# Characterizing integral domains by semigroups of ideals

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# Contents

## Introduction

Throughout this paper $R$ will denote an integral domain with quotient field $K$ and we will assume that $R \neq K$. An *overring* of $R$ is a domain $T$ such that $R \subseteq T \subseteq K$.

A *fractional ideal* of $R$ (or simply an *ideal*) is an $R$-submodule $I$ of $K$ such that $dI \subseteq R$ for some nonzero element $d \in R$. If $I \subseteq R$, we say that $I$ is an *integral ideal*. Any finitely generated $R$-submodule of $K$ is a fractional ideal. We denote by $F(R)$ the set of nonzero fractional ideals of $R$ and by $f(R)$ its subset of finitely generated ideals. Both $F(R)$ and $f(R)$ are semigroups with respect to multiplication of ideals. An invertible element of $F(R)$ is called an *invertible* ideal. Principal ideals are invertible and invertible ideals are finitely generated. Denoting by $\mathrm{Inv}(R)$ the group of invertible ideals and by $P(R)$ its subgroup of principal ideals, we have a chain of semigroups

$$P(R) \subseteq \mathrm{Inv}(R) \subseteq f(R) \subseteq F(R).$$

Two wide classes of integral domains, namely Noetherian and Prüfer domains, may be defined by the property that one of these inclusions becomes an equality. In fact $R$ is a *Noetherian domain* (a domain satisfying the ascending chain condition on integral ideals) if and only if each ideal is finitely generated, that is if $f(R) = F(R)$, and $R$ is a *Prüfer domain* (a domain whose localizations at maximal ideals are valuation domains) if and only if each finitely generated ideal is invertible, that is $\mathrm{Inv}(R) = f(R)$. A *Dedekind domain* is a Noetherian Prüfer domain, that is a domain such that each ideal is invertible, or $\mathrm{Inv}(R) = F(R)$. If each ideal is principal, that is $P(R) = F(R)$, $R$ is called a *principal ideal domain*, for short a $PID$.

In this course we will give similar characterizations of more general classes of domains by introducing other semigroups of ideals; this kind of study is particularly useful for investigating some arithmetical properties. For example it is well known that a $PID$ is a *unique factorization domain*, for short a $UFD$, and that it is precisely a one-dimensional (equivalently a Dedekind) $UFD$. To study $UFD$s in dimension greater than one, it is necessary to enlarge the class of Dedekind domain to the class of *Krull domains* (classically defined as intersection of $DVR$s with finite character) and consider the semigroup of *$v$-ideals*, or *divisorial ideals*. In fact P. Samuel proved that a $UFD$ is precisely a Krull domain such that each divisorial ideal is principal [39]. In the same vein, A. Bouvier and M. Zafrullah showed that to study the existence of the greatest common divisor one has to enlarge the class of Krull domains to the class of *Prüfer $v$-multiplication domains*, for short $PvMD$s (domains whose localizations at $t$-maximal ideals are valuation domains), and consider the semigroup of *$t$-ideals*. Indeed they proved that a domain with the greatest common divisor, for short a *$GCD$-domain*, is precisely a $PvMD$ such that each $t$-invertible ideal is principal [8].

Divisorial and $t$-ideals are examples of *star ideals*, that is ideals closed under a *star operation*. In order to unify the notations, we will firstly introduce star operations and survey their main properties. Then we will use $t$-ideals to study *Mori domains* (domains satisfying the ascending chain condition on integral divisorial ideals) and $PvMD$s. The reason for introducing Mori domains in this context is that a Krull domain is precisely a Mori $PvMD$. Finally we will introduce the *Class Group of $t$-ideals* and show that a $UFD$ (respectively a $GCD$-domain) is a Krull domain (respectively a $PvMD$) with trivial class group.

We will assume a good knowledge of the basic properties of Noetherian and valuation domains; for these one can refer to [20, 13, 3, 7].

# 1 Star operations

A *star operation* is a map $I \to I^*$ from the set $F(R)$ of nonzero fractional ideals of $R$ to itself such that:

(1) $R^* = R$ and $(aI)^* = aI^*$, for all $a \in K \setminus \{0\}$;

(2) $I \subseteq I^*$ and if $I \subseteq J$ then $I^* \subseteq J^*$;

(3) $I^{**} = I^*$.

General references for star operations and systems of ideals are [20, 24, 25, 32].

A nonzero fractional ideal $I$ is called a $*$-*ideal* if $I = I^*$; the set of $*$-ideals of $R$ will be denoted by $F_*(R)$. The identity is a star operation, called the $d$-*operation*; thus $F_d(R) = F(R)$.

The following properties can be easily proved. Recall that for two ideals $I$ and $J$ of $R$, the set $(I : J) := \{x \in K \,;\, xJ \subseteq I\}$ is an ideal of $R$.

**Proposition 1.1** *Let $*$ a star operation on $R$. For all $I$, $J \in F(R)$ and any family of nonzero ideals $\{I_\alpha\}$*

(a) *If $\sum I_\alpha \in F(R)$, $(\sum I_\alpha)^* = (\sum I_\alpha^*)^*$; in particular $(I + J)^* = (I^* + J^*)^*$.*

(b) *If $\cap I_\alpha \neq (0)$, $\cap I_\alpha^* = (\cap I_\alpha^*)^*$; in particular, $F_*(R)$ is closed under finite intersections.*

(c) $(IJ)^* = (I^*J)^* = (I^*J^*)^*$.

(d) $(I^* : J^*) = (I^* : J)$ *is a $*$-ideal.*

A star operation $*$ is of *finite type* if $I^* = \cup\{J^* \,;\, J \subseteq I$ and $J$ is finitely generated$\}$, for each $I \in F(R)$. To any star operation $*$, we can associate a star operation $*_f$ of finite type by defining $I^{*_f} = \cup J^*$, with the union taken over all finitely generated ideals $J$ contained in $I$. Clearly $I^{*_f} \subseteq I^*$ and the equality holds if $I$ is finitely generated. Note that $I = I^{*_f}$ if and only if $(x_1, \dots, x_n)^* \subseteq I$, for every finite set $\{x_1, \dots, x_n\} \subseteq I$.

The set of star operations defined on $R$ can be ordered. We say that $*_2$ is *coarser* than $*_1$ and write $*_1 \leq *_2$, if $I^{*_1} \subseteq I^{*_2}$, for all $I \in F(R)$. Hence, if $*_1 \leq *_2$, a $*_2$-ideal is also a $*_1$-ideal. Note that $*_f \leq *$ and, when $*_1 \leq *_2$, we also have $(*_1)_f \leq (*_2)_f$.

A nonzero ideal $I$ is called $*$-*finite* if $I^* = J^*$ for some finitely generated ideal $J$ and *strictly* $*$-*finite* if $I^* = J^*$ for some finitely generated ideal $J \subseteq I$. The set of $*$-finite $*$-ideals will be denoted by $f_*(R)$.

**Proposition 1.2** *Let $I$ be a nonzero ideal of $R$. The following conditions are equivalent:*

(i) *$I$ is strictly $*$-finite;*

(ii) *$I$ is $*_f$-finite;*

(iii) *$I$ is strictly $*_f$-finite.*

*Under these conditions, $I^* = J^* = J^{*_f} = I^{*_f}$ for some finitely generated ideal $J \subseteq I$.*

**Proof.** (i) $\Rightarrow$ (iii) Let $I^* = J^*$, with $J \subseteq I$ finitely generated. Hence $I^* = J^* = J^{*_f} \subseteq I^{*_f} \subseteq I^*$. Whence $I^* = J^{*_f} = I^{*_f}$.

(ii) $\Rightarrow$ (iii) Let $I^{*_f} = H^{*_f}$, where $H := x_1R + \cdots + x_nR$. Since $*_f$ is of finite type, for each $i = 1, \ldots, n$, there exists a finitely generated ideal $F_i \subseteq I$ such that $x_i \in F_i^{*_f}$. Hence $J := F_1 + \cdots + F_n \subseteq I$ and $I^{*_f} = H^{*_f} \subseteq J^{*_f} \subseteq I^{*_f}$. Thus $J^{*_f} = I^{*_f}$.

(iii) $\Rightarrow$ (ii) is clear and (iii) $\Rightarrow$ (i) because $*_f \leq *$. $\square$

A prime $*$-ideal is called a $*$-*prime* ideal. A $*$-*maximal* ideal is an ideal that is maximal in the set of the proper $*$-ideals. It is easy to check that a $*$-maximal ideal (if it exists) is a prime ideal. The set of the $*$-maximal ideals of $R$ will be denoted by $*\text{-Max}(R)$.

**Proposition 1.3** *If $* = *_f$ is a star operation of finite type, the set of the $*$-maximal ideals of $R$ is not empty. Moreover, for each $I \in F(R)$,*

$$I^* = \cap_{M \in *\text{-Max}(R)} I^* R_M;$$

*in particular $R = \cap_{M \in *\text{-Max}(R)} R_M$.*

**Proof.** Let $I$ be the union of an ascending chain $\{I_\alpha\}$ of $*$-ideals. If $x_1, \ldots, x_n \in I$, then $(x_1, \ldots, x_n)^*$ is contained in some $I_\alpha$ and so in $I$. It follows that $I$ is a $*$-ideal and by applying Zorn's Lemma, $*\text{-Max}(R) \neq \emptyset$.

For the second part, it is enough to consider integral $*$-ideals $I = I^* \subseteq R$ and show that $\cap_{M \in *\text{-Max}(R)} IR_M \subseteq I$. Let $x \in IR_M$, for each $M \in *\text{-Max}(R)$ and write $x = a(M)/t(M)$, with $a(M) \in I$ and $t(M) \in R \setminus M$.

Hence $t(M) \in (I : x) \cap R$ and $(I :_R x) = (I : x) \cap R \nsubseteq M$. It follows that $(I :_R x)^* = (I :_R x) = R$ and so $x \in I$. $\square$

Since $*_f \leq *$, each $*$-ideal is contained in a $*_f$-maximal ideal. However we will see later that, if $*$ is not of finite type, the set of $*$-maximal ideals may be empty (Remark 1.8).

**Proposition 1.4** *For any star operation $*$, a minimal prime of a $*$-ideal (in particular of a nonzero principal ideal) is a $*_f$-prime.*

**Proof.** Let $I = I^*$ and $P$ a prime minimal over $I$. We have to show that $J^* \subseteq P$, for each finitely generated ideal $J \subseteq P$. Since $\sqrt{IR_P} = PR_P$, $J^n R_P \subseteq IR_P$ for some $n \geq 1$. Let $s \in R \setminus P$ be such that $sJ^n \subseteq I$. Then

$$s(J^*)^n \subseteq (s(J^*)^n)^* = (sJ^n)^* \subseteq I \subseteq P$$

and so $J^* \subseteq P$. $\square$

## 1.1 The $v$-operation and the $t$-operation

The $v$- and the $t$-operations are the best known nontrivial star operations. The *$v$-operation*, or *divisorial closure*, is defined by setting

$$I_v := (R\colon (R\colon I))$$

for each $I \in F(R)$. A $v$-ideal is also called a *divisorial ideal*.

**Proposition 1.5** *For each $I \in F(R)$, $I_v = \cap\{yR\,;\ y \in K\,,\ I \subseteq yR\}$.*

**Proof.** Let $J := \cap\{yR\,;\ y \in K\,,\ I \subseteq yR\}$. Since principal ideals are divisorial, $I_v \subseteq J$. Assume that $I_v \subsetneq J$ and let $x \in J \setminus I_v$. Then $x(R : I) \nsubseteq R$ and $x \notin z^{-1}R$ for some $z \in (R : I)$. This is a contradiction, since $I \subseteq z^{-1}R$, and so $J \subseteq z^{-1}R$. $\square$

**Proposition 1.6** *The following statements are equivalent for a nonzero ideal $I$ of $R$:*

(i) *$I$ is divisorial (i.e. $I = (R : (R : I))$;*

(ii) *$I = (R : J)$, for some ideal $J$;*

(iii) *$I = \cap\{yR\,;\ y \in K\,,\ I \subseteq yR\}$.*

**Proof.** (i) $\Rightarrow$ (ii) Take $J := (R : I)$.

(ii) $\Rightarrow$ (iii) follows from Proposition 1.5, since $(R : J)$ is divisorial (Proposition 1.1, (d)).

(iii) $\Rightarrow$ (i) Because a nonzero intersection of divisorial ideals is a divisorial ideal (Proposition 1.1, (b)). $\square$

The $t$-operation is the finite type star operation associated to the $v$-operation: it is therefore defined by

$$I_t := \cup\{J_v;\ J \subseteq I \text{ and } J \text{ is finitely generated}\},$$

for each $I \in F(R)$, and $J_v = J_t$ whenever $J$ is finitely generated.

Since $(R : I) = (R : I^*)$ (Proposition 1.1 (d)), we have $I^* \subseteq I_v$, for each ideal $I \in F(R)$ and each star operation $*$. It follows that a divisorial ideal is a $*$-ideal and a $t$-ideal is a $*_f$-ideal. In other words, the $v$-operation is the coarsest star operation on $R$ and the $t$-operation is the coarsest star operations of finite type.

**Proposition 1.7** *An height-one prime is a $t$-ideal.*

**Proof.** It is enough to observe that an height-one prime is minimal over a principal ideal and apply Proposition 1.4. $\square$

**Remark 1.8** Let $V$ be a valuation domain. Then:

(a) Each nonzero ideal of $V$ is a $t$-ideal, because each finitely generated ideal is principal.

(b) A nonzero principal prime ideal of $V$ is maximal. In fact, let $P = xR$ and $y \in R \setminus P$. Then $P \subseteq yR$ and $z := xy^{-1} \in R$. Since $x = zy$, $P$ is prime and $y \notin P$, we have $z := xy^{-1} \in P = xR$. Hence $y^{-1} \in R$ and $y$ is invertible.

(c) If $P$ is a nonprincipal prime ideal of $V$, then $V_P = (P : P) = (V : P)$. To show this, we first note that $V_P \subseteq (P : P)$. This is clearly true if $P$ is maximal, that is $V = V_P$; thus we can assume that $V \subsetneq V_P$. Let $x \in V_P \setminus V$. Then $y := x^{-1} \in V \setminus P$ and so $P \subseteq yV$. It follows that $xP \subseteq V$. Also $P = xyP$ and $y \notin P$ implies $xP \subseteq P$. Hence $x \in (P : P)$. Finally $(V : P) \subseteq V_P$. In fact, let $z \in (V : P)$. Since $V_P$ is a valuation domain, if $z \notin V_P$, then $z^{-1} \in PV_P \subseteq P(P : P) \subseteq P$. Hence $1 = z^{-1}z \in P(V : P)$ and $P$ is invertible, that is principal. A contradiction.

(d) The maximal ideal $M$ of $V$ is divisorial if and only if it is principal. Indeed, if $M$ is divisorial, $V \subsetneq (V : M)$. Thus, if $x \in (V : M) \setminus V$, we have $M_v = M \subseteq x^{-1}V \subseteq V$, whence $M = x^{-1}V$.

(e) If $P$ is a nonzero nonmaximal prime ideal of $V$, then $P$ is divisorial. In fact, if $P$ is nonmaximal, $V_P$ is a proper overring of $V$ and $P_v = (V : (V : P)) = (V : V_P)$ is a proper ideal of $V_P$. Since $PV_P = P(P : P) = P \subseteq P_v \subsetneq V_P$, we conclude that $P = P_v$.

We see that when $M$ is not principal $v$-Max$(V)$ is empty, while $t$-Max$(V) = \{M\}$. Also, if $V$ is a one-dimensional non-discrete valuation domain, $M$ is an height-one prime that is not divisorial.

A $v$-maximal divisorial ideal $M$ may be properly contained in a $t$-maximal ideal; examples are given in [19] or [31]. However the next result shows that this cannot happen if $M$ is $v$-finite.

**Proposition 1.9** *A $v$-finite $v$-maximal ideal is a $t$-maximal ideal.*

**Proof.** Assume that $J$ is finitely generated and $J_v := M \in v\text{-Max}(R)$. Since $I_v = I_t$ for any $I$ finitely generated, for each $x \notin M$ we have $(M + xR)_t = (J_t + xR)_t = (J_v + xR)_v = R$. Hence $M \in t\text{-Max}(R)$. $\square$

## 1.2 The $w$-operation

If $\{R_\alpha\}$ is a family of overrings of $R$ such that $R \subseteq R_\alpha \subseteq K$ and $R = \cap R_\alpha$, for all $I \in F(R)$, the map $I \mapsto I^* = \cap IR_\alpha$ is a star operation *induced* by the family $\{R_\alpha\}$ [20]. Then, by Proposition 1.3, one can consider the star operation induced by the family $\{R_M\}_{M\in t\text{-Max}(R)}$. This operation is called the *$w$-operation* and is defined by setting

$$I_w := \cap_{M\in t\text{-Max}(R)} IR_M,$$

for each nonzero ideal $I$. An equivalent definition is obtained by setting

$$I_w := \cup\{(I\colon J);\ J \text{ is finitely generated and } (R\colon J) = R\}.$$

By using the latter definition, one can see that the notion of $w$-ideal coincides with the notion of *semi-divisorial* ideal introduced by S. Glaz and W. Vasconcelos in 1977 [22]. As a star-operation, the $w$-operation was first considered by E. Hedstrom and E. Houston in 1980 under the name of $F_\infty$-operation [26]. Since 1997 this star operation was intensely studied by Wang Fanggui and R. McCasland in a more general context. In particular they showed that the notion of $w$-closure is a very useful tool in the study of Strong Mori domains [41, 42].

The $w$-operation is of finite type and distributes over intersections, that is $(I \cap J)_w = I_w \cap J_w$, for all $I$, $J \in F(R)$. This is equivalent to say that $(I : J)_w = (I_w : J) = (I_w : J_w)$ for all $I$, $J \in F(R)$ and $J$ finitely generated. We have $w\text{-Max}(R) = t\text{-Max}(R)$ and $IR_M = I_wR_M \subseteq I_tR_M$, for each $I \in F(R)$ and $M \in t\text{-Max}(R)$. Thus $w \leq t \leq v$. In addition, $w$ is the coarsest star operation is of finite type which distributes over intersections.

## 1.3 $t$-compatible extensions of domains

The $v$-operation, $t$-operation and $w$-operation are intrinsecally defined on any domain, so it makes sense to study the behaviour of these operations with respect to extensions of domains $R \subseteq T$. The general problem of comparing a star operation on $R$ with a star operation on $T$ is studied in [11].

Given an extension of domains $R \subseteq T$, we will denote by $v'$ and $t'$ respectively the divisorial closure and the $t$-operation in $T$.

**Proposition 1.10** *Let $R \subseteq T$ be an extension of domains. The following conditions are equivalent:*

(i) *$J_v \subseteq (JT)_{v'}$, for each nonzero finitely generated ideal $J$ of $R$;*

(ii) *$(I_tT)_{t'} = (IT)_{t'}$, for each nonzero ideal $I$ of $R$;*

(iii) *$I' \cap R$ is a $t$-ideal, for each $t'$-ideal $I'$ of $T$ such that $I' \cap R \neq 0$;*

(iv) *$(IT)_{t'} \cap R$ is a $t$-ideal, for each nonzero ideal $I$ of $R$.*

**Proof.** (i) $\Rightarrow$ (ii). From the definition of the $t$-closure, condition (i) implies that $I_t \subseteq (IT)_{t'}$, for each ideal $I$ of $R$. Whence $(I_tT)_{t'} = (IT)_{t'}$.

(ii) $\Rightarrow$ (iii) If $J$ is a finitely generated ideal, the $v$- and the $t$-operation coincide on $J$ and so, if (ii) holds, $J_v \subseteq (J_vT)_{v'} = (JT)_{v'}$. Then, if $J \subseteq I' \cap R$, we have $J_v \subseteq (JT)_{v'} \cap R \subseteq I' \cap R$ and $I' \cap R$ is a $t$-ideal.

(iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are clear. $\square$

Under the equivalent conditions of Proposition 1.10, we say that the extension $R \subseteq T$ is *$t$-compatible.*

We now show that flat extensions and generalized rings of fractions are $t$-compatible.

**Proposition 1.11** *Let $R \subseteq T$ be a flat extension of domains. Then:*

(a) *$(R : J)T = (T : JT)$, for each nonzero finitely generated ideal $J$ of $R$.*

(b) *The extension is $t$-compatible.*

(c) *If $J$ is a nonzero finitely generated ideal of $R$ and $(R : J)$ is $v$-finite, $(J_vT)_{v'} = J_vT$.*

**Proof.** (a) Let $J := x_1R + \cdots + x_nR$. Then $(R : J) = x_1^{-1}R \cap \cdots \cap x_n^{-1}R$ and, by flatness, $(R : J)T = x_1^{-1}T \cap \cdots \cap x_n^{-1}T = (T : JT)$.

(b) If $J$ is finitely generated, by (a) we have $J_v \subseteq J_vT \subseteq (T : (R : J)T) = (T : (T : JT)) = (JT)_{v'}$.

(c) Let $J$ be finitely generated and $(R:J) = H_v$, with $H$ finitely generated. By part (b), $(HT)_{v'} = (H_vT)_{v'}$. Hence

$$\begin{aligned} J_vT &= (R:H)T = (T:HT) = (T:H_vT) \\ &= (T:(R:J)T) = (T:(T:JT)) = (JT)_{v'}. \end{aligned}$$

□

If $\mathcal{S}$ is a multiplicative set of nonzero integral ideals of $R$, the *generalized ring of fractions* of $R$ with respect to $\mathcal{S}$ is the overring

$$R_{\mathcal{S}} := \cup\{(R:J)\,;\ J \in \mathcal{S}\}.$$

When $S \subseteq R$ is a multiplicative part of $R$, setting $\mathcal{S} := \{xR\,; x \in S\}$ we have $R_{\mathcal{S}} = R_S$.

For any ideal $I$ of $R$, the set $I_{\mathcal{S}} := \cup\{(I:J)\,;\ J \in \mathcal{S}\}$ is an ideal of $R_{\mathcal{S}}$ such that $IR_{\mathcal{S}} \subseteq I_{\mathcal{S}}$.

The *saturation* of $\mathcal{S}$ is the multiplicative system of ideals

$$\overline{\mathcal{S}} := \{I \subseteq R\,;\ \text{there exists } J \in \mathcal{S} \text{ such that } J \subseteq I\}.$$

Clearly $I_{\mathcal{S}} = I_{\overline{\mathcal{S}}}$ for each ideal $I$, thus if necessary we can assume that $\mathcal{S} = \overline{\mathcal{S}}$, in other words that $\mathcal{S}$ is *saturated.*

We will denote by $v_{\mathcal{S}}$ and $t_{\mathcal{S}}$ respectively the divisorial closure and the $t$-operation in $R_{\mathcal{S}}$.

**Proposition 1.12** *Let $I$, $J$ be nonzero ideals of $R$ and $\mathcal{S}$ a multiplicative set of ideals. Then:*

(a) *$I_{\mathcal{S}} = R_{\mathcal{S}}$ if and only if $I \in \overline{\mathcal{S}}$;*

(b) *$(I:J)_{\mathcal{S}} \subseteq (I_{\mathcal{S}} : J_{\mathcal{S}})$ and if $J$ is finitely generated, $(I:J)_{\mathcal{S}} = (I_{\mathcal{S}} : J_{\mathcal{S}}) = (I_{\mathcal{S}} : JR_{\mathcal{S}})$.*

(c) *If $J$ is finitely generated, $(J_v)_{\mathcal{S}} \subseteq (J_{\mathcal{S}})_{v_{\mathcal{S}}} = (JR_{\mathcal{S}})_{v_{\mathcal{S}}}$; whence the extension $R \subseteq R_{\mathcal{S}}$ is $t$-compatible.*

(d) *If $J$ is finitely generated and $(R:J)$ is $v$-finite, $(J_vR_{\mathcal{S}})_{v_{\mathcal{S}}} = (J_v)_{\mathcal{S}} = (JR_{\mathcal{S}})_{v_{\mathcal{S}}}$.*

**Proof.** (a) Let $I \in \overline{\mathcal{S}}$. If $x \in R_{\mathcal{S}}$, then $xH \subseteq R$ for some $H \in \mathcal{S}$ and $xHI \subseteq I$. Since $IH \in \overline{\mathcal{S}}$, then $x \in I_{\overline{\mathcal{S}}} = I_{\mathcal{S}}$ and so $I_{\mathcal{S}} = R_{\mathcal{S}}$.

Conversely, if $I_{\mathcal{S}} = R_{\mathcal{S}}$, then $1 \in I_{\mathcal{S}}$ and so $H = 1H \subseteq I$, for some $H \in \mathcal{S}$. It follows that $I \in \overline{\mathcal{S}}$.

(b) If $x \in (I:J)_{\mathcal{S}}$ and $y \in J_{\mathcal{S}}$, then $xy \in I_{\mathcal{S}}$. Indeed, if $H$, $L \in \mathcal{S}$ are such that $xH \subseteq (I:J)$ and $yL \subseteq J$, then $HL \in \mathcal{S}$ and $xyHL \subseteq I$. Hence $(I:J)_{\mathcal{S}} \subseteq (I_{\mathcal{S}} : J_{\mathcal{S}}) \subseteq (I_{\mathcal{S}} : JR_{\mathcal{S}})$.

Now assume that $J := x_1R + \cdots + x_nR$ is finitely generated. Let $y \in (I_{\mathcal{S}} : JR_{\mathcal{S}})$, so that $yJ \subseteq I_{\mathcal{S}}$, and let $H_i \in \mathcal{S}$ be such that $yx_iH_i \subseteq I$, $i = 1, \ldots, n$. Hence $H := H_1 \ldots H_n \in \mathcal{S}$ and $yJH \subseteq I$. Whence $y \in (I : J)_{\mathcal{S}}$ and $(I_{\mathcal{S}} : JR_{\mathcal{S}}) \subseteq (I : J)_{\mathcal{S}}$.

(c) By applying (a), for $I = R$, we have that $(J_{\mathcal{S}})_{v_{\mathcal{S}}} = (JR_{\mathcal{S}})_{v_{\mathcal{S}}}$. In addition

$$(J_v)_{\mathcal{S}} \subseteq (R_{\mathcal{S}} : (R : J)_{\mathcal{S}}) = (R_{\mathcal{S}} : (R_{\mathcal{S}} : J_{\mathcal{S}})) = (J_{\mathcal{S}})_{v_{\mathcal{S}}} = (JR_{\mathcal{S}})_{v_{\mathcal{S}}}.$$

(d) Let $J$ be finitely generated and $(R : J) = H_v$, with $H$ finitely generated. By using (a),

$$(J_v)_{\mathcal{S}} = (R : H)_{\mathcal{S}} = (R_{\mathcal{S}} : H_{\mathcal{S}}) = (R_{\mathcal{S}} : (R_{\mathcal{S}} : JR_{\mathcal{S}})) = (JR_{\mathcal{S}})_{v_{\mathcal{S}}}.$$

In addition $J_vR_{\mathcal{S}} \subseteq (J_v)_{\mathcal{S}} = (JR_{\mathcal{S}})_{v_{\mathcal{S}}}$, whence $(J_vR_{\mathcal{S}})_{v_{\mathcal{S}}} = (JR_{\mathcal{S}})_{v_{\mathcal{S}}}$. □

**Remark 1.13** (1) An overring $T$ of $R$ is flat if and only if $T_M = R_{M\cap R}$ for each maximal ideal $M$ of $T$ [15, Lemma 6.5].

(2) It is well known that a ring of fractions of $R$ is a flat overring. In addition, a flat overring is an intersection of localizations of $R$ (by the previous remark) and an intersection of localizations is a generalized ring of fractions (indeed, if $\mathcal{Z} \subseteq \mathrm{Spec}(R)$ and $T = \cap\{R_P\,; P \in \mathcal{Z}\}$, then $T = R_{\mathcal{S}}$ with $\mathcal{S} = \overline{\mathcal{S}} = \{\text{ideals } I \subseteq R\,; I \nsubseteq P, \text{ for each } P \in \mathcal{Z}\}$). On the other hand, none of the previous implications reverses: an example of a generalized ring of fractions that is not an intersection of localizations is given in [13, Example 8.4.6] and an example of an intersection of localizations that is not flat is given in [15, pag. 32]. Prüfer domains are characterized by the property that each overring is flat [13, Theorem 1.1.1].

(3) Polynomial rings are flat extensions; hence they are $t$-compatible. Moreover, if $I \subseteq R$ is any divisorial ideal (respectively a $t$-ideal, a $w$-ideal) its extension $I[X]$ is still a divisorial ideal (respectively a $t'$-ideal, a $w'$-ideal) of $R[X]$ [26, Proposition 4.3].

(4) The ring $Int(R) := \{f(X) \in K[X]\,;\ f(R) \subseteq R\}$ of the integer valued polynomials over $R$ is $t$-compatible [9, Lemma 4.1]. But the extension of a divisorial ideal of $R$ is not necessarily divisorial [9, Example 4.2].

### 1.4 $*$-invertibility

The set $F_*(R)$ of $*$-ideals of $R$ is a semigroup (with unity $R$) with respect to the $*$-*multiplication*, defined by

$$(I^*, J^*) \longrightarrow (IJ)^*$$

and the set $f_*(R)$ of $*$-finite $*$-ideals is a subsemigroup of $F_*(R)$.

We say that an ideal $I \in F(R)$ is *$*$-invertible* if $I^*$ is invertible in the semigroup $F_*(R)$. It is easy to check that in this case the *$*$-inverse* of $I^*$ in $F_*(R)$ is $(R : I)$. Hence $I \in F(R)$ is $*$-invertible if and only if $(I(R : I))^* = R$. It follows that if $* = *_f$ is of finite type, $I$ is $*$-invertible if and only if $I(R : I)$ is not contained in any $*$-maximal ideal of $R$. When $* = d$ is the identity, we obtain the usual notion of *invertible ideal*. The group of $*$-invertible $*$-ideals will be denoted by $\mathrm{Inv}_*(R)$. It is easy to see that when $*_1 \leq *_2$, a $*_1$-invertible ideal is also $*_2$-invertible.

**Proposition 1.14** *An invertible ideal is divisorial.*

**Proof.** If $I(R : I) = R$, the inverse of $(R : I)$ is $I = (R : (R : I)) = I_v$. $\square$

We are particularly interested in $v$-invertibility and $t$-invertibility: a suitable reference for these notions is [43].

**Proposition 1.15** *A nonzero ideal $I$ of $R$ is $v$-invertible if and only if $(I_v : I_v) = R$.*

**Proof.** It is enough to note that $(I_v : I_v) = (R : I(R : I))$. $\square$

It is well known that a nonzero ideal is invertible if and only if it is finitely generated and locally principal. This characterization can be extended to $t$-invertible ideals in the following way.

**Proposition 1.16** *The following conditions are equivalent for a nonzero ideal $I$ of $R$.*

(i) *$I$ is $t$-invertible;*

(ii) *$I$ is $t$-finite and $I_tR_M$ is principal, for each $M \in t\text{-Max}(R)$;*

(iii) *$I$ is $v$-invertible and $I$, $(R : I)$ are $t$-finite.*

**Proof.** (i) $\Rightarrow$ (iii) Since $t \leq v$, $I$ is $v$-invertible. If $(I(R : I))_t = R$, $1 \in H_v$ for some finitely generated ideal $H \subseteq I(R : I)$. Let $H := x_1R + \cdots + x_sR$ and write $x_i = \sum_{j=1}^{n_i} a_{ij}b_{ij}$, with $a_{ij} \in I$ and $b_{ij} \in (R : I)$. Letting $F := \sum a_{ij}R \subseteq I$ and $G := \sum b_{ij}R \subseteq (R : I)$, where $i = 1, \ldots, s$ and $j = 1, \ldots n_s$, we have $R = H_v \subseteq (FG)_v \subseteq (IG)_v \subseteq (I(R : I))_v = R$. Thus $(FG)_v = (IG)_v = R$ and it follows that $G_v = (R : I)$ and $F_v = (R : G) = I_v$.

(iii) $\Rightarrow$ (ii) Let $I_v = J_v$ and $(R : I) = H_v$, with $J$, $H$ finitely generated. Since $(JH)_t = (JH)_v = R$, $JH \nsubseteq M$ for each $M \in t\text{-Max}(R)$. Hence $(JH)R_M = (JR_M)(HR_M) = R_M$. It follows that $JR_M$ is invertible and finitely generated, hence principal. Finally $I_t = J_t$ by Proposition 1.2 and $I_tR_M = J_tR_M = (JR_M)_{t_M} = JR_M$ by Proposition 1.12 (d).

(ii) $\Rightarrow$ (i) Let $I_t = J_t$ with $J \subseteq I$ finitely generated (Proposition 1.2). By (ii) and Proposition 1.12 (c), $I_tR_M = (I_tR_M)_{t_M} = (J_tR_M)_{t_M} = (JR_M)_{t_M}$ is principal, for each $M \in t\text{-Max}(R)$. Hence

$$\begin{aligned} R_M \supseteq & (I(R:I))_t R_M = (I_t(R:I))_t R_M \\ & \supseteq (I_t(R:J))R_M = I_tR_M(R_M : JR_M) = R_M. \end{aligned}$$

It follows that $(I(R:I))_tR_M = R_M$, for each $M \in t\text{-Max}(R)$, and so $(I(R:I))_t = R$ (Proposition 1.3). $\square$

**Corollary 1.17** *A $t$-invertible $t$-ideal is divisorial and $v$-finite.*

**Proof.** If $I = I_t$ is $t$-invertible it is strictly $v$-finite by Proposition 1.16. Hence $I = I_v$ by Proposition 1.2. $\square$

If the intersection $R = \cap_{M \in *_f\text{-Max}(R)} R_M$ has finite character, that is each nonzero noninvertible element of $R$ is contained only in finitely many $*_f$-maximal ideals, we say that $R$ has *$*_f$-finite character*.

The following result holds more in general for any star operation of finite type (in particular for the identity); we give a proof for the $t$-operation.

**Proposition 1.18** *Assume that $R$ has $t$-finite character. Then a nonzero ideal $I$ of $R$ is $t$-invertible if and only if $I_tR_M$ is principal, for each $M \in t\text{-Max}(R)$.*

**Proof.** By Proposition 1.16 we have only to prove that if $R$ has $t$-finite character and $I_tR_M$ is principal, for each $M \in t\text{-Max}(R)$, then $I$ is $t$-finite. It is enough to consider integral ideals.

Let $M_1, \dots, M_n$ be the $t$-maximal ideals containing $I$ and let $I_tR_{M_i} = x_iR_{M_i}$, $x_i \in I$, $i = 1, \dots, n$. Set $J := x_1R + \cdots + x_nR$. If $J_t \subsetneq I_t$, let $M_1, \dots, M_n, N_1, \dots, N_h$ be the $t$-maximal ideals containing $J$, take $y \in I \setminus (N_1 \cup \cdots \cup N_h)$ and set $H := J + yR$. Then we have $x_iR_{M_i} \subseteq H_tR_{M_i} \subseteq I_tR_{M_i} = x_iR_{M_i}$, so that $H_tR_{M_i} = I_tR_{M_i}$ for $i = 1, \dots, n$ and moreover $H_tR_N = I_tR_N = R_N$ for each $t$-maximal ideal $N \neq M_i$. It follows that $H_t = I_t$ with $H \subseteq I$. $\square$

## 1.5 $t$-invertible $t$-prime ideals

We now study $t$-invertibility of $t$-prime ideals.

**Proposition 1.19** *A proper integral divisorial ideal $M$ of $R$ is a $v$-maximal ideal if and only if $M = x^{-1}R \cap R$, for each $x \in (R:M) \setminus R$.*

**Proof.** If $I \subseteq R$ is a proper divisorial ideal, then $I \subseteq x^{-1}R \cap R \neq R$, for each $x \in (R:I) \setminus R$.

Since an ideal of type $zR \cap R$ is divisorial, If $M \in v\text{-Max}(R)$ we have $M = x^{-1}R \cap R$, for each $x \in (R : M) \setminus R$. Conversely, assume that $I = x^{-1}R \cap R$, for each $x \in (R : I) \setminus R$. If $I \subseteq J$ for some proper divisorial ideal $J \subseteq R$, we have $I \subseteq J \subseteq y^{-1}R \cap R \neq R$, for each $y \in (R : J) \setminus R \subseteq (R : I) \setminus R$. It follows that $I = y^{-1}R \cap R = J$ is a $v$-maximal ideal. $\square$

**Proposition 1.20** *A $v$-invertible $v$-prime ideal is a $v$-maximal ideal.*

**Proof.** Let $P$ be a $v$-invertible $v$-prime ideal and let $y \in (R : P) \setminus R$. If $P \subsetneq y^{-1}R \cap R$, then $xy^{-1} \in R \setminus P$ for some $x \in R$. So that $x^{-1}y \in R_P$ and $y \in xR_P \subseteq R_P$. Let $z \in R \setminus P$ be such that $zy \in R$. Then $zyP \subseteq P$ and, since $yP \subseteq R$ and $z \notin P$, we get $yP \subseteq P$, that is $y \in (P : P)$. Since $P$ is divisorial and $v$-invertible, by Proposition 1.15 we have $(P : P) = R$; thus $y \in R$. This contradiction shows that $P = y^{-1}R \cap R$ for each $y \in (R : P) \setminus R$ and thus $P$ is a $v$-maximal ideal (Proposition 1.19). $\square$

**Proposition 1.21** *The following conditions are equivalent for a prime ideal $P$ of $R$:*

(i) *$P$ is a $t$-invertible $t$-ideal;*

(ii) *$P$ is a $t$-invertible $t$-maximal ideal;*

(iii) *$P$ is a $v$-finite $v$-invertible $v$-prime ideal;*

(iv) *$P$ is a $v$-finite $v$-invertible $v$-maximal ideal.*

**Proof.** (i) $\Rightarrow$ (iii) follows from Proposition 1.16.

(iii) $\Rightarrow$ (iv) follows from Proposition 1.20.

(iv) $\Rightarrow$ (ii) $P$ is $t$-maximal by Proposition 1.9. We have $P \subseteq P(R : P) \subseteq (P(R : P))_t \subseteq R$. If $P$ is not $t$-invertible, then $P = P(R : P)$ and $(P : P) = (R : P)$. It follows that $(P(R : P))_v = (P(P : P))_v = P$, while $P$ is $v$-invertible.

(ii) $\Rightarrow$ (i) is clear. $\square$

**Remark 1.22** (1) A $v$-finite $v$-invertible ideal $I$ of $R$ need not be $t$-invertible; in fact $(R : I)$ need not be $v$-finite (see the next Remark 2.13).

(2) A $v$-invertible $v$-maximal ideal need not be $t$-maximal; in fact it need not be $v$-finite (see the next Remark 2.10).

(3) If $M \in t\text{-Max}(R)$, $MR_M$ need not be a $t_M$-ideal of $R_M$ [43]. However, $MR_M$ is a $t_M$-ideal when $M$ is $t$-invertible (in this case $MR_M$ is principal by Proposition 1.16) or when $M$ is divisorial (in this case $M = yR \cap R$ for some $y \in K$ by Proposition 1.19 and, by flatness, $MR_M = yR_M \cap R_M$ is divisorial in $R_M$).

(4) If $IR_M$ is $t_M$-invertible, for each $M \in t\text{-Max}(R)$, the ideal $I$ of $R$ need not be $t$-invertible. In fact, when $M \in t\text{-Max}(R)$ and $MR_M$ is not a $t_M$-ideal of $R_M$, $M$ is not $t$-invertible and $(MR_N)_{t_N} = R_N$ is clearly $t_N$-invertible, for each $N \in t\text{-Max}(R)$.

(5) If $M$ is an height-one $t$-invertible ideal, then $R_M$ is a $DVR$. In fact it is a local one-dimensional domain with principal maximal ideal.

# 2 Relevant classes of domains

In this section we introduce the main classes of domains we need to know in order to study arithmetical properties.

## 2.1 Completely integrally closed domains

Dealing with non-Noetherian domains, it is useful to weaken the notion of integral dependence by considering almost integral elements. A nonzero element $x$ of $K$ is said to be *almost integral* over $R$ if there exists a nonzero element $d$ of $R$ such that $dx^i \in R$, for $i \geq 0$.

**Proposition 2.1** *The following conditions are equivalent:*

(i) *$x \in K$ is almost integral over $R$;*

(ii) *The sub-$R$-module $R[x]$ of $K$ is a fractional ideal of $R$;*

(iii) *$R[x] \subseteq \frac{1}{d}R$, for some $d \in R$;*

(iv) *$R[x]$ is contained in a finitely generated sub-$R$-module of $K$;*

(v) *$R[x]$ is contained in a fractional ideal of $R$;*

(vi) *$x \in (I : I)$ for some fractional ideal $I$ of $R$.*

**Proof.** (i) $\Leftrightarrow$ (ii) by definition, since $R[x]$ is generated over $R$ by the powers of $x$.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (ii) are clear.

(ii) $\Rightarrow$ (vi) Take $I := R[x]$.

(vi) $\Rightarrow$ (ii) Since $(I : I)$ is an overring of $R$, $R[x] \subseteq (I : I)$. But $(I : I)$ is also a fractional ideal of $R$. Hence $R[x]$ too is a fractional ideal of $R$. $\square$

**Corollary 2.2** *If $x_1, \ldots, x_n \in K$ are quasi integral over $R$, then $R[x_1, \ldots, x_n]$ is contained in a finitely generated sub-$R$-module of $K$.*

**Proof.** If $M_i := m_{i1}R + \cdots + m_{ik_i}R$ is a finite sub-$R$-module of $K$ containing $R[x_i]$ (Proposition 2.1), then $R[x_1, \ldots, x_n]$ is contained in the finite sub-$R$-module $M_1 \ldots M_n$ of $K$, generated by all the products $m_{1j_1} \ldots m_{nj_n}$, $1 \leq j_i \leq k_i$. $\square$

We recall that an element $x \in K$ is *integral* over $R$ if and only if $R[x]$ is a finitely generated sub-$R$-module of $K$, if and only if $x \in (J : J)$ for some finitely generated ideal $J$. Hence integral elements are almost integral and the converse holds when $R$ is Noetherian.

The *complete integral closure* of $R$ in $K$, here denoted by $\widetilde{R}$, is the set of the elements of $K$ that are almost integral over $R$. It follows from Corollary 2.2 that $\widetilde{R}$ is an overring of $R$. One says that $R$ is *completely integrally closed* if $R = \widetilde{R}$.

Denoting by $R'$ the integral closure of $R$, one has $R \subseteq R' \subseteq \widetilde{R}$ and $R' = \widetilde{R}$ when $R$ is Noetherian.

**Proposition 2.3** *The complete integral closure of a domain $R$ is an integrally closed domain.*

**Proof.** Let $x \in K$ be integral over $\widetilde{R}$ and $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ be an equation of integral dependance for $x$ over $\widetilde{R}$. Then $x$ is integral over the subring $S := R[a_{n-1}, \ldots, a_0]$ of $\widetilde{R}$ and $S[x] = S + xS + \cdots + x^{n-1}S$ is an $S$-module finitely generated. On the other hand, $S$ is contained in a finite sub-$R$-module $M = y_1R + \cdots + y_hR$ of $K$ (Corollary 2.2). Hence $S[x] = S + xS + \cdots + x^{n-1}S$ is also contained in a finite sub-$R$-module of $K$, namely $S[x] \subseteq M + xM + \cdots + x^{n-1}M = \sum_{i=0}^{n-1} \sum_{j=1}^{h} x^i m_j R$. By Proposition 2.1, we conclude that $x$ is almost integral over $R$ and so $x \in \widetilde{R}$. $\square$

**Remark 2.4** (1) The complete integral closure is not a proper closure operation. In fact many examples have been given to show that the complete integral closure of a domain might not be completely integrally closed [21, 27, 30, 34, 40].

(2) A localization of a completely integrally closed domain is not necessarily completely integrally closed; an example is given by the ring of entire functions over the complex field [29].

**Proposition 2.5** *For any domain $R$,*

$$\widetilde{R} = \cup\{(I : I); I \in F(R)\} = \cup\{(I_v : I_v); I \in F(R)\}.$$

**Proof.** By Proposition 2.1, (i) $\Leftrightarrow$ (vi), $\widetilde{R} = \cup\{(I : I); I \in F(R)\}$. Now observe that $(I : I) \subseteq (I_v : I) = (I_v : I_v) \subseteq \widetilde{R}$. $\square$

**Proposition 2.6** *The following conditions are equivalent for a domain $R$.*

(i) *$R$ is completely integrally closed;*

(ii) *$R = (I : I)$, for each ideal $I$ of $R$;*

(iii) *Each nonzero ideal of $R$ is $v$-invertible.*

**Proof.** It follows from Proposition 2.5 and Proposition 1.15. □

If each finitely generated ideal of $R$ is $v$-invertible we say that $R$ is a *$v$-domain*. A recent nice survey on $v$-domains is [14].

**Proposition 2.7** *A completely integrally closed domain is a $v$-domain and a $v$-domain is integrally closed.*

**Proof.** A completely integrally closed domain is a $v$-domain by Proposition 2.6. Assume that $R$ is a $v$-domain and let $J$ be a finitely generated ideal of $R$. Since $J$ is $v$-invertible, we have $(J_v : J_v) = R$ (Proposition 1.15) and since $R \subseteq (J : J) \subseteq (J_v : J_v) = R$, we also have $(J : J) = R$. It follows that a $v$-domain is integrally closed. □

**Proposition 2.8** *A valuation domain is completely integrally closed if and only if it has dimension one.*

**Proof.** We apply Proposition 2.6. Let $V$ be a valuation domain. If $V$ is one-dimensional, then $V$ does not have proper overrings [20, Theorem 17.6]. Hence $V = (I : I)$, for each ideal $I$ of $R$, and $V$ is completely integrally closed.

Conversely, let $P$ a nonzero nonmaximal prime ideal of $V$. Then $V_P = (P : P)$ is a proper overring of $R$ (Remark 1.8) and $V$ is not completely integrally closed. □

**Proposition 2.9** *Let $R$ be a completely integrally closed domain and $P$ a divisorial prime ideal of $R$. Then $P$ is a $v$-maximal ideal and $R_P$ is a $DVR$, in particular $P$ has height one.*

**Proof.** Since $P$ is $v$-invertible (Proposition 2.6), $P$ is $v$-maximal by Proposition 1.20. Let $Q$ be a nonzero prime ideal such that $Q \subsetneq P$ and let $x \in P \setminus Q$. Then $x(R : P) \subseteq R$ and $xQ(R : P) \subseteq Q$; whence $Q(R : P) \subseteq Q$ and $(R : P) \subseteq (Q : Q) \subseteq \widetilde{R} = R$ (Proposition 2.5). This is a contradiction since $P$ is divisorial. It follows that $P$ has height one. To show that $R_P$ is a $DVR$, it is enough to show that $PR_P$ is principal. Since $(P : P) \subseteq \widetilde{R} = R \subsetneq (R : P)$, there exists $x \in (R : P) \setminus (P : P)$. Then $xPR_P \subseteq R_P$ and $xPR_P \nsubseteq PR_P$; whence $xPR_P = R_P$ and $PR_P = x^{-1}R_P$ is principal. □

**Remark 2.10** (1) A $v$-maximal ideal $P$ of a completely integrally closed domain need not be $t$-maximal; indeed, being $v$-invertible, $P$ is $t$-maximal if and only if it is $v$-finite if and only if it is $t$-invertible (Proposition 1.21). An example is given in [19, Example 3.1].

(2) An height-one prime of a completely integrally closed domain need not be divisorial. It is enough to consider the maximal ideal of a non-discrete one-dimensional valuation domain (Proposition 2.8 and Remark 1.8).

(3) It is easily seen that any intersection of completely integrally closed domains with the same quotient field is a completely integrally closed domain. However intersections of one-dimensional valuation domains do not exhaust the class of completely integrally closed domains [35].

## 2.2 Prüfer $v$-multiplication domains

A domain whose localizations at maximal ideals are valuation domains is called a *Prüfer domain* and a domain whose localizations at $t$-maximal ideals are valuation domains is called a *Prüfer v-multiplication domain*, for short a $PvMD$. Clearly a Prüfer domain is a $PvMD$.

In the next proposition, the equivalence (i) $\Leftrightarrow$ (ii) is due to Griffin [24, Theorem 5] and the equivalence (i) $\Leftrightarrow$ (iii) is due to Kang [33].

If $f(X) \in R[X]$ is a polynomial, we denote by $c_f$ the *content* of $f(X)$, that is the ideal of $R$ generated by the coefficients of $f(X)$. Given two nonzero polynomials $f(X), g(X) \in R[X]$ with $d := \deg f(X)$, the *Dedekind-Mertens formula* says that $c_f^{d+1}c_g = c_f^d c_{fg}$ [20, Theorem 28.1].

**Theorem 2.11** *The following statements are equivalent for a domain $R$:*

(i) *$R$ is a $PvMD$ (i.e. $R_M$ is a valuation domain for each $M \in t\text{-Max}(R)$);*

(ii) *Each finitely generated ideal of $R$ is $t$-invertible;*

(iii) *$R$ is integrally closed and each $w$-ideal is a $t$-ideal.*

**Proof.** (i) $\Rightarrow$ (ii) Let $J$ be a finitely generated ideal of $R$ and $M \in t\text{-Max}(R)$. Since $R_M$ is a valuation domain, $JR_M$ is a principal ideal. Hence, denoting by $t_M$ the $t$-operation on $R_M$, we have $(J_tR_M)_{t_M} = (JR_M)_{t_M} = JR_M$. From this, we get that $J_tR_M = JR_M$ is principal and so $J_t$ is $t$-invertible (Proposition 1.16).

(ii) $\Rightarrow$ (iii) Since $t$-invertible ideals are $v$-invertible (Proposition 1.16), $R$ is a $v$-domain and so is integrally closed (Proposition 2.7). To show that $w = t$, let $I \in F(R)$ and $H \subseteq I$ be a finitely generated ideal. Since $H$ is $t$-invertible, there exists a finitely generated ideal $L$ such that $L \subseteq H(R : H)$ and $L_v = R$. Now $H_v(R : H) \subseteq R = (H : H) \subseteq (I : H)$, whence $H_vL \subseteq H_v((R : H)H) \subseteq I$. Then $H_v \subseteq (I : L) \subseteq I_w$ and so $I_t = I_w$.

(iii) $\Rightarrow$ (i) Given $a, b \in R\backslash\{0\}$, we have to show that the ideal $aR_M + bR_M$ is invertible, for each $t$-maximal ideal $M$.

Since $R$ is integrally closed, for each nonzero finitely generated ideal $J$, we have $(J : J) = R$. Since in addition $w = t$, we also have $R = (J : J)_w = (J_w : J_w) = (J_t : J_t) = (J_v : J_v)$. It follows that each $J$ is $v$-invertible.

By applying the Dedekind-Mertens formula for the content to the polynomials $f := aX + b$, $g := aX - b \in R[X]$, we get

$$(aR + bR)^3 = c_f^2 c_g = c_f c_{fg} = (aR + bR)(a^2R + b^2R),$$

whence

$$((aR + bR)^2)_t = ((aR + bR)^2)_v = (a^2R + b^2R)_v = (a^2R + b^2R)_t,$$

because the ideal $(aR + bR)$ is $v$-invertible. Localizing at a $t$-maximal ideal $M$, we get

$$(aR + bR)^2 R_M = (a^2R + b^2R)R_M,$$

because $w = t$. It follows that $ab \in a^2R_M + b^2R_M$ and this is enough to conclude that the ideal $aR_M + bR_M$ is principal. In fact, if $ab = xa^2 + yb^2$, $x, y \in R$, multiplying by $y/a^2$, we obtain that $z := yb/a$ is integral over $R_M$. Then $z \in R_M$ and $(aR_M + bR_M)(yR_M + (1 - z)R_M) = aR_M$. $\square$

As a consequence, we recover some well known characterizations of Prüfer domains.

**Theorem 2.12** *The following statements are equivalent for a domain $R$:*

(i) *$R$ is a Prüfer domain (i.e. $R_M$ is a valuation domain for each maximal ideal $M$);*

(ii) *$R$ is a $PvMD$ and each ideal is a $t$-ideal;*

(iii) *$R$ is integrally closed and each ideal is a $t$-ideal;*

(iv) *Each finitely generated ideal of $R$ is invertible.*

**Proof.** (i) $\Rightarrow$ (ii) follows from the fact that each ideal of a valuation domain is a $t$-ideal (Remark 1.8, (a)). Indeed, let $I \subseteq R$ be an ideal and $J \subseteq I$ any finitely generated ideal. By $t$-compatibility (Proposition 1.12, (c)), for each maximal ideal $M$ we have $J_tR_M \subseteq (JR_M)_{t_M} = JR_M \subseteq IR_M$. Hence $J_t \subseteq I$ and $I$ is a $t$-ideal.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) by Theorem 2.11.

(iv) $\Rightarrow$ (ii) $R$ is a $PvMD$ by Theorem 2.11. To show that each ideal is a $t$-ideal, it is enough to observe that each finitely generated ideal, being invertible, is a $t$-ideal (Proposition 1.14).

(ii) $\Rightarrow$ (i) is clear. $\square$

**Remark 2.13** A $v$-domain is not necessarily a $PvMD$; indeed a $v$-invertible finitely generated ideal $J$ is $t$-invertible if and only if its $v$-inverse $(R : J)$ is $v$-finite (Proposition 1.16). The first example of a non-$PvMD$ $v$-domain was given by J. Dieudonné in 1941 [10].

A domain with the property that $(R : J)$ is $v$-finite whenever $J$ is finitely generated was named in [12] a *$v$-coherent* domain. With this terminology, a $PvMD$ is precisely a $v$-coherent $v$-domain.

## 2.3 Mori domains

A *Mori domain* is a domain satisfying the ascending chain condition on integral divisorial ideals (for short *acc on divisorial ideals*). Clearly the class of Mori domains includes Noetherian domains. A good reference for Mori domains is [4].

**Theorem 2.14** *The following statements are equivalent for a domain $R$:*

(i) *$R$ is a Mori domain (i.e. $R$ satisfies the acc on divisorial ideals);*

(ii) *Any descending chain of divisorial ideals $\{I_\alpha\}$ of $R$ such that $\cap I_\alpha \neq (0)$ stabilizes;*

(iii) *Each nonzero ideal of $R$ is $t$-finite.*

**Proof.** (i) $\Rightarrow$ (ii) Let $\{I_\alpha\}$ be a descending chain of divisorial ideals of $R$. Assume that $I := \cap I_\alpha \neq (0)$ and let $x \in I$ be a nonzero element. Then $(R : I_\alpha) \subseteq (R : I) \subseteq (R : xR) = x^{-1}R$ and so $\{x(R : I_\alpha)\}$ is an ascending chain of proper integral divisorial ideals of $R$. By the acc, $x(R : I_\beta) = x(R : I_\gamma)$, that is $I_\beta = I_\gamma$, for $\gamma \geq \beta$.

(ii) $\Rightarrow$ (i) Let $\{I_\alpha\}$ be an ascending chain of proper integral divisorial ideals of $R$ and $I := \cup I_\alpha$. Since $R \subseteq (R : I) \subseteq (R : I_\alpha)$, we have $\cap(R : I_\alpha) \neq (0)$. Hence the descending chain of ideals $\{(R : I_\alpha)\}$ stabilizes and, by divisoriality, so does the chain $\{I_\alpha\}$.

(i) $\Rightarrow$ (iii) It is enough to consider integral ideals. Let $I \subseteq R$ be a nonzero ideal and let $\Sigma := \{H_v\, ; H$ is a nonzero finitely generated ideal and $H \subseteq I\}$. By the acc on divisorial ideals, $\Sigma$ has a maximal element $J_v$. We show that $J_v = I_v$. Clearly $J_v \subseteq I_v$. If $J_v \neq I_v$, then also $J_v \subsetneq I$. Let $x \in I \setminus J_v$ and consider the ideal $F := J + xR$. Then $F_v \in \Sigma$ and $J_v \subsetneq F_v = (J_v + xR)_v$. By this contradiction, $J_v = I_v$.

(iii) $\Rightarrow$ (i) Let $\{I_\alpha\}$ be an ascending chain of proper integral divisorial ideals of $R$ and $I := \cup I_\alpha$. Since $I$ is $t$-finite, we have $I = (x_1R + \cdots + x_nR)_v$ with $x_j \in I$, $j = 1, \ldots, n$. Hence $x_j \in I_\beta$ for some index $\beta$ and all $j$ and it follows that $I_\beta = I\gamma = I$ for $\gamma \geq \beta$. $\square$

**Proposition 2.15** *A valuation domain is a Mori domain if and only if it is a $DVR$.*

**Proof.** Since each ideal of a valuation domain is a $t$-ideal, by Theorem 2.14, a valuation domain is Mori if and only if it is Noetherian, equivalently a $DVR$. $\square$

**Proposition 2.16** *A generalized ring of fractions of a Mori domain is a Mori domain.*

**Proof.** Let $R$ be a Mori domain and let $\mathcal{S}$ be a multiplicative set of nonzero integral ideals of $R$. We show that each nonzero ideal of $R_{\mathcal{S}}$ is $t_{\mathcal{S}}$-finite. For this it is enough to consider integral ideals. Let $I' \subseteq R_{\mathcal{S}}$ be a nonzero ideal and set $I := I' \cap R$. Then $IR_{\mathcal{S}} \subseteq I' \subseteq I_{\mathcal{S}}$. In fact, for the second inclusion, if $x \in I'$, we have $xH \in I' \cap R = I$ for some $H \in \mathcal{S}$. Hence $x \in (I : H) \subseteq I_{\mathcal{S}}$.

Since $I$ is $t$-finite, $I_v = J_v$ with $J \subseteq I$ finitely generated. By applying Proposition 1.12, we get

$$I_{\mathcal{S}} \subseteq (I_v)_{\mathcal{S}} = (J_v)_{\mathcal{S}} = (JR_{\mathcal{S}})_{v_{\mathcal{S}}} \subseteq (IR_{\mathcal{S}})_{v_{\mathcal{S}}} \subseteq (I')_{v_{\mathcal{S}}} \subseteq (I_{\mathcal{S}})_{v_{\mathcal{S}}},$$

whence $(I')_{v_{\mathcal{S}}} = (JR_{\mathcal{S}})_{v_{\mathcal{S}}}$. $\square$

**Proposition 2.17** *An intersection with finite character of Mori domains is Mori.*

**Proof.** Let $\{A_\lambda\}$ be a family of Mori domains such that $R = \cap A_\lambda$ with finite character. To prove that $R$ is Mori, we show that each ascending chain of integral divisorial ideals $\{I_j\}$ of $R$ stabilizes. Writing $I_j = (R : H_j)$, we have $I_j = (\cap A_\lambda : H_j) = \cap(A_\lambda : H_j A_\lambda)$. Since by finite character $H_j A_\lambda = A_\lambda$ for all indexes $j$ and almost all indexes $\lambda$ and since the ascending chain of ideals $\{(A_\lambda : H_j A_\lambda)\}$ stabilizes for each $\lambda$, we conclude that the chain $\{I_j\}$ stabilizes. $\square$

In the study of Mori domains, it is useful to know their $t$-invertible $t$-primes [5, 6].

**Proposition 2.18** *Let $R$ be a Mori domain.*

(1) *Each $t$-ideal of $R$ is divisorial.*

(2) *The following statements are equivalent for a divisorial prime $P$ of $R$:*

   (i) *$P$ is $t$-invertible;*

   (ii) *$(P : P) \subsetneq (R : P)$;*

   (iii) *$R_P$ is a $DVR$.*

   *Under these conditions, $P$ is a $t$-maximal ideal of height one.*

**Proof.** (1) follows from Proposition 1.2, because each ideal of $R$ is $t$-finite (Theorem 2.14).

(2) (i) $\Rightarrow$ (ii) $(P : P) \subsetneq (R : P)$ because $(P : P) = R$ (Proposition 1.15) while $P(R : P) = R$.

(ii) $\Rightarrow$ (iii) If $x \in (R : P) \setminus (P : P)$, $xPR_P = R_P$ and so $PR_P = x^{-1}R_P$ is principal. Since $R_P$ is a Mori domain (Proposition 2.16), the strictly decreasing chain of principal ideals $\{x^{-n}R_P\}_{n\geq 0}$ ha zero intersection. It follows that $P$ has height one and then $R_P$ is a $DVR$.

(iii) $\Rightarrow$ (i) $P$ is $t$-finite because $R$ is Mori. Since $PR_P$ is principal, $P$ is $v$-invertible by Proposition 1.16.

Condition (i) implies that $P$ is a $t$-maximal ideal by Propositions 1.20 and part (1). $\square$

By Proposition 2.16, each localization of a Mori domain is Mori. For a converse we need the $t$-finite character.

**Proposition 2.19** *Each proper divisorial ideal of a Mori domain is contained only in finitely many divisorial primes. In particular a Mori domain has t-finite character.*

**Proof.** Let $\mathcal{F}$ be the family of divisorial prime ideals containing the nonzero ideal $I$. By the acc, we can choose a sequence of divisorial primes $P_1, P_2, \ldots, P_n, \ldots$ such that $P_1$ is maximal in $\mathcal{F}$ and $P_n$ is maximal in $\mathcal{F} \setminus \{P_1, \ldots, P_{n-1}\}$ for $n \geq 2$. Setting $I_n = P_1 \cap \cdots \cap P_n$, we obtain a strictly descending chain of divisorial ideals $\{I_j\}_{j\geq 1}$ whose intersection contains $I$. Hence the chain stabilizes and it follows that $\mathcal{F}$ is finite. $\square$

**Proposition 2.20** *The following statements are equivalent for a domain R:*

(i) *$R$ is Mori;*

(ii) *$R_M$ is Mori for each $t$-maximal ideal $M$ and $R$ has $t$-finite character.*

**Proof.** (i) $\Rightarrow$ (ii) by Propositions 2.16 and 2.19.

(ii) $\Rightarrow$ (i) follows from Proposition 2.17. $\square$

**Remark 2.21** (1) If $R$ is locally Noetherian and the intersection $R = \cap\{R_M; M \in \mathrm{Max}(R)\}$ has finite character, then $R$ is Noetherian. However a locally Noetherian domain need not be Noetherian [28, Example 2.2] and a Noetherian domain need not have finite character (just take the polynomial ring $k[X, Y]$, $k$ a field).

(2) A finite intersection of Noetherian domains is Mori by Proposition 2.17, but need not be Noetherian. For example, let $F$ be a field and let $X$, $Y$ be independent indeterminate over $F$. Then, by properties of

the $D+M$ construction [18, Theorem 4.6], $R_1 := F(X^2)+YF(X)[[Y]]$ and $R_2 := F(X+X^2)+YF(X)[[Y]]$ are Noetherian domains, but $R := R_1 \cap R_2 = F + YF(X)[[Y]]$ is not Noetherian.

(3) A Mori domain such that $R_M$ is Noetherian for each $t$-maximal ideal $M$ is called a *strong Mori domain*. These domains were firstly studied in [41, 42]; they are precisely the domains satisfying the acc on $w$-ideals. A polynomial ring in infinitely many indeterminates over a Strong Mori domain is a Strong Mori domain that is not Noetherian [36].

## 2.4 Krull domains

A *Krull domain* is classically defined as an intersection of $DVR$s with finite character. We denote by $X^1(R)$ the set of the height-one prime ideals of $R$.

**Theorem 2.22** *The following statements are equivalent for a domain $R$:*

(i) *$R$ is a Krull domain (i.e. $R$ is an intersection of $DVR$s with finite character);*

(ii) *$R_P$ is a $DVR$ for each $P \in X^1(R)$ and $R = \cap_{P\in X^1(R)} R_P$ with finite character;*

(iii) *$R_M$ is a $DVR$ for each $M \in t\text{-Max}(R)$ and $R$ has $t$-finite character;*

(iv) *$R$ is a completely integrally closed Mori domain;*

(v) *$R$ is completely integrally closed and each $t$-maximal ideal is divisorial;*

(vi) *$R$ is a $PvMD$ and a Mori domain;*

(vii) *Each nonzero ideal of $R$ is $t$-invertible.*

**Proof.** (i) $\Rightarrow$ (iv) Since $DVRs$ are completely integrally closed (Proposition 2.8) and Noetherian, $R$ is completely integrally closed and Mori (Proposition 2.17).

(iv) $\Rightarrow$ (iii) $R$ has $t$-finite character and each $M \in t\text{-Max}(R)$ is divisorial, because $R$ is a Mori domain. Since $R$ is completely integrally closed, $R_M$ is a $DVR$ by Proposition 2.9.

(iii) $\Rightarrow$ (ii) It is enough to observe that $X^1(R) = t\text{-Max}(R)$. In fact, height-one primes are $t$-ideals by Proposition 1.7. Conversely, if $R_M$ is a $DVR$ for each $M \in t\text{-Max}(R)$, each $M$ has height one.

(ii) $\Rightarrow$ (i) is clear.

(iv) $\Rightarrow$ (v) Because in Mori domains $t = v$ (Proposition 2.18).

(v) $\Rightarrow$ (vii) If $I$ is not $t$-invertible, $I(R : I) \subseteq M$ for some $t$-maximal ideal $M$ of $R$. Thus $(R : M) \subseteq (R : I(R : I)) = ((R : I) : (R : I)) = R$, where the last equality holds because $R$ is completely integrally closed (Proposition 2.6). Then $M_v = R$, which is a contradiction.

(vii) $\Rightarrow$ (iv) Since $t$-invertible ideals are $v$-invertible and $t$-finite (Proposition 1.16), $R$ is completely integrally closed by Proposition 2.6 and is Mori by Proposition 2.14.

(iii) $\Rightarrow$ (vi) $R$ is a $PvMD$ by Theorem 2.11 and is Mori by Proposition 2.17.

(vi) $\Rightarrow$ (vii) Since $R$ is a $PvMD$ each finitely generated ideal is $t$-invertible and since $R$ is Mori each ideal is $t$-finite, hence $t$-invertible. $\square$

**Proposition 2.23** *Noetherian integrally closed domains are Krull domains.*

**Proof.** Since for Noetherian domains the integral closure and the complete integral closure coincide (Section 2.1), we can apply Theorem 2.22. $\square$

A celebrated theorem of Mori-Nagata, proved in the middle fifties, asserts that more generally the integral closure of a Noetherian domain is Krull. A proof can be found in [15, Theorem 4.3].

An integrally closed Noetherian domain of dimension one is called a *Dedekind domain*.

**Theorem 2.24** *The following statements are equivalent for a domain $R$:*

(i) *$R$ is a Dedekind domain (i.e. $R$ is an integrally closed Noetherian domain of dimension one);*

(ii) *$R$ is a one-dimensional Krull domain;*

(iii) *$R_M$ is a $DVR$ for each $M \in \mathrm{Max}(R)$ and $R = \cap_{M \in \mathrm{Max}(R)} R_M$ with finite character;*

(iv) *$R$ is a Prüfer and a Mori domain;*

(v) *$R$ is a Prüfer and a Krull domain;*

(vi) *Each nonzero ideal of $R$ is invertible.*

**Proof.** It follows from Theorem 2.22, recalling that each ideal of a Prüfer domain is a $t$-ideal and that a Mori valuation domain is a $DVR$ (Proposition 2.15). $\square$

**Remark 2.25** (1) An integrally closed Mori domain need not be Krull, in fact it need not be completely integrally closed. For example we can take $R := \overline{\mathbb{Q}} + X\mathbb{C}[[X]]$, where $\overline{\mathbb{Q}}$ is the field of algebraic numbers [18, Theorems 1.2 and 4.11].

(2) The complete integral closure $\widetilde{R}$ of a Mori domain $R$ is not always a Krull domain. In fact there are examples of Mori domains $R$ such that $\widetilde{R}$ is not Mori or is not completely integrally closed [37]. On the other hand, if $R$ is seminormal or $(R : \widetilde{R}) \neq (0)$, then $\widetilde{R}$ is a Krull domain [4, Section 7].

(3) If $R$ is a completely integrally closed domain (i.e. each nonzero ideal is $v$-invertible), to check that $R$ is a Krull domain (i.e. each nonzero ideal is $t$-invertible) it is enough to check that each divisorial prime is $t$-finite. However in general a domain whose divisorial primes are all $t$-finite is not necessarily a Mori domain, for example consider $R := \mathbb{Z}+X\mathbb{Q}[[X]]$ [4, Section 2].

By Theorem 2.22 a domain $R$ is a Krull domain if and only if the semigroup $F_t(R)$ of its $t$-ideals is a group. We now show that this is equivalent to say that the semigroup $F_v(R)$ of divisorial ideals is a free abelian group, generated by the height-one prime ideals. Recall that, for a prime ideal $P$ and an integer $e \geq 1$, $P^{(e)} := P^e R_P \cap R$.

**Proposition 2.26** *Let $R$ be a Krull domain and $I \subseteq R$ a nonzero integral ideal. Then $I$ is divisorial if and only if $I = (P_1^{e_1} \dots P_n^{e_n})_v = P_1^{(e_1)} \cap \dots \cap P_n^{(e_n)}$, where $P_1, \dots, P_n \in X^1(R)$ and $e_1, \dots, e_n \geq 0$ are uniquely determined.*

**Proof.** Since $R$ is a Krull domain, the $t$-operation and the $v$-operation coincide, $X^1(R) = t\text{-Max}(R)$ and $R = \cap_{P \in X^1(R)} R_P$ with finite character (Theorem 2.22). Hence an integral divisorial ideal $I$ of $R$ is contained at most in finitely many height-one primes $P_1, \dots, P_n$. Since $R_{P_i}$ is a $DVR$, $IR_{P_i} = P_i^{e_i}$ for some $e_i \geq 0$, $i = 1, \dots, n$, and

$$I = \cap_{P \in X^1(R)} IR_P = P_1^{e_1} \cap \dots \cap P_n^{e_n} \cap R = P_1^{(e_1)} \cap \dots \cap P_n^{(e_n)}.$$

Conversely, by $t$-compatibility, $P^{(e)} := P^e R_P \cap R$ is a $t$-ideal, for each height-one prime $P$ and a finite intersection of divisorial ideals is a divisorial ideal.

Now let $J := P_1^{e_1} \dots P_n^{e_n}$. Since $J$ is strictly $v$-finite (being $t$-invertible), we can write $J_v = H_v$, with $H \subseteq J$ and $H$ finitely generated. By $t$-compatibility, applying Proposition 1.12 (c), we have $J_v R_P = H_v R_P \subseteq (HR_P)_{v_P} = HR_P \subseteq JR_P$, because $R_P$ is a $DVR$ for each $P \in X^1(R)$. Whence $J_v R_P = IR_P$ for each $P \in X^1(R) = t\text{-Max}(R)$ and it follows that $J_v = I$. Since $I$ is $t$-invertible, the height-one prime ideals $P_1, \dots, P_n$ and the exponents $e_1, \dots, e_n$ are uniquely determined. $\square$

In the rest of this section, for an ideal $I$ of $R$ and $n \geq 0$, we set $I^{-n} := (R : I^n)$.

**Theorem 2.27** *The following statements are equivalent for a domain $R$:*

(i) *$R$ is a Krull domain;*

(ii) *A nonzero ideal $I$ of $R$ is divisorial if and only if $I = (P_1^{t_1} \dots P_n^{t_n})_v$, where $P_1, \dots, P_n \in X^1(R)$ and $t_1, \dots, t_n \in \mathbb{Z}$ are uniquely determined.*

**Proof.** (i) $\Rightarrow$ (ii) By Proposition 2.26 each integral divisorial ideal of $R$ is of the form $(P_1^{e_1} \dots P_n^{e_n})_v$, where $P_1, \dots, P_n \in X^1(R)$ and $e_i \geq 1$, $i = 1, \dots, n$ are uniquely determined.

Let $J$ be any divisorial ideal and let $d \in R \setminus \{0\}$ be such that $I := dJ \subseteq R$. Write the integral principal ideal $dR$ as a $v$-product of height-one primes, say $dJ = (Q_1^{a_1} \dots Q_n^{a_n})_v$, $a_i \geq 1$. Since $Q_i$ is $v$-invertible and the $v$-inverse of $Q_i^{a_i}$ is $(R : Q_i^{a_i}) = ((R : Q_i)^{a_i})_v = Q_i^{-a_i}$, we have $d^{-1}R = (R : Q_1^{a_1} \dots Q_n^{a_n}) = (Q_1^{-a_1} \dots Q_n^{-a_n})_v$. Since $J = (d^{-1}R)dJ$ and the integral ideal $dJ$ is also a $v$-product of height-one primes, we get that $J$ is uniquely expressible as a $v$-product $J = (P_1^{t_1} \dots P_n^{t_n})_v$, where $P_1, \dots, P_n \in X^1(R)$ and $t_i \in \mathbb{Z}$, $i = 1, \dots, n$.

(ii) $\Rightarrow$ (i) For each $x \in K \setminus \{0\}$, we have $xR = (\prod_{P \in X^1(R)} P^{t_P(x)})_v$, where $t_P(x) \in \mathbb{Z}$ is uniquely determined and $t_P(x) = 0$ for almost all $P \in X^1(R)$. In addition $xR \subseteq R$ if and only if $t_P(x) \geq 0$ for each $P \in X^1(R)$. It is easy to check that the map

$$v_P : K \setminus \{0\} \longrightarrow \mathbb{Z}\,; \quad x \mapsto t_P(x)$$

is a discrete valuation for all $P \in X^1(R)$. Denoting by $V_P$ the valuation domain associated to $v_P$, we have $xR \subseteq R$ if and only if $x \in V_P$ for all $P \in X^1(R)$. Hence $R = \cap_{P \in X^1(R)} V_P$ with finite character and it follows that $R$ is a Krull domain. $\square$

**Proposition 2.28** *$R$ is a Dedekind domain if and only if each nonzero ideal $I$ of $R$ can be written in the form $I = P_1^{t_1} \dots P_n^{t_n}$, where $P_1, \dots, P_n$ are prime ideals and $t_1, \dots, t_n \in \mathbb{Z}$ are uniquely determined.*

**Remark 2.29** The abelian free group $\mathbb{Z}^{(X^1(R))} := \bigoplus_{P \in X^1(R)} P\mathbb{Z}$ is a lattice ordered group with respect to the product order. In addition, each nonempty set of positive elements has (at least) a minimal element and the set of the minimal positive elements is precisely $X^1(R)$.

On the other hand, also the set of divisorial ideals $F_v(R)$ can be endowed with a structure of lattice ordered semigroup setting $J_v \leq I_v$ if $J \subseteq I$ (with $\inf(I_v, J_v) = I_v \cap J_v$ and $\sup(I_v, J_v) = (I + J)_v$).

The previous theorem says that $R$ is a Krull domain if and only if $F_v(R)$ is a group and there is an isomorphim of lattice groups $F_v(R) \longrightarrow \mathbb{Z}^{(X^1(R))}$ which reverse the order, given by

$$F_v(R) \longrightarrow \mathbb{Z}^{(X^1(R))} := \bigoplus_{P \in X^1(R)} P\mathbb{Z}\,, \quad (P_1^{t_1} \dots P_n^{t_n})_v \mapsto t_1 P_1 + \dots + t_n P_n.$$

The additive group $\mathbb{Z}^{(X^1(R))}$ is called the *divisor group* of the Krull domain $R$.

## 3 Arithmetical properties

In this section we show how some arithmetical properties of a domain $R$, like the existence of a greatest common divisor or of a factorization into prime elements, are reflected by the properties of the semigroup of $t$-ideals of $R$, or more precisely by the properties of the Class Group of $R$.

The *$t$-Class Group* or simply the *Class Group* of a domain $R$ is the quotient group $C(R) := \mathrm{Inv}_t(R)/P(R)$ of the $t$-invertible $t$-ideals modulo the principal ideals. If $R$ is a Prüfer domain each ideal is a $t$-ideal and so the Class Group coincides with the Class Group of the invertible ideals, also called the *Picard Group* of $R$. If $R$ is a Krull domain, since each $t$-ideal is divisorial and $v$-invertible, the Class Group coincides with the classical *Divisor Class Group* [15]. A recent survey on the Class Group is [2].

We recall some basic definitions. Given two elements $x$, $y$ of a domain $R$, we say that $y$ *divides* $x$ in $R$ if there exists an element $z \in R$ such that $x = yz$; hence $y$ divides $x$ if and only if $xR \subseteq yR$. The zero element of $R$ does not divide any other element, but it is divided by any $y \in R$. Hence we will consider nonzero elements.

The invertible elements of $R$ (also called the *units* of $R$) are precisely the divisors of 1; they form a multiplicative group, that we denote by $\mathcal{U}(R)$. We say that $y$ is *associated* to $x$ in $R$ if there exists $u \in \mathcal{U}(R)$ such that $y = ux$. It is clear that this is an equivalence relation on $R$ and that $x$ and $y$ are associated if and only if $x$ divides $y$ and $y$ divides $x$.

Any element $x \in R$ is divided by all the units of $R$ and by all its associated elements: in fact $x = 1x = u(u^{-1}x)$, for each $u \in \mathcal{U}(R)$. We say that $d$ is a *proper divisor* of $x$ if $d$ divides $x$ and it is neither invertible nor associated to $x$. A nonzero noninvertible element is called *irreducible* if it has no proper divisors and it is called a *prime element* if it generates a prime ideal. Hence $p$ is a prime element if and only if, when $p$ divides a product $xy$, necessarily either $p$ divides $x$ or $p$ divides $y$. By using this, it is easy to check that any prime element is irreducible.

### 3.1 $GCD$-domains

Given two nonzero elements $x$, $y$ of a domain $R$, we say that $d$ is a *greatest common divisor*, for short a $GCD$, of $x$ and $y$ and write $d = (x, y)$ if

(a) $d$ divides $x$ and $y$; (b) if $d'$ divides $x$ and $y$ then $d'$ divides $d$.

Dually, we say that $m$ is a *lowest common multiple*, for short an $lcm$, of $x$ and $y$ and write $m = [x, y]$ if

(a) $x$ and $y$ divide $m$; (b) if $x$ and $y$ divide $m'$, then $m$ divides $m'$.

It is clear that if $d$ is a greatest common divisor of $x$ and $y$ (resp. $m$ is a lowest common multiple), then also $ud$ is a greatest common divisor (resp. $um$ is a lowest common multiple), for each unit $u \in \mathcal{U}(R)$. If $x$ and $y$ have no proper common divisors, we write $(x, y) = 1$ and say that $x$ and $y$ are *coprime*.

If any two nonzero elements of $R$ have a greatest common divisor (resp. a lowest common multiple), we say that $R$ is a *GCD-domain* (resp. an *lcm-domain*). We will see soon that these two properties are equivalent. Recent results on $GCD$-domains and their generalizations can be found in [1].

An important property of $GCD$-domains is that irreducible and prime elements coincide.

**Lemma 3.1 (Euclide)** *Let $R$ be a $GCD$-domain and let $x$, $y$, $z \in R$ be nonzero elements. If $x$ divides $yz$ and $(x, y) = 1$, then $x$ divides $z$.*

**Proof.** $(xz, yz)$ exists because $R$ is a $GCD$ domain and it is easy to see that $(xz, yz) = z(x, y)$. Then, if $(x, y) = 1$ and $x$ divides $yz$, it follows that $x$ divides $(xz, yz) = z(x, y) = z$. □

**Proposition 3.2** *If $R$ is a $GCD$-domain, then each irreducible element is prime.*

**Proof.** Let $p \in R$ be an irreducible element and assume that $p$ divides $xy$. If $p$ does not divide $x$, then $(p, x) = 1$ and so $p$ divides $y$ by Lemma 3.1. □

If $d = (x, y)$ is a $GCD$ of $x$ and $y$ and it is possible write $d = ax + by$, for some $a$, $b \in R$, we say that this expression is a *Bezout identity*.

**Proposition 3.3** *Given two nonzero elements $x$, $y$ of a domain $R$, the following conditions are equivalent:*

(i) *$xR + yR = dR$ is a principal ideal;*

(ii) *$(x, y) = d$ and $d = ax + by$, for some $a$, $b \in R$ (i.e. there exists a Bezout identity).*

**Proof.** Recall that $d$ divides $x$ and $y$ if and only if $xR + yR \subseteq dR$.

(i) $\Rightarrow$ (ii) If $xR + yR = dR$, $d$ divides $x$, $y$ and $d = ax + by$, for some $a$, $b \in R$. Hence each $d'$ dividing $x$ and $y$ divides $d$ and it follows that $(x, y) = d$.

(ii) $\Rightarrow$ (i) If $(x, y) = d$, $xR + yR \subseteq dR$ and if $d = ax + by$, $dR \subseteq xR + yR$. □

A domain satisfying the equivalent conditions of Proposition 3.3 is called a *Bezout domain* and if each ideal of $R$ is a principal ideal, $R$ is called a *principal ideal domain*, for short a $PID$.

**Proposition 3.4** *The following conditions are equivalent for a domain $R$:*

(i) *$R$ is a $PID$;*

(ii) *$R$ is a Noetherian Bezout domain.*

**Proof.** By induction, it follows directly from Proposition 3.3. $\square$

We now characterize $GCD$-domains by means of $t$-ideals.

**Proposition 3.5** *Given two nonzero elements $x$, $y$ of a domain $R$, the following conditions are equivalent:*

(i) *$x$ and $y$ have a lowest common multiple and $[x, y] = m$;*

(ii) *$xR \cap yR = mR$ is a principal ideal;*

(iii) *$(xR + yR)_v = dR$ is a principal ideal.*

*In addition, under (any of) these conditions, $(x, y) = d$ and $dm = xy$.*

**Proof.** (i) $\Leftrightarrow$ (ii) follows directly from the definition of $lcm$.

(ii) $\Leftrightarrow$ (iii) We have $(R : xR + yR) = x^{-1}R \cap y^{-1}R$ and so $xy(R : xR + yR) = xR \cap yR$. It follows that $(xR+yR)_v$ is principal if and only if $xR \cap yR$ is principal. In addition, $xR \cap yR = mR$ if and only if $(xR + yR)_v = dR$, with $d = xy/m$.

To finish, $d'$ divides $x$ and $y$ if and only if $xR + yR \subseteq (xR + yR)_v \subseteq d'R$. Thus, if $(xR + yR)_v = dR$, we have $dR \subseteq d'R$. It follows that $d'$ divides $d$ and so $(x, y) = d$. $\square$

**Remark 3.6** By Proposition 3.5, the existence of a lowest common multiple of two elements implies the existence of a greatest common divisor. But the converse is not true. Thus $(x, y) = d$ does not necessarily imply $(xR+yR)_v = dR$. For example, in $R := k[X^2, X^3]$, $k$ a field and $X$ an indeterminate over $k$, we have $(X^2, X^3) = 1$, but $X^2$, $X^3$ do not have a lowest common multiple. In fact, up to units, the $lcm$ of $X^2$ and $X^3$ in $k[X]$ is $X^5$, but $X^5$ does not divide any other common multiple of $X^2$ and $X^3$ in $R$ (for example does not divide $X^6$).

**Proposition 3.7** *The following conditions are equivalent for a domain $R$:*

(i) *$R$ is an lcm-domain;*

(ii) *$R$ is a $GCD$-domain;*

(iii) *$xR \cap yR$ is a principal ideal, for any $x$, $y \in R \setminus \{0\}$;*

(iv) *$(xR + yR)_v = dR$ is a principal ideal, for any $x$, $y \in R \setminus \{0\}$.*

**Proof.** (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) $\Rightarrow$ (ii) follow from Proposition 3.5.

(ii) $\Rightarrow$ (i) Let $x$, $y \in R \setminus \{0\}$ and $(x, y) = d$. Set $m := xy/d$. Clearly $x$ and $y$ divide $m$. We prove that $[x, y] = m$ by using Euclide's Lemma. We have $x = da$ and $y = db$, with $a$, $b \in R$ and $(a, b) = 1$. If $x$ and $y$ divide an element $m'$, then $m' = xa' = yb'$. Whence $daa' = dbb'$ and $aa' = bb'$. Since $a$ divides $bb'$ and $(a, b) = 1$, by Lemma 3.1 $a$ divides $b'$. Hence $m = xy/d = ya$ divides $m' = yb'$. $\square$

The following theorem for $PvMD$s is due to Bouvier and Zafrullah [8, Corollary 1.5].

**Theorem 3.8** *The following conditions are equivalent for a domain $R$:*

(i) *$R$ is a $GCD$-domain (resp. a Bezout domain);*

(ii) *$R$ is a $PvMD$ (resp. a Prüfer domain) with trivial Class Group.*

**Proof.** By induction, it follows from Proposition 3.7, recalling that $R$ is a $PvMD$ if and only if each finitely generated ideal is $t$-invertible (Theorem 2.11) and a Prüfer domain is a $PvMD$ such that $d = t$ (Theorem 2.12). $\square$

## 3.2 Unique factorization domains

A domain $R$ is called *atomic* if any nonzero noninvertible element $x \in R$ is a product of irreducible elements. This property is granted by the *ascending chain condition on integral principal ideals* (for short *accp*).

**Proposition 3.9** *If $R$ is a domain satisfying accp, then $R$ is atomic.*

**Proof.** Let $S$ be the set of principal proper ideals $xR$ of $R$ such that $x$ is not a product of irreducible elements. If $S$ is not empty, $S$ has a maximal element $mR$ (by the accp). Since $m$ is not irreducible, $m$ has proper divisors. Hence we can write $m = xy$ with $mR \subsetneq xR \subsetneq R$ and $mR \subsetneq yR \subsetneq R$. By the maximality of $mR$, $x$ and $y$ can be factorized into irreducible elements; but then also $m$ can be factorized. This contradiction shows that $S$ must be empty. $\square$

**Remark 3.10** (1) By the proposition above, any Mori (in particular Noetherian) domain is atomic.

(2) There exist atomic domains not satisfying accp [23].

We say that $R$ is a *unique factorization domain*, for short a $UFD$, if

(a) $R$ is atomic; (b) If $x = p_1 \dots p_n = q_1 \dots q_m$ are two factorizations of $x$ in irreducible elements, then $n = m$ and, after a reordering, $p_i$ is associated to $q_i$, for $i = 1, \dots, n$.

**Proposition 3.11** *A UFD is a GCD-domain.*

**Proof.** If either $x$ or $y$ is invertibile, we have $(x, y) = 1$. Otherwise we can write $x = p_1^{a_1} \dots p_n^{a_n}$, $y = p_1^{b_1} \dots p_n^{b_n}$, where $p_1, \dots, p_n$ are distinct prime elements of $R$ and $a_i$, $b_i \geq 0$, for $i = 1, \dots, n$. Then it is easy to check that $(x, y) = p_1^{m_1} \dots p_n^{m_n}$, where $m_i := \min\{a_i, b_i\}$, $i = 1, \dots, n$. □

**Proposition 3.12** *Let $R$ be an atomic domain. The following conditions are equivalent:*

(i) *$R$ is a UFD;*

(ii) *Each irreducible element of $R$ is prime;*

(iii) *$R$ is a GCD-domain.*

**Proof.** (i) $\Rightarrow$ (iii) by Proposition 3.11.

(iii) $\Rightarrow$ (ii) by Proposition 3.2.

(ii) $\Rightarrow$ (i) Let $x = p_1 \dots p_n = q_1 \dots q_m$ be two factorizations of $x$ into irreducible elements. Since $p_1$ is prime, it divides one of the $q_i$. By reordering, we can assume that $p_1$ divides $q_1$ and so, since also $q_1$ is prime, $p_1 = u_1 q_1$, with $u_1 \in \mathcal{U}(R)$. Canceling $p_1$, we get $p_2 \dots p_n = q_2 \dots q_m$. Repeating this process, we obtain that, after a reordering, $n = m$ and $p_i$ is associated to $q_i$, for $i = 1, \dots, n$. □

**Corollary 3.13** *A domain $R$ is a UFD if and only if each nonzero noninvertible element of $R$ is a product of finitely many prime elements.*

We end by the theorem of Samuel [39].

**Theorem 3.14** *The following conditions are equivalent for a domain $R$:*

(i) *$R$ is a UFD;*

(ii) *$R$ is Krull domain with trivial Class Group;*

(iii) *$R$ is Krull domain and each height-one prime ideal is principal.*

**Proof.** (i) $\Rightarrow$ (iii) Let $x \in R$ be a nonzero element and let $P$ be a minimal prime of $xR$. Then $P$ contains a prime $p$ dividing $x$ and $xR \subseteq pR \subseteq P$. It follows that $pR = P$. Since an height-one prime is minimal over a principal ideal, we get that each height-one prime is principal. Hence the localizations of $R$ at the height-one prime ideals are $DVR$s. In addition each nonzero $x \in R$ is contained at most in finitely many height-one primes because it has at most finitely many prime divisors. We conclude that $R$ is a Krull domain (Theorem 2.22).

(iii) $\Rightarrow$ (ii) By Theorem 2.27, each $t$-ideal of $R$ can be written in the form $I_t = (P_1^{e_1} \dots P_n^{e_n})_t$, where $P_1, \dots, P_n$ are height-one primes and $e_1, \dots, e_n \geq 0$ are uniquely determined. If $P_i = p_i R$, with $p_i$ a prime element, we get that $I_t = p_1^{e_1} \dots p_n^{e_n} R$ is principal.

(ii) $\Rightarrow$ (i) Each height-one prime ideal of a Krull domain is a $t$-invertible $t$-prime, hence it is principal. By Theorem 2.27, each nonzero principal ideal of $R$ can be written in the form $xR = (P_1^{e_1} \dots P_n^{e_n})_t$, where $P_1, \dots, P_n$ are height one primes and $e_1, \dots, e_n \geq 0$ are uniquely determined. If $P_i = p_i R$, with $p_i$ a prime element, we get that $xR = p_1^{e_1} \dots p_n^{e_n} R$ and $x = u p_1^{e_1} \dots p_n^{e_n}$, $u \in \mathcal{U}(R)$, is a factorization of $x$ into prime elements. $\square$

As a consequence, we get a classical characterization of principal ideal domains.

**Theorem 3.15** *The following conditions are equivalent for a domain $R$:*

(i) *$R$ is a $PID$;*

(ii) *$R$ is a one-dimensional $UFD$;*

(iii) *$R$ is a Dedekind $UFD$;*

(iv) *$R$ is a Dedekind domain with trivial Class Group.*

**Proof.** (i) $\Rightarrow$ (ii) $R$ is atomic by Proposition 3.9, because a $PID$ is Noetherian. In addition $R$ is a Bezout domain, hence a $GCD$-domain (Proposition 3.3). We conclude that $R$ is a $UFD$ by Proposition 3.12.

(ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) by Theorem 3.14, because a one-dimensional Krull domain is Dedekind (Theorem 2.24).

(iv) $\Rightarrow$ (i) because each ideal of a Dedekind domain is invertible (Theorem 2.24). $\square$

# 4 Extension theorems

In general, if $R \subseteq T$ is an extension of domains, there is no way of comparing the Class Groups of $R$ and $T$, unless the extension is $t$-compatible. We denote by $v'$ (resp. $t'$) the $v$-operation (resp. $t$-operation) on $T$.

**Proposition 4.1** *Let $R \subseteq T$ be a $t$-compatible extension of domains. If $I$ is a $v$-invertible (respectively $t$-invertible) ideal of $R$, $IT$ is a $v'$-invertible (respectively $t'$-invertible) ideal of $T$.*

**Proof.** If $I$ is $v$-invertible, the ideal $(I(R:I))_v = R$ is clearly $v$-finite. Hence, by Proposition 1.10,

$$T = (I(R:I))_v T \subseteq (I(R:I)T)_{v'} \subseteq (IT(T:IT))_{v'} \subseteq T$$

and so $(IT(T:IT))_{v'} = T$. The proof for the $t$-operation is the same. $\square$

**Proposition 4.2** *Let $R \subseteq T$ be a $t$-compatible extension of domains. Then the map*

$$\theta : \mathrm{Inv_t}(R) \longrightarrow \mathrm{Inv}_{t'}(T)\,; \quad I \mapsto (IT)_{t'}$$

*is a homomorphism of groups which induces a homomorphism of Class Groups*

$$\overline{\theta} : C(R) \longrightarrow C(T)\,; \quad \overline{I} \mapsto \overline{(IT)_{t'}}.$$

**Proof.** It follows directly from Proposition 4.1, noting that if $I$ is principal, also $\theta(I)$ is principal. □

**Remark 4.3** An extension $R \subseteq T$ of Krull domains is $t$-compatible if and only if $T$ satisfies the so called *Condition PDE* (for *pas d'eclatement*): if $Q$ is an height-one prime of $T$, either $Q \cap R = (0)$ or $Q \cap R$ is an height-one prime of $R$.

We are interested in the two classical cases where $T = R_{\mathcal{S}}$ is a generalized ring of fractions or $T = R[X]$ is a polynomial ring. Note that if $I \in \mathrm{Inv_t}(R)$, since $I$ and $(R : I)$ are $t$-finite, we have $(IR_{\mathcal{S}})_{t'} = I_{\mathcal{S}}$ (Proposition 1.12 (c)) and $(IR[X])_{t'} = I[X]$ (Proposition 1.11).

## 4.1 Generalized rings of fractions

We start by showing that a generalized ring of fractions of a $PvMD$ is still a $PvMD$. It is interesting to observe that, if $R$ is a $PvMD$, a generalized ring of fractions of $R$ is an intersection of localizations; the converse is always true (Remark 1.13 (2)).

**Proposition 4.4** *Let $R$ be a $PvMD$ (resp. a Krull domain) and let $\mathcal{S}$ be a multiplicative system of nonzero ideals of $R$. Then:*

(1) $R_{\mathcal{S}} = \cap\{R_{(N \cap R)}\,;\ N \in t'\text{-}\mathrm{Max}(R_{\mathcal{S}})\}$.

(2) *$R_{\mathcal{S}}$ is a $PvMD$ (resp. a Krull domain).*

**Proof.** Set $T := R_{\mathcal{S}}$.

(1) Let $N$ be a $t'$-maximal ideal of $T$. Then by $t$-compatibility $P := N \cap R$ is a $t$-prime ideal of $R$ (Proposition 1.10) and $R_P \subseteq T_N$. Since $R$ is a $PvMD$, $R_P$ is a valuation domain. Hence $T_N = R_P$ and finally $T := \cap\{T_N\,;\ N \in t'\text{-}\mathrm{Max}(R_{\mathcal{S}})\} = \cap\{R_{(N \cap R)}\,;\ N \in t'\text{-}\mathrm{Max}(T)\}$.

(2) As in (1), if $R$ is a $PvMD$, $T_N$ is a valuation domain, for each $N \in t'\text{-}\mathrm{Max}(T)$. Hence $T$ is a $PvMD$.

By Theorem 2.22, a Krull domain is precisely a Mori PvMD. If $R$ is Krull, $T$ is a $PvMD$ and is also a Mori domain by Proposition 2.16. Hence $T$ is a Krull domain. □

In general the canonical homomorphism $C(R) \longrightarrow C(R_{\mathcal{S}})$, $\overline{I} \mapsto \overline{I_{\mathcal{S}}}$, is neither injective nor surjective, even when $\mathcal{S}$ is particularly good [2, Section 6]. However we will show soon that it does be surjective when $R$ is a $PvMD$, in particular a Krull domain. Before, we consider the interesting case where each $t$-ideal in $\mathcal{S}$ is $t$-invertible. This condition is always satisfied when $R$ is a Krull domain.

**Proposition 4.5** *Let $\mathcal{S}$ be a saturated multiplicative system of nonzero ideals of $R$. Then the following conditions are equivalent:*

(i) $\mathcal{S} \cap F_t(R) \subseteq \mathrm{Inv_t}(R)$;

(ii) $\mathcal{S} \cap t\text{-Spec}(R) \subseteq \mathrm{Inv_t}(R)$;

(iii) $\mathcal{S} \cap F_t(R) = \{(M_1^{e_1} \dots M_n^{e_n})_t;\ M_i \in \mathcal{S}$ *is a $t$-invertible $t$-prime ideal*, $e_i \geq 0, i = 1, \dots, n\}$.

**Proof.** (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) are clear.

(ii) $\Rightarrow$ (iii) Since $\mathcal{S}$ is saturated, if $I \neq R$ is a $t$-ideal in $\mathcal{S}$, any $t$-prime $P$ containing $I$ is in $\mathcal{S}$; thus $P$ is $t$-invertible and hence $t$-maximal. In addition $P$ is minimal over $I$, because a minimal prime of a $t$-ideal is a $t$-prime.

We now show that $I$ is contained in finitely many $t$-primes. By saturation, given any chain of ideals $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_s \subseteq \cdots$ in $\mathcal{S}$, their union $I := \bigcup\{I_j; j \geq 0\}$ is an ideal in $\mathcal{S}$. Since $(I_j)_t$ and $I_t$ are $t$-invertible (being in $\mathcal{S}$), they are $t$-finite. Hence any chain of $t$-ideals in $\mathcal{S}$ stabilizes and any nonempty subset of $t$-ideals of $\mathcal{S}$ has a maximal element. Assume that the set of proper $t$-ideals of $\mathcal{S}$ contained in infinitely many $t$-primes is not empty and let $J$ be a maximal element of this set. Let $P$ be a $t$-prime containing $J$. Since $P \neq J$ and $P$ is $t$-invertible, we have that $J \subsetneq ((R\colon P)J)_t \subsetneq R$. The ideal $J' := ((R\colon P)J)_t$ is in $\mathcal{S}$ and, by the maximality of $J$, is contained in finitely many $t$-primes. Hence the ideal $J = (PJ')_t$ is also contained in finitely many $t$-primes, which is a contradiction.

Let $M$ be a $t$-maximal ideal containing $I$. Since $M$ is minimal over $I$, $IR_M$ is $MR_M$-primary and since $M$ is $t$-invertible $MR_M = xR_M$ is principal. Hence $IR_M = x^e R_M = M^e R_M$ for some $e \geq 1$ [20, Theorem 7.6].

To finish, if $M_1, \cdots, M_n$ are the $t$-maximal ideals containing $I$, then

$$I = \bigcap\{IR_M; M \in t\text{-Max}(R))\} = \bigcap\{IR_{M_i} \cap R; I \subseteq M_i\} = M_1^{(e_1)} \cap \cdots \cap M_n^{(e_n)}.$$

Like in the proof of Proposition 2.26, we conclude that

$$I = M_1^{(e_1)} \cap \cdots \cap M_n^{(e_n)} = (M_1^{(e_1)} \cdots M_n^{(e_n)})_t$$

and that $\mathcal{S} \cap F_t(R) \subseteq \{(M_1^{e_1} \dots M_n^{e_n})_t; M_i \in \mathcal{S}$ $t$-invertible $t$-maximal ideals, $e_i \geq 0\}$.

The other inclusion is clear. $\square$

**Theorem 4.6** *Let $R$ be a PvMD and let $\mathcal{S}$ be a saturated multiplicative system of nonzero ideals of $R$. Then:*

(1) *The canonical homomorphism $\theta : \mathrm{Inv_t}(R) \longrightarrow \mathrm{Inv}_{t'}(R_{\mathcal{S}})$, $I \mapsto I_{\mathcal{S}}$, is surjective;*

(2) $\mathrm{Ker}\,\theta = \mathrm{Inv_t}(R) \cap \mathcal{S}$*;*

(3) *If $R$ is a Krull domain,* $\mathrm{Ker}\,\theta$ *is the free abelian group generated by the height-one primes in $\mathcal{S}$.*

**Proof.** (1) Let $I' := x_1R_{\mathcal{S}} + \cdots + x_nR_{\mathcal{S}}$, $x_i \in K$, be a $t'$-invertible ideal of $R_{\mathcal{S}}$. Setting $I := x_1R + \cdots + x_nR$, we have that $I$ is $t$-invertible and $I' = IR_{\mathcal{S}}$. Hence $(I')_{t'} = (IR_{\mathcal{S}})_{t'} = (I_t)_{\mathcal{S}} = \theta(I_t)$.

(2) holds because $I_{\mathcal{S}} = R_{\mathcal{S}}$ if and only if $I \in \mathcal{S}$ (Proposition 1.12 (a)).

(3) holds by (2), Proposition 4.5 ((i) $\Rightarrow$ (iii)) and Theorem 2.27. □

**Remark 4.7** If each $t$-prime in the saturation $\overline{\mathcal{S}}$ of $\mathcal{S}$ is principal, by Proposition 4.5 ((ii) $\Rightarrow$ (iii)) each $t$-ideal $I \in \overline{\mathcal{S}}$ is a product of principal prime ideals, in particular it is principal. In this case, since $(R : I) = (R : I_t)$, $R_{\mathcal{S}}$ is a ring of fractions of $R$ with respect to a multiplicative part $S \subseteq R$ generated by prime elements.

Conversely, let $S \subseteq R$ be a multiplicative part generated by a set of prime elements $\{p_\alpha\}$ and $\mathcal{S} = \{xR\,;\ x \in S\}$ be the correspondent system of principal ideals. Then the $t$-primes in $\overline{\mathcal{S}}$ are precisely the principal primes $\{p_\alpha R\}$. In fact, if $x \in S$ and $P$ is a $t$-prime containing $x$, $P$ contains a prime factor $p$ of $x$ and, since $pR$ is $t$-maximal, we have that $P = pR \in \mathcal{S}$. From Proposition 4.5 ((ii) $\Rightarrow$ (iii)), it follows that each $t$-ideal in $\overline{\mathcal{S}}$ is principal, generated by a product $p_{\alpha_1} \dots p_{\alpha_n}$, and so $\overline{\mathcal{S}} \cap F_t(R) = \mathcal{S}$.

**Theorem 4.8** *Let $R$ be a PvMD and let $S \subseteq R$ be a multiplicative part generated by prime elements. Then the canonical homomorphism $\overline{\theta} : C(R) \longrightarrow C(R_S)$, $\overline{I} \mapsto \overline{I_S}$, is bijective.*

**Proof.** Setting $\mathcal{S} = \{xR\,;\ x \in S\}$, we have $R_S = R_{\mathcal{S}}$. By Theorem 4.6, the homomorphism $\theta : \mathrm{Inv_t}(R) \longrightarrow \mathrm{Inv}_{t'}(R_{\mathcal{S}})$ is surjective and $\mathrm{Ker}\,\theta = \mathrm{Inv_t}(R) \cap \overline{\mathcal{S}}$. But, as in Remark 4.7, $\mathrm{Ker}\,\theta = \mathcal{S} \subseteq P(R)$. Hence $\overline{\theta}$ is bijective. □

**Theorem 4.9** *Assume that $R$ is a PvMD (resp. a Krull domain).*

(1) *For any multiplicative system of ideals $\mathcal{S}$ of $R$, if $R$ is a GCD-domain (resp. a UFD) then $R_{\mathcal{S}}$ is a GCD-domain (resp. a UFD).*

(2) [Nagata's Theorem] *If $S \subseteq R$ is a multiplicative part generated by prime elements, $R$ is a GCD-domain (resp. a UFD) if and only if $R_S$ is a GCD-domain (resp. a UFD).*

**Proof.** It follows from Theorems 4.4 and 4.8, recalling that a $GCD$-domain (resp. a $UFD$) is a $PvMD$ (resp. a Krull domain) with trivial class group (Theorem 3.8). $\square$

### 4.2 Polynomial extensions

We recall that $I[X] \cap R = I$, for any integral ideal $I$ of a domain $R$. We will use repeatedly the following properties.

**Lemma 4.10** *Let $R$ be a domain. Then:*

(1) $(J : I)[X] = (J[X] : I[X])$*, for any nonzero ideals $I$, $J$ of $R$. In particular* $I_v[X] = (I[X])_{v'}$*.*

(2) *There is an inclusion preserving injective correspondence $I \mapsto I[X]$ between the set of integral $t$-ideals of $R$ and the set of integral $t'$-ideals of $R[X]$, whose left inverse is the intersection. In addition, $I$ is $t$-invertible if and only if $I[X]$ is $t'$-invertible.*

(3) *Let $N$ be a $t'$-maximal ideal of $R[X]$. Then either $N = M[X]$, with $M := N \cap R \in t\text{-Max}(R)$ or $N = fK[X] \cap R[X]$ for some irreducible polynomial $f \in K[X]$. In the second case, $R[X]_N = K[X]_{fK[X]}$ is a $DVR$.*

**Proof.** (1) It is enough to consider integral ideals. First note that if $u \in K(X)$ is such that $uI \subseteq J[X] \subseteq R[X]$, then $u \in K[X]$. Let $c_u$ be the content of $u$. Then $u \in (J[X] : I[X])$ if and only if $c_u \in (J : I)$, if and only if $u \in (J : I)[X]$.

(2) If $I$ is a $t$-ideal, $I[X]$ is a $t'$-ideal by part (1). In fact, let $J' \subseteq I[X]$ be a finitely generated ideal and $J := J' \cap R$. Then $J'_{v'} = J[X]_{v'} = J_v[X] \subseteq I[X]$. In addition $\theta$ is injective because $I = I[X] \cap R$. If $I$ is $t$-invertible, $I[X]$ is $t'$-invertible by Proposition 4.1. Conversely, if $R[X] = (R[X] : (R[X] : I[X]))_{t'} = (R : (R : I))_t[X]$, then $R = (R : (R : I))_t$ by the injectivity.

(3) If $N \cap R := M \neq (0)$, $N$ is $t'$-maximal if and only if $M$ is $t$-maximal because the map $I \mapsto I[X]$ is inclusion preserving. If $N \cap R = (0)$, since $K[X] = R[X]_S$ is a ring of fractions of $R[X]$, with $S := R \setminus \{0\}$, $NK[X] = N_S$ is a prime ideal of $K[X]$. Hence $NK[X] = N_S = fK[X]$, with $f \in K[X]$ irreducible and $N = N_S \cap R[X] = fK[X] \cap R[X]$. Finally, $R[X]_N = (R[X]_S)_{N_S} = K[X]_{fK[X]}$ is $DVR$, because $K[X]$ is a Dedekind domain (being a $PID$). $\square$

**Proposition 4.11** *The following conditions are equivalent for a domain $R$.*

(i) *$R$ is a $PvMD$ (resp. a Krull domain);*

(ii) *The polynomial ring $R[X]$ is a $PvMD$ (resp. a Krull domain).*

**Proof.** (i) $\Rightarrow$ (ii) Assume that $R$ is a $PvMD$ and let $N \in t'$-Max$(R[X])$. If $N = M[X]$, with $M \in t$-Max$(R)$, let $v_M$ be the valuation on $K$ associated to the valuation domain $R_M$. Then $v_M$ can be extended to $K(X)$ by setting $v'_M(\sum a_i X^i) = \inf\{v_M(a_i)\}$, for each polynomial $\sum a_i X^i \in K[X]$. It is easy to verify that the valuation domain associated to $v'_M$ is $R[X]_N$. If $N \cap R = (0)$, $R[X]_N$ is a $DVR$. It follows that $R[X]_N$ is a valuation domain for each $N \in t$-Max$(R[X])$ and so $R[X]$ is a $PvMD$.

If $R$ is a Krull domain, $R_M$ is a $DVR$, for each $M \in t$-Max$(R)$, and so also $R[X]_{M[X]}$ is a $DVR$. Since the intersections $R = \cap\{R_M\,;\ M \in t\text{-Max}(R)\}$ and $K[X] = \cap\{K[X]_{fK[X]}\,;\ f \in K[X] \text{ irreducible}\}$ have finite character, also the intersection $R[X] = \cap\{R[X]_N\,;\ N \in t\text{-Max}(R[X])\}$ has finite character. We conclude that $R[X]$ is a Krull domain.

(ii) $\Rightarrow$ (i) If $M$ is a $t$-maximal ideal of $R$, $M[X]$ is a $t'$-maximal ideal of $R[X]$. If $R[X]$ is a $PvMD$, then $R[X]_{M[X]}$ is a valuation domain and so $R_M = R[X]_{M[X]} \cap K$ is a valuation domain too. It follows that $R$ is a $PvMD$.

In addition, if $R[X]$ is a Mori domain, $R = R[X] \cap K$ is a Mori domain by Proposition 2.17. Hence, if $R[X]$ is a Krull domain, also $R$ is a Krull domain. $\square$

**Remark 4.12** If $R$ is a Mori domain, $R[X]$ need not be Mori [38]. However $R[X]$ is Mori when $R$ is Mori integrally closed [4, Section 6].

**Theorem 4.13** *Let $\overline{\theta} : C(R) \longrightarrow C(R[X])$, $\overline{I} \mapsto \overline{I[X]}$ be the canonical homomorphism of Class Groups. Then*

(1) *$\overline{\theta}$ is injective.*

(2) *If $R$ is integrally closed, $\overline{\theta}$ is bijective.*

**Proof.** It is enough to consider integral ideals.

(1) If $I \subseteq R$ is an integral divisorial ideal such that $I[X] = fR[X]$ is principal, then $I = I[X] \cap R = fR[X] \cap R = f(0)R$ is principal.

(2) Let $I' \subseteq R[X]$ be a $t'$-invertible $t'$-ideal. If $I' \cap R = I \neq (0)$, then $I$ is a $t$-invertible $t$-ideal and $I' = I[X]$. If $I' \cap R = (0)$, then $I'K[X]$ is a proper ideal of $K[X]$. Hence $I'K[X] = fK[X]$ is principal, because $K[X]$ is a $PID$. Since $R$ is integrally closed, choosing $f \in R[X]$, we have $fK[X] \cap R[X] = f(R : c_f)[X]$, where $c_f$ denotes the content of $f$ [20, Corollary 34.9]. Hence $I' \subseteq fK[X] \cap R[X] = f(R : c_f)[X]$. If $d \in R$ is such that $d(R : c_f) \subseteq R$, we have $H' := df^{-1}I' \subseteq d(R : c_f)[X] \subseteq R[X]$ and $H'K[X] = dK[X] = K[X]$. Hence $H' \cap R := H \neq (0)$ and $H' = HR[X]$. Finally $I' = d^{-1}fH' = d^{-1}fH[X]$ and $\overline{I'} = \overline{\theta}(\overline{H})$. $\square$

**Corollary 4.14 (Gauss' Lemma)** *$R$ is a $GCD$-domain (resp. a $UFD$) if and only if $R[X]$ is a $GCD$-domain (resp. a $UFD$).*

**Proof.** It follows from Proposition 4.11 and Theorems 3.8, 3.14 and 4.13. $\square$

**Remark 4.15** (1) The converse of Theorem 4.13 (2) is also true, that is, if $\overline{\theta} : C(R) \longrightarrow C(R[X])$ is surjective then $R$ is integrally closed [16, Theorem 3.6].

(2) By induction on $n$, we get that if $R$ is a $GCD$-domain (resp. a $UFD$) the polynomial ring $R[X_1, \ldots, X_n]$ is a $GCD$-domain (resp. a $UFD$).

## References


[1] D. D. Anderson, *GCD domains, Gauss' lemma, and contents of polynomials*, Non-Noetherian Commutative Ring Theory, 1-31, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[2] D. F. Anderson, *The Class Group and local Class Group of an integral domain*, Non-Noetherian Commutative Ring Theory, 33-55, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[3] M. F. Atiyah and I. G. Macdonald, Introduction to Commutative Algebra, Addison-Wesley,1969.

[4] V. Barucci, *Mori domains*, Non-Noetherian Commutative Ring Theory, 57-74, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[5] V. Barucci and S. Gabelli, *How far is a Mori domain from being a Krull domain?*, J. Pure Appl. Algebra **45** (1987), 101-112.

[6] V. Barucci and S. Gabelli, *On the Class Group of a Mori domain*, J. Algebra **108** (1987), 161-173.

[7] N. Bourbaki, Algèbre Commutative, Chap.1–7, Hermann, 1964.

[8] A. Bouvier and M. Zafrullah, *On some Class Group of an integral domain* Bull. Soc. Math. Grece **29** (1988), 45-59.

[9] P.-J. Cahen, S. Gabelli, and E. Houston, *Mori domains of integer-valued polynomials*, J. Pure Appl. Algebra **153** (2000), 1-15.

[10] J. Dieudonné, *Sur la théorie de la divisibilité*, Bull. Soc. Math. France **69** (1941), 133-143.

[11] J. Elliot, *Functorial properties of star operations*, manuscript.

[12] M. Fontana and S. Gabelli, *On the class group and the local class group of a pull-back*, J.Algebra, **181** (1996), 803-835.

[13] M. Fontana, J. Huckaba, and I. Papick, Prüfer domains, Monographs and Textbooks in Pure and Applied Mathematics, 203, M. Dekker, New York, 1997.

[14] M. Fontana and M. Zafrullah, *$v$-domains: a survey*, Commutative Algebra; Noetherian and non-Noetherian Perspectives, 145-180, Springer, 2011.

[15] R. M. Fossum, The divisor class group of a Krull domain, Springer-Verlag, 1973.

[16] S. Gabelli, *On divisorial ideals in polynomial rings over Mori domains*, Comm. Algebra, **15** (1987), 2349-2370.

[17] S. Gabelli, *Completely integrally closed domains and $t$-ideals*, Bollettino UMI **3**-B (1989), 327–342.

[18] S. Gabelli and E.G. Houston, *Ideal theory in pullbacks*, Non-Noetherian Commutative Ring Theory, 199-227, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

[19] S. Gabelli and M. Roitman, *Maximal divisorial ideals and $t$-maximal ideals*, J. Algebra Number Theory Appl., **2** (2004), 323-336,

[20] R. Gilmer, Multiplicative ideal theory, Dekker, New York, 1972.

[21] R. G. Gilmer and W. Heinzer, *On the complete integral closure of an integral domain*, J. Austral. Math. Soc. **6** (1966), 351–361.

[22] S. Glaz and W. Vasconcelos, *Flat ideals, II*, Manuscripta Math. **22** (1977), 325-341.

[23] A. Grams, *Atomic rings and the ascending chain condition for principal ideals*, Proc. Cambridge Philos. Soc. **75** (1974), 321-329.

[24] M. Griffin, *Some results on $v$-multiplication rings*, Canad. J. Math. **19** (1967), 710-722.

[25] F. Halter-Koch, Ideal systems. An introduction to multiplicative ideal theory, Monographs and Textbooks in Pure and Applied Mathematics, 211, M. Dekker, New York, 1998.

[26] J. R. Hedstrom and E. G. Houston, *Some remarks on star-operations*, J. Pure Appl. Algebra **18** (1980), 37-44.

[27] W. Heinzer, *Some remarks on complete integral closure*, J. Austral. Math. Soc. **9** (1969), 310–314.

[28] W. Heinzer and J. Ohm, *Locally Noetherian Commutative Rings*, Trans. Amer. Math. Society **158** (1971), 273-284.

[29] M. Henriksen, *On the ideal structure of the ring of the entire functions*, Pacific J. Math. **2** (1952), 179–184.

[30] P. Hill, *On the complete integral closure of a domain*, Proc. Amer. Math. Soc. **36** (1) (1972), 26–30.

[31] E. G. Houston and M. Zafrullah, *$UMV$-domains*, Arithmetical properties of commutative rings and monoids, 304–315, Lect. Notes Pure Appl. Math., 241, Chapman & Hall/CRC, Boca Raton, FL, 2005.

[32] P. Jaffard, Les Systèmes d'Idéaux, Dunod, Paris, 1970.

[33] B.G. Kang, *Prüfer $v$-multiplication domains and the ring $R[X]_{N_v}$*, J. Algebra **123** (1989), 151-170.

[34] D. Lantz, *Finite Krull dimension, complete integral closure and GCD-domains*, Comm. Algebra **3** (1975), 951–958.

[35] T. Nakayama, *On Krull's conjecture concerning completely integrally closed integrity domains I*, Proc. Imp. Acad. Tokyo **18** (1942), 185–187; *II*, Proc. Imp. Acad. Tokyo **18** (1942), 233–236; *III*, Proc. Imp. Acad. Tokyo **18** (1946), 249–250.

[36] M. H. Park, *Group rings and semigroup rings over strong Mori domains*, J. Pure Applied Algebra **163** (2001), 301–318.

[37] M. Roitman, *On the complete integral closure of a Mori domain*, J. Pure Appl. Algebra **66** (1990), 55-79.

[38] M. Roitman, *On polynomial extensions of Mori domains over countable fields*, J. Pure Appl. Algebra **64** (1990), 315-328.

[39] P. Samuel, *Sur les anneaux factoriels*, Bull. Soc. Math. France **89** (1961), 155-173.

[40] P. B. Sheldon, *Two counterexamples involving complete integral closure in finite-dimensional Prüfer domains*, J. Algebra **27** (1973), 462–474.

[41] Wang Fanggui and R.L. McCasland, *On $w$-modules over Strong Mori domains*, Comm. Algebra **25** (1997), 1285-1306.

[42] Wang Fanggui and R.L. McCasland, *On Strong Mori domains*, J. Pure Appl. Algebra **135** (1999), 155-165.

[43] M. Zafrullah, *Putting $t$-invertibility to use*, Non-Noetherian Commutative Ring Theory, 429-458, Math. Appl., 520, Kluwer Acad. Publ., Dordrecht, 2000.

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