrak{a})$ and let $a \in \mathfrak{a}$. Then $V(a) \cap Y = Y \in \mathscr{U}$, that is, $a \in Y_{\mathscr{U}}$. This proves that, $\mathfrak{a} \subseteq Y_{\mathscr{U}}$, i.e., $Y_{\mathscr{U}} \in Y$.

(b) For any element $f \in A$, then $D(f)$ is ultrafilter closed. Indeed, if $\mathscr{U}$ is an ultrafilter on $Y := D(f)$, then $f \notin Y_{\mathscr{U}}$ (otherwise, $\emptyset = V(f) \cap Y \in \mathscr{U}$, contradiction), that is, $Y_{\mathscr{U}} \in Y$.

(c) If $A := \mathbb{Z}$ and $Y := \mathrm{Max}(A)$, then Y is not ultrafilter closed, by (10.4b).

The proof of the following result is straightforward and it is left to the reader as an exercise.

(10.7) Lemma. *Let A be a ring, $X \subseteq \mathrm{Spec}(A)$ and $\mathscr{U}$ be an ultrafilter on X.*

(a) *If $U \in \mathscr{U}$, then $\mathscr{U}|_U := \{V \subseteq U : V \in \mathscr{U}\}$ is an ultrafilter on U and the equality of prime ideals $X_{\mathscr{U}} = U_{\mathscr{U}|_U}$ holds.*

(b) *If $X \subseteq Y \subseteq \mathrm{Spec}(A)$, then $\mathscr{U}^Y := \{T \subseteq Y : T \cap X \in \mathscr{U}\}$ is an ultrafilter on Y and $X_{\mathscr{U}} = Y_{\mathscr{U}^Y}$.*

(10.8) Proposition ([7] and [3, Remark 2.7(3)]). *Let A be a ring. Then the ultrafilter closed subsets of* $\mathrm{Spec}(A)$ *form the collection of the closed sets for a (unique) topology on* $\mathrm{Spec}(A)$. *We will call such a topology* the ultrafilter topology.

Proof. Clearly, $\emptyset$ and $\mathrm{Spec}(A)$ are ultrafilter closed.

Now, suppose that $Y, Z \subseteq \mathrm{Spec}(A)$ are ultrafilter closed and let $\mathscr{U}$ be an ultrafilter on $Y \cup Z$. By definition, $T := Y \cup Z \in \mathscr{U}$ and thus, by (9.5), we can assume, without loss of generality, that $Y \in \mathscr{U}$. By (10.7a), $\mathscr{U}|_Y$ is an ultrafilter on Y and $T_{\mathscr{U}} = Y_{\mathscr{U}|_Y}$, and $Y_{\mathscr{U}|_Y} \in Y \subseteq T$ since Y is ultrafilter closed. Thus T is ultrafilter closed.

Let $\mathcal{G}$ be a nonempty collection of ultrafilter closed subsets of $\mathrm{Spec}(A)$, let $X := \bigcap \mathcal{G}$ and let $\mathscr{U}$ be an ultrafilter on X. By (10.7b), for any $Y \in \mathcal{G}$, $\mathscr{U}^Y$ is an ultrafilter on Y and $X_{\mathscr{U}} = Y_{\mathscr{U}^Y} \in Y$, since Y is ultrafilter closed. It follows $X_{\mathscr{U}} \in \bigcap G =: X$, proving that X is ultrafilter clsoed. □

We recall now the following useful and basic fact about General Topology.

(10.9) Remark. Let X be a set and let $\mathcal{T}, \mathcal{U}$ be topologies on X, such that $(X, \mathcal{T})$ is compact, $(X, \mathcal{U})$ is Hausdorff and $\mathcal{U}$ is coarser than $\mathcal{T}$. Then, $\mathcal{T} = \mathcal{U}$. Indeed, any continuous function from a compact space to a Hausdorff space is closed, and thus the identity $(X, \mathcal{T}) \longrightarrow (X, \mathcal{U})$, which is continuous since $\mathcal{U}$ is coarser than $\mathcal{T}$, is a closed map, i.e., it is a homeomorphism. In other words, $\mathcal{U} = \mathcal{T}$.

(10.10) Theorem ([7, Theorem 8] and [3, Corollary 2.17]). *Let A be a ring. Then the constructible topology and the ultrafilter topology on* $\mathrm{Spec}(A)$ *are the same topology.*

Proof. Let Ω an open and compact subspace of $\mathrm{Spec}(A)$, with respect to the Zariski topology. As we saw in (10.2a), $\Omega = \bigcup_{i=1}^n (D(f_i))$, for some $f_1, \dots, f_n \in A$, and thus $\mathrm{Spec}(A) - \Omega = \bigcap_{i=1}^n V(f_i) = V(\mathfrak{a})$ where $\mathfrak{a} = (f_1, \dots, f_n)A$. In view of (10.6) and of the fact that the ultrafilter topology is a topology (see (10.8)), it follows that $\Omega, \mathrm{Spec}(A) - \Omega$ are ultrafilter closed, that is, that Ω is clopen, with respect to the ultrafilter topology. Then, by definition, the ultrafilter topology is finer than the constructible topology. By (10.2c) and (10.9), the conclusion will follow from the following claim.

Claim. Spec(A) is compact, with respect to the ultrafilter topology.

Let $\mathcal{G}$ be a nonempty collection of ultrafilter closed subsets of $X :=$ Spec(A) and assume that $\mathcal{G}$ has the finite intersection property. By (9.4) we can extend $\mathcal{G}$ to an ultrafilter $\mathscr{U}$ on X. We claim that the ultrafilter limit point $X_{\mathscr{U}}$ belongs to $\bigcap \mathcal{G}$. To prove this, take any set $Y \in \mathcal{G}$ and note that $Y \in \mathscr{U}$, since $\mathscr{U}$ extends $\mathcal{G}$. Then, by (10.7), $\mathscr{U}|_Y$ is an ultrafilter on Y and $X_{\mathscr{U}} = Y_{\mathscr{U}|_Y} \in Y$, since Y is ultrafilter closed. The proof is now complete. □

From the previous theorem and the claim in its proof the following corollary immediately follows.

(10.11) Corollary. *Let A be a ring. Then* $\mathrm{Spec}(A)^{\mathrm{cons}}$ *is compact.*

(10.12) Corollary. *Let A be a ring. Then the constructible topology is equal to the Zariski topology on* Spec(A) *if and only if A is zero-dimensional.*

Proof. Note that A is zero-dimensional if and only if the Zariski topology on Spec(A) is compact and Hausdorff, by (3.3) and (3.9). Apply (10.9) and (10.11) to get the conclusion. □

(10.13) Proposition. *Let $f : A \longrightarrow B$ be a ring homomorphism. Then the canonical map $f^\star$:* Spec(B) $\longrightarrow$ Spec(A) *is continuous and closed, when* Spec(A) *and* Spec(B) *are endowed with the constructible topology.*

Proof. Since, by (10.2c) and (10.11), $\mathrm{Spec}(A)^{\mathrm{cons}}, \mathrm{Spec}(B)^{\mathrm{cons}}$ are compact and Hausdorff spaces, it suffices to show that $f^\star : \mathrm{Spec}(B)^{\mathrm{cons}} \longrightarrow \mathrm{Spec}(A)^{\mathrm{cons}}$ is continuous. Let Ω be an open and compact subspace of Spec(A), with the Zariski topology, say $\Omega = \bigcup_{i=1}^n D(a_i)$, for some $a_1, \ldots, a_n \in A$. In view of (3.5a), we have $f^{\star -1}(\Omega) = \displaystyle\bigcup_{i=1}^n D(f(a_i))$ is open and compact (with respect to the Zariski topology) and $\mathrm{Spec}(B) - f^{\star -1}(\Omega) = V(f(a_1), \ldots, f(a_n))$. By definition, we infer that both $f^{\star -1}(\Omega)$ and $\mathrm{Spec}(B) - f^{\star -1}(\Omega)$ are open in $\mathrm{Spec}(B)^{\mathrm{cons}}$. Hence $f^\star : \mathrm{Spec}(B)^{\mathrm{cons}} \longrightarrow \mathrm{Spec}(A)^{\mathrm{cons}}$ is continuous, by (10.2b). □

In the following $\overline{Y}^{\mathrm{c}}$ will denote the closure of a subset $Y \subseteq$ Spec(A), with respect to the constructible topology.

(10.14) Proposition ([3, Proposition 2.13])**.** *Let A be a ring and let $Y \subseteq$* Spec(A)*. Then,*

$$\overline{Y}^{\mathrm{c}} = \{Y_{\mathscr{U}} : \mathscr{U} \text{ ultrafilter on } Y\}.$$

Proof. By (10.3b) and (10.13), the set $Y' := \{Y_{\mathscr{U}} : \mathscr{U}$ ultrafilter on $Y\}$ is closed in $\mathrm{Spec}(A)^{\mathrm{cons}}$. Furthermore, by (10.4a), $Y \subseteq Y'$. Take now any closed subset $C \subseteq \mathrm{Spec}(A)^{\mathrm{cons}}$ such that $Y \subseteq C$ and take an ultrafilter limit point $Y_{\mathscr{U}}$ of Y, for some ultrafilter $\mathscr{U}$ on Y. By (10.7b), $\mathscr{U}^C$ is an ultrafilter on C and $Y_{\mathscr{U}} = C_{\mathscr{U}^C}$. By (10.10), C is ultrafilter closed and thus $Y_{\mathscr{U}} \in C$. The conclusion is now clear. □

(10.15) Corollary. *Let A be a ring and let $Y \subseteq$* Spec(A)*. Then, the following conditions are equivalent.*

(i) *Y is closed in* $\mathrm{Spec}(A)^{\mathrm{cons}}$.

(ii) *There is a ring homomorphism* $f : A \longrightarrow B$, *for some ring* B, *such that* $Y = f^\star(\mathrm{Spec}(B))$.

Proof. (ii)$\Longrightarrow$(i) follows from the fact that $f^\star : \mathrm{Spec}(B)^{\mathrm{cons}} \longrightarrow \mathrm{Spec}(A)^{\mathrm{cons}}$ is a closed map (see (10.13)).

(i)$\Longrightarrow$(ii). Apply (10.14) and (10.3b). □

Recall that a subset of a topological space X is called to be locally closed if it is intersection of an open subset and a closed subset of X. Following Chevalley, we say that a subset of a topological space is *constructible* if it is a finite union of locally closed subsets. The following remark will justify the terminology choosen for the constructible topology.

(10.16) Remark. Let A be a ring. In (10.2b) we observed that a basis of clopen subsets for the constructible topology of Spec(A) is

$$\mathcal{B}_{\mathrm{cons}} := \{D(a) \cap V(\mathfrak{i}) : a \in A, \mathfrak{i} \text{ finitely generated ideal of } A\}.$$

(a) Any clopen subset of $\mathrm{Spec}(A)^{\mathrm{cons}}$ is constructible, with respect to the Zariski topology. As a matter of fact, a clopen Ω of $\mathrm{Spec}(A)^{\mathrm{cons}}$ is a union of a suitable subfamily of $\mathcal{B}_{\mathrm{cons}}$, being it open and it is compact, because it is closed in the compact space $\mathrm{Spec}(A)^{\mathrm{cons}}$ (see (10.11)). It follows that Ω is a finite union of members of $\mathcal{B}_{\mathrm{cons}}$. Thus the conclusion follows by noting that $\mathcal{B}_{\mathrm{cons}}$ consists of locally closed subspaces of Spec(A) (with the Zariski topology).

(b) If Spec(A), with the Zariski topology, is a Noetherian space, then the constructible subsets of Spec(A) are precisely the clopen subsets of $\mathrm{Spec}(A)^{\mathrm{cons}}$. Indeed, for any ideal $\mathfrak{a}$ of A, the open subset $\Omega := \mathrm{Spec}(A) - V(\mathfrak{a})$ of Spec(A) is compact, by noetherianity, and thus $\Omega = \bigcup_{i=1}^{n} D(f_i)$, for some $f_1, \ldots, f_n \in A$. It follows $V(\mathfrak{a}) = V(f_1, \ldots, f_n)$. Hence, a locally closed subset of Spec(A) is of the form $\Gamma := V(\mathfrak{a}) \cap (\mathrm{Spec}(A) - V(\mathfrak{b}))$, for some finitely generated ideals $\mathfrak{a}, \mathfrak{b}$ of A. In view of (10.2b), Γ is clopen, with respect to the constructible topology. Finally, it is enough to note that a finite union of clopen subsets of a topological space is clopen.

11. Spectral spaces.

(11.1) Definition ([12])**.** *A topological space is* spectral *if it is homeomorphic to the prime spectrum of a ring, endowed with the Zariski topology.*

By (3.2f), the collection $\mathcal{B} := \{D(f) : f \in A\}$ of all principal open subsets of the prime spectrum of a ring A is a basis for the Zariski topology, consisting of compact subspaces, by (3.6b). Note that $\mathcal{B}$ is closed under finite intersections, since we have $D(f) \cap D(g) = D(fg)$, for any $f, g \in A$. Thus, any spectral space has a basis of open and compact subspaces which is closed under finite intersections. Moreover, in view of (3.3), a spectral space is compact. Finally, given a ring A and an irreducible closed subset C of Spec(A), there is a unique prime ideal $\mathfrak{p}$ of A such that $C = V(\mathfrak{p}) = \overline{\{\mathfrak{p}\}}$, in view of (3.2d,e,g). We infer that any irreducible closed subspace of a spectral space X is the closure of a unique point $x \in X$ (the topological spaces satisfying this condition are called *sober spaces*). In particular, a spectral space is T_0.

Thus, any spectral space is compact, sober and has a basis of open and compact subspaces which is closed under finite intersections. In his thesis, M. Hochster showed much more.

(11.2) Theorem ([12, Proposition 4]). *For a topological space X the following conditions are equivalent.*

(i) *X is a spectral space.*
(ii) *X is compact, sober and has a basis of open and compact subspaces which is closed under finite intersections.*

The proof of the nontrivial part of this theorem ((ii)$\Longrightarrow$(i)) is hard and requires several difficult techniques which seem not so close to the goals of this course. For this reason we will not present the proof of (ii)$\Longrightarrow$(i).

We now classify Hausdorff spectral spaces.

(11.3) Corollary. *Let X be a Hausdorff space. Then, the following conditions are equivalent.*

(i) *X is a spectral space.*
(ii) *X is compact and it admits a basis consisting of clopen subsets.*

Proof. (i)$\Longrightarrow$(ii). It is sufficient to note that an open and compact subspace of a Hausdorff space is clopen and apply the remarks above or the trivial part of (11.2).

(ii)$\Longrightarrow$(i). Since X is compact, a clopen subspace of X is compact too. Thus X has a basis of open and compact subspaces closed under finite intersections, namely the collection of all clopen subspaces of X (note that the intersection of finitely many clopen sets is clopen). Moreover X is clearly sober since the irreducible subspaces of a Hausdorff space are the singletons. Then, it is sufficient to apply (11.2). $\square$

(11.4) Lemma. *Let X be a compact and Hausdorff space, and let $x \in X$. Then*

$$\bigcap\{C : C \text{ is clopen in } X \text{ and } x \in C\}$$

is a connected subspace of X.

Proof. We argue by contradiction, and take nonempty disjoint closed subspaces Γ, Δ of $Q := \bigcap\{C : C \text{ is clopen in } X \text{ and } x \in C\}$, with respect to the subspace topology, such that $Q = \Gamma \cup \Delta$. Note that Γ, Δ are closed in X, since Q is closed in X. Since X is a T_4 space, being it compact and Hausdorff, there are open and disjoint subsets U, V of X such that $U \supseteq \Gamma$ and $V \supseteq \Delta$. Consider now the collection

$$\mathcal{G} := \{C : C \text{ is clopen in } X \text{ and } x \in C\} \cup \{X - (U \cup V)\}$$

of closed sets of X. Since $X - (U \cup V) \subseteq X - Q$, we have $\bigcap \mathcal{G} = \emptyset$ and thus, by compactness, $\mathcal{G}$ has not the finite intersection property. Hence, there are finitely many clopen subspaces $C_1, \ldots, C_n$ of X such that $x \in \bigcap_{i=1}^{n} C_i$ and satisfying

$$\bigcap_{i=1}^{n} C_i \cap (X - (U \cup V)) = \emptyset,$$

that is, $x \in Q \subseteq C := \bigcap_{i=1}^{n} C_i \subseteq U \cup V$. Assume, without loss of generality, that $x \in U$. We have immediately

$$\overline{U \cap C} \subseteq \overline{U} \cup C = \overline{U} \cap C \cap (U \cup V) = U \cap C.$$

This proves that $U \cap C$ is a clopen set of X and, since clearly $x \in U \cap C$, we have $Q \subseteq U \cap C$. Thus $\Delta \subseteq V \cap Q \subseteq V \cap (U \cap C) \subseteq V \cap U$ and, since $\Delta \neq \emptyset$, we deduce $V \cap U \neq \emptyset$, a contradiction. □

(11.5) Proposition. *Let X be a compact Hausdorff and totally disconnected space. Then X has a basis of clopen sets. In particular, X is a spectral space.*

Proof. Let Ω be an open subset of X and let $x \in \Omega$. Consider the collection

$$\mathcal{G} := \{C : C \text{ is clopen in } X \text{ and } x \in C\} \cup \{X - \Omega\}$$

of closed subsets of X. Since X is totally disconnected and

$$\bigcap \{C : C \text{ is clopen in } X \text{ and } x \in C\}$$

is connected, in view of (11.4), we infer that

$$\{x\} = \bigcap \{C : C \text{ is clopen in } X \text{ and } x \in C\}.$$

Thus $\bigcap \mathcal{G} = \emptyset$ and, by compactness, there are finitely many clopen subsets $C_1, \ldots, C_n$ of X such that $x \in C := \bigcap_{i=1}^{n} C_i \subseteq \Omega$. From the fact that C is clopen the conclusion immediately follows. The last statement is a consequence of (11.3). □

The next goal is to find explicitly a ring whose prime spectrum is homeomorphic to a given compact Hausdorff totally disconnected space.

(11.6) Theorem. *Let X be a topological space and let $A(X)$ be the ring of all continuous functions $X \longrightarrow \mathbb{F}_2$, where $\mathbb{F}_2$ is equipped with the discrete topology. The following properties hold.*

(a) *$A(X)$ is zero-dimensional.*

(b) *For any $x \in X$, consider the maximal ideal $\mathfrak{m}_x := \{f \in A(X) : f(x) = 0\}$ of $A(X)$ ($\mathfrak{m}_x$ is the kernel of the canonical surjective ring homomorphism $A(X) \longrightarrow \mathbb{F}_2$, $f \mapsto f(x)$). Then, the map $\tau : X \longrightarrow \mathrm{Spec}(A(X))$, $x \mapsto \mathfrak{m}_x$ is continuous.*

(c) *If X is compact, then τ is surjective.*

(d) *If X is compact, Hausdorff and totally disconnected, then τ is a homeomorphism.*

Proof. Since any element of $A(X)$ is idempotent, statement (a) follows immediately from (9.13).

(b). Take any function $f \in A(X)$. Then clearly $\tau^{-1}(D(f)) = f^{-1}(\{1\})$ is open, since f is continuous. This proves that τ is continuous.

(c). Take any prime ideal $\mathfrak{p}$ of $A(X)$ and let $\mathcal{F} := \{f^{-1}(\{0\}) : f \in \mathfrak{p}\}$. If $\bigcap \mathcal{F} = \emptyset$, then $\{f^{-1}(\{1\}) : f \in \mathfrak{p}\}$ is an open cover of X. By compactness, there are finitely many $f_1, \ldots, f_n \in \mathfrak{p}$ such that $X = \bigcup_{i=1}^{n} f^{-1}(\{1\})$. It follows immediately that $\prod_{i=1}^{n}(1 - f_i) = 0 \in \mathfrak{p}$ and, by primality, $1 - f_i \in \mathfrak{p}$, for some $1 \leq i \leq n$. Then $1 \in \mathfrak{p}$, a contradiction.

(d) Take distinct points $x, y \in X$. Since X is, in particular, a T_1 space, then $X - \{y\}$ is an open neighborhood of x. Since X is compact, Hausdorff and totally disconnected, there is a clopen set C of X such that $x \in C \subseteq X - \{y\}$, by (11.5). It follows that the function $f : X \longrightarrow \mathbb{F}_2$ such that

$$f(z) := \begin{cases} 0 & \text{if } z \in C \\ 1 & \text{if } z \in X - C \end{cases}$$

is, by definition, continuous, that is, $f \in A(X)$. Moreover $f \in \mathfrak{m}_x - \mathfrak{m}_y$. This proves that τ is bijective. The fact that τ is a homeomorphism is an immediate consequence of the previous parts, of the fact that X is, in particular, compact, and that $\mathrm{Spec}(A(X))$ is Hausdorff, by part (a) and (3.9). □

In the following definition, we essentially extend the notion of the constructible topology to any topological space.

(11.7) Definition ([12]). *Let X be a topological space.* The patch topology *on X is the coarsest topology for which the open and compact subspaces of X are clopen sets. We shall denote by X^{patch} the set X equipped with the patch topology.*

By definition, if $X := \mathrm{Spec}(A)$, then the patch topology and the constructible topology are equal.

(11.8) Remark. Let X be a T_0 space with a basis of open and compact subspaces. Then, X^{patch} is a Hausdorff space and the patch topology is finer than the given topology on X. As a matter of fact, take $x, y \in X$ and $x \neq y$. By assumption, there is an open and compact subspace U of X such that $x \in U$ and $y \notin U$. By definition, $U, X - U$ are clopen disjoint neighborhoods of x, y, respectively. The last statement follows by definition.

(11.9) Lemma. *Let X be a spectral space and let $f : \mathrm{Spec}(A) \longrightarrow X$ be a homeomorphism. Then, $f : \mathrm{Spec}(A)^{\text{cons}} \longrightarrow X^{\text{patch}}$ is a homeomorphism. In particular, X^{patch} is a compact space.*

Proof. By definition, a subbasis of open sets for X^{patch} is

$$\mathcal{S}_{\text{patch}} := \{\Omega, X - \Omega : \Omega \text{ open and compact in } X\}.$$

Since f is a homeomorphism, $f^{-1}(\Omega)$ is open and compact in $\mathrm{Spec}(A)$, for any open and compact subspace Ω of X, and thus $f^{-1}(\Omega)$ is clopen in $\mathrm{Spec}(A)^{\text{cons}}$. This proves that $f^{-1}(U)$ is open in $\mathrm{Spec}(A)^{\text{cons}}$, for any $U \in \mathcal{S}_{\text{patch}}$, that is, the map $f : \mathrm{Spec}(A)^{\text{cons}} \longrightarrow X^{\text{patch}}$ is continuous. Moreover, $f : \mathrm{Spec}(A)^{\text{cons}} \longrightarrow X^{\text{patch}}$ is bijective and closed, since $\mathrm{Spec}(A)^{\text{cons}}$ is compact (see (10.11)) and X^{patch} is Hausdorff, by (11.8). Thus $f : \mathrm{Spec}(A)^{\text{cons}} \longrightarrow X^{\text{patch}}$ is a homeomorphism. □

Surprisingly, compactness of certain topological spaces, equipped with the patch topology, is crucial to show if such spaces are spectral, as the following result proves.

(11.10) Theorem ([12, Corollary to Proposition 7]). *For a topological space X, the following conditions are equivalent.*

(i) *X is a spectral space.*
(ii) *X is T_0, has a basis of open and compact subspaces and X^{patch} is compact.*

Proof. (i)$\Longrightarrow$(ii). Apply the trivial part of (11.2) and (11.9).

(ii)$\Longrightarrow$(i). Let $\mathcal{B}$ be the collection of all open and compact subspaces of X. By assumption, $\mathcal{B}$ is a basis of X. By definition, $\mathcal{B}$ is a collection of clopen sets in the compact space X^{patch}. Thus, in particular, an intersection of finitely many members of $\mathcal{B}$ is closed in X^{patch} and thus it is compact both in the compact space X^{patch} and in X since, by (11.8), the topology of X is coarser than the patch topology. This proves that $\mathcal{B}$ is closed under finite intersection. Moreover, X is compact, since X^{patch} is compact. Now let C be an irreducible closed subspace of X and let

$$\mathcal{G} := \{U \cap C : U \text{ open and compact in } X \text{ and } U \cap C \neq \emptyset\}$$

Since the patch topology is finer than the given topology on X, then $\mathcal{G}$ is a collection of closed subsets of X^{patch}. Moreover, since C is irreducible, then $\mathcal{G}$ has the finite intersection property. Since X^{patch} is compact, there is a point $x_0 \in \bigcap \mathcal{G}$, and clearly $\overline{\{x_0\}} \subseteq C$. Conversely, take a poin $x \in C$ and let V be an open neighborhood of x. By assumption, there is an open and compact subspace U of X such that $x \in U \subseteq V$. Then $U \cap C \neq \emptyset$, that is, $U \cap C \in \mathcal{G}$, and thus $x_0 \in U \cap C$, in particular. This proves that $C = \overline{\{x_0\}}$. The conclusion follows from (11.2). ☐

(11.11) Corollary. *Let X be a spectral space and let $\Omega_1, \ldots, \Omega_n$ be open and compact subspaces of X. Then $\bigcap_{i=1}^n \Omega_i$ is (open) and compact. Thus the basis of all open and compact subspaces of X is closed under finite intersections.*

Proof. Each Ω_i is clopen in X^{patch} and thus $\Omega := \bigcap_{i=1}^n \Omega_i$ is closed in X^{patch}, in particular, and thus it is compact in the compact space X^{patch} (in view of 11.10). Since, by (11.8), the patch topology on X is finer than the given spectral topology of X, it follows that Ω is compact in X. ☐

In the applications it may be not so easy to discuss compactness of X^{patch}, thus we will provide now a more direct and powerfur criterion for deciding when a topological space X is spectral, based on ultrafilters.

Take any set X and fix a nonempty collection $\mathcal{S}$ of subsets of X. For any ultrafilter $\mathscr{U}$ on X and any subset Y of X, set

$$Y(\mathscr{U}) := Y_{\mathcal{S}}(\mathscr{U}) := \{x \in X : [\forall S \in \mathcal{S}, (x \in S \iff S \cap Y \in \mathscr{U})]\}.$$

The set $Y(\mathscr{U})$ is called *the ultrafilter limit set of Y, with respect to $\mathscr{U}$*. Since $\mathcal{S}$ will be always fixed, we will not mention it in the terminology.

(11.12) Example ([3, Example 2.1(2)]). Let A be a ring, $X := \text{Spec}(A)$ and $\mathcal{S} := \{D(f) : f \in A\}$ be the collection of principal open subsets of X. Take any subset Y of X and any ultrafilter $\mathscr{U}$ on Y and consider the ultrafilter limit point

$$Y_{\mathscr{U}} := \{x \in A : V(x) \cap Y \in \mathscr{U}\}$$

of Y, with respect to $\mathscr{U}$ (see (10.3)). Then, by definition $Y_{\mathcal{S}}(\mathscr{U}) = \{Y_{\mathscr{U}}\}$.

(11.13) Theorem ([3, Corollary 3.3]). *Let X be a topological space. Then, the following conditions are equivalent.*

(i) *X is a spectral space.*
(ii) *X is a T_0 space and has a subbasis $\mathcal{S}$ of open sets such that, for any ultrafilter $\mathscr{U}$ on X, the ultrafilter limit set $X_{\mathcal{S}}(\mathscr{U})$ is nonempty.*

Proof. (i)$\Longrightarrow$(ii). Since X is a spectral space, the collection $\mathcal{S}$ of all open and compact subspaces of X is a basis of X (in particular, a subbasis), by the trivial part of (11.2). For any ultrafilter $\mathscr{U}$ on X set

$$\mathscr{U}^* := \{X - S : S \in \mathcal{S} - \mathscr{U}\} \cup (\mathcal{S} \cap \mathscr{U}).$$

By (9.5), $\mathscr{U}^* \subseteq \mathscr{U}$ and thus it has the finite intersection property. Moreover, by definition, $\mathscr{U}^*$ consists of clopen sets of X^{patch}. Since, in view of (11.10), X^{patch} is compact, there is a point $x \in \bigcap \mathscr{U}^*$. We claim that $x \in X_{\mathcal{S}}(\mathscr{U})$. To prove this, fix a set $S \in \mathcal{S}$. If $x \in S$ and $S - X \in \mathscr{U}$, then $X - S \in \mathscr{U}^*$ and thus $x \in X - S$, a contradiction. Thus $x \in S$ implies $S \in \mathscr{U}$. The converse is trivial, since $\mathcal{S} \cap \mathscr{U} \subseteq \mathscr{U}^*$.

(ii)$\Longrightarrow$(i). By definition, the collection of sets $\mathcal{B} := \{\bigcap \mathcal{U} : \mathcal{U} \subseteq \mathcal{S}, \mathcal{U} \text{ finite}\}$ is a basis of open sets for X.

Claim 1. $\mathcal{B}$ consists of compact subspaces of X.

As a matter of fact, the generic element of $\mathcal{B}$ is of the form $B := \bigcap_{i=1}^n \Sigma_i$, for some $\Sigma_1, \ldots, \Sigma_n \in \mathcal{S}$. By the Alexander subbasis Theorem it suffices to prove that any open cover of B consisting of sets of $\mathcal{S}$ admits a finite subcover. Take a subcollection $\mathcal{V}$ of $\mathcal{S}$ such that $B \subseteq \bigcup \mathcal{V}$ and assume that $\mathcal{V}$ has not finite subcovers. Then the collection of sets $\{B - V : V \in \mathcal{V}\}$ has the finite intersection property and, by (9.4), it can be extended to an ultrafilter $\mathscr{U}$ on X. By assumption, there is an element $x_0 \in X_{\mathcal{S}}(\mathscr{U})$. Choose any set $V \in \mathcal{V}$ and note that from $B - V \subseteq B$ and $B - V \in \mathscr{U}$ it follows $B \in \mathscr{U}$ and, a fortiori, $\Sigma_1, \ldots, \Sigma_n \in \mathscr{U}$. Since $\Sigma_1, \ldots, \Sigma_n \in \mathcal{S}$, $x_0 \in X_{\mathcal{S}}(\mathscr{U})$ implies $x_0 \in \bigcap_{i=1}^n \Sigma_i =: B$. Furthermore, since $\mathcal{V} \subseteq \mathcal{S}$ and $B - V \in \mathscr{U}$, for any $V \in \mathcal{V}$, $x_0 \in X_{\mathcal{S}}(\mathscr{U})$ implies $x_0 \notin V$, for any $V \in \mathcal{V}$. This contradicts the inclusion $B \subseteq \bigcup \mathcal{V}$. The proof of Claim 1 is now complete.

Claim 2. X^{patch} is compact.

Recall that a subbasis of open sets for the patch topology on X is, by definition, the collection

$$\mathcal{S}_{\text{patch}} := \{\Omega, X - \Omega : \Omega \text{ open and compact in } X\}.$$

Again by the Alexander subbasis theorem, it suffices to prove that if $\mathcal{H}$ is a subcollection of $\mathcal{S}_{\text{patch}}$ and an (open) cover of X, then $\mathcal{H}$ has a finite subcover. Suppose this is not the case. As before, the collection $\mathcal{G} := \{X - H : H \in \mathcal{H}\}$ has the finite intersection property and can be extended to an ultrafilter $\mathscr{U}$ on X. By assumption, there are a point $x_0 \in X_{\mathcal{S}}(\mathscr{U})$ and a set $H_0 \in \mathcal{H}$ such that $x_0 \in H_0$. Since $\mathcal{H} \subseteq \mathcal{S}_{\text{patch}}$, there is an open and compact subspace Ω of X such that either $H_0 = \Omega$ or $H_0 = X - \Omega$. Since Ω is open and compact and $\mathcal{B}$ is a basis of X, there are finitely many sets $B_1, \ldots, B_n \in \mathcal{B}$ such that $\Omega = \bigcup_{i=1}^n B_i$. If $H_0 = \Omega$, take an index i such that $x_0 \in B_i$ and, since $B_i \in \mathcal{S}$, take finitely many $S_1, \ldots, S_m \in \mathcal{S}$ such that $B_i = \bigcap_{j=1}^m S_j$. The fact that $x_0 \in S_j$, for $1 \leq j \leq m$, and that $x_0 \in X_{\mathcal{S}}(\mathscr{U})$ imply that $S_j \in \mathscr{U}$, for $1 \leq j \leq m$, and thus $H_0 \in \mathscr{U}$, since $B_i = \bigcap_{j=1}^m S_j \in \mathscr{U}$ and $B_i \subseteq H_0$. On the other hand, from $H_0 \in \mathcal{H}$ and $\mathcal{G} \subseteq \mathscr{U}$ it follows $X - H_0 \in \mathscr{U}$, and thus $\emptyset \in \mathscr{U}$, a contradiction.

Suppose now that $H_0 = X - \Omega = \bigcap_{i=1}^n (X - B_i)$. Since $x_0 \in H_0$ and each B_i is a finite intersection of members of $\mathcal{S}$, for any $i \in \{1, \ldots, n\}$ there is a set $T_i \in \mathcal{S}$ such that $x_0 \in X - T_i \subseteq X - B_i$. Since $x_0 \in X_{\mathcal{S}}(\mathscr{U})$, we have $X - T_i \in \mathscr{U}$ and, a fortiori, $X - B_i \in \mathscr{U}$, for $1 \leq i \leq n$. Thus $H_0 = \bigcap_{i=1}^n (X - B_i) \in \mathscr{U}$ and this leads to a contradiction, as before.

Now the conclusion is an immediate consequence of Claim 1, Claim 2 and (11.10).

□

(11.14) Example ([3, Proposition 3.5]). Let A, B rings such that A is a subring of B, and let $\mathcal{R}(B|A)$ be the set of the subrings C of B such that A is a subring of C. Consider the natural topology on $\mathcal{R}(B|A)$ whose subbasic open sets are the sets of the type $U(x) := \{C \in \mathcal{R}(B|A) : x \in C\}$. Then $\mathcal{R}(B|A)$ is spectral. As a matter of fact, we first observe that $X := \mathcal{R}(B|A)$ is a T_0 space. This is obvious, because if $C, D \in X$ and $x \in C - D$ then $C \in U(x)$ and $D \notin U(x)$. Take now any ultrafilter $\mathscr{U}$ on X and set

$$A_{\mathscr{U}} := \{x \in B : U(x) \in \mathscr{U}\}.$$

We claim that $A_{\mathscr{U}} \in X$. Indeed, $0 \in A_{\mathscr{U}}$, since $U(0) = X \in \mathscr{U}$ and, if $x, y \in A_{\mathscr{U}}$, then $U(x) \cap U(y) \in \mathscr{U}$, by definition. Since clearly $U(x) \cap U(y) \subseteq U(x - y)$, $U(x) \cap U(y) \subseteq U(xy)$, we have, a fortiori, $U(x - y), U(xy) \in \mathscr{U}$ and, in other words, $x - y, xy \in A_{\mathscr{U}}$. Thus, $A_{\mathscr{U}}$ is a subring of B. Finally, $U(a) = X$, for any $a \in A$, and thus $A \subseteq A_{\mathscr{U}}$. This proves that $A_{\mathscr{U}} \in X$. Moreover, by definition, if $\mathcal{S} := \{U(x) : x \in B\}$, then $A_{\mathscr{U}} \in X_{\mathcal{S}}(\mathscr{U})$. Thus the conclusion follows from (11.13).

(11.15) Example. Let K be a field, D be any subring of K and let $\mathrm{Zar}(K|D)$ be the set of all valuation rings V of K (i.e., K is the quotient field of V) such that D is a subring of V. In the particular case $D := (0)$ $\mathrm{Zar}(K|D)$ will be denoted by $\mathrm{Zar}(K)$ and it consists of all valuation domains of K. Clearly, $\mathrm{Zar}(K|D)$ is a subset of $\mathcal{R}(K|D)$ and thus it can take the subspace topology induced by the topology on $\mathcal{R}(K|D)$ defined in (11.14). Such a topology on $\mathrm{Zar}(K|D)$ is called *the Zariski topology* and clearly a subbasis of open sets of $\mathrm{Zar}(K|D)$ consists of the sets of the form $B^K(x) := B(x) := \mathrm{Zar}(K|D[x])$, for $x \in K$. The set $\mathrm{Zar}(K|D)$, equipped with the Zariski topology, is usually called *the Riemann-Zariski space of K over D*. Being it a subspace of the T_0 space $\mathcal{R}(K|D)$, the space $Z := \mathrm{Zar}(K|D)$ is T_0. Take any ultrafilter $\mathscr{U}$ on Z and define $A_{\mathscr{U}} := \{x \in K : B(x) \in \mathscr{U}\}$. The same argument given in (11.14) proves that $A_{\mathscr{U}} \in \mathcal{R}(K|D)$. Furthermore, we claim that $A_{\mathscr{U}}$ is a valuation domain of K. To do this, take any nonzero element $x \in K$ and assume that $x \notin A_{\mathscr{U}}$. By definition, $B(x) \notin \mathscr{U}$ and thus $Z - B(x) \in \mathscr{U}$. Since the elements of Z are, in particular, valuation domains we have $Z - B(x) \subseteq B(x^{-1})$ and thus $B(x^{-1}) \in \mathscr{U}$, meaning that $x^{-1} \in A_{\mathscr{U}}$. This shows that $A_{\mathscr{U}} \in Z$ and, by definition, for any $x \in K$, $A_{\mathscr{U}} \in B(x)$ if and only if $B(x) \in \mathscr{U}$. This proves that $A_{\mathscr{U}} \in Z_{\mathcal{S}}(\mathscr{U})$, where $\mathcal{S} := \{B(x) : x \in K\}$. By (11.13), $\mathrm{Zar}(K|D)$ is a spectral space.

12. A ring whose prime spectrum is homeomorphic to $\mathrm{Zar}(K|D)$.

This section is motivated by the last example of the previous one: in (11.15) we showed that, for any field K and any subring D of K, the Riemann-Zariski space $\mathrm{Zar}(K|D)$, endowed with the Zariski topology, is a spectral space. The proof we gave is based on the ultrafilter criterion (11.13) and thus it is not constructive. Our aim is to give now a constructive proof of (11.15): we will find a ring B, namely a Bézout domain, such that $\mathrm{Spec}(B)$ is homeomorphic to $\mathrm{Zar}(K|D)$. First, we will need some tool on valuation theory.

(12.1) Definition. *For any local ring A, let $\mathfrak{m}_A$ denote the maximal ideal of A.*

If A, B are local rings, we say that B dominates *A, and we write $A \leq_d B$, if A is a subring of B and $\mathfrak{m}_A = A \cap \mathfrak{m}_B$ (i.e., $\mathfrak{m}_A \subseteq \mathfrak{m}_B$).*

(12.2) Proposition. *Let L be a field and let*

$$\mathcal{L}_L := \mathcal{L} := \{D : D \text{ is a local subring of } L\}.$$

The following properties hold.

(a) *$\leq_d$ is a partial order on $\mathcal{L}$.*

(b) *Suppose that* V *is a maximal element of* $(\mathcal{L}, \leq_d)$. *If* A *is a subring of* K *such that* $V \subseteq A$ *and there exixts a prime ideal of* A *lying over the maximal ideal of* V, *then* $A = V$.

(c) *For any* $A \in \mathcal{L}$ *there is a maximal element* $B \in \mathcal{L}$, *under* $\leq_d$, *such that* $A \leq_d B$.

(d) *Valuation domains of* L *are precisely the maximal elements of* $(\mathcal{L}, \leq_d)$.

Proof. Part (a) is straightforward.

(b) Let $\mathfrak{m}$ be the maximal ideal of V and let $\mathfrak{p}$ be a prime ideal of A such that $\mathfrak{p} \cap V = \mathfrak{m}$. Then $A_\mathfrak{p} \in \mathcal{L}$, since $A_\mathfrak{p}$ is local and contains V, and moreover

$$\mathfrak{p}A_\mathfrak{p} \cap V = \mathfrak{p}A_\mathfrak{p} \cap A \cap V = \mathfrak{p} \cap V = \mathfrak{m}.$$

It follows $V \leq_D A_\mathfrak{p}$ and, by maximality of V, $V = A = A_\mathfrak{p}$.

(c) Restrict the partial order $\leq_d$ to the nonempty set $\mathcal{L}_A := \{B \in \mathcal{L} : A \leq_d B\}$, And consider a chain $\mathcal{C} := \{B_i : i \in I\} \subseteq \mathcal{L}_A$, with respect to $\leq_d$. For any $i \in I$, let $\mathfrak{m}_i$ be the maximal ideal of B_i, and set $B := \bigcup_{i \in I} B_i, \mathfrak{m} := \bigcup_{i \in I} \mathfrak{m}_i$. Thus B is a ring, since it is the union of a chain of subrings of L and $\mathfrak{m}$ is an additive subgroup of L, being it the union of a chain of additive subgroups of L. Take elements $m \in \mathfrak{m}$, $b \in B$, take indexes $i, j \in I$ such that $m \in \mathfrak{m}_i$ and $b \in B_j$. If $B_i \leq_d B_j$, then $m \in \mathfrak{m}_i \subseteq \mathfrak{m}_j$. It follows $bm \in \mathfrak{m}_j \subseteq \mathfrak{m}$. If $B_j \leq_d B_i$, then $b \in B_j \subseteq B_i$, and thus $bm \in \mathfrak{m}_i \subseteq \mathfrak{m}$. This proves that $\mathfrak{m}$ is an ideal of B, and clearly $\mathfrak{m} \neq B$. Take a non invertible element $x \in B$ and an index $i \in I$ such that $x \in B_i$. A fortiori, x is not invertible in B_i, and thus $x \in \mathfrak{m}_i \subseteq \mathfrak{m}$, since B_i is local. This proves that $\mathfrak{m}$ is the set of all non invertible elements of B, that is, that B is a local ring and $B \in \mathcal{L}$. By definition $B_i \leq_d B$, for each $i \in I$, that is, B is an upper bound for the chain $\mathcal{C}$. Then the conclusion follows by Zorn's Lemma.

(d). Suppose that V is a maximal element of $(\mathcal{L}, \leq_d)$, and take an element $x \in L - V$. Then $V[x]$ is a subring of L and properly contains V. If $\mathfrak{m}$ is the maximal ideal of V and the ideal $\mathfrak{m}[x]$ of $V[x]$ is proper, then there is a maximal ideal $\mathfrak{n}$ of $V[x]$ such that $\mathfrak{m}[x] \subseteq \mathfrak{n}$, and this implies $\mathfrak{m} \subseteq \mathfrak{m}[x] \cap V \subseteq \mathfrak{n} \cap V \subsetneq V$, that is, $\mathfrak{n}$ lies over $\mathfrak{m}$. By part (b) it follows $V = V[x]$, a contradiction. This proves that $\mathfrak{m}[x] = V[x]$ and then $1 = m_0 + m_1 x + \ldots + m_h x^h$, for some $m_0, \ldots, m_h \in \mathfrak{m}$. The element $1 - m_0$ is clearly invertible in V. If $\lambda := (1 - m_0)^{-1}$, then $1 = \lambda m_1 x + \ldots + \lambda m_h x^h$. By dividing both sides of the previous equality for x^h we get $(x^{-1})^h = \lambda m_1 (x^{-1})^{h-1} + \ldots + \lambda m_h$. This proves that x^{-1} is integral over V, that is, $V \subseteq V[x^{-1}]$ is an integral extension. By the lying over Theorem, there is a prime ideal of $V[x^{-1}]$ lying over $\mathfrak{m}$. Then, by part (b), $V = V[x^{-1}]$, that is, $x^{-1} \in V$. This proves that V is a valuation domain of L.

Conversely, assume that V is a valuation domain of L. Thus, in particular, $V \in \mathcal{L}$. Take a local ring $A \in \mathcal{L}$ such that $V \leq_d A$ and assume, by contradiction, that $A \neq V$, that is, $V \subsetneq A$. Take an element $x \in A - V$ and note that, since V is a valuation domain, $x^{-1} \in \mathfrak{m}_V$. Moreover we have $\mathfrak{m}_V \subseteq \mathfrak{m}_A$, since $V \leq_d A$, and thus $x^{-1} \in \mathfrak{m}_A$. Since $x \in A$, it follows $1 \in \mathfrak{m}_A$, a contradiction. The proof is now complete. □

(12.3) Corollary. *Let* A *be a subring of a field* L *and let* $\mathfrak{p} \in \mathrm{Spec}(A)$. *Then* $A_\mathfrak{p}$ *is dominated by some valuation domain of* L.

Proof. It is enough to apply parts (c) and (d) of (12.2). □

(12.4) Remark. Let $K \subseteq L$ be a field extension and let W be a valuation domain of L. Then it is easily seen that $W \cap K$ is a valuation domain of K. Furthermore, W dominates $W \cap K$.

(12.5) Corollary. *Let $K \subseteq L$ be a field extension and let V be a valuation domain of K. Then there is a valuation domain W of L such that $W \cap K = V$. Such a valuation domain W is called* an extension of V to L.

Proof. By (12.2c,d), V is dominated by some valuation domain W of L. Keeping in mind that $\mathfrak{m}_V \subseteq \mathfrak{m}_W$, it easily follows that the valuation domain $W \cap K$ of K dominates V. Since both V and $W \cap K$ are valuation domains of K, then (12.2d) implies $V = W \cap K$. □

Let A be an integral domain with quotient field L. Then the canonical map $\delta : \mathrm{Zar}(L|A) \longrightarrow \mathrm{Spec}(A)$ defined by $\delta(V) := \mathfrak{m}_V \cap A$, where $\mathfrak{m}_V$ is the maximal ideal of V, is called *the domination map of A*.

(12.6) Proposition ([1, Lemma 2.1 and Proposition 2.2])**.** *Let A be an integral domain with quotient field L. The following properties hold.*

(a) *The domination map $\delta : \mathrm{Zar}(L|A) \longrightarrow \mathrm{Spec}(A)$ is continuous and surjective.*
(b) *If A is a Prüfer domain, then δ is a homeomorphism.*

Proof. (a). Let $\mathfrak{p}$ be a prime ideal of A. By (12.3), the local overring $A_\mathfrak{p}$ of A is dominated by some valuation domain V, i.e., $A_\mathfrak{p} \subseteq V$ and $\mathfrak{m}_V \cap A_\mathfrak{p} = \mathfrak{p}A_\mathfrak{p}$. It follows immediately $\delta(V) := \mathfrak{m}_V \cap A = \mathfrak{p}$. This proves that δ is surjective. Now, consider a principal open set $D(a)$ of $\mathrm{Spec}(A)$, for a fixed element $a \in A$. If $V \in \mathrm{Zar}(L|A)$, then clearly $\delta(V) := \mathfrak{m}_V \cap A \in D(a)$ if and only if $a^{-1} \in V$. This proves that $\delta^{-1}(D(a)) = B(a^{-1})$ and thus, by (3.2f), δ is continuous.

(b). Assume that A is a Prüfer domain, and let $V, W \in \mathrm{Zar}(L|A)$ such that $\delta(V) = \delta(W) =: \mathfrak{p}$. It follows immediately that $A_\mathfrak{p}$ is dominated by both V and W. Since A is a Prüfer domain, $A_\mathfrak{p}$ is a valuation domain, and thus it is maximal under domination, by (12.2d). It follows $A_\mathfrak{p} = V = W$, proving that δ is injective. Finally, we show that δ is open. By definition, a basis of open sets for $\mathrm{Zar}(L|A)$ consists of finite intersections of sets of the type $B(x)$, where $x \in L$. Since we know that δ is injective, it suffices to prove that $\delta(B(x))$ is open in $\mathrm{Spec}(A)$, for any $x \in L$. We claim that $\delta(B(x)) = \mathrm{Spec}(A) - V(\mathfrak{a})$, where $\mathfrak{a} := \{a \in A : ax \in A\}$. As a matter of fact, let $\mathfrak{p} \in \mathrm{Spec}(A)$. If $\mathfrak{p} \in \delta(B(x))$, we have $A_\mathfrak{p} \in B(x)$, since $A_\mathfrak{p}$ is the unique point of the space $\mathrm{Zar}(L|A)$ which is mapped to $\mathfrak{p}$ by δ, being A a Prüfer domain. It follows $x = \dfrac{a}{s}$, for some $a \in A, s \in A - \mathfrak{p}$, and thus $s \in \mathfrak{a} - \mathfrak{p}$. Thus $\mathfrak{a} \nsubseteq \mathfrak{p}$. The previous implications can be easily reversed, and thus the equality $\delta(B(x)) = \mathrm{Spec}(A) - V(\mathfrak{a})$ is proved. □

The following notion will be crucial to provide a representatio of a Riemann-Zariski space as a spectrum of a ring.

(12.7) Definition ([11])**.** *Let K be a field and let T be an indeterminate over K. A subring H of $K(T)$ is called to be* an K-Halter-Koch ring *if the following properties are satisfied:*

(a) *$T, T^{-1} \in H$;*
(b) *for any nonzero polynomial $f \in K[T]$, then $\dfrac{f(0)}{f} \in H$ (i.e., $f(0) \in fH$).*

(12.8) Remark. Let K be a field and let T be an indeterminate over K. The following properties immediately follow from the definition.

(a) If H is a K-Halter-Koch ring and H' is a subring of $K(T)$ such that $H \subseteq H'$, then H' is a K-Halter-Koch ring.
(b) If $\mathcal{H}$ is a nonempty collection of K-Halter-Koch rings, then $\bigcap \mathcal{H}$ is a K-Halter-Koch ring.

We now investigate about some fundamental properties of K-Halter-Koch rings.

(12.9) Theorem ([11, Theorem 2.2]). *Let K be a field, let T be an indeterminate over K and let $H \subseteq K(T)$ be a K-Halter-Koch ring. The following properties hold.*

(a) *The quotient field of H is $K(T)$.*
(b) *If $f := f_0 + f_1T + \ldots + f_nT^n \in K[T]$, then the equality $fH = f_0H + \ldots + f_nH$ of H-submodules of $K(T)$ holds.*
(c) *H is a Bézout domain.*

Proof. (a). It is sufficient to show that $K[T]$ is contained in the quotient field of H. In order to prove this, take a polynomial $f \in K[T]$ and set $F_f := 1 + Tf \in K[T]$. By condition (b) of the definition, we have $h := \dfrac{1}{1+Tf} = \dfrac{F_f(0)}{F_f} \in H$. It follows that $f = \dfrac{1-h}{Th}$ is an element of the quotient field of H, keeping in mind that $T \in H$, by definition.

(b). The fact that $T \in H$ easily implies $fH \subseteq f_0H + \ldots + f_nH$. Conversely, it is sufficient to show, by induction, that $f_i \in fH$, for any $0 \leq i \leq n$. By part (b) of definition we have $f_0 = \dfrac{f(0)}{f} \in H$, that is, $f_0 \in fH$. Assume now that $0 < h \leq n$ and that $f_j \in fH$, for any $0 \leq j < h$, and set $g := f_h + f_{h+1}T + \ldots + f_nT^{n-h} \in K[T]$. Clearly we have

$$g = T^{-h}(f - \sum_{j=0}^{h-1} f_jT^j) \in fH$$

since $T^{-h} \in H$, by definition, and $f_j \in fH$, for $0 \leq j \leq h-1$. Thus we have $gH \subseteq fH$. Again by part (b) of the definition, $f_h = g(0) \in gH \subseteq fH$. Now the statement follows by induction.

(c). Take two rational functions $\alpha, \beta \in H$. It suffices to show that $(\alpha, \beta)H$ is principal. Take polynomials $f, g, h \in K[T]$, where $h \neq 0$, such that $\alpha := \dfrac{f}{h}, \beta := \dfrac{g}{h}$. If n is a natural number greater than the degree of f, then part (b) easily implies $(f, g)H = (f + T^ng)H$, and thus $(\alpha, \beta)H = (\alpha + T^n\beta)H$. The proof is now complete. □

(12.10) Example. Let K be a field, T be an indeterminate over K, and let V be a valuation domain of K. Let v denote any valuation on K determining the ring V. For any nonzero polynomial $f := f_0 + f_1T + \ldots + f_nT^n \in K[T]$ define $v'(f) := \inf\{v(f_0), v(f_1), \ldots, v(f_n)\}$. It is easily seen that v' extends to a valuation v_g of $K(T)$ defined by setting $v_g(\dfrac{f}{h}) := v'(f) - v'(h)$, for any $f, h \in K[T]$ with $h \neq 0$. The valuation domain of v_g is called *the Gaussian extension of V to $K(T)$* and it is usually denoted by $V(T)$. It clear that $V(T) \cap K = V$, i.e, $V(T)$ is an extension of V to $K(T)$.

(a) Let $\mathfrak{m}$ be the maximal ideal of V. Then $V(T) = V[T]_{\mathfrak{m}[T]}$. As a matter of fact, for any polinomial $f := f_0 + \ldots + f_n T^n \in V[T] - \mathfrak{m}[T]$ we have $v(f_i) \geq 0$, for $0 \leq i \leq n$, since $f \in V[T]$, and moreover $v(f_j) = 0$ for some j, since $f \notin \mathfrak{m}[T]$. It follows $v_g(f) = 0$, that is, f is invertible in $V(T)$, and thus $V[T]_{\mathfrak{m}[T]} \subseteq V(T)$. Take now a rational function

$$\alpha := \frac{f_0 + f_1 T + \ldots + f_n T^n}{h_0 + h_1 T + \ldots + h_m T^m} \in V(T),$$

where $f := f_0 + f_1 T + \ldots + f_n T^n, h := h_0 + h_1 T + \ldots + h_m T^m \in K[T]$, and thus, by definition, $\inf\{v(f_i) : 0 \leq i \leq n\} \geq \inf\{v(h_j) : 0 \leq j \leq m\} =: v(h_l)$. It immeditaly follows that $\frac{f_i}{h_l}, \frac{h_j}{h_l} \in V$, for $0 \leq i \leq n$ and $0 \leq j \leq m$, and thus $\frac{f}{h_l} \in V[T], \frac{h}{h_l} \in V[T] - \mathfrak{m}[T]$, and this proves that $\alpha \in V[T]_{\mathfrak{m}[T]}$.

(b) We prove now that $V(T)$ is a K-Halter-Koch ring. Clearly we have $v_g(T) = v_g(T^{-1}) = v(1) = 0$, and thus $T, T^{-1} \in V(T)$. Moreover, for any nonzero polynomial $f := f_0 + f_1 T + \ldots + f_n T^n \in K[T]$, we have

$$v_g(\frac{f(0)}{f}) := v(f_0) - \inf\{v(f_0), v(f_1), \ldots, v(f_n)\} \geq 0,$$

that is, $\frac{f(0)}{f} \in V(T)$. The conclusion is now clear.

(12.11) Example. Let K be a field, T be an indeterminate over K and let Y be a nonempty subset of $\mathrm{Zar}(K)$. In view of (12.8b) and (12.10b), $H_Y := \bigcap_{V \in Y} V(T)$ is a K-Halter-Koch ring.

(12.12) Remark. Let $K \subseteq L$ be a field extension. In view of (12.4) and (12.5), we can define a natural surjection $\pi : \mathrm{Zar}(L) \longrightarrow \mathrm{Zar}(K)$ by setting $\pi(W) := W \cap K$, for any $W \in \mathrm{Zar}(L)$. Then π is continuous, since clearly $\pi^{-1}(B^K(x)) = B^L(x)$, for any $x \in K$.

(12.13) Proposition ([4, Proposition 3.1])**.** *Let K be a field, T be an indeterminate over K and $\pi : \mathrm{Zar}(K(T)) \longrightarrow \mathrm{Zar}(K)$ be the continuous surjection defined in (12.12). Then the restriction of π to $Z_0 := \{V(T) : V \in \mathrm{Zar}(K)\}$ is a homeomorphism of Z_0 with* $\mathrm{Zar}(K)$.

Proof. The restriction $\varphi : Z_0 \longrightarrow \mathrm{Zar}(K)$ is clearly a continuous bijection since, for any $V \in \mathrm{Zar}(K)$, $\varphi(V(T)) = V$. We have to show that φ is open. In order to prove this, keeping in mind that φ is bijective, it is sufficient to verify that, for any $0 \neq \alpha \in K(T)$, $\varphi(Z_0 \cap B(\alpha))$ is open in $\mathrm{Zar}(K)$. Set

$$\alpha := \frac{a_0 + a_1 T + \ldots + a_r T^r}{b_0 + b_1 T + \ldots + b_s T^s},$$

where $a_i, b_j \in K$ for $0 \leq i \leq r, 0 \leq j \leq s$. Fix a valuation ring $V(T) \in Z_0$, for some $V \in \mathrm{Zar}(K)$, let v be a valuation defining v and v_g the corresponding valuation defining $V(T)$. Then $V(T) \in B(\alpha)$ if and only if $v_g(\alpha) \geq 0$, that is, if and only if

$$(\star) \qquad \inf\{v(a_i) : 0 \leq i \leq r\} \geq \inf\{v(b_j) : 0 \leq j \leq s\}.$$

Now let $M := \{(i,j) : 0 \le i \le r, 0 \le j \le s, a_j, b_j \ne 0\}$ and, for any $(i,j) \in M$, set

$$F_{ij} := \{\frac{a_i}{b_j}, \frac{a_\lambda}{a_i}, \frac{b_\mu}{b_j} : 0 \le \lambda \le r, 0 \le \mu \le s\}.$$

Then $(\star)$ easily implies that $\varphi(Z_0 \cap B(\alpha)) = \bigcup_{(i,j)\in M}(\bigcap_{x\in F_{ij}} B(x))$. The proof is now complete. □

(12.14) Proposition ([4, Proposition 3.3])**.** *Let K be a field, T be an indeterminate over K, and let H be a K-Halter-Koch ring. Then $\mathrm{Zar}(K(T)|H)$ consists of Gaussian extensions of valuation domains of K. Precisely, if $W \in \mathrm{Zar}(K(T)|H)$ and $V := W \cap K$, then $W = V(T)$.*

Proof. Let w be a valuation on $K(T)$ defining W. By definition, $v := w|_K$ is a valuation on K defining V. Fix a nonzero polynomial $f := f_0 + f_1T + \ldots + f_rT^r \in K[T]$. Since $H \subseteq W$, then W is a K-Halter-Koch ring, by (12.8a). Since $T, T^{-1} \in W$ we have $w(T) = 0$. Thus

$$w(f) \ge \inf\{w(f_i) : 0 \le i \le r\} = \inf\{v(f_i) : 0 \le i \le r\} =: v_g(f)$$

On the other hand, by (12.9b), $fH = f_0H + \ldots + f_rH$, and thus $f_i \in fH$, for $0 \le i \le r$. Take elements $h_i \in H$ such that $f_i = fh_i$, for $0 \le i \le r$. Thus $w(h_i) \ge 0$, for any $0 \le i \le r$, since $H \subseteq W$. It follows that $v(f_i) = w(f_i) = w(f) + w(h_i) \ge w(f)$, for $0 \le i \le r$, and thus $v_g(f) \ge w(f)$. This proves that $w|_{K[T]} = v_g|_{K[T]}$, that is, $w = v_g$. The conclusion is now clear. □

(12.15) Corollary. *Let K be a field, T be an indeterminate over K and let H be a subring of $K(T)$. Then, the following conditions are equivalent.*

(i) *H is a K-Halter-Koch ring.*
(ii) *H is integrally closed and $\mathrm{Zar}(K(T)|H)$ consists of Gaussian extensions of valuation domains of K.*

Proof. (i)⟹(ii). Apply (12.9c) and (12.14), keeping in mind that any Bézout domain is integrally closed, being it a Prüfer domain, in particular.

(ii)⟹(i). By assumption, H is the intersection of a collection of Gaussian extensions of valuation domains of K. Then, it suffices to apply (12.11). □

(12.16) Theorem ([4, Corollary 3.6])**.** *Let K be a field, D be a subring of K and T be an indeterminate over K. If $H := \bigcap_{V\in\mathrm{Zar}(K|D)} V(T)$, then the following properties hold.*

(a) *The canonical map $\eta : \mathrm{Zar}(K(T)|H) \longrightarrow \mathrm{Zar}(K|D)$, $W \mapsto W \cap K$, is a homeomorphism.*
(b) *The canonical map $\sigma : \mathrm{Zar}(K|D) \longrightarrow \mathrm{Spec}(H)$, $V \mapsto \mathfrak{m}_{V(T)} \cap H$, is a homeomorphism.*

Proof. (a). As in (12.13), let $Z_0 := \{V(T) : V \in \mathrm{Zar}(K)\}$. In view of (12.14), we have $\mathrm{Zar}(K(T)|H) = \{V(T) : V \in \mathrm{Zar}(K|D)\} \subseteq Z_0$. Thus η is the restriction to $\mathrm{Zar}(K(T)|H)$ of the homeomorphism $\varphi : Z_0 \longrightarrow \mathrm{Zar}(K)$ presented in (12.13). The conclusion immediately follows by noting that $\varphi(\mathrm{Zar}(K(T)|H)) = \mathrm{Zar}(K|D)$.

(b). Let $\delta : \mathrm{Zar}(K(T)|H) \longrightarrow \mathrm{Spec}(H)$ be the domination map. Clearly we have $\sigma = \delta \circ \eta^{-1}$. Being H a K-Halter-Koch ring (12.11), H is a Bézout domain (see (12.9c)) and, a fortiori, a Prüfer domain. Thus δ is a homeomorphism, by (12.6b). Then it suffices to apply part (a). □

13. The order induced by a topology.

(13.1) Definition. *Let $(X,\mathcal{T})$ be a topological space. Then the preorder $\leq_{\mathcal{T}}$ (which will be denoted by $\leq$ when there is no danger of confusion) on X defined by setting*

$$x \leq y :\iff y \in \overline{\{x\}}$$

is called the preorder induced by the topology of X. *Note that $\leq$ is a partial order on X if and only if X is a T_0 space.*

(13.2) Example. Let X be a topological space.

(a) If X is a T_1 space, then the partial order $\leq$ is trivial, by definition.

(b) If A is a ring and $\mathfrak{p},\mathfrak{q} \in X := \mathrm{Spec}(A)$, endowed with the Zariski topology, then $\mathfrak{p} \leq \mathfrak{q}$ if and only if $\mathfrak{p} \subseteq \mathfrak{q}$, by (3.2d).

(c) If A is a subring of B and $C,D \in X := \mathcal{R}(B|A)$, with the topology defined in (11.14), then $C \leq D$ if and only if $D \subseteq C$. Indeed, if $D \in \overline{\{C\}}$ and $d \in D$, then $U(d)$ is an open neighborhood of D and thus $C \in U(d)$, i.e., $d \in C$. Conversely, assume that $D \subseteq C$ and take an open neighborhood of D. By definition, there are finitely many elements $x_1,\ldots,x_n \in B$ such that $D \in \bigcap_{i=1}^n U(x_i) \subseteq U$. It follows, $x_1,\ldots,x_n \in D \subseteq C$, i.e., $C \in \bigcap_{i=1}^n U(x_i) \subseteq U$. This proves that $D \in \overline{\{C\}}$.

(13.3) Definition. *Let $(X,\preceq)$ be a partially ordered set. We say that a topology $\mathcal{T}$ on X is* order-compatible with $\preceq$, *or that* $\preceq$ and $\mathcal{T}$ are order compatible, *provided that the preorder $\leq_{\mathcal{T}}$ induced by $\mathcal{T}$ is an order (i.e., X is T_0) and it coincides with $\preceq$.*

We will provide a classification of all partial orders on a set X which are order-compatible with some spectral topology on X. First, we will characterize all topologies which are order-compactible with a fixed partial order.

If $(X,\preceq)$ is a partially ordered set and $x \in X$, the set $x^+ := \{y \in X : x \preceq y\}$ is called *specialization of x*.

(13.4) Proposition. *Let $(X,\preceq)$ be a partially ordered set and let $\mathcal{T}$ be a topology on X. Then, the following conditions are equivalent.*

(i) *$\mathcal{T}$ is order-compatible with $\preceq$.*

(ii) *The following properties are satisfied:*

  (a) *For any $x \in X$, the set x^+ is closed in $(X,\mathcal{T})$.*

  (b) *If C is a closed subset of $(X,\mathcal{T})$ and $x \in C$, then $x^+ \subseteq C$.*

Proof. Let $\leq$ denote the order induced by $\mathcal{T}$.

(i)$\Longrightarrow$(ii). Fix a point $x \in X$ and and element $y \in X - x^+$. By assumption, $\leq$ and $\preceq$ are the same order, and thus $y \notin \overline{\{x\}}$. Take any open set Ω_y of $(X,\mathcal{T})$ such that $y \in \Omega_y$ and $x \notin \Omega_y$. Then clearly $y \in \Omega_y \subseteq X - x^+$. It follows $X - x^+ = \bigcup_{y \in X - x^+} \Omega_y$, that is, $X - x^+$ is open. This proves statement (a). Take now a closed subset C of $(X,\mathcal{T})$ and elements $x \in C, y \in x^+$. Since $\preceq$ is the order induced by the topology $\mathcal{T}$, we infer $y \in \overline{\{x\}} \subseteq C$, and thus $y \in C$. Thus statement (b) is proved.

(ii)$\Longrightarrow$(i). Assume (ii) and that $x \preceq y$, that is $y \in x^+$. For any closed subset C of $(X,\mathcal{T})$ such that $x \in C$ we have, by statement (b), $x^+ \subseteq C$ and, a fortiori, $y \in C$. This shows that $y \in \overline{\{x\}}$, that is $x \leq y$. Conversely, assume that $y \in \overline{\{x\}}$. Since, by statement (a), x^+ is a closed subset of $(X,\mathcal{T})$ and $x \in x^+$, it follows $\overline{\{x\}} \subseteq x^+$ and, a fortiori, $y \in x^+$, that is, $x \preceq y$. The proof is now complete. □

(13.5) Example. Let $(X, \preceq)$ be a partially ordered set.

(a) Consider the topology $\mathcal{T}_l$ on X for which the collection $\{x^+ : x \in X\}$ is a subbasis for the closed sets. Then, $\mathcal{T}_l$ is the coarsest topology on X which is order-compatible with $\preceq$. As a matter of fact, take a closed subset C of $(X, \mathcal{T}_l)$. By definition, $C = \bigcap_{i \in I} D_i$ where $D_i = \bigcup_{j=1}^{n_i} y_{ij}^+$, for suitable elements $y_{ij} \in X$. Thus, if $x \in C$, for any $i \in I$ there is an index $j(i) \in \{1, \ldots, n_j\}$ such that $x \in y_{ij(i)}^+$, i.e., $y_{ij(i)} \preceq x$. It immediately follows that $x^+ \subseteq C$. By (13.4), $\mathcal{T}_l$ is order-compatible with $\preceq$. By statement (a) of (13.4), any topology on X which is order compatible with $\preceq$ is finer that $\mathcal{T}_l$.

(b) It is easy to verify that the collection of subsets

$$\{C \subseteq X : \text{for any } x \in C, x^+ \subseteq C\}$$

of X is the family of closed sets for a topology on X, and we will denote such a topology by $\mathcal{T}_L$. Clearly, for any $x \in X$ and any $y \in x^+$, we have $y^+ \subseteq x^+$, that is, x^+ is closed in $(X, \mathcal{T}_L)$. Thus, by (13.4), $\mathcal{T}_L$ is order compatible with $\preceq$ and clearly any topology on X which is order-compatible with $\preceq$ is coarser than $\mathcal{T}_L$.

(c) Let $\mathcal{T}$ be any topology on X which is order compatible with $\preceq$. Then, for any $x, y \in X$,

$$y \in \overline{\{x\}} \iff x \leq_{\mathcal{T}} y \iff x \preceq y \iff y \in x^+,$$

that is $\overline{\{x\}} = x^+$.

(d) If X is finite and and $C := \{x_1, \ldots, x_n\} \subseteq X$ is closed, with respect to $\mathcal{T}_L$, we have ${x_i}^+ \subseteq C$, for any $1 \leq i \leq n$. It follows $C = \bigcup_{i=1}^n {x_i}^+$, that is, C is closed, with respect to T_l. It follows that $\mathcal{T}_l = \mathcal{T}_L$, that is, there is a unique topology on X which is order-compatible with $\preceq$.

(13.6) Proposition. *Let X be a finite T_0 space. Then X is spectral.*

Proof. Clearly any subset of X is compact. Moreover X^{patch} is compact, being it finite. Thus the conclusion follows from (11.10). □

(13.7) Corollary. *Let $(X, \preceq)$ be a finite partially ordered set. Then there is a unique spectral topology on X which is order-compatible with $\preceq$.*

Proof. By (13.5d) $\mathcal{T}_L$ is the unique topology on X which is order compatible with $\preceq$ and, by definition, $(X, \mathcal{T}_L)$ is T_0. Then it suffices to apply (13.6). □

(13.8) Proposition ([12, Proposition 14])**.** *Let $(X, \preceq)$ be a partially ordered set. Then, there is at most one Noetherian spectral topology on X which is order compatible with $\preceq$.*

Proof. Let $\mathcal{T}, \mathcal{T}'$ be Noetherian spectral topologies on X which are order-compatible with $\preceq$, and let C be a closed subset of $(X, \mathcal{T})$. Since $(X, \mathcal{T})$ is Noetherian, C is union of only finitely many irreducible components, say $C_1, \ldots, C_n$, in view of (2.17), and they are clearly closed in $(X, \mathcal{T})$. Since the topology $\mathcal{T}$ is spectral, for each $1 \leq i \leq n$ there is a point $x_i \in C_i$ such that $C_i = \overline{\{x_i\}}$ (any spectral space is sober). Thus, since $\mathcal{T}, \mathcal{T}'$ are order-compatible with $\preceq$ and applying (13.5c), $C_i = {x_i}^+$ is the closure of $\{x_i\}$ also with respect to the topology $\mathcal{T}'$. Thus C is closed in $(X, \mathcal{T}')$, being it finite union of closed sets. The converse part is done by exchanging the role of $\mathcal{T}$ and $\mathcal{T}'$. □

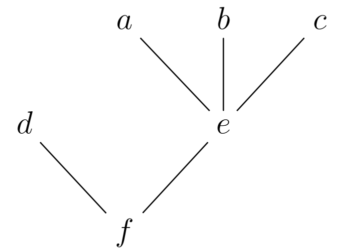
(13.9) Proposition. *Let X, Y be Noetherian spectral spaces. Let X, Y be endowed by their natural structures of partially ordered sets induced by the topologies. If $f : X \longrightarrow Y$ is an order-isomorphism (i.e., f is bijective and both f, f^{-1} are order preserving), then f is a homeomorphism.*

Proof. Let C be a closed subset of Y. Keeping in mind the proof of (13.8), there are elements $c_1, \ldots, c_n \in C$ such that $C = \bigcup_{i=1}^n {c_i}^+$. Keeping in mind that f is an order-isomorphism and (13.5c), it follows that $f^{-1}(C) = \bigcup_{i=1}^n f^{-1}(c_i)^+ = \bigcup_{i=1}^n \overline{\{f^{-1}(c_i)\}}$, and thus f is continuous. Apply the same argument to f^{-1} to show that f^{-1} is continuous. □

The problem of finding explicitly a ring whose prime spectrum is order isomorphic (and then homeomorphic, by 13.9) to a given finite partially ordered set was solved by Lewis in 1973. In the following we will present an easy example which will put in evidence the crucial role of fiber products for producing such constructions. We start from an easy remark.

(13.10) Remark. Let K be any field, let T be an indeterminate over K and, for any $\lambda \in K$ let $\mathfrak{p}_\lambda := (T - \lambda)K[T]$. Given pairwise distinct elements $\alpha_1, \ldots, \alpha_n \in K$, consider the multiplicative subset $S := K[T] - \bigcup_{i=1}^n \mathfrak{p}_{\alpha_i}$ and let $A := K[T]_S$. Keeping in mind the Prime avoidance Lemma and applying properties of localization, we infer that $\mathrm{Spec}(A)$ is homeomorphic and order isomorphic to $\{(0), \mathfrak{p}_{\alpha_i} : 1 \leq i \leq n\}$. Thus A has precisely n maximal ideals, namely $\mathfrak{m}_i := \mathfrak{p}_{\alpha_i}A$, for $1 \leq i \leq n$. Furthermore, we easily infer that $A/\mathfrak{m}_i \cong K$, for $1 \leq i \leq n$ (an isomorphim is induced by the extension to A of the canonical ring homomorphism $K[T] \longrightarrow K$, $f \mapsto f(\alpha_i)$).

(13.11) Example. Consider the finite $X := \{a, b, c, d, e, f\}$, partially orderd by an order $\preceq$ whose Hasse diagram diagram is



By (13.5d) and (13.7), the unique topology $\mathcal{T}_L$ which is order compatible with the given partial order is spectral. We will find a ring D such that $\mathrm{Spec}(D)$ is order isomorphic $(X, \preceq)$ (and, a fortiori, $\mathrm{Spec}(D)$ is homeomorphic to X, endowed with the unique topology inducing $\preceq$, by (13.9). Take a field K, an indeterminate T over K, distinct elements $\alpha, \beta \in K$ and consider the multiplicative subset

$$S := K[T] - ((T - \alpha)K[T] \cup (T - \beta)K[T])$$

of $K[T]$. By (13.10), the prime spectrum of the ring $B := K[T]_S$ is order isomorphic to the subset $\{d, e, f\}$ of X, and the residue fields of the two maximal ideals $\mathfrak{m}_\alpha, \mathfrak{m}_\beta$ of B are isomorphic to K. Now choose $K := L(U)$, where L is any field and U is an indeterminate over L. Consider pairwise distinct elements $\lambda_1, \lambda_2, \lambda_3 \in L$ and set $A := L[U]_\Sigma$, where

$$\Sigma := L[U] - \bigcup_{i=1}^{3}((U - \lambda_i)L[U]),$$

and note that $A \subseteq K$. Again by (13.10), $\mathrm{Spec}(A)$ is order isomorphic to the subset $\{a,b,c,e\}$ of X. By the Chinese Remainder Theorem, the canonical ring homomorphism $\pi : B \longrightarrow (B/\mathfrak{m}_\alpha)\times(B/\mathfrak{m}_\beta) = K\times K$ is surjective. Since $K\times A$ is a subring of $K\times K$, we can consider the ring $D := \pi^{-1}(K\times A)$. By applying (4.2) and (4.11) it follows that $\mathrm{Spec}(D)$ is order isomorphic to X.

(13.12) Proposition. *Let X be a spectral space and let*

$$\mathcal{O} := \{X-\Omega : \Omega \text{ open and compact in } X\} \cup \{X\}.$$

Then there is a (unique) topology on X for which $\mathcal{O}$ is a basis of open sets. This topology is called the inverse topology (with respect to the given spectral topology) on X. *We will denote by X^{inv} the set X, endowed with the inverse topology. Moreover, if $Y \subseteq X$, we will denote by $\overline{Y}^{\mathrm{i}}$ the closure of Y, with respect to the inverse topology.*

Proof. It suffices to note that $\mathcal{O}$ is closed under finite intersection. As a matter of fact, if Ω,Ω' are open and compact in X, then $\Omega\cup\Omega'$ is open and compact too, and thus $(X-\Omega)\cap(X-\Omega') = X-(\Omega\cup\Omega') \in \mathcal{O}$. ☐

The following result justifies the choice of the terminology.

(13.13) Proposition. *Let X be a spectral space. The following properties hold.*

(a) *X^{inv} is a T_0 space.*

(b) *If $x,y\in X$, $\leq$ is the order induced by the topology of X and $\leq_{\mathrm{i}}$ is the order induced by the inverse topology, then*

$$x\leq y \iff y \leq_{\mathrm{i}} x.$$

Proof. (a). Take distinct points $x,y\in X$. Since X is spectral, it is T_0, and thus there is an open set U of X such that $x\in U$, $y\notin U$. Moreover, since X has a basis of open and compact subspaces (being it spectral), there is an open and compact subspace of X such that $x\in\Omega\subseteq U$. Since $y\notin U$, then $y\notin\Omega$, a fortiori. Thus $V := X-\Omega$ is, by definition, an open neighborhood of y, with respect to the inverse topology, and $x\notin V$. It follows that X^{inv} is a T_0 space.

(b). Assume $x\leq y$ and let U be an open neighborhood of x, with respect to the inverse topology. By definition, there is an open and compact subspace Ω of X such that $x\in X-\Omega\subseteq U$. Since $x\leq y$ and $X-\Omega$ is closed in X, it follows $y\in\overline{\{x\}}\subseteq X-\Omega\subseteq U$. It follows $x\in\overline{\{y\}}^{\mathrm{i}}$. Conversely, assume that $y\leq_{\mathrm{i}} x$ and let V be an open neighborhood of y, with respect to the given spectral topology of X. Since X is spectral, there is an open and compact subspace Ω of X such that $y\in\Omega\subseteq V$. By definition, the set Ω is closed in X^{inv} and, keeping in mind that $y\leq_{\mathrm{i}} x$, we have $x\in\overline{\{y\}}^{\mathrm{i}}\subseteq\Omega\subseteq V$, that is, $y\in\overline{\{x\}}$. ☐

(13.14) Theorem (Hochster duality [12, Proposition 8])**.** *Let X be a spectral space. Then, the following properties hold.*

(a) *The collection*

$$\mathcal{O} := \{X-\Omega : \Omega \text{ open and compact in } X\} \cup \{X\}$$

is a basis of open and compact subspaces of X^{inv}.

(b) *X^{inv} is a spectral space and the patch topology of X^{inv} is equal to the patch topology of X.*

(c) $(X^{\mathrm{inv}})^{\mathrm{inv}}$ *is* X.

Proof. (a). The inverse topology on X is coarser than the patch topology of X, since, by definition, the members of $\mathcal{O}$ are clopen in X^{patch}. Since X is spectral, then X^{patch} is compact, by (11.10) and thus the members of $\mathcal{O}$ are compact in X^{patch}, being them, in particular, closed in X^{patch}. A fortiori, they are compact as subspaces of X.

(b) Let V be an open and compact subspace of X^{inv}. Being it open, V is a union of a subcollection of $\mathcal{O}$. By compactness, we can assume that such a subcollection is finite. Thus $V = \bigcup_{i=1}^{n}(X - \Omega_i)$, for some open and compact subspaces $\Omega_1, \ldots, \Omega_n$ of X. Since X is spectral, $\bigcap_{i=1}^{n} \Omega_i$ is compact in X, in view of (11.11), and thus $V = \bigcup_{i=1}^{n}(X - \Omega_i) = X - \bigcap_{i=1}^{n} \Omega_i \in \mathcal{O}$. It follows that $\mathcal{O}$ is precisely the set of all open and compact subspaces of X^{inv}.

Now, let $\mathcal{B}$ be the basis of open and compact subspaces of X, and let $\mathcal{T}$ be a topology on X. Since $\mathcal{B} := \{X - U : U \in \mathcal{O}\}$, $\mathcal{B}$ is a collection of clopen sets for $\mathcal{T}$ if and only if $\mathcal{O}$ is a collection of clopen sets for $\mathcal{T}$. It follows that the patch topology of X and the patch topology on X^{inv} are the same topology. Keeping in mind part (a) and (11.10), it follows that X^{inv} is spectral.

(c). By the proof of part (b), $\mathcal{O}$ consists precisely of the open and compact subspaces of X^{inv}. By definition, a basis of open sets for the inverse topology of X^{inv} is given by the complements of the members of $\mathcal{O}$, that is, such a basis is precisely $\mathcal{B}$. The proof is now complete. □

14. Topology and irredundant intersections.

(14.1) Definition ([14]). *Let D be a set and let A, C be subsets of D such that $A \subsetneq C$. If X is a fixed collection of subsets of D and $F \subseteq D$, set*

$$\mathcal{V}(F) := \{B \in X : F \subseteq B\} \qquad \mathcal{U}(F) := X - \mathcal{V}(F)$$

and, with a small abuse of notation, set $\mathcal{V}(d) := \mathcal{V}(\{d\}), \mathcal{U}(d) := \mathcal{U}(\{d\})$, for any $d \in D$.

*We say that X is C-*representation of A *if $A = \bigcap_{B \in X} B \cap C$. Moreover, we will say that a C-representation X is* spectral *if X is a spectral space and the collection of sets $\{\mathcal{U}(d) : d \in D\}$ is a subbasis of open and compact subspaces of X. When X is a C-representation of A and $C := D$, we will simply say that X is a* representation *of A.*

(14.2) Remark. Preserve the notation of (14.1) and let X be a topological space, whose points are subsets of D, such that $\{\mathcal{U}(d) : d \in D\}$ is a subbasis of open sets for X. Then, the order induced by the topology is the inclusion $\subseteq$.

As a matter of fact, let $G, H \in X$ and assume that $G \leq H$, i.e., $H \in \overline{\{G\}}$. Then $g \in G$ is equivalent to $G \in \mathcal{V}(g)$ and, since $\mathcal{V}(g)$ is, by definition, closed in X, it follows $H \in \overline{\{G\}} \subseteq \mathcal{V}(g)$ and, in particular, $g \in H$. This proves that $G \subseteq H$.

Conversely, assume that $G \subseteq H$ and let Ω be an open neighborhood of H. By definition, there is a finite subset F of D such that $H \in \bigcap_{f \in F} \mathcal{U}(f) \subseteq \Omega$. Since $G \subseteq H$, we infer that $G \in \bigcap_{f \in F} \mathcal{U}(f)$. It follows $G \in \Omega$, and thus $H \in \overline{\{G\}}$.

(14.3) Proposition ([14, Lemma 3.2]). *We preserve the notation of (14.1) and assume that X is a spectral C-representation of A. Then X contains* a minimal closed C-representation of A *(i.e., a closed subset of X that is minimal, under inclusion, among closed C-representations of A).*

Proof. Let Σ be the collection of all closed (in X) C-representations of A, partially ordered by inclusion $\supseteq$. The set Σ is clearly nonempty, since $X \in \Sigma$. Let $\mathcal{C} \subseteq \Sigma$ be a chain and let $Y := \bigcap \mathcal{C}$. Then Y is clearly closed in X (being it intersection of closed subsets of X). We want to show that Y is a C-representation of A. The inclusion $A \subseteq \bigcap_{B \in Y} B \cap C$ is obvious. Conversely, take an element $d \in \bigcap_{B \in Y} B \cap C$. It follows $Y \subseteq \mathcal{V}(d)$ and, in other words, $\mathcal{U}(d) \subseteq \bigcup_{T \in \mathcal{C}}(X - T)$. Since, by assumption, $\mathcal{U}(d)$ is compact, the open cover $\{X - T : T \in \mathcal{C}\}$ of $\mathcal{U}(d)$ has a finite subcover and, keeping in mind that $\mathcal{C}$ is a chain, it follows that there exists a set $T^* \in \mathcal{C}$ such that $\mathcal{U}(d) \subseteq X - T^*$. In other words, $T^* \subseteq \mathcal{V}(d)$ and, since T^* is a C-representation of A, being $T^* \in \Sigma$, we have $d \in \bigcap_{B \in T^*} B \cap C = A$. This argument proves that $A = \bigcap_{B \in Y} B \cap C$, that is, $Y \in \Sigma$. The conclusion follows immediately by applying Zorn's Lemma. $\square$

(14.4) Definition. *We preserve the notation of (14.1) and let Z be a C-representation of A. We say that a set $B \in Z$ is* irredundant in Z *if* $A \subsetneq \bigcap_{H \in Z, H \neq B} H \cap C$.

(14.5) Remark. We preserve the notation of (14.1) and assume that $Z \subseteq Z'$ are C-representations of A. If $B \in Z$ is irredundant in Z', then B is irredundant in Z too. Indeed we have

$$A = \bigcap_{H \in Z} H \cap C = \bigcap_{H \in Z'} H \cap C \subsetneq \bigcap_{H \in Z', H \neq B} H \cap C \subseteq \bigcap_{H \in Z, H \neq B} H \cap C$$

If X is a spectral space and $Y \subseteq X$, we will denote the closure of Y in the patch topology by $\overline{Y}^{\mathrm{p}}$.

(14.6) Proposition ([14, Lemma 3.3]). *We preserve the notation of (14.1) and assume that X is a spectral C-representation of A. If $Z \subseteq X$ is a C-representation of A and $B \in Z$, then B is irredundant in Z if and only if B is irredundant in $\overline{Z}^{\mathrm{p}}$.*

Proof. First, assume that B is irredundant in Z. By definition, there exists an element $d \in D - B$ such that $d \in H$, for any $H \in Z - \{B\}$. It follows that $Z \subseteq \mathcal{V}(d) \cup \{B\}$. By (11.8), the patch topology of X is finer than the given spectral topology of X and Hausdorff. It follows immediately that $\mathcal{V}(d) \cup \{B\}$ is closed in the patch topology of X, and thus $Z \subseteq \mathcal{V}(d) \cup \{B\}$ implies $\overline{Z}^{\mathrm{p}} \subseteq \mathcal{V}(d) \cup \{B\}$. We infer that $\overline{Z}^{\mathrm{p}} - \{B\} \subseteq \mathcal{V}(d)$, and thus, since $d \notin B$,

$$\bigcap_{H \in \overline{Z}^{\mathrm{p}}, H \neq B} H \cap C \supsetneq \bigcap_{H \in \overline{Z}^{\mathrm{p}}} H \cap C,$$

meaning that B is irredundant in $\overline{Z}^{\mathrm{p}}$.

The converse part is obvious, keeping in mind that $Z \subseteq \overline{Z}^{\mathrm{p}}$ and (14.5). $\square$

(14.7) Proposition ([14, Lemma 3.3]). *We preserve the notation of (14.1) and assume that X is a spectral C-representation of A. If $Z \subseteq X$ is a C-representation of A and $B \in Z$ is irredundant in Z, then B is an isolated point of Z, equipped with both the spectral and the patch subspace topology.*

Proof. Take an element $d \in \bigcap_{H \in Z, H \neq B} H \cap C$ and $d \notin B$. Then we easily obtain $Z - \{B\} = Z \cap \mathcal{V}(d)$. Keeping in mind that $\mathcal{V}(d)$ is closed both in the spectral topology and in the patch topology of X, in view of (11.8), the conclusion follows. $\square$

Let X be a topological space, $Y \subseteq X$ and $\leq$ be the preorder induced by the topology. Then the set

$$Y^{\mathrm{sp}} := \bigcup_{y \in Y} y^{+} = \{x \in X : x \geq y, \text{ for some } y \in Y\}$$

is called *the specialization of* Y.

(14.8) Remark. Let X be a topological space and let $Y \subseteq X$. Then clearly $Y \subseteq Y^{\mathrm{sp}}$ and the equality holds whenever Y is closed. Indeed, if Y is closed and $x \in Y^{\mathrm{sp}}$, take an element $y \in Y$ such that $x \geq y$. It follows $x \in \overline{\{y\}} \subseteq \overline{Y} = Y$.

(14.9) Proposition ([6, Lemma 1.1])**.** *Let X be a spectral space and let Y be a subset of X. Then the following equality $\overline{Y} = (\overline{Y}^{\mathrm{p}})^{\mathrm{sp}}$ holds.*

Proof. By (11.8), the patch topology of X is finer than the given spectral topology, and thus we have $\overline{Y}^{\mathrm{p}} \subseteq \overline{Y}$. By (14.8) we immediately infer that $(\overline{Y}^{\mathrm{p}})^{\mathrm{sp}} \subseteq \overline{Y}$. Conversely, take a point $x \in \overline{Y}$ and let

$$\mathcal{F} := \{\Omega \cap \overline{Y}^{\mathrm{p}} : \Omega \text{ open and compact in } X, x \in \Omega\}$$

Since $x \in \overline{Y}$, $\mathcal{F}$ consists of nonempty sets and, in view of (11.11), it is closed under finite intersections. It follows that $\mathcal{F}$ is a collection of closed subsets of X^{patch} with the finite intersection property. Since X is spectral, X^{patch} is compact, by (11.10), and thus there is a point $x_0 \in \bigcap \mathcal{F}$. Since open and compact subspaces of X form a basis for the topology of X, it immediately follows that $x \in \overline{\{x_0\}}$, that is, $x \geq x_0$. The conclusion is now clear. □

If $(X, \leq)$ is a partially ordered set, let $\mathrm{Min}(X)$ (resp., $\mathrm{Max}(X)$) denote the set of all minimal (resp., maximal) elements of X.

(14.10) Proposition. *Let X be a spectral space and let $x \in X$. Then, there are elements $y \in \mathrm{Min}(X), z \in \mathrm{Max}(X)$ such that $y \leq x \leq z$ (where $\leq$ is the order induced by the topology of X).*

Proof. By assumption, there are some ring A, a homeomorphism $f : \mathrm{Spec}(A) \longrightarrow X$ and consider the prime ideal $\mathfrak{p} := f^{-1}(x)$ of A. By Zorn's Lemma, there are a maximal ideal $\mathfrak{m}$ of A and a minimal prime ideal $\mathfrak{n}$ of A such that $\mathfrak{n} \subseteq \mathfrak{p} \subseteq \mathfrak{m}$. It is easily verified that f is an isomorphism of partially ordered sets, being it a homeomorphism, and thus it is sufficient to take $y := f(\mathfrak{n}), z := f(\mathfrak{m})$ to get the conclusion, keeping in mind that the order of $\mathrm{Spec}(A)$ is the inclusion (see (13.2b)). □

(14.11) Proposition. *Preserve the notation of (14.1), let X be a spectral C-representation of A. Then, $\mathrm{Min}(X) \neq \emptyset$ and it is a C-representation of A.*

Proof. By (14.10), $\mathrm{Min}(X)$ is nonempty. Clearly, $A \subseteq C \cap \bigcap_{B \in \mathrm{Min}(X)} B$. Conversely, take an element $d \in C \cap \bigcap_{B \in \mathrm{Min}(X)} B$ and fix a set $H \in X$. Keeping in mind (14.2) and (14.10), we can pick a set $B \in \mathrm{Min}(X)$ such that $B \subseteq H$. Since $d \in \bigcap_{B \in \mathrm{Min}(X)} B$ we have $d \in H$. Since X is a C-representation of A, we infer that $A = C \cap \bigcap_{H \in X} H \supseteq C \cap \bigcap_{B \in \mathrm{Min}(X)} B$. The conclusion is now clear. □

(14.12) Corollary. *Preserve the notation of (14.1), let X be a spectral C-representation of A, and let $Y \subseteq X$ be a nonempty closed set. Then Y is a C-representation of A if and only if $\mathrm{Min}(Y)$ is a C-representation of A.*

Proof. Keeping in mind (3.6a), Y is a spectral subspace of X. Then, it is sufficient to apply (14.11). □

(14.13) Definition. *We preserve the notation of (14.1) and let X be a spectral C-representation of A. A subspace of X of the form* $\mathrm{Min}(Y)$*, for some minimal closed C-representation Y of A, is a C-representation of A, by (14.12), and it is called* a minimal C-representation of A.

In view of (14.11), any spectral C-representation of A contains a minimal C-representation of A.

(14.14) Proposition ([14, Lemma 3.5])**.** *Preserve the notation of (14.1), let X be a spectral C-representation of A and let $Z \subseteq X$ be a minimal C-representation of A. Then, the following statements hold.*

(a) *$\overline{Z}$ is a minimal closed C-representation of A.*
(b) *$\overline{Z}^{\mathrm{p}}$ is a minimal C-representation of A, among the the closed C-representations of A, with respect to the patch topology of X.*
(c) *We have $Z = \mathrm{Min}(\overline{Z}) = \mathrm{Min}(\overline{Z}^{\mathrm{p}})$ and $\overline{Z} = Z^{\mathrm{sp}}$.*

Proof. By definition, there exists a minimal closed C-representation Γ of A such that $Z = \mathrm{Min}(\Gamma)$. Keeping in mind that Γ is closed, we have $\overline{Z} \subseteq \Gamma$. Since $\overline{Z}$ is a closed C-representation of A, the minimality of Γ implies $\overline{Z} = \Gamma$, proving that $\overline{Z}$ is a minimal closed C-representation of A. Furthermore, the last equality implies $Z = \mathrm{Min}(\overline{Z})$.

Since $Z \subseteq \overline{Z}^{\mathrm{p}}$, it follows that $\overline{Z}^{\mathrm{p}}$ is a C-representation of A. Now, let Y be a C-representation of A such that $Y \subseteq \overline{Z}^{\mathrm{p}}$ and Y is closed, with respect to the patch topology of X. We want to show that $Y = \overline{Z}^{\mathrm{p}}$. By (14.9), Y^{sp} is closed in X and it is a C-representation of A, since $Y \subseteq Y^{\mathrm{sp}}$. Moreover, again by (14.9), we have $Y^{\mathrm{sp}} \subseteq (\overline{Z}^{\mathrm{p}})^{\mathrm{sp}} = \overline{Z}$. Since, by part (a) (which we proved before), $\overline{Z}$ is a minimal closed C-representation of A, we infer that $Y^{\mathrm{sp}} = \overline{Z}$. By applying (14.10) to the spectral subspace $\overline{Z}$ of X, it is easily shown that $(\mathrm{Min}(\overline{Z}))^{\mathrm{sp}} = \overline{Z} = Y^{\mathrm{sp}}$ and, since we have already proved that $Z = \mathrm{Min}(\overline{Z})$, we deduce that

$$Y^{\mathrm{sp}} = (\mathrm{Min}(\overline{Z}))^{\mathrm{sp}} = Z^{\mathrm{sp}}.$$

Moreover, since, by definition, the elements of Z are pairwise not comparable, we have $Z = \mathrm{Min}(Z^{\mathrm{sp}})$. It follows that

$$\mathrm{Min}(Y) = \mathrm{Min}(Y^{\mathrm{sp}}) = \mathrm{Min}(Z^{\mathrm{sp}}) = Z \qquad (\star)$$

and, in particular, $Z \subseteq Y$. Since, by assupmption, $Y \subseteq \overline{Z}^{\mathrm{p}}$ and Y is closed in the patch topology of X, it finally follows $Y = \overline{Z}^{\mathrm{p}}$ and, by $(\star)$, $Z = \mathrm{Min}(\overline{Z}^{\mathrm{p}})$. It remains to show that $\overline{Z} = Z^{\mathrm{sp}}$. Since $Z = \mathrm{Min}(\overline{Z})$, then $Z^{\mathrm{sp}} = (\mathrm{Min}(\overline{Z}))^{\mathrm{sp}} = \overline{Z}$. The proof is now complete. □

If X is a topological space and $Y \subseteq X$, we will denote by $\overset{\circ}{Y}$ the interior of Y.

(14.15) Proposition ([15, Lemma 2.5])**.** *Let X be a spectral space and let $x \in X$. Then, the following conditions are equivalent.*

(i) *x is a minimal point of X (with respect to the order induced by the topology).*
(ii) *If Y is an open neighborhood of x in X^{patch}, then $x \in \overset{\circ}{Y}$.*

Proof. (i)$\Longrightarrow$(ii). Let Y be an open neighborhood of x in X^{patch} and assume, by contradiction, that $x \notin \overset{\circ}{Y}$. By definition, for any open and compact subset Ω of X such that $x \in \Omega$ we have $\Omega \nsubseteq Y$. Keeping in mind (11.11), it follows that

$$\mathcal{F} := \{\Omega \cap (X - Y) : \Omega \subseteq X \text{ open and compact}, x \in \Omega\}$$

has the finite intersection property and, by definition, it consists of closed subsets of X^{patch}. Then (11.10) implies that there is a point $x_0 \in \bigcap \mathcal{F}$. Since open and compact subspaces of X form a basis for the topology, it easily follows that $x_0 \leq x$ and, since x is minimal, $x_0 = x$, a contradiction, since $x_0 \in \bigcap \mathcal{F} \subseteq X - Y$, but $x \in Y$.

(i)$\Longrightarrow$(i). By contradiction, assume that there exists a point $y \in X$ such that $y < x$. Thus, $x \in \overline{\{y\}}$ and there exists an open and compact subspace U of X such that $y \in U$ and $x \notin U$. By definition, $Y := X - U$ is an open neighborhood of x in X^{patch}, and thus, by assumption $x \in \overset{\circ}{Y}$. Then, there is an open neighborhoof V of x (in X) such that $V \subseteq Y$. Since $x \in \overline{\{y\}}$, it follows $y \in V$ and, a fortiori, $y \in X - U$, a contradiction. $\square$

(14.16) Proposition. *Let X be a spectral space. Then, the subspace topologies induced on* $\operatorname{Min}(X)$ *by the given spectral topology and by the patch topology are the same.*

Proof. Let U be an open set of $\operatorname{Min}(X)$, equipped with the subspace topology induced by the patch topology of X. Then, there is an open subset Y of X^{patch} such that $U = \operatorname{Min}(X) \cap Y$. By (14.15), we have $U = \overset{\circ}{Y} \cap \operatorname{Min}(X)$ and thus U is open in $\operatorname{Min}(X)$, with respect to the subspace topology induced by the given spectral topology of X. $\square$

(14.17) Proposition. *Let X be a spectral space and let Y be a closed subset of X, with respect to the patch topology. Then the following properties hold.*

(a) *Y is a spectral space, endowed with the subspace topology induced by the given spectral topology of X.*
(b) *The patch topology of Y is equal to the subspace topology induced by the patch topology of X.*
(c) *The inverse topology of Y is equal to the subspace topology induced by the inverse topology of X.*

Proof. Let $\mathcal{T}_1$ (resp., $\mathcal{T}_2$) denote the patch topology of Y (resp., the subspace topology induced on Y by the given spectral topology of X). Let $\mathcal{A}$ be the collection of all open and compact subspaces of $(Y, \mathcal{T}_2)$. We want to show the following claim.

Claim. $\mathcal{A} = \{\Omega \cap Y : \Omega \subseteq X \text{ open and compact}\}$.

The inclusion $\supseteq$ is easy: indeed, any set of the form $\Omega \cap Y$ (Ω open and compact in X) is closed, with respect to the patch topology of X. Since X^{patch} is compact, in view of (11.10), the set $Y \cap \Omega$ is compact in X^{patch} and, a fortiori, it is compact in X, by (11.8). Conversely, let U be an open and compact subspace of $(Y, \mathcal{T}_2)$, and let V be an open set of X such that $U = V \cap Y$. Since open and compact subspaces form a basis of X, for any $u \in U$ there is an open and compact subspace Ω_u of X such that $u \in \Omega_u \subseteq V$. Since U is compact and $U \subseteq \bigcup\{\Omega_u : u \in U\}$, there is a finite set F of U such that $U \subseteq \Omega^* := \bigcup_{u \in F} \Omega_u \subseteq V$. Then Ω^* is clearly open and compact and $U = Y \cap \Omega^*$, proving the claim.

Now, let $\mathcal{T}_3$ denote the subspace topology induced on Y by the patch topology of X. By the claim, $\mathcal{A}$ is a collection of clopen sets of $(Y, \mathcal{T}_3)$ and thus, by definition, $\mathcal{T}_1$ is coarser that $\mathcal{T}_3$. Since Y is closed in the compact space X^{patch}, then $(Y, \mathcal{T}_3)$ is compact and, a fortiori, $(Y, \mathcal{T}_1)$ is compact. Moreover, since $\mathcal{A}$ is clearly a basis for $(Y, \mathcal{T}_2)$, (11.10) implies that $(Y, \mathcal{T}_2)$ is a spectral space, proving (a). Keeping in mind that both $(Y, \mathcal{T}_1), (Y, \mathcal{T}_3)$ are compact and Hausdorff spaces and that $\mathcal{T}_1$ is coarser than $\mathcal{T}_3$, it follows $\mathcal{T}_1 = \mathcal{T}_3$, in view of (10.9). Part (c) is an immediate consequence of the Claim. The proof is now complete. □

(14.18) Proposition. *Preserve the notation of (14.1), let $Z \subseteq X$ be C-representations of A, and assume that X is spectral and Z is minimal (see (14.13)). Then, the spectral and the patch topologies of X induce the same topology on Z.*

Proof. By definition, there is a minimal closed C-representation Y of A such that $Z = \text{Min}(Y)$. By (11.8), Y is closed in X^{patch}. Thus the conclusion follows immediately from (14.16) and (14.17). □

(14.19) Definition. *Preserve the notation of (14.1), let $Z \subseteq X$ be C-representations of A, and assume that X is spectral. We say that a set $B \in Z$ is* strongly irredundant *in Z if the unique closed subset Y of $\mathcal{V}(B)$ such that $Y \cup (Z - \{B\})$ is a C-representation of A is $Y = \mathcal{V}(B)$.*

(14.20) Remark. Preserve the notation of (14.1), let $Z \subseteq X$ be C-representations of A, and assume that X is spectral. If $B \in Z$ is strongly irredundant in Z, then B is irredundant in Z. Indeed, $Y := \emptyset$ is a proper closed subset of $\mathcal{V}(B)$ and thus, by definition, $Y \cup (Z - \{B\}) = Z - \{B\}$ is not a C-representation of A.

Clearly, the notion of irredundance and strong irredundance are not equivalent. In the next result a topological criterion for irredundance in minimal representations is given. For such representations, irredundance and strong irredundance will turn out to be equivalent.

(14.21) Theorem ([14, Theorem 3.6])**.** *We preserve the notation of (14.1). Let $Z \subseteq X$ be C-representations of A, and assume that X is spectral and Z is minimal (see (14.13)). If $B \in Z$, then the following conditions are equivalent.*

(i) *B is irredundant in Z.*
(ii) *B is strongly irredundant in Z.*
(iii) *B is isolated in Z, endowed with the subspace topology induced by the spectral topology of X.*

Proof. (ii)$\Longrightarrow$(i) and (i)$\Longrightarrow$(iii) follow from (14.20) and (14.7), respectively (and without any extra assumption on Z).

(iii)$\Longrightarrow$(ii). First, note that $\mathcal{V}(B) = \overline{\{B\}}$, in view of (14.2), and thus $B \in Z$ implies $\mathcal{V}(B) \subseteq \overline{Z}$. Let Y be a closed subset of $\mathcal{V}(B)$ such that $Y \cup (Z - \{B\})$ is a C-representation of A. A fortiori, $Y \cup \overline{Z - \{B\}}$ is a closed C-representation of A and $Y \cup \overline{Z - \{B\}} \subseteq \overline{Z}$. Since Z is a minimal C-representation of A, then $\overline{Z}$ is a minimal closed C-representation of A, in view of (14.14). Then we have $Y \cup \overline{Z - \{B\}} = \overline{Z}$. Moreover, since B is isolated in Z, by assumption, we have $B \notin \overline{Z - \{B\}}$. Since $B \in \overline{Z}$, it follows $B \in Y$ and thus $\mathcal{V}(B) = \overline{\{B\}} \subseteq Y$, because Y is closed. It follows $\mathcal{V}(B) = Y$. The conclusion is now clear. □

(14.22) Definition. *We preserve the notation of (14.1). Let X be a spectral C-representation of A. We say that a C-representation $Z \subseteq X$ is* irredundant *(resp.,* strongly irredundant*) if any set $B \in Z$ is irredundant (resp., strongly irredundant) in Z.*

(14.23) Proposition. *We preserve the notation of (14.1). Let $Z \subseteq X$ be C-representations of A, and assume that X is spectral and Z is minimal (see (14.13)). Then $\overline{Z}^{\mathrm{p}}$ contains at most one irredundant C-representation of A. More precisely, if $\Gamma \subseteq \overline{Z}^{\mathrm{p}}$ is an irredundant C-representation of A, then Γ is the set of isolated points of Z (endowed with the subspace topology induced by the spectral topology of Z).*

Proof. $\overline{\Gamma}^{\mathrm{p}}$ is a C-representation of A, being $\Gamma \subseteq \overline{\Gamma}^{\mathrm{p}}$, and thus, by (14.14b), we have $\overline{\Gamma}^{\mathrm{p}} = \overline{Z}^{\mathrm{p}}$. Take a set $B \in \Gamma$. Since B is irredundant in Γ, (14.6) implies that B is irredundant in $\overline{Z}^{\mathrm{p}}$ and hence B is isolated in $\overline{Z}^{\mathrm{p}}$, with respect to the patch topology. Thus, since $\{B\}$ is open in $\overline{Z}^{\mathrm{p}}$ and Z is dense in $\overline{Z}^{\mathrm{p}}$, we have $B \in Z$. This shows that $\Gamma \subseteq Z$. Keeping in mind (14.21) and (14.6), a set $B \in Z$ is isolated in Z if and only if B is irredundant in $\overline{Z}^{\mathrm{p}}$. Thus, if B is isolated in Z, then it is isolated in $\overline{Z}^{\mathrm{p}}$ (with respect to the patch topology) and, since Γ is dense in $\overline{Z}^{\mathrm{p}}$, we infer $B \in \Gamma$. Conversely, if $B \in \Gamma$, then it is irredundant in Γ, by assumption, and thus B is irredundant in $\overline{\Gamma}^{\mathrm{p}} = \overline{Z}^{\mathrm{p}}$, and this is equivalent to state that B is isolated in Z. □

We now provide a topological criterion for the existence of an irredundant representation in a minimal representation.

(14.24) Theorem ([14, Corollary 3.7])**.** *We preserve the notation of (14.1). Let $Z \subseteq X$ be C-representations of A, and assume that X is spectral and Z is minimal (see (14.13)). Then, the following conditions are equivalent.*

- (i) *Z contains an irredundant C-representation of A (or, equivalently, a strongly irredundant C-representation of A, in view of (14.21)).*
- (ii) *The set of isolated points of Z is dense in Z, with respect to the topology induced by the spectral topology of X.*

Proof. (i)$\Longrightarrow$(ii). Let $\Gamma \subseteq Z$ be an irredundant C-representation of A. Then, $\overline{\Gamma}^{\mathrm{p}}$ is a C-representation of A such that $\overline{\Gamma}^{\mathrm{p}} \subseteq \overline{Z}^{\mathrm{p}}$ and, in view of (14.14b), $\overline{\Gamma}^{\mathrm{p}} = \overline{Z}^{\mathrm{p}}$. Keeping in mind (14.18), it follows that Γ is dense in Z, with respect to the subspace topology induced by the spectral topology of X. Then condition (ii) follows from (14.23).

(ii)$\Longrightarrow$(i). Let Y be the set of isolated points of Z and assume, by contradiction, that $A = C \cap \bigcap_{B \in Z} B \subsetneq C \cap \bigcap_{B \in Y} B$. Take an element $d \in C \cap \bigcap_{B \in Y} B$ such that $d \notin A$. Then, there is a set $B' \in Z$ such that $d \notin B'$. Since, by assumption, Y is dense in Z, the nonempty open set $\mathcal{U}(d) \cap Z$ of Z must intersect Y. On the other hand, $d \in C \cap \bigcap_{B \in Y} B$ implies $Y \subseteq \mathcal{V}(d)$, a contradiction. It follows that Y is a C-representation of A and, by (14.21), it is irredundant. □

(14.25) Definition. *Let X be a topological space. We say that X is* scattered *if any nonempty subset Y of X contains a point that is isolated in Y.*

We start with some simple property that characterize scattered spaces.

(14.26) Proposition. *Let X be a topological space. Then, the following conditions are equivalent.*

- (i) *X is scattered.*

(ii) *Any nonempty closed subset C of X contains a point that is isolated in C.*
(iii) *For any nonempty subset Y of X, the set $\{y \in Y : y \text{ is isolated in } Y\}$ is dense in Y.*

Proof. Clearly, it suffices to show that (ii)$\Longrightarrow$(i) and that (i)$\Longrightarrow$(iii).

(ii)$\Longrightarrow$(i). Take any nonempty subset Y of X and, by assumption, pick a point $x \in \overline{Y}$ that is isolated in $\overline{Y}$. Since $\{y\}$ is open in $\overline{Y}$ and Y is dense in $\overline{Y}$, we have $y \in Y$. It follows that y is isolated in Y.

(i)$\Longrightarrow$(iii). Take any nonempty subset Y of X, and let Ω be an open subset of X such that $\Omega \cap Y \neq \emptyset$. By assumption, there is a point $y \in \Omega \cap Y$ such that y is isolated in $\Omega \cap Y$. In other words, there is an open set Ω' of X such that $\{y\} = \Omega' \cap \Omega \cap Y$. It follows that y is isolated in Y, proving condition (iii). $\square$

(14.27) Theorem (Mazurkiewicz-Sierpinski, 1920)**.** *Any compact Hausdorff and countable space is scattered.*

Proof. Let X be a compact, Hausdorff and countable space. Keeping in mind that any closed subspace of X is compact, Hausdorff and at most countable, it suffices to show that X has some isolated point, in view of condition (ii) of (14.26). We argue by contradiction, and assume that X has no isolated points. We are going to show the following claim.

Claim. For any nonempty open set U of X and any $x \in X$, there exists a nonempty open set V of X such that $V \subseteq U$ and $x \notin \overline{V}$.

As a matter of fact, we can choose a point $y \in U$ such that $x \neq y$ (this is obvious if $x \notin U$; otherwise, since x is not isolated in X, we have $\{x\} \subsetneq U$). Since X is Hausdorff, there are disjoint open sets Ω_1, Ω_2 such that $x \in \Omega_1, y \in \Omega_2$. Then the set $V := \Omega_2 \cap U \subseteq U$ is nonempty (it contains y) and open and $x \notin \overline{V}$, since W_1 is an open neighborhood of x disjoint from V. This proves the claim.

By assumption, $X = \{x_0, x_1, \ldots, x_n, \ldots\}$. In view of the claim, it is easily proved, by induction, that there exists a sequence $\{V_n : n \in \mathbb{N}\}$ of nonempty open sets of X such that $V_i \supseteq V_{i+1}$ and $x_i \notin \overline{V_i}$, for any $i \in \mathbb{N}$. By construction, $\mathcal{F} := \{\overline{V_i} : i \in \mathbb{N}\}$ is a collection of closed subsets of X with the finite intersection property. Since X is compact, there is a point $x \in \bigcap \mathcal{F}$, and we clearly have $x \neq x_i$, for any $i \in \mathbb{N}$, a contradiction. $\square$

(14.28) Theorem ([14, Corollary 3.8])**.** *We preserve the notation of (14.1). Let X be a spectral C-representation of A and assume that X is scattered with respect to either the spectral topology or the patch topology. Then, X contain a strongly irredundant C-representation of A.*

Proof. By (11.8), the patch topology is finer than the given spectral topology of X. Thus, if X is scattered in the spectral topology, it is scattered in the patch topology. Thus it is enough to prove the statement when X is endowed with the patch topology. By (14.3), X contains a minimal C-representation Z of A. Since X is scattered in the patch topology, the set of isolated points of Z is dense in Z, with respect to the subspace patch topology, i.e., is dense in Z with respect to the subspace spectral topology, by (14.18). Thus, in view of (14.24), Z contains a strongly irredundant representation of A. $\square$

(14.29) Corollary. *We preserve the notation of (14.1). If X is a spectral and countable C-representation of the set A, then X contains a strongly irredundant C-representation of A.*

Proof. X^{patch} is compact, Hausdorff and countable and thus it is scattered, in view of (14.27). Thus, the conclusion follows from (14.28). □

(14.30) Definition. *We preserve the notation of (14.1). Let X be a spectral C-representation of A, and let Y be the intersection of all closed C-representations of A in X. We say that a set $B \in X$ is* critical in X *if $B \in Y$. In the following we will set $\mathscr{C}(X) := \operatorname{Min}(Y)$.*

(14.31) Proposition ([14, Proposition 3.11])**.** *We preserve the notation of (14.1). If X is a spectral C-representation of A and $B \in X$, then the following conditions are equivalent.*

(i) *B is critical in X.*

(ii) *Whenever $A = A_1 \cap \ldots \cap A_n \cap C$, where each A_i is an intersection of members of X, then $A_i \subseteq B$, for some $i \in \{1, \ldots, n\}$.*

Proof. (i)$\Longrightarrow$(ii). Assume that $A = A_1 \cap \ldots \cap A_n \cap C$, where each A_i is an intersection of members of X. It immediately follows that each closed subset $\mathcal{V}(A_i)$ of X is such that $A_i = \bigcap_{H \in \mathcal{V}(A_i)} H$. This proves that $Z := \bigcup_{i=1}^{n} \mathcal{V}(A_i)$ is a closed C-representation of A. Since, by assumption, B is critical in X, we have $B \in Z$, and thus $B \in \mathcal{V}(A_i)$, for some $1 \leq i \leq n$, that is, $B \supseteq A_i$.

(ii)$\Longrightarrow$(i). Assume condition (ii) and let Z be any closed C-representation of A. Since $\{\mathcal{U}(d) : d \in D\}$ is a subbasis of open and compact subspace of X, being X spectral, it follows that Z is intersection of sets of the form $\mathcal{V}(d_1) \cup \ldots \cup \mathcal{V}(d_n)$, where $d_1, \ldots, d_n \in D$. Then, in order to prove that $B \in Z$, it suffices to show that, if $Z \subseteq \mathcal{V}(d_1) \cup \ldots \cup \mathcal{V}(d_n)$, then $B \in \mathcal{V}(d_1) \cup \ldots \cup \mathcal{V}(d_n)$. For any $1 \leq i \leq n$, set $A_i := \bigcap_{H \in \mathcal{V}(d_i)} H$ and note that $d_i \in A_i$. Moreover, $Z' := \mathcal{V}(d_1) \cup \ldots \cup \mathcal{V}(d_n)$ is a C-representation of A, being $Z \subseteq Z'$, and thus

$$A = C \cap \bigcap_{H \in Z'} H = C \cap A_1 \cap \ldots \cap A_n.$$

By assumption, there is some $1 \leq i \leq n$ such that $A_i \subseteq B$. Then $d_i \in B$, i.e., $B \in \mathcal{V}(d_i) \subseteq Z'$. The conclusion is now clear. □

We note now that for a critical set of a spectral representation, the notions of irredundance and strongly irredundance in some representation turn out to be equivalent.

(14.32) Proposition ([14, Corollary 3.12])**.** *We preserve the notation of (14.1). If $Z \subseteq X$ are C-representations of A, X is spectral, and $B \in Z$ is critical in X, then B is irredundant in Z if and only if B is strongly irredundant in Z.*

Proof. Assume that $B \in Z$ is irredundant in Z, and let Y be a closed subset of $\mathcal{V}(B)$ such that $(Z - \{B\}) \cup Y$ is a C-representation of A. We have to show that $Y = \mathcal{V}(B)$ and, since $\mathcal{V}(B) = \overline{\{B\}}$ and Y is closed, what we need to prove is that $B \in Y$. Since X is spectral, Y is intersection of sets of the form $\mathcal{V}(d_1) \cup \ldots \cup \mathcal{V}(d_n)$, for some $d_1, \ldots, d_n \in D$. Thus, take arbitrary elements $d_1, \ldots, d_n \in D$ such that $Y \subseteq Z' := \mathcal{V}(d_1) \cup \ldots \cup \mathcal{V}(d_n)$. Since $Y \cup (Z - \{B\}) \subseteq Z' \cup (Z - \{B\})$, it follows that $Z' \cup (Z - \{B\})$ is a spectral C-representation of A. Then, if $A_i := \bigcap_{H \in \mathcal{V}(d_i)} H$, for $1 \leq i \leq n$, and $A^* := \bigcap_{H \in Z, H \neq B} H$, we have $A = C \cap A_1 \cap \ldots \cap A_n \cap A^*$. Since B is irredundant in Z, we have $A^* \nsubseteq B$ and thus, keeping in mind (14.31) and the fact that B is critical in X, it follows $d_i \in A_i \subseteq B$, for some $1 \leq i \leq n$. This proves that $B \in \mathcal{V}(d_i) \subseteq Z'$. The proof is now complete. $\square$

(14.33) Theorem ([14, Theorem 3.13]). *Let $A \subsetneq C \subseteq D$ be sets, let X be a spectral C-representation of A and let $\mathscr{C}(X)$ be as in (14.30). Then $\mathscr{C}(X)$ is a C-representation of A if and only if X contains a unique minimal C-representation of A, if and only if X contains a unique minimal closed C-representation. In this case, if*

$$S := \{B \in X : B \text{ is strongly irredundant in some } C\text{-representation of } A \text{ in } X\},$$

then the following properties hold.

(a) *$S \subseteq \mathscr{C}(X)$.*
(b) *Any $B \in S$ is strongly irredundant in $\mathscr{C}(X)$.*
(c) *If Z is a strongly irredundant C-representation of X, then $Z = S$. In particular, X contains at most one strongly irredundant C-representation of A.*

Proof. By definition, $\mathscr{C}(X) := \mathrm{Min}(Y)$, where Y is the intersection of all closed C-representations of A. Since Y is closed in the spectral space X, (14.12) implies that $\mathscr{C}(X)$ is a C-representation of A if and only if Y is a (closed) C-representation of A, i.e., there exists a unique minimal closed C-representation of A. Since, in view of (14.10), for any closed subset of Γ of X, $(\mathrm{Min}(\Gamma))^{\mathrm{sp}} = \Gamma$, it follows that $\mathrm{Min}(\Gamma)$ determines Γ. Thus, X contains a unique minimal closed C-representation of A if and only if X contains a unique minimal C-representation of A.

(a). Take a set B in S. By definition, there exists a C-representation Z of A such that $B \in Z$ and B is strongly irredundant in Z. Set $Z' := Z - \{B\}$ and note that $A = C \cap B \cap \bigcap_{H \in Z'} H$. Consider the closed subset $\Gamma := Y \cap \mathcal{V}(B)$ of $\mathcal{V}(B)$, take an element $d \in \bigcap_{H \in Z'} H \cap \bigcap_{E \in \Gamma} E \cap C$ and fix a set $E^* \in \mathscr{C}(X)$. If $E^* \supseteq B$, then $E^* \in \Gamma$ and thus $d \in E^*$. If $E^* \not\supseteq B$, then the equality $A = C \cap B \cap \bigcap_{H \in Z'} H$, the fact that E^* is critical and (14.31) imply $E^* \supseteq \bigcap_{H \in Z'} H$, and thus $d \in E^*$. Then $d \in \bigcap_{E \in \mathscr{C}(X)} E \cap C = A$, since, by assumption, $\mathscr{C}(X)$ is a C-representation of A. This proves that $\Gamma \cup Z'$ is a C-representation of A and, since $B \in Z$ is strongly irredundant in Z, we must have $\Gamma = \mathcal{V}(B)$ and, in particular, $B \in Y$, proving that $S \subseteq Y$. We show now that B is minimal in Y (that is, $B \in \mathscr{C}(X)$). Fix a set $E \in Y$ such that $E \subseteq B$. If $\bigcap_{H \in Z'} H \subseteq E$, it would follow $\bigcap_{H \in Z'} H \subseteq B$, against the fact that B is irredundant in Z, being it strongly irredundant. Then, the equality $A = C \cap B \cap \bigcap_{H \in Z'} H$ and (14.31) imply $B \subseteq E$, that is $B = E$.

(b). Let $B \in S$ and let Z be a C-representation of A such that $B \in Z$ and B is strongly irredundant in Z. Since, by part (a), $B \in \mathscr{C}(X)$, any $E \in \mathscr{C}(X)$ such that $E \neq B$ is not comparable with B. Thus, keeping in mind the equality $A = C \cap B \cap \bigcap_{H \in Z, H \neq B} H$, (14.31) implies that for any $E \in \mathscr{C}(X) - \{B\}$ is such that $E \supseteq \bigcap_{H \in Z, H \neq B} H$. Then, since B is irredundant in Z, being it strongly irredundant, we have

$$C \cap \bigcap_{E \in \mathscr{C}(X), E \neq B} E \supseteq C \cap \bigcap_{H \in Z, H \neq B} H \supsetneq A.$$

This proves that B is irredundant in the C-representation $\mathscr{C}(X)$ of A and, since B is critical in X, by part (a), (14.32) implies that B is strongly irredundant in $\mathscr{C}(X)$.

(c). Let Z be a strongly irredundant C-representation of A. By definition and part (a) we have $Z \subseteq S \subseteq \mathscr{C}(X)$. Assume, by contradiction, that there is a set $B \in S - Z$. Since $\mathscr{C}(X)$ is a C-representation of A, we have $A = C \cap B \cap \bigcap_{E \in \mathscr{C}(X), E \neq B} E$. Since

$\mathscr{C}(X)$ is the collection of minimal elements of Y, any set $H \in Z$ is not comparable with B, since $B \notin Z$. Then, the equality $A = C \cap B \cap \bigcap_{E \in \mathscr{C}(X), E \neq B} E$ and (14.31) imply that $H \supseteq \bigcap_{E \in \mathscr{C}(X), E \neq B} E$, for any $H \in Z$. Thus, keeping in mind that B is irredundant in $\mathscr{C}(X)$, by part (b), we have

$$C \cap \bigcap_{H \in Z} H \supseteq C \cap \bigcap_{E \in \mathscr{C}(X), E \neq B} E \supsetneq A,$$

against the fact that Z is a C-representation of A. It follows $Z = S$. □

15. More on Riemann-Zariski spaces.

Let K be a field and let D be any subring of K. As we proved in (11.15) and, in a constructive way, in (12.16), $\mathrm{Zar}(K|D)$ is a spectral space. In this section we are going to provide further applications of Riemann-Zariski spaces of valuation rings in Multiplicative Ideal Theory. We start with some easy remarks.

(15.1) Remark. (a) In view of (12.6b), if D is a Prüfer domain with quotient field L, then $\mathrm{Zar}(L|D) = \{D_{\mathfrak{p}} : \mathfrak{p} \in \mathrm{Spec}(D)\}$.

(b) In particular, keeping in mind that the polynomial ring $K[T]$ in the indeterminate T over a field K is a Dedekind domain, we have

$$\mathrm{Zar}(K(T)|K[T]) = \{K[T]_{(f)} : f \in K[T] \text{ irreducible over } K\} \cup \{K(T)\},$$

where, as usual, $(f) := fK[T]$.

(c) If K is a field and T is an indeterminate over K, then

$$\mathrm{Zar}(K(T)|K) = \mathrm{Zar}(K(T)|K[T]) \cup \{K[T^{-1}]_{(T^{-1})}\}$$

The inclusion $\supseteq$ is trivial, by (b) and the fact that $K[T^{-1}]_{(T^{-1})}$ is a valuation overring of the Dedekind domain $K[T^{-1}]$. Conversely, take a valuation domain $V \in \mathrm{Zar}(K(T)|K)$. If $T \in V$, then $V \in \mathrm{Zar}(K(T)|K[T])$. If $T \notin V$, then $T^{-1} \in \mathfrak{m}_V$ (as in (12.1), $\mathfrak{m}_V$ is the maximal ideal of V). It follows that $K[T^{-1}] \subseteq V$ and that the center of V in $K[T^{-1}]$ is (T^{-1}). In view of (12.6b), we have $V = K[T^{-1}]_{(T^{-1})}$.

(15.2) Lemma. *Let X be a spectral space and let $\mathcal{S}$ be a subbasis of open and compact subspaces of X. Then $\mathcal{S}^{\mathrm{i}} := \{X - S : S \in \mathcal{S}\}$ is a subbasis of open and compact subspaces of X^{inv} (recall that X^{inv} denotes the set X, endowed with the inverse topology).*

Proof. In view of (13.14a), $\mathcal{S}^{\mathrm{i}}$ is a collection of open and compact subspaces of X^{inv}. Take now an open proper subset U of X^{inv} and take a point $x \in U$. Again by (13.14a), there is an open and compact subspace Ω of X such that $x \in X - \Omega \subseteq U$. Since $\mathcal{S}$ is a subbasis of open sets of X and Ω is compact, we have $\Omega = \bigcup_{i=1}^{n} \bigcap_{j=1}^{n_i} S_{ij}$, for suitable sets $S_{ij} \in \mathcal{S}$. It follows that for any $1 \leq i \leq n$ there is some $j_i \in \{1, \ldots, n_i\}$ such that $x \in \bigcap_{i=1}^{n} (X - S_{ij_i}) \subseteq X - \Omega \subseteq U$. The proof is now complete. □

(15.3) Corollary. *Let K be a field, let D be a subring of K, and let $\mathrm{Zar}(K|D)$ be endowed with the Zariski topology. As in (11.15), set*

$$B^K(x) := B(x) := \mathrm{Zar}(K|D[x]),$$

for any $x \in K$. Then $\{\mathrm{Zar}(K|D) - B(x) : x \in K\}$ is a subbasis of open and compact subspaces of $\mathrm{Zar}(K|D)^{\mathrm{inv}}$.

Proof. By definition, any $B(x)$ is a subbasic open set of $\mathrm{Zar}(K|D)$, with respect to the Zariski topology. Moreover, $B(x)$ is compact, being it spectral (it is a Riemann-Zariski space). Then the conclusion follows from (15.2). □

(15.4) Remark. Let D be an integral domain which is integrally closed domain in some field K. Then, according to (14.1) and (15.3), $\mathrm{Zar}(K|D)^{\mathrm{inv}}$ is a spectral representation of D. This suggests that the topology that we have to use in order to apply the results of the previous section to spaces of valuation domains is the inverse topology.

Let X be a topological space and $\leq$ be the canonical preorder induced by the topology. For any subset Y of X, the set

$$Y^{\mathrm{gen}} := \{x \in X : x \leq y, \text{ for some } y \in Y\}$$

is called the *closure under generization of* Y (or the *generic closure of* Y). It is immediately seen that $Y \subseteq Y^{\mathrm{gen}}$ and that, if Y is open, then $Y = Y^{\mathrm{gen}}$. Whenever the previous equality is true, we say that Y is *closed under generizations.*

(15.5) Example. Let K be a field and D be a subring of K. Keeping in mind (13.2c), it easily follows that if $Y \subseteq \mathrm{Zar}(K|D)$, then

$$Y^{\mathrm{gen}} = \{V \in \mathrm{Zar}(K|D) : V \supseteq W, \text{ for some } W \in Y\}.$$

(15.6) Proposition ([4, Remark 2.2])**.** *Let X be a spectral space and let Y be a subset of X. Then $\overline{Y}^{\mathrm{i}} = (\overline{Y}^{\mathrm{p}})^{\mathrm{gen}}$, i.e., the closure of Y in the inverse topology is the generic closure (in the given spectral topology) of the closure of Y in the patch topology. In particular, if Y is closed in the patch topology of X, then $\overline{Y}^{\mathrm{i}} = Y^{\mathrm{gen}}$.*

Proof. In view of the Hochster duality (13.14), X^{inv} is a spectral space and the patch topology of X and that of X^{inv} are the same. Thus, by (14.9), $\overline{Y}^{\mathrm{inv}}$ is the specialization, in the inverse topology, of $\overline{Y}^{\mathrm{p}}$. Finally, in view of (13.13), the spacialization of any subset, in the inverse topology, is equal to the generization in the given spectral topology. □

(15.7) Proposition. *Let X be a spectral space and let $Y \subseteq X$. Then, the following conditions are equivalent.*

(i) *Y is closed, with respect to the inverse topology.*
(ii) *Y is compact, in the given spectral topology, and closed under generizations.*

Proof. (i)$\Longrightarrow$(ii). Suppose that Y is closed in the inverse topology. By (15.6) a closed set in the inverse topology is closed under generization. Moreover, the given spectral topology and the inverse topology have the same patch topology. Since the patch topology is compact and finer than both the spectral topology and the inverse topology, it follows that Y is closed, and hence compact, in the patch topology. Finally, Y is compact in given spectral topology.

(ii)$\Longrightarrow$(i). Suppose that Y is compact and closed under generization. We argue by contradiction, and take a point $x \in \overline{Y}^{\mathrm{i}} - Y$. Since $Y = Y^{\mathrm{gen}}$ it happens, for any $y \in Y$, that $x \not\leq y$, that is, $y \notin \overline{\{x\}}$. Since open and compact subsets form a basis for the given spectral topology of X, for any $y \in Y$ there is an open and compact neighborhood Ω_y of y such that $x \notin \Omega_y$. It follows that $Y \subseteq \bigcup_{y \in Y} \Omega_y$ and, since Y is compact, there are finitely many elements $y_1, \ldots, y_n \in Y$ such that

$Y \subseteq \bigcup_{i=1}^{n} \Omega_{y_i} =: \Omega$. The set Ω is open and compact, being it a finite union of open and compact sets, and thus, by definition, it is closed in the inverse topology. It follows $x \in \overline{Y}^{\mathrm{i}} \subseteq \Omega$, a contradiction. □

(15.8) Proposition. *Let K be a fiend and let $D \subseteq E$ be subrings of K. Then* $\mathrm{Zar}(K|E)$ *is closed in* $\mathrm{Zar}(K|D)$*, with respect to the inverse topology.*

Proof. $\mathrm{Zar}(K|E)$ is compact, being it spectral, and, by (15.5), it is closed under generizations. Then, the conclusion follows from (15.7). □

(15.9) Proposition. *Let K be a field, D be a subring of K and C be a subset of K such that $D \subsetneq C$. If $Z \subseteq \mathrm{Zar}(K|D)^{\mathrm{inv}}$ is a closed C-representation of D, then Z contains a minimal closed C-representation of D.*

Proof. Apply (14.3). □

(15.10) Proposition ([14, page 292])**.** *Let K be a field, D be a subring of K and C be a subset of K such that $D \subsetneq C$. If $Z \subseteq X \subseteq \mathrm{Zar}(K|D)$ are C-representations of D, X is spectral and $V \in Z$, then the following properties holds.*

(a) *V is irredundant in Z if and only if V is irredundant in the closure of Z in the patch topology of X.*
(b) *If V is one-dimensional, then V is irredundant in Z if and only if V is irredundant in the closure $\overline{Z}^{\mathrm{i}}$ of Z in X.*

Proof. (a) is a consequence of (14.6).

(b). Assume that V is irredundant in Z and take an element $k \in K$ such that $k \in C \cap \bigcap_{W \in Z, W \neq V} W$ and $k \notin W$. Since V is one-dimensional, the unique nontrivial valuation overring of V is K and thus $(X \cap B(k)) \cup \overline{\{V\}}^{\mathrm{i}} = (X \cap B(k)) \cup \{V\} \supseteq Z$. Since X is a spectral C-representation of D, $X \cap B(k)$ is closed in X (equipped with the inverse subspace topology induced by the inverse topology of $\mathrm{Zar}(K|D)$), and thus $\overline{Z}^{\mathrm{i}} \subseteq (X \cap B(k)) \cup \{V\}$. This proves that $k \in \bigcap_{W \in \overline{Z}^{\mathrm{i}}, W \neq V} W$. Since $k \notin V$, it follows that V is irredundant in $\overline{Z}^{\mathrm{i}}$. The converse part is obvious. □

(15.11) Proposition ([14, (4.5)])**.** *Let D be an integral domain which is integrally closed in some field $K \supseteq D$, let C be a set such that $D \subsetneq C \subseteq K$ and let $X \subseteq \mathrm{Zar}(K|D)^{\mathrm{inv}}$ be a minimal C-representation of D.*

(a) *If $V \in X$, then following conditions are equivalent.*
  (i) *V is irredundant in X.*
  (ii) *V is strongly irredundant in X.*
  (iii) *V is isolated in X (with respect to the subspace inverse topology or, equivalenty, to the subspace patch topology).*
(b) *X contains a strongly irredundant C-representation of D if and only if the set of isolated points in X is dense in X, with respect to the subspace inverse topology.*

Proof. Apply (14.21) and (14.24). □

(15.12) Proposition ([14, (4.8)])**.** *Let K be a field, D be a subring of K and C be a set such that $D \subsetneq C \subseteq K$. If $X \subseteq \mathrm{Zar}(K|D)$ is a countable closed subspace, with respect to the patch topology, and $D = C \cap \bigcap_{V \in X} V$, then X contains a strongly irredundant C-representation of D.*

Proof. Apply (14.29). □

If D is an integral domain with quotient field K and $\mathfrak{a}$ is a subset of D, we set

$$\mathfrak{a}^{-1} := (D : \mathfrak{a}) := \{x \in K : x\mathfrak{a} \subseteq D\}.$$

The subset $\mathfrak{a}_v := (\mathfrak{a}^{-1})^{-1}$ of K is called the *divisorial closure of* $\mathfrak{a}$. Note that, if $\mathfrak{a}$ is an ideal of D, then so is $\mathfrak{a}_v$. We say that an ideal $\mathfrak{a}$ of D is *divisorial* if $\mathfrak{a} = \mathfrak{a}_v$. If $\mathfrak{i}$ is a nonzero ideal of D, set

$$\mathfrak{i}_t := \sum\{\mathfrak{a}_v : 0 \neq \mathfrak{a} \subseteq \mathfrak{i}, \mathfrak{a} \text{ finitely generated ideal of } D\}.$$

The ideal $\mathfrak{i}$ is called *a t-ideal of D* if either $\mathfrak{i} = 0$ or $\mathfrak{i}_t = \mathfrak{i}$.

(15.13) Remark (See, for instance, [9])**.** Let D be an integral domain with quotient field K.

(a) A map $F \mapsto F_*$ from the set of nonzero fractional ideals of D it itself is called to be *a star operation on D* if
- $D_* = D$ and $kF_* = (kF)_*$,
- $F \subseteq F_*$ and $F \subseteq G$ implies $F_* \subseteq G_*$.
- $(F_*)_* = F_*$,

for any $k \in K - \{0\}$ and nonzero fractional ideals F, G of D.

(b) It is straightforward that the maps $F \mapsto F_v$, $F \mapsto F_t$ are both star operations on D.

(c) A star operation $*$ is called to be *of finite type* if

$$F_* = \bigcup\{G_* : G \subseteq F, G \text{ finitely generated nonzero fractional ideal of } D\}$$

for any nonzero fractional ideal F of D. Thus, by definition, the t operation is of finite type.

(d) Given a star operation $*$ on D, an ideal $\mathfrak{a}$ of D is called to be *a $*$-ideal* if either $\mathfrak{a} = 0$ or $\mathfrak{a} = \mathfrak{a}_*$. It is straightforward that, if $\mathcal{F}$ is a collection of nonzero fractional ideals of D with nonzero intersection, then $\bigcap_{F\in\mathcal{F}} F_* = (\bigcap_{F\in\mathcal{F}} F_*)_*$. In particular, a finite intersection of $*$-ideals is a $*$-ideal.

(e) By the first axion of a star operation, $kD = (kD)_*$, for any $k \in K$ and any star operation $*$ on D. In particular, any principal ideal is a $*$-ideal.

(15.14) Definition. *Let $*$ be a star operation on an integral domain D. A prime ideal of D is* $*$-prime *if it is a $*$-ideal. A maximal element, under inclusion, of the set of proper $*$-ideals of D is called* a $*$-maximal ideal. *We shall denote by* $\operatorname{Spec}^*(D)$ *(resp.,* $\operatorname{Max}^*(D)$*) the collection of all $*$-prime (resp., $*$-maximal) ideals of D.*

(15.15) Proposition. *Let D be an integral domain and let $*$ be a star operation on D. The following properties hold.*

(a) $\operatorname{Max}^*(D) \subseteq \operatorname{Spec}^*(D)$.

(b) *If $*$ is of finite type, then any proper $*$-ideal is contained in a $*$-maximal ideal. In particular,* $\operatorname{Max}^*(D) \neq \emptyset$.

(c) *If $*$ is of finite type, then* $D = \bigcap_{\mathfrak{m}\in\operatorname{Max}^*(D)} D_{\mathfrak{m}}$.

Proof. (a). Let $\mathfrak{m}$ be a $*$-maximal ideal of D, and let $x, y \in D$ be such that $xy \in \mathfrak{m}$ and $x \notin \mathfrak{m}$. Since $\mathfrak{m} \subsetneq xD + \mathfrak{m} \subseteq (xD + \mathfrak{m})_*$ we have $(xD + \mathfrak{m})_* = D$, since $\mathfrak{m} \in \operatorname{Max}^*(D)$. It follows

$$y \in y(xD + \mathfrak{m})_* = [y(xD + \mathfrak{m})]_* \subseteq (xyD + \mathfrak{m})_* = \mathfrak{m}_* = \mathfrak{m},$$

proving that $\mathfrak{m}$ is a prime ideal (and thus a $*$-prime ideal).

(b). Let $\mathfrak{a}$ ba proper $*$-ideal of D, let $\mathcal{C}$ be a chain of proper $*$-ideals of D containing $\mathfrak{a}$, and let $\mathfrak{b}$ be the union of $\mathcal{C}$. Take an element $x \in \mathfrak{b}_*$. Since $*$ is of finite type, there is a finitely generated ideal $\mathfrak{b}_0$ of D such that $\mathfrak{b}_0 \subseteq \mathfrak{b}$ and $x \in \mathfrak{b}_{0*}$. Since $\mathfrak{b}_0$ is finitely generated and $\mathcal{C}$ is a chain, there is an ideal $\mathfrak{c} \in \mathcal{C}$ such that $\mathfrak{b}_0 \subseteq \mathfrak{c}$. Keeping in mind that $\mathcal{C}$ consists of $*$-ideals, it follows $x \in \mathfrak{b}_{0*} \subseteq \mathfrak{c}_* = \mathfrak{c} \subseteq \mathfrak{b}$, proving that $\mathfrak{b}$ is a $*$-ideal. Then the conclusion follows from Zorn's Lemma.

(c) Take an element $x \in \bigcap_{\mathfrak{m}\in \mathrm{Max}^*(D)} D_{\mathfrak{m}}$. For a fixed $\mathfrak{m} \in \mathrm{Max}^*(D)$, take elements $a, s \in D$, $s \notin \mathfrak{m}$ such that $x = \dfrac{a}{s}$, and note that $D \cap x^{-1}D \nsubseteq \mathfrak{m}$ (since $s \in x^{-1}D$). Moreover, by (15.13d,e), $D \cap x^{-1}D$ is a $*$-ideal. Thus, in view of part (b), we infer $D = D \cap x^{-1}D$, that is, $x \in D$. □

(15.16) Proposition. *Any invertible ideal of an integral domain is divisorial.*

Proof. Let $\mathfrak{a}$ be an invertible ideal of an integral domain D, i.e., $\mathfrak{a}\mathfrak{a}^{-1} = D$. It follows that the inverse $(\mathfrak{a}^{-1})^{-1} =: \mathfrak{a}^v$ of $\mathfrak{a}^{-1}$ is $\mathfrak{a}$. The conclusion follows. □

(15.17) Corollary. *Any ideal of a Prüfer domain is a t-ideal.*

Proof. Since any nonzero finitely generated ideal of a Prüfer domain is invertible, and a fortiori divisorial, by (15.16), we have

$$\mathfrak{i}^t = \sum \{\mathfrak{a} : 0 \neq \mathfrak{a} \subseteq \mathfrak{i}, \mathfrak{a} \text{ finitely generated ideal of } D\} = \mathfrak{i}.$$

□

(15.18) Definition. *Let D be an integral domain.*

(a) *Let V be a valuation overring of D. We say that V is* essential for D *if $V = D_{\mathfrak{p}}$, for some $\mathfrak{p} \in \mathrm{Spec}(D)$. Such a prime ideal $\mathfrak{p}$ is called* an essential prime ideal of D. *Set*

$$\mathcal{E}(D) := \{\textit{essential prime ideals of } D\},$$

$$\mathcal{V}(D) := \{\textit{essential valuation overrings of } D\}.$$

(b) *We say that D is* an essential domain *if D is intersection of a nonempty collection $\mathcal{V}$ of essential valuation overrings of D. Such a family $\mathcal{V}$ is called* an essential representation of D.

(15.19) Example. Any Prüfer domain D is an essential domain; indeed, the family $\{D_{\mathfrak{m}} : \mathfrak{m} \in \mathrm{Max}(D)\}$ is an essential representation of D.

(15.20) Theorem (See [4, Proposition 4.5])**.** *Let D be an essential domain with quotient field K and let $Y \subseteq \mathrm{Zar}(K|D)$ be an essential representation of D. Then*

$$\mathrm{Max}^t(D) \subseteq \overline{\{\mathfrak{m}_V \cap D : V \in Y\}}^{\mathrm{i}}.$$

In particular, if D is Prüfer, then $\mathrm{Max}(D) \subseteq \overline{\{\mathfrak{m}_V \cap D : V \in Y\}}^{\mathrm{i}}$.

Proof. The last statement follows from the first one, in view of (15.17). According to (10.14) and (15.6), it suffices to show that any t-maximal ideal of D is contained in an ultrafilter limit point of $Y_0 := \{\mathfrak{m}_V \cap D : V \in Y\}$. Fix a t-maximal ideal $\mathfrak{m}$ of D and consider the collection of subsets $\mathcal{F} := \{V(x) \cap Y_0 : x \in \mathfrak{m}\}$ of Y_0. Assume that there are elements $x_1, \ldots, x_n \in \mathfrak{m}$ such that $V(x_1, \ldots, x_n) \cap Y_0 = \emptyset$, let $\mathfrak{a} := (x_1, \ldots, x_n)D$ and take an element $x \in \mathfrak{a}^{-1}$. For any $V \in Y$, there is an element $d_V \in \mathfrak{a} - \mathfrak{m}_V$, since

$V(\mathfrak{a}) \cap Y_0 = \emptyset$. It follows $xd_V \in D$, that is, $x \in \dfrac{1}{d_V} D \subseteq D_{\mathfrak{m}_V \cap D} = V$ (note that the last equality is a consequence of the fact that V is essential for D). Keeping in mind that Y is a representation of D, we have $\mathfrak{a}^{-1} \subseteq \bigcap_{V \in Y} V = D$ and, since the inclusion $D \subseteq \mathfrak{a}^{-1}$ is always true, it follows $\mathfrak{a}^{-1} = D$, and a fortiori $\mathfrak{a}^v = D^{-1} = D \subsetneq \mathfrak{m}^t = \mathfrak{m}$, a contradiction. This argument shows that $\mathcal{F}$ has the finite intersection property, and thus there exists an ultrafilter $\mathscr{U}$ on Y_0 extending $\mathcal{F}$, by (9.4). It immediately follows $\mathfrak{m} \subseteq (Y_0)_{\mathscr{U}} := \{d \in D : V(d) \cap Y_0 \in \mathscr{U}\}$. □

(15.21) Lemma. *Let $f : X \longrightarrow Y$ be a homeomorphism of spectral spaces. Then $f : X^{\rm patch} \longrightarrow Y^{\rm patch}$ and $f : X^{\rm inv} \longrightarrow Y^{\rm inv}$ are homeomorphisms.*

Proof. It is sufficient to note that a subset Ω of X is open and compact if and only if $f(\Omega)$ is open and compact, since $f : X \longrightarrow Y$ is a homeomorphism. Then the conclusion follows from the definition of the patch topology and of the inverse topology. □

(15.22) Lemma. *Let A be a ring. Then* $\mathrm{Max}(A)$ *is dense in* $\mathrm{Spec}(A)$*, with respect to the inverse topology.*

Proof. Take an open and compact subspace Ω of $\mathrm{Spec}(A)$. Then $\Omega = D(\mathfrak{a})$, for some finitely generated ideal $\mathfrak{a}$ of A. Then $\mathrm{Max}(A) \subseteq \Omega$ if and only if $\mathfrak{a} = A$, if and only if $\Omega = \mathrm{Spec}(A)$. □

(15.23) Corollary. *Let D be a Prüfer domain with quotient field K and let $Y \subseteq$* $\mathrm{Zar}(K|D)$ *be a representation of D. Then Y is dense in* $\mathrm{Zar}(K|D)$*, with respect to the inverse topology. In particular, the unique representation of D which is closed in the inverse topology is* $\mathrm{Zar}(K|D)$.

Proof. Since D is Prüfer, the domination map $\delta : \mathrm{Zar}(K|D) \longrightarrow \mathrm{Spec}(D)$ is a homeomorphism, with respect to the Zariski topology. In view of (15.21), $\delta : \mathrm{Zar}(K|D)^{\rm inv} \longrightarrow \mathrm{Spec}(D)^{\rm inv}$ is a homeomorphism. By (15.20) and (15.22), $\{\mathfrak{m}_V \cap D : V \in Y\}$ is dense in $\mathrm{Spec}(D)^{\rm inv}$. Since the domination map is a homeomorphism, Y is dense in $\mathrm{Zar}(K|D)^{\rm inv}$. □

(15.24) Definition. *Let K be a field, let D be a subring of K, let C be a subset of K such that $D \subsetneq C$ and let T be an indeterminate over K. For any nonempty subset X of* $\mathrm{Zar}(K|D)$*, consider the K-Halter-Koch ring* $\mathrm{Kr}(X) := \bigcap_{V \in X} V(T)$ *(see (12.11)). If X is a C-representation (resp., a representation) of D we say that* $\mathrm{Kr}(X)$ *is a* C-Kronecker function ring of D *(resp.,* a Kronecker function ring of D*).*

In the setting of (15.24), endow $\mathrm{Zar}(K|D)$ with the inverse topology and set

$$\mathcal{K}_C(D) := \{C\text{-Kronecker function rings of } D\}$$

$$\mathcal{R}_C(D) := \{\text{closed } C\text{-representations of } D\}.$$

The set $\mathcal{K}_C(D)$ will be called *the C-Kronecker space of A*. By definition, $\mathcal{K}_C(D) \neq \emptyset$ if and only if $\mathcal{R}_C(D) \neq \emptyset$. Indeed, if $\mathcal{K}_C(D) \neq \emptyset$, there is some C-representation $X \subseteq \mathrm{Zar}(K|D)$ of D. A fortiori, $\mathrm{Zar}(K|D)$ is a closed C-representation of D.

(15.25) Corollary. *Let K be a field, D be a subring of K and T be an indeterminate over K. If $H :=$* $\mathrm{Kr}(\mathrm{Zar}(K|D))$*, then the natural map*

$$\eta : \mathrm{Zar}(K(T)|H)^{\rm inv} \longrightarrow \mathrm{Zar}(K|D)^{\rm inv} \qquad W \mapsto W \cap K$$

is a homeomorphism.

Proof. Apply (12.16) and (15.21). □

(15.26) Theorem ([14, (4.15)])**.** *We preserve the setting of (15.24) and assume that* Zar$(K|D)$ *is a* C*-representation of* D*. Consider the map* $\beta : \mathcal{R}_C(D) \longrightarrow \mathcal{K}_C(D)$ *defined by setting* $\beta(X) := \mathrm{Kr}(X)$*, for any* $X \in \mathcal{R}_C(A)$*. The following properties hold.*

(a) *β is an inclusion reversing bijection.*
(b) *Minimal closed C-representations of D bijectively corresponds, via β, to maximal C-Kronecker function rings of D.*

Proof. (a) The fact that β is inclusion reversing is trivial. Assume now that $X, Y \subseteq \mathrm{Zar}(K|D)$ are closed C-representations of D and that $\beta(X) = \beta(Y)$, that is, $R := \mathrm{Kr}(X) = \mathrm{Kr}(Y)$. By (15.25), $\eta^{-1}(X) = \{V(T) : V \in X\}, \eta^{-1}(Y) = \{V(T) : V \in Y\}$ are closed subspaces of $\mathrm{Zar}(K(T)|H)^{\mathrm{inv}}$ and, a fortiori, of $\mathrm{Zar}(K(T)|R)^{\mathrm{inv}}$, by (14.17c) and (15.8). Since $\eta^{-1}(X), \eta^{-1}(Y)$ are, by definition, representations of the Bézout domain R (see (12.9c)), then (15.23) implies $\eta^{-1}(X) = \eta^{-1}(Y)$, that is, $X = Y$. For surjectivity, take any C-Kronecker function ring R of D. Thus $R = \mathrm{Kr}(X)$, for some C-representation $X \subseteq \mathrm{Zar}(K|D)$ of D. By (15.25) and (15.8), $X' := \eta(\mathrm{Zar}(K(T)|R))$ is closed in $\mathrm{Zar}(K|D)^{\mathrm{inv}}$ and, since R is integrally closed, being it Bézout, we have, by definition,

$$\mathrm{Kr}(X') := \bigcap_{V \in X'} V(T) = \bigcap_{W \in \mathrm{Zar}(K(T)|R)} W = R.$$

(b) is an immediate consequence of part (a). □

(15.27) Corollary. *We preserve the setting of (15.24) and assume that* Zar$(K|D)$ *is a C-representation of D. If $\mathcal{K}_C(D)$ has a unique maximal point (i.e., there is a unique maximal C-Kronecker function ring), then D has at most one strongly irredundant C-representation in* Zar$(K|D)$*.*

Proof. Apply (15.26b) and (14.33c). □

(15.28) Lemma. *Let V be a valuation domain with quotient field K, and let M, N be V-submodules of K containing V. Then, M, N are comparable.*

Proof. Suppose $C \nsubseteq D$ and take an element $c \in C - D$. Then, for any $d \in D$, we have $\dfrac{d}{c} \in V$ (otherwise, $\dfrac{c}{d} \in V$ and thus $c \in dV \subseteq D$, a contradiction). It follows $d \in cV \subseteq C$. □

(15.29) Definition. *Let K be a field, D be a subring of K and C be a subset of K such that $D \subsetneq C$. Adapting the terminology of the previous section, we say that a valuation domain $V \in$* Zar$(K|D)$ *is* C-critical for D *if $V \in \Gamma$, for any C-representation Γ of D which is closed in the inverse topology of* Zar$(K|D)$*. A K-critical valuation domain will be simply called* critical*.*

Keeping in mind (14.31), the following result is clear.

(15.30) Proposition. *Let K be a field, D be a subring of K and C be a subset of K such that $D \subsetneq C$. If $V \in$* Zar$(K|D)$*, then the following conditions are equivalent.*

(i) *V is C-critical for D.*
(ii) *Whenever $A_1, \ldots, A_n$ are subrings of K that are integrally closed in K and $D = A_1 \cap \ldots \cap A_n \cap C$, then $A_i \subseteq V$, for some $i \in \{1, \ldots, n\}$.*

(15.31) Proposition ([14, Proposition 4.11]). *Let D be an integral domain with quotient field K, and let C be a D-submodule of K such that $D \subsetneq C$. If V is an essential valuation overring of D such that $C \nsubseteq V$, then V is C-critical for D.*

Proof. By definition, there is some prime ideal $\mathfrak{p}$ of D such that $V = D_{\mathfrak{p}}$. Take integrally closed overrings $A_1, \ldots, A_n$ of D such that $D = A_1 \cap \ldots \cap A_n \cap C$. Keeping in mind that localization commutes with finite intersections, we infer that $V = D_{\mathfrak{p}} = (A_1)_{\mathfrak{p}} \cap \ldots \cap (A_n)_{\mathfrak{p}} \cap C_{\mathfrak{p}}$. By (15.28), the collection $\mathcal{M} := \{C_{\mathfrak{p}}, (A_i)_{\mathfrak{p}} : 1 \leq i \leq n\}$ of V-submodules of K (containing V) is totally ordered by inclusion and, since $C \nsubseteq V$, the minimum of $\mathcal{M}$ cannot be $C_{\mathfrak{p}}$ (otherwise $V = C_{\mathfrak{p}} \supseteq C$). Thus the minimum of $\mathcal{M}$ is some $(A_i)_{\mathfrak{p}}$, that is, $V = (A_i)_{\mathfrak{p}} \supseteq A_i$. The conclusion follows from (15.30). □

(15.32) Corollary. *Any essential domain admits at most one strongly irredundant representation (in the space of its valuation overrings).*

Proof. Let D be an essential domain. In view of (15.31), D is intersection of critical valuation overrings. The conclusion follows immediately from (14.33). □

(15.33) Definition ([2]). *An integral domain is called to be* vacant *if it admits a unique Kronecker function ring.*

Note that, by definition, any vacant domain is integrally closed. The following result gives a topological criterion to decide if an integral domain is vacant.

(15.34) Theorem. *Let D be an integral domain with quotient field K. Then the following conditions are equivalent.*

(i) *D is vacant.*
(ii) *D is integrally closed and any representation of D is dense in* $\mathrm{Zar}(K|D)$, *with respect to the inverse topology.*
(iii) *D is integrally closed and any valuation overring of D is critical for D.*
(iv) *D is integrally closed and, whenever $A_1, \ldots, A_n$ are integrally closed overrings of D such that $D = A_1 \cap \ldots \cap A_n$, then* $\mathrm{Zar}(K|D) = \bigcup_{i=1}^{n} \mathrm{Zar}(K|A_i)$.

Proof. It is enough to apply (15.26), (14.33) and (15.30). □

(15.35) Remark. The equivalence of (i) and (ii) of (15.34) is [4, Corollary 4.16]. The equivalence of (i) and (iii) is [14, Example 4.12]. The equivalence of (i) and (iv) was proved in [2, Theorem 3.1].

(15.36) Corollary. *Any Prüfer domain is vacant.*

Proof. Apply (15.23) and (15.34). □

(15.37) Proposition. *Any vacant domain (in particular, any Prüfer domain) admits at most one strongly irredundant representation (in the space of its valuation overrings).*

Proof. It follows by definition, keeping in mind (15.27). □

The following remark is straightforward and its proof is left to the reader.

(15.38) Remark. Let D be an integral domain with quotient field K, and let X be a D-submodule of K.

(a) Then $(X : X) := \{k \in K : kX \subseteq X\}$ is an overring of D.
(b) If X is an overring of D, then $(X : X) = X$.

(c) If X is an ideal of D, then X is an ideal of $(X:X)$.
(d) If D is a valuation domain and X is its maximal ideal, then $(X:X)=D$.

(15.39) Lemma ([2, Lemma 4.3]). *Let D be an integral domain and let $\mathfrak{p}$ be a prime ideal of D such that $V:=(\mathfrak{p}:\mathfrak{p})$ is a valuation overring of D with maximal ideal $\mathfrak{p}$. Then, any valuation overring of D is comparable with V.*

Proof. Let W be a valuation overring of D. First, assume that $\mathfrak{p}W=W$. If $x\in V$, then $x\mathfrak{p}\subseteq\mathfrak{p}$ and, a fortiori $x\mathfrak{p}W\subseteq\mathfrak{p}W$, that is, $x\in(\mathfrak{p}W:\mathfrak{p}W)=(W:W)=W$, in view of (15.38b). This proves that, if $\mathfrak{p}W=W$, then $V\subseteq W$.

Assume now that $\mathfrak{p}W$ is a proper ideal of W and that $W\nsubseteq V$, and take an element $w\in W-V$. Since V is a valuation domain and $\mathfrak{p}$ is its maximal ideal, it follows $w^{-1}\in\mathfrak{p}$. Then, $1=w^{-1}w\in\mathfrak{p}W$, against the fact that $\mathfrak{p}W\neq W$. This proves that, if $\mathfrak{p}W\neq W$, then $W\subseteq V$. The conclusion is now clear. □

(15.40) Proposition. *Let K be a field and let T be an indeterminate over K. Then,* $\mathrm{Zar}(K(T)|K)-\{K(T)\}$ *is an irredundant representation of K.*

Proof. By (15.1c), we have

$$\mathrm{Zar}(K(T)|K)=\mathrm{Zar}(K(T)|K[T])\cup\{K[T^{-1}]_{(T^{-1})}\}.$$

It immediately follows that $K[T^{-1}]_{(T^{-1})}$ is irredundant in $\mathrm{Zar}(K(T)|K)$. Now, take any irreducible polynomial $f:=a_0+\ldots+a_nT^n\in K[T]$ (where $a_n\neq 0$), and consider the valuation domain $V:=K[T]_{(f)}$. Clearly, $\dfrac{1}{f}\in K[T]_{(g)}$, for any irreducible polynomial $g\in K[T]$ such that $(g)\neq(f)$. Furthermore

$$\frac{1}{f}=\frac{T^{-n}}{a_0T^{-n}+\ldots+a_n}\in K[T^{-1}]_{(T^{-1})},$$

proving that $\dfrac{1}{f}\in\bigcap\{W:W\in\mathrm{Zar}(K(T)|K),W\neq V\}-K$. The conclusion is now clear. □

We now provide an example of a vacant domain that is not Prüfer.

(15.41) Example. Let K be a field, let T,U be indeterminates over K, consider the valuation domain $V:=K(T)[U]_{(U)}$ and let $\pi:V\longrightarrow K(T)$ be the canonical projection (of V onto its residue field). In view of (4.2) and that (4.8), the ring $D:=\pi^{-1}(K)=K+UK(T)[U]_{(U)}$ is a local domain and, since $\mathfrak{m}:=UK(T)[U]_{(U)}$ is a common prime ideal of V and D, V is a valuation overring of D, by (8.1). In view of (8.3), D is not a valuation domain, and thus it is not a Prüfer domain. Now, note that D is integrally closed. Indeed, by (15.40) and (8.2), $\{\pi^{-1}(V'):V'\in\mathrm{Zar}(K(T)|K)\}$ is a representation of D. Moreover, note that $(\mathfrak{m}:\mathfrak{m})=V$, by (15.38d), and thus, keeping in mind (15.39), any valuation overring of D is comparable with V. It follows that, if $\Gamma\subseteq\mathrm{Zar}(K(T,U)|D)$ is any representation of D that is closed in the inverse topology, then $\Gamma':=\{W\in\Gamma:W\subseteq V\}$ is nonempty, contains V (being Γ closed under generizations) and it is a representation of D. In view of (8.3), $\Gamma'':=\{\pi(W):W\in\Gamma'\}\subseteq\mathrm{Zar}(K(T)|K)$ is a representation of K. Furthermore, $\Gamma''=\mathrm{Zar}(K(T)|K)$, again by (15.40). By applying (8.3), we easily infer that $\Gamma'=\{W\in\mathrm{Zar}(K(T,U)|D):W\subseteq V\}$. Since V is a DVR and any valuation overring of D is comparable with V, it immediately follows that $\Gamma=\mathrm{Zar}(K(T,U)|D)$. Thus, by (15.34), D is vacant.

(15.42) Remark. (a) Let D be an integral domain such that any valuation overring of D is essential. Then D is a Prüfer domain. As a matter of fact, let $\mathfrak{m}$ be a maximal ideal of D and let V be a valuation overring of D such that V dominates $D_\mathfrak{m}$ (see (12.3)). By assumption, $V = D_\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of D. It immediately follows $\mathfrak{m} = \mathfrak{p}$, and thus $D_\mathfrak{m}$ is a valuation domain.

(b) By part (a) and (15.34), any vacant domain that is not a Prüfer domain admits a critical valuation overring that is not essential. Thus, in general, for an integral domain D, the inclusion

$$\mathcal{V}(D) := \{\text{essential valuation overrings of } D\} \subseteq \{\text{critical valuation overrings of } D\}$$

proved in (15.31) can be proper. In the following, we will study a class of integrally closed domains for which the converse of (15.31) holds.

(15.43) Definition. *Let D be an integral domain. We say that D is a PvMD if $D_\mathfrak{m}$ is a valuation domain, for any t-maximal ideal $\mathfrak{m}$ of D.*

(15.44) Remark. (1) Any PvMD is always essential, in view of (15.15c). In particular, any PvMD has at most one strongly irredundant representation consisting of valuation overrings (see (15.32)).

(2) Any Prüfer domain is a PvMD, by (15.17). We will give examples of PvMDs that are not Prüfer.

(15.45) Proposition ([13, Proposition 2.9])**.** *Let D be an integral domain and let $*$ be a star operation of finite type on D. Then* $\operatorname{Spec}^*(D)$ *is closed in* $\operatorname{Spec}(D)$*, with respect to the constructible topology.*

Proof. Let $\mathscr{U}$ be an ultrafilter on $X := \operatorname{Spec}^*(D)$ and let

$$\mathfrak{p} := X_\mathscr{U} := \{d \in D : V(d) \cap X \in \mathscr{U}\}.$$

By (10.10), it is enough to show that $\mathfrak{p}$ is a $*$-prime ideal. Take an element $x \in \mathfrak{p}_*$. Since $*$ is of finite type, there exists a finitely generated ideal $\mathfrak{a} \subseteq \mathfrak{p}$ of D such that $x \in \mathfrak{a}_*$. Since $\mathfrak{a}$ is finitely generated, we have $V(\mathfrak{a}) \cap X \in \mathscr{U}$ and, since X consists of $*$-ideals, $V(\mathfrak{a}) \cap X = V(\mathfrak{a}_*) \cap X \subseteq V(x) \cap X$. Hence, $V(x) \cap X \in \mathscr{U}$, that is, $x \in \mathfrak{p}$. □

(15.46) Remark. Let D be an integral domain.

(a) $\mathcal{E}(D)$ is closed under generizations. Indeed, if $\mathfrak{p} \in \mathcal{E}(D)$ and $\mathfrak{q} \subseteq \mathfrak{p}$, then $D_\mathfrak{q} \supseteq D_\mathfrak{p}$ and thus $D_\mathfrak{q}$ is a valuation domain, since $D_\mathfrak{p}$ is.

(b) If δ is the domination map of D and $\mathcal{V}(D)$ is the collection of the essential valuation overrings of D, then $\mathcal{V}(D) = \delta^{-1}(\mathcal{E}(D))$. The inclusion $\subseteq$ is trivial. Conversely, let $V \in \delta^{-1}(\mathcal{E}(D))$ and let $\mathfrak{p} := \mathfrak{m}_V \cap D$. It follows that V dominates $D_\mathfrak{p}$ and then, by (12.2), $V = D_\mathfrak{p}$, since $\mathfrak{p}$ is essential.

(15.47) Theorem ([5, Theorem 2.4])**.** *Let D be an integral domain. Then, the following conditions are equivalent.*

(i) *D is a PvMD.*

(ii) *D is an essential domain and admits an essential representation $\mathcal{V}$ such that*

$$\overline{\{\mathfrak{m}_V \cap D : V \in \mathcal{V}\}}^{\mathrm{c}} \subseteq \mathcal{E}(D)$$

Proof. (i)$\Longrightarrow$(ii). We know, by definition, that $\{D_{\mathfrak{m}} : \mathfrak{m} \in \operatorname{Max}^t(D)\}$ is an essential representation of D. If $\mathfrak{p}$ is any t-prime ideal of D, let $\mathfrak{m}$ be a t-maximal ideal of D such that $\mathfrak{p} \subseteq \mathfrak{m}$. Then, $D_{\mathfrak{p}}$ is a valuation domain, being it an overring of the valuation domain $D_{\mathfrak{m}}$. We infer that $\mathcal{V} := \{D_{\mathfrak{p}} : \mathfrak{p} \in \operatorname{Spec}^t(D)\}$ is an essential representation of D and, by (15.45),

$$\overline{\{\mathfrak{m}_V \cap D : V \in \mathcal{V}\}}^{\mathrm{c}} = \overline{\operatorname{Spec}^t(D)}^{\mathrm{c}} = \operatorname{Spec}^t(D) \subseteq \mathcal{E}(D).$$

(ii)$\Longrightarrow$(i). Assume that $\mathcal{V}$ is an essential representation of D and that

$$\overline{\{\mathfrak{m}_V \cap D : V \in \mathcal{V}\}}^{\mathrm{c}} \subseteq \mathcal{E}(D).$$

If $\mathfrak{m}$ is any t-maximal ideal of D, there exists a prime ideal $\mathfrak{p} \in \overline{\{\mathfrak{m}_V \cap D : V \in \mathcal{V}\}}^{\mathrm{c}}$ such that $\mathfrak{m} \subseteq \mathfrak{p}$, by (15.6) and (15.20). Since $\mathfrak{p} \in \mathcal{E}(D)$, by assumption, $D_{\mathfrak{p}}$ is a valuation domain and thus $D_{\mathfrak{m}}$ is a valuation domain too, being $D_{\mathfrak{p}} \subseteq D_{\mathfrak{m}}$. The conclusion follows. □

(15.48) Proposition. *Any essential prime ideal of an integral domain is a t-prime ideal.*

Proof. Let $\mathfrak{p}$ be an essential prime ideal of an integral domain D, that is, $D_{\mathfrak{p}}$ is a valuation domain, and take an element $x \in \mathfrak{p}_t$. Take a finitely generated ideal $\mathfrak{a} \subseteq \mathfrak{p}$ of D such that $x \in \mathfrak{a}_t$. Since $D_{\mathfrak{p}}$ is valuation domain, the ideal $\mathfrak{a}D_{\mathfrak{p}}$ of $D_{\mathfrak{p}}$ is principal, say $\mathfrak{a}D_{\mathfrak{p}} = \alpha D_{\mathfrak{p}}$, for some $\alpha \in \mathfrak{a}$. Then, since $\mathfrak{a}$ is finitely generated, there is an element $s \in D - \mathfrak{p}$ such that $s\mathfrak{a} \subseteq \alpha D$. It follows $s\mathfrak{a}_t \subseteq \alpha D \subseteq \mathfrak{a} \subseteq \mathfrak{p}$, and thus $sx \in s\mathfrak{a}_t \subseteq \mathfrak{p}$. Finally $x \in \mathfrak{p}$, since $s \in D - \mathfrak{p}$. □

(15.49) Corollary ([5, Corollary 2.6])**.** *Let D be an integral domain. The following conditions are equivalent.*

(i) *D is a PvMD.*
(ii) *D is essential and $\mathcal{E}(D)$ is closed in* $\operatorname{Spec}(D)$*, with respect to the inverse topology.*
(iii) *D is essential and $\mathcal{E}(D)$ is closed in* $\operatorname{Spec}(D)$*, with respect to the constructible topology.*

Proof. (i)$\Longrightarrow$(ii). By assumption, D is essential. By the proof of (15.47,(i)$\Longrightarrow$(ii)),

$$\mathcal{V} := \{D_{\mathfrak{p}} : \mathfrak{p} \in \operatorname{Spec}^t(D)\}$$

is an essential representation of D. Thus, keeping in mind (15.48), $\mathcal{E}(D) = \operatorname{Spec}^t(D)$. The conclusion follows by (15.45) and (15.6).

(ii)$\Longrightarrow$(iii) is trivial, since the inverse topology is coarser than the constructible topology, by (13.14).

(iii)$\Longrightarrow$(i). We know that $\mathcal{V} := \{D_{\mathfrak{p}} : \mathfrak{p} \in \mathcal{E}(D)\}$ is an essential representation of D. Since $\mathcal{E}(D)$ is closed in the constructible topology, it suffices to apply condition (ii) of (15.47). □

(15.50) Proposition. *Let D be an integral domain with quotient field K. Then, the domination map $\delta : \operatorname{Zar}(K|D)^{\mathrm{inv}} \longrightarrow \operatorname{Spec}(D)^{\mathrm{inv}}$, is continuous. Moreover, $\delta : \operatorname{Zar}(K|D)^{\mathrm{patch}} \longrightarrow \operatorname{Spec}(D)^{\mathrm{const}}$ is continuous and closed.*

Proof. Keeping in mind the proof of (12.6), we have $\delta^{-1}(D(d)) = B(d^{-1})$, for each element $d \in D - \{0\}$. It follows that the inverse image of an open and compact subspace of $\operatorname{Spec}(D)$ is open and compact in $\operatorname{Zar}(K|A)$. The conclusion follows, by definition. For the last part, keep in mind that a continuous map from a compact space to a Hausdorff space is closed. □

(15.51) Remark. Let D be a PvMD with quotient field K. Keeping in mind (15.46) and (15.48), the set $\mathcal{V}(D)$ of essential valuation overrings of D is

$$\mathcal{V}(D) = \{D_{\mathfrak{p}} : \mathfrak{p} \in \operatorname{Spec}^t(D)\}.$$

By (15.50), the domination map $\delta : \operatorname{Zar}(K|D)^{\text{patch}} \longrightarrow \operatorname{Spec}(D)^{\text{cons}}$ restricts to a continuous bijection $\delta_{\mathcal{V}} := \delta|_{\mathcal{V}(D)} : \mathcal{V}(D) \longrightarrow \operatorname{Spec}^t(D)$ (with respect to the subspace topologies induced by the patch topologies). Since δ is closed, it is not hard to infer that $\delta_{\mathcal{V}}$ is a homeomorphism. In particular, the space $\operatorname{Min}(\mathcal{V}(D))$ of minimal elements of $\mathcal{V}(D)$ is homeomorphic to $\operatorname{Max}^t(D)$. Keeping in mind (14.16) and the fact that $\operatorname{Max}^t(D)$ is the minimal space of $\operatorname{Spec}^t(D)$, with respect to the inverse topology, it follows that $\operatorname{Min}(\mathcal{V}(D))$ and $\operatorname{Max}^t(D)$ are homeomorphic, via δ, with respect to the subspace inverse topology.

(15.52) Corollary ([14, Lemma 6.3])**.** *Let D be an integral domain. Then, the following conditions are equivalent.*

(i) *D is a PvMD.*
(ii) *D is essential and every critical valuation overring of D is essential.*

Proof. (i)$\Longrightarrow$(ii). D is essential, being it a PvMD. Thus, by (15.46b), (15.49) and (15.50), it follows that the collection $\mathcal{V}(D)$ forms a closed representation of D in the space of valuation overrings of D, endowed with the inverse topology. Then, by definition, any critical valuation overring of D is essential.

(ii)$\Longrightarrow$(i). By assumption, $\mathcal{V}(D)$ is a representation of D. Moreover, in view of (15.31), $\mathcal{V}(D)$ is precisely the set of all critical valuation overrings of D; thus it is closed in the inverse topology and, a fortiori, in the patch topology. Since the domination map δ is closed, with respect to the patch topology (see (15.50)), then (15.46b) implies that $\mathcal{E}(D)$ is closed in $\operatorname{Spec}(D)^{\text{cons}}$. The conclusion follows from (15.49). □

(15.53) Theorem ([14, Theorem 6.4])**.** *Let D be a PvMD, let V be a valuation overring of D and let $\mathfrak{p}$ be the center of V in D. Then, the following conditions are equivalent.*

(i) *V is strongly irredundant in some representation of D (consisting of valuation overrings of D).*
(ii) *$V = D_{\mathfrak{p}}$, $\mathfrak{p} \in \operatorname{Max}^t(D)$ and $\mathfrak{p}$ is isolated in $\operatorname{Max}^t(D)$, endowed with the subspace topology induced by the inverse (or the patch) topology.*

Proof. First, note that, since any PvMD is essential, then D is intersection of all critical valuation overrings, by (15.31). Thus, in view of (14.33), $\operatorname{Zar}(K|D)$ contains a unique minimal closed representation of D, in the inverse topology. Let $\mathscr{C} := \mathscr{C}(\operatorname{Zar}(K|D))$ denote the space consisting of minimal critical valuation overrings of D. By (15.31) and (15.52), $\mathscr{C} = \operatorname{Min}(\mathcal{V}(D))$.

(i)$\Longrightarrow$(ii). If V is strongly irredundant in some representation of D, then $V \in \mathscr{C}$, by (14.33a), and $\mathfrak{p} \in \operatorname{Max}^t(D)$, by (15.51). Moreover V is strongly irredundant in $\mathscr{C}$ (14.33b), and thus isolated in $\mathscr{C}$, by (14.21), since $\mathscr{C}$ is the uniquel minimal representation of D. Since $\mathscr{C}$ is homeomorphic to $\operatorname{Max}^t(D)$, by (15.51), it follows that $\mathfrak{p}$ is isolated in $\operatorname{Max}^t(D)$.

(ii)$\Longrightarrow$(i). Again by (15.51) we infer that V is isolated in $\mathscr{C}$. By (14.21), V is strongly irredundant in $\mathscr{C}$. □

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