Ten problems on stability of domains

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Abstract We survey the notions of (finite) stability, quasi-stability and Clifford regularity of domains and illustrate some open problems.

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Introduction

Stability of ideals was explicitly introduced by J. Lipman in 1971, in order to study Arf rings [30]. Even though this notion was already known and widely used in the context of one-dimensional Noetherian rings, in particular in relation with reflexive rings and decomposition of torsion free modules [5, 31].

A stable ideal of a Noetherian ring is defined as an ideal that is projective over its ring of endomorphisms [51, 52]; extending this definition to arbitrary integral domains, one says that a nonzero ideal I of a domain R is stable if I is invertible in the overring $E(I) := (I : I)$ of R [1]. If each nonzero ideal (respectively, finitely generated ideal) of R is stable, one says that R itself is stable (respectively, finitely stable).

Since 1998, stability of domains has been thoroughly investigated by B. Olberding. In [39] he illustrated several ideal-theoretic and module-theoretic applications of this concept and announced some new results, then published in [40, 41, 42].

Invertible ideals are clearly stable, thus stability finds interesting applications in the setting of Prüfer domains (i. e., domains in which nonzero finitely generated ideals are invertible). Stable Noetherian domains are one-dimensional [52]. However,

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as showed by Olberding, a stable domain need not be coherent, nor one-dimensional, nor integrally closed [41, Section 3].

Weakening the notion of stability, in more recent years, there were introduced other classes of domains, like Rutliff-Rush domains [33], quasi-stable domains [46] and Clifford regular domains [6]. All these notions coincide with stability in the Noetherian case, but not in general.

In this short survey, leaving aside the module-theoretic point of view, we focus on some ideal-theoretic aspects of stability and discuss some unresolved problems.

All the rings considered are commutative rings with unity that are not fields. A *local ring* is a ring with a unique maximal ideal and a *semilocal ring* is a ring with finitely many maximal ideals, not necessarily Noetherian.

If R is a ring with total quotient ring K, an *overring* of R is a ring between R and K. If I, J are R-submodules of K, we set $(I:J) := \{x \in K\,;\ xJ \subseteq I\}$ and $(I:_R J) := \{x \in R\,;\ xJ \subseteq I\}$.

1 Stable Notherian rings

Stable ideals were introduced in 1971 by J. Lipman, in his paper [30] on *Arf rings*, which are local Noetherian rings satisfying certain conditions studied by Arf in [3].

Lipman worked in the setting of semilocal one-dimensional Macaulay rings, that is, semilocal one-dimensional Noetherian rings whose Jacobson radical contains a regular element. If R is such a ring, Lipman defined a regular ideal $I \subseteq R$ to be *stable* if $IR^I = I$, or, equivalently, $R^I = (I:I)$ [30, Definition 1.3], where $R^I := \bigcup_{n\geq 1}(I^n : I^n)$ is the ring obtained by *blowing up* I.

The main motivation for introducing this notion is that it furnishes a useful characterization of Arf rings.

Recall that if I is an ideal of the ring R, an element $x \in R$ is said to be *integral over* I if there exist a positive integer n and elements $a_k \in I^k$, $k = 1, \dots, n$, such that

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0.$$

The ideal I is called *integrally closed* if all the elements of R which are integral over I belong to I.

Theorem 1.1. [30, Theorem 2.2] *A local one-dimensional Macaulay ring is an Arf ring if and only if each integrally closed regular ideal of R is stable.*

To the extent of proving this result, Lipman gave several characterizations of stable ideals. In particular, he proved the following:

Proposition 1.2. [30, Lemma 1.11] *Let R be a semilocal one-dimensional Macaulay ring and $I \subseteq R$ a regular ideal. The following conditions are equivalent:*

(i) I is stable (i.e., $IR^I = I$);

(ii) *There exists an element* $x \in I$ *such that* $I^2 = xI$;
(iii) *There exists a regular element* $x \in I$ *such that* $I = xR^I$;
(iv) *There exists a regular element* $x \in I$ *such that* $I = x(I : I)$.

In order to solve a problem posed by Bass, the notion of stability was then extended to Noetherian rings by J. Sally and W. Vasconcelos, in a paper appeared in 1973. They called an ideal I of a Noetherian ring R *stable* if I is projective over its endomorphism ring $End_R(I)$ and called R stable if each ideal is stable [51, Section 1], [52, Section 2].

Note that when I is a regular ideal of the ring R, $End_R(I)$ is isomorphic to the overring $E(I) := (I : I)$ of R and so I is stable if and only if it is an invertible ideal of $E(I)$. In particular, when R is a semilocal Noetherian ring, I is stable if and only if I is principal in $E(I)$. It follows from Proposition 1.2 that this more general notion of stability coincides with the one introduced by Lipman for regular ideals of semilocal one-dimensional Macaulay ring [52, Proposition 2.2].

Nevertheless, a local one-dimensional Noetherian ring which is stable according to Sally-Vaconcelos need not be a Macaulay ring. For example, one can take $R := k[[X,Y]]/\langle X^2, XY\rangle$, where k is a field and X, Y are indeterminates over k [52, page 324].

Proposition 1.3. [52, Proposition 2.1] *Let R be a Noetherian ring. If R is stable (i.e., each ideal is projective over its endomorphism ring), R has dimension at most one.*

Stability is related to the 2-generator property. An ideal of R is *2-generated* if it is generated by 2 elements and R is 2-generated, or it has the 2-*generator property*, if each *finitely generated* ideal is 2-generated. The 2-generator property plays an important part in the decomposition of torsion-free modules [32].

Bass proved that if R is a one-dimensional reduced Noetherian ring whose integral closure is a finitely generated R-module, the 2-generator property implies stability [5, Proposition 7.1 and Corollary 7.3]. Sally and Vasconcelos showed that, as conjectured by Bass, also the converse holds.

Theorem 1.4. *Let R be a Noetherian ring.*

(1) [52, Theorem 3.4] *Assume that R is a one-dimensional Macaulay ring whose maximal ideals are not minimal primes. If each regular ideal is 2-generated, then R is stable.*
(2) [51, Theorem 2.4] *Assume that R is one-dimensional reduced and that its integral closure is a finitely generated R-module. If R is stable, then R is 2-generated.*

However, even for Noetherian domains, the 2-generator property is strictly stronger than stability. The first example of a local Noetherian domain that is stable and not 2-generated was given in [52, Example 5.4]; several other examples are collected in [44, Section 3].

The relationships among the 2-generator property, stability of finitely generated regular ideals and decomposition of finitely generated torsion free modules were further investigated by D. Rush in two papers of 1991 and 1995 [49, 50]. In particular he extended Bass' result to local rings.

Theorem 1.5. [50, Proposition 2.5] *Let R be a local ring. If R is 2-generated, each finitely generated regular ideal is stable.*

Rush gave also the following characterization of stability for Noetherian rings.

Theorem 1.6. [49, Theorem 2.4] *Let R be a one-dimensional Noetherian ring with integral closure R'. Then each regular ideal of R is stable if and only if the following conditions hold:*

(a) Each (or each finitely generated) R-submodule of R' containing R is a ring;
(b) Each maximal ideal of R has at most two maximal ideals of R' lying over it.

Together with other results of Rush, the last two theorems were later extended to general integral domains by B. Olberding in [40]. For an extension to rings, see Olberding's paper in this volume.

A notion weaker than stability, still useful to bound the number of generators of ideals, was introduced by P. Eakin and P. Sathaye in 1976. They observed that part of the Lipman's result given in Proposition 1.2 can be extended in the following way:

Proposition 1.7. [15, Lemma, page 447] *Let R be a local ring and I a finitely generated regular ideal of R. The following conditions are equivalent:*

(i) There exists an element $x \in I$ such that $I^2 = xI$;
(ii) There exists a regular element $x \in I$ such that $I = x(I : I)$.

Thus Eakin and Sathaye defined an ideal I of a *semilocal* ring to be *stable* if there is an element $x \in I$ such that $I^2 = xI$ and say that I is *prestable* if some power of I is stable, that is, for some $k \geq 1$ there is an $x \in I$ such that $I^{2k} = xI^k$ [15, Section 3].

Proposition 1.8. [15, Corollary 1, page 446] *Let R be a local ring and I a finitely generated ideal. The following conditions are equivalent:*

(i) I is prestable (i.e., $I^{2k} = xI^k$, for some $k \geq 1$);
(ii) There is a positive integer $b := b(I)$ such that I^n has b generators, for each $n \geq 1$;
(iii) There is a positive integer n such that I^n has n generators, for some $n \geq 1$.

Moreover, if I is regular and I^n has n generators, then $I^{2(n-1)} = xI^{n-1}$, for some $x \in I$.

2 Stable domains

In a note of 1987, D.D. Anderson, J. Huckaba and I. Papick considered the notion of stability for arbitrary integral domains [1]. Given a nonzero ideal I of a domain R, they say that I is *Lipman-stable* (for short, *L-stable*), if $R^I = (I : I)$ and say that I is stable according to Sally-Vasconcelos, or is *SV-stable*, if I is invertible in the overring $E(I) := (I : I)$. The domain R is called *L-stable* (respectively, *SV-stable*) if each nonzero ideal of R is L-stable (respectively, SV-stable).

As in [14, Section 7.4], one can also say that I is stable according to Eakin-Sathaye, or that I is *ES-stable*, if $I^2 = JI$ for some invertible ideal J contained in I. The ideal I is called *ES-prestable* (respectively, *SV-prestable*) if some power of I is *ES*-stable (respectively, SV-stable).

Proposition 2.1. [1, Lemmas 2.1 and 2.2] *Let R be a domain and I a nonzero ideal. Then:*

$$I \text{ ES-stable } \Rightarrow I \text{ SV-stable} \Rightarrow I \text{ L-stable} .$$

If I is finitely generated, often all these notions coincide.

Proposition 2.2. [14, Corollary 7.4.2 and Proposition 7.4.3] *Let R be a domain and I a nonzero finitely generated ideal. The following conditions are equivalent:*

(i) I is SV-stable;
(ii) IR_M is SV-stable, for each maximal ideal $M \subseteq R$;
(iii) IR_M is ES-stable, for each maximal ideal $M \subseteq R$.

In particular, if R is local, I is SV-stable if and only if I is ES-stable

A domain R is integrally closed if and only if $R = (I : I)$ for each nonzero finitely generated ideal I. If $R = (I : I)$ for each nonzero ideal I, R is called *completely integrally closed*. Hence, if R is completely integrally closed, $R = (I : I) = R^I$, for each nonzero ideal; similarly, if R is integrally closed, $R = (I : I) = R^I$, for each nonzero finitely generated ideal. This shows that for completely integrally closed domains L-stability is a trivial concept.

Proposition 2.3. *(1) Let R be a completely integrally closed domain and I a nonzero ideal of R. Then I is L-stable; in addition,*

$$I \text{ is invertible } \Leftrightarrow I \text{ is SV-stable}.$$

(2) [14, Proposition 7.4.4] *Let R be an integrally closed domain and I a nonzero finitely generated ideal of R. Then I is L-stable; in addition,*

$$I \text{ is invertible } \Leftrightarrow I \text{ is ES-(pre)stable } \Leftrightarrow I \text{ is SV-(pre)stable}.$$

SV-stability implies L-stability (Proposition 2.1). The converse is not true, even in the Noetherian case. In fact, by the proposition above, any Noetherian integrally closed domain is L-stable, but is SV-stable only if it is Dedekind. More generally, we have:

Proposition 2.4. [1, Proposition 2.4] *A Noetherian domain is SV-stable if and only if it is L-stable and one-dimensional.*

Proposition 2.3(2) furnishes also a characterization of Prüfer domains in terms of SV-stability. Recall that a domain R is called a *Prüfer domain* if R_P is a valuation domain, for each nonzero prime ideal P; this is equivalent to say that each nonzero finitely generated ideal is invertible. A Prüfer domain such that PR_P is a principal ideal, for each nonzero prime ideal P, is called *strongly discrete*.

Proposition 2.5. *(1) R is a Prüfer domain if and only if R is integrally closed and each finitely generated nonzero ideal of R is SV-stable (equivalently, ES-stable).*
(2) [1, Lemma 2.7] *Each Prüfer domain is L-stable.*
(3) [1, Proposition 2.10] *A semilocal Prüfer domain (in particular, a valuation domain) is SV-stable if and only it is strongly discrete.*

If the integral closure of R is a Prüfer domain, R is called *quasi-Prüfer* [14, Corollary 6.5.14]. Quasi-Prüfer domains can be characterized by the property that each nonzero finitely generated ideal is locally ES-prestable.

Theorem 2.6. [15, Theorem 2] *Let R be a local domain with integral closure R'. The following conditions are equivalent:*

(i) R' is a Prüfer domain;
(ii) Each nonzero finitely generated ideal of R is ES-prestable.

A global version of Theorem 2.6 is the following.

Theorem 2.7. [14, Theorem 7.4.6] *Let R be a domain with integral closure R'. The following conditions are equivalent:*

(i) R' is a Prüfer domain;
(ii) Each nonzero finitely generated ideal of R is SV-prestable;
(iii) Each nonzero 2-generated ideal of R is SV-prestable.

Since 1998, SV-stable domains have been thoroughly investigated by B. Olberding in a series of papers [38, 40, 41, 42]. Olberding calls an SV-stable ideal of a domain R simply a *stable ideal* and he says that R is *stable* (respectively, *finitely stable*) if each nonzero ideal (respectively, finitely generated ideal) of R is stable. We keep this notation; thus from now on "stable" means "SV-stable".

Stability and finite stability transfer to overrings. In addition, their study can be reduced to the local case.

Proposition 2.8. [42, Lemma 2.4 and Theorem 5.1] *Let S be an overring of R. If R is (finitely) stable, then S is (finitely) stable.*

By the result above and Proposition 2.5, we get that finitely stable domains are quasi-Prüfer.

Corollary 2.9. [50, Proposition 2.1] *If R is finitely stable (respectively, stable), its integral closure is a Prüfer domain (respectively, a strongly discrete Prüfer domain).*

Theorem 2.10. *Let R be a domain.*

(1) [42, Theorem 3.3] *R is stable if and only if R_M is stable, for each maximal ideal M, and R has finite character (i.e., each nonzero element is contained at most in finitely many maximal ideals).*
(2) [14, Proposition 7.3.4] *R is finitely stable if and only if R_M is finitely stable.*

Since any Prüfer domain is finitely stable, finitely stable domains need not have finite character. Also, finitely stable domains with finite character need not be stable. For example, any valuation domain with nonprincipal maximal ideal is finitely stable but not stable (Proposition 2.5).

Theorem 2.11. [41, Theorem 2.3] *A domain R is stable if and only if the following conditions hold: (a) R is finitely stable; (b) PR_P is a stable ideal of R_P, for each nonzero prime P; (c) R_P is a valuation domain for each nonzero nonmaximal prime P, and (d) R has finite character.*

Thus the semilocal case given in [1] (Proposition 2.5(3)) can be generalized in the following:

Proposition 2.12. [38, Theorem 4.6] *Let R be an integrally closed domain. Then R is stable if and only if it is a strongly discrete Prüfer domain with finite character.*

A strongly discrete Prüfer domain such that each noninvertible element has finitely many minimal primes is called a *generalized Dedekind domain*. These domains were introduced by N. Popescu in [47] and have very good ring-theoretic and ideal-theoretic properties; an overview is given in [18].

By the previous proposition, integrally closed stable domains are generalized Dedekind. More precisely, we have:

Corollary 2.13. *The following conditions are equivalent for a domain R:*

(i) R is a generalized Dedekind domain with finite character;
(ii) R is integrally closed and stable.

It is also interesting to observe that stability of nonzero prime ideals forces a Prüfer domain to be generalized Dedekind.

Theorem 2.14. [17, Theorem 5], [38, Theorem 4.7] *The following conditions are equivalent for a domain R:*

(i) R is a generalized Dedekind domain;
(ii) R is a Prüfer domain and each nonzero prime ideal of R is stable.

However, a domain whose nonzero prime ideals are all stable need not be stable, as [41, Example 3.4] shows. An example of generalized Dedekind domain that is not stable is $R := \mathbb{Z} + X\mathbb{Q}[[X]]$; in fact R does not have finite character.

Olberding gave a complete characterization of stable domains [41]. In particular, he proved that each local stable domain is a suitable pullback of a local stable ring of dimension at most one.

Theorem 2.15. [41, Corollary 2.7], [43] *A local domain R is stable if and only if one of the following conditions is satisfied (a) R is one-dimensional stable; (b) R is a strongly discrete valuation domain; (c) R arises from a pullback diagram of type:*

$$\begin{array}{ccc} R & \longrightarrow & D \\ \downarrow & & \downarrow \\ V & \xrightarrow{\varphi} & \frac{V}{I} \end{array}$$

where V is a strongly discrete valuation domain, I is an ideal of V, D is a local stable ring of dimension at most one having a prime ideal P such that P contains all the zero-divisors of D and $P^2 = (0)$, and V/I is isomorphic to the total quotient ring of D.

For example, let W be a one-dimensional discrete valuation domain with quotient field F and X be an indeterminate over F. With the notations of Theorem 2.15(c), setting $V := F[[X]]$, $I := X^2V$ and $D := W[[X]]/X^2W[[X]]$, we get that $R = W + XW + X^2F[[X]]$ is a stable domain [43].

A stable Noetherian ring is one-dimensional (Proposition 1.3). An example of a local one-dimensional domain which is stable and not Noetherian was constructed by Olberding in [41, Proposition 5.2]. Generalizing this construction, Olberding then exhibited a whole class of examples, as a particular class of one-dimensional domains whose integral closure is not a finitely generated module [45, Theorems 4.1 and 4.4] (see also [44, Theorem 3.10]). In fact Theorem 1.4(2) can be extended in the following way:

Theorem 2.16. *Let R be a stable domain with integral closure R'.*

(1) [41, Proposition 4.5] *If R is one-dimensional and $(R : R') \neq (0)$, R is Noetherian 2-generated and R' is a finitely generated R-module.*
(2) [42, Corollary 4.17] *If R is local and $(R : R') = (0)$, R' is a one-dimensional discrete valuation ring (in particular R is one-dimensional).*

On the other hand, it is also possible for stable Noetherian domains, even Noetherian 2-generated domains, to have $(R : R') = (0)$ [44, Section 3].

By using Theorem 2.16, M. Roitman and the author of this paper recently proved that a stable one-dimensional domain is Mori and is precisely a Mori finitely stable domain.

Recall that if I is a nonzero ideal of the domain R, the *divisorial closure* of I is the ideal $I_v := (R : (R : I))$ and that I is called *divisorial* if $I = I_v$. A *Mori domain* is a domain satisfying the ascending chain condition on divisorial ideals. Clearly Noetherian domains are Mori. For the main properties of Mori domains the reader is referred to [4].

Theorem 2.17. *[24] The following conditions are equivalent for a domain R:*

(i) R is stable and one-dimensional;
(ii) R is Mori and stable;

(iii) R is Mori and finitely stable.

In addition, under the previous conditions, for each nonzero ideal I of R, $I_v = \langle x, y \rangle_v$, for some $x, y \in I$.

This result shows that the local one-dimensional domains that are stable and not Noetherian constructed by Olberding in [45] are new examples of Mori domains.

It also shows that a one-dimensional stable domain R cannot arise a pullback like in Theorem 2.15, unless $R = V$ is a discrete valuation domain. In fact, in a pullback of that type, R is Mori if and only if V is a one-dimensional discrete valuation domain and D is a field [35, Theorem 9].

3 Divisorial domains

The class of domains in which each ideal is divisorial has been investigated in the sixties of the last century by several authors and with different methods. Following S. Bazzoni and L. Salce, these domains are now called *divisorial domains*. If each overring of R is divisorial, R is called *totally divisorial* [12].

As for (finite) stability, the study of divisorial domains can be reduced to the local case. We recall that, with a terminology introduced by Matlis, a domain is called *h-local* if it has finite character and each nonzero prime ideal is contained in a unique maximal ideal.

Proposition 3.1. [12, Proposition 5.4] *A domain R is divisorial if and only if it is h-local and R_M is divisorial, for each maximal ideal M.*

The local Noetherian case was independently studied by H. Bass [5] and E. Matlis [31].

Theorem 3.2. [5, Theorems 6.2, 6.3], [31, Theorem 3.8] *Let R be a local Noetherian domain, with maximal ideal M. Then R is divisorial if and only if R is one-dimensional and $(R : M)$ is a 2-generated R-module.*

It was already known to W. Krull that an integrally closed domain such that each nonzero finitely generated ideal is divisorial is Prüfer. The following characterization of integrally closed divisorial domains was given by W. Heinzer in [25].

Theorem 3.3. [25, Theorem 5.1] *Let R be an integrally closed domain. Then R is divisorial if and only if R is an h-local Prüfer domain with invertible maximal ideals.*

In the general case, local divisorial domains were studied in [12, Section 5], [7, Section 2], [20, Section 1]. Recall that a nonzero ideal I of a domain R is called *m-canonical* if $(I : (I : J)) = J$, for each nonzero ideal J of R. With this terminology, the domain R is divisorial if and only if R itself is an m-canonical ideal.

Theorem 3.4. *Let R be a local domain, with maximal ideal M. Then:*

(1) [12, Lemma 5.5] *If M is principal, R is divisorial if and only if it is a valuation domain.*

(2) [20, Theorem 1.2] *If M is not principal and R is not a valuation domain, R is divisorial if and only if $(R:M) = (M:M)$ is a 2-generated R-module and M is an m-canonical ideal of $(M:M)$.*

Divisoriality and stability are strictly related.

Theorem 3.5. [40, Theorem 3.12 and Corollary 3.13] *The following conditions are equivalent for a domain R:*

(i) R is stable and divisorial;
(ii) R is totally divisorial;
(iii) R is h-local and R_M is totally divisorial, for each maximal ideal M.

As always, the Noetherian case and the integrally closed case are of particular interest.

Theorem 3.6. [12, Proposition 7.1 and Theorem 7.3] *Let R be a Noetherian domain. The following conditions are equivalent:*

(i) R is stable and divisorial;
(ii) R is totally divisorial and one-dimensional;
(iii) R is 2-generated.

Thus a Noetherian stable domain is divisorial if and only if it is 2-generated. It follows from Theorem 2.16 that:

Corollary 3.7. *Assume that R is a one-dimensional stable domain whose integral closure is a finitely generated R-module. Then R is (totally) divisorial.*

By Proposition 2.5(3) and Theorem 3.3, a stable valuation domain is (totally) divisorial. Globalizing we obtain:

Theorem 3.8. [40, Theorem 3.1], [12, Proposition 7.6] *Let R be an integrally closed domain. The following conditions are equivalent:*

(i) R is stable and divisorial;
(ii) R is an h-local strongly discrete Prüfer domain;
(iii) R is a divisorial generalized Dedekind domain.

In the local case, totally divisorial domains can be completely classified by using Theorem 2.15: they are either Noetherian 2-generated domains, or strongly discrete valuation domains, or arise from a suitable pullback diagram [40, Corollary 3.16].

4 Ratliff-Rush domains

In a paper of 1978, L. Ratliff and D. Rush associated to a regular ideal I of a Noetherian ring, the ideal $\widetilde{I} := \bigcup_{n\geq 0}(I^{n+1} :_R I^n)$ [48]. W. Heinzer, D. Lantz and K. Shah called $\widetilde{I}$ the *Ratliff-Rush ideal associated to* I [27]. If $I = \widetilde{I}$, I is called a *Ratliff-Rush ideal* and we can say that R is a *Ratliff-Rush ring* if each regular ideal is Ratliff-Rush.

Among other results, Ratliff and Rush proved that, for any regular ideal I of a Noetherian ring, there is a positive integer n such that, for $k \geq n$, $\widetilde{I^k} = I^k$; so that all sufficiently hight powers of I are Rutliff-Rush. Indeed, if R is local with maximal ideal M and I is an M-primary ideal, $\widetilde{I}$ is the unique largest ideal containing I and having the same Hilbert polynomial as I [48]. Stable ideals and integrally closed ideals of Noetherian rings are Ratliff-Rush [27].

An early survey on Ratliff-Rush ideals is [28]. In the setting of integral domains, Ratliff-Rush ideals were studied by A. Mimouni in 2009 [33, 34].

Proposition 4.1. [33, Proposition 2.3 and Theorem 2.5] *Let R be a domain. Then:*

$$R \text{ stable } \Rightarrow R \text{ Ratliff-Rush } \Rightarrow R \text{ L-stable}.$$

In addition:

Proposition 4.2. [27, Proposition 3.1 and Theorem 2.9], [33, Corollary 2.8] *A Noetherian domain is Ratliff-Rush if and only if it is stable.*

Ratliff-Rush domains are quasi-Prüfer.

Proposition 4.3. [33, Lemma 2.4] *Let R be a domain. If each nonzero finitely generated ideal of R is Rutliff-Rush, R' is a Prüfer domain.*

Theorem 4.4. [33, Theorem 2.6] *Let R be an integrally closed domain. The following conditions are equivalent:*

(i) $I = \widetilde{I}$ for each finitely generated nonzero ideal (respectively, each nonzero ideal) I of R;
(ii) R is a Prüfer domain (respectively, a strongly discrete Prüfer domain).

Since any Prüfer domain is L-stable (Proposition 2.5(2)), an L-stable domain need not be Ratliff-Rush. Also, by Theorems 2.12 and 4.4, we get:

Proposition 4.5. *An integrally closed Rutliff-Rush domain is stable if and only if it has finite character.*

5 Quasi-stable domains

G. Picozza and F. Tartarone weakened the notion of stability in the following way. Observing that an invertible ideal of a domain is flat, they define a nonzero ideal I

of a domain R to be *quasi-stable* if I is flat in its endomorphism ring $E(I) := (I : I)$ and they say that R is *quasi-stable* if each nonzero ideal is quasi-stable [46, Section 2].

A stable domain is clearly quasi-stable. In addition, quasi-stable domains are finitely stable. More precisely:

Proposition 5.1. [46, Proposition 2.4] *A domain R is finitely stable if and only if each nonzero finitely generated ideal of R is quasi-stable.*

Thus (finite) stability and quasi-stability coincide for Mori domains (Theorem 2.17). The next result says that if R is integrally closed, quasi-stability is equivalent to R being Prüfer, so that a quasi-stable domain need not be stable (Proposition 2.5(3)).

Proposition 5.2. [46, Proposition 2.7] *The following conditions are equivalent for an integrally closed domain R.*

(i) R is quasi-stable;
(ii) R is finitely stable
(iii) R is Prüfer.

A very tricky example of a finitely stable domain that is not quasi-stable is given in [46, Example 2.8]. Examples of local quasi-stable domains that are not integrally closed nor stable are constructed as pseudo-valuation domains in [46, Example 2.6(2)]. Precisely, let R be a pseudo-valuation domain with maximal ideal M and associated valuation domain $V := (M : M) = (R : M)$ and assume that $R \neq V$. If V is a 2-generated R-module, then R is quasi-stable and not integrally closed. If, in addition, M is not principal in V, then R is not stable.

By Theorem 2.16, a one-dimensional stable domain such that $(R : R') \neq (0)$ is Noetherian. This result cannot be extended to quasi-stable domains. Indeed, let R be a one-dimensional pseudo-valuation domain that is quasi-stable and not stable, as above. Then R is necessarily not Noetherian (nor Mori), but $V = R'$ and $(R : V) = M \neq (0)$.

It is not clear whether quasi-stability passes to overrings. However this happens in several cases: for example quasi-stability transfers to localizations, fractional, flat and Noetherian overrings. More generally:

Proposition 5.3. [46, Corollary 3.7] *Let S be an overring of R and assume that each ideal of S is extended from a fractional ideal of R. If R is quasi-stable, then S is quasi-stable.*

Proposition 5.4. [46, Corollary 3.8] *Let R be a quasi-stable domain with integral closure R'. If $(R : R') \neq (0)$, each overring of R is quasi-stable.*

6 Clifford regular domains

Let S be a multiplicative commutative semigroup. An element $x \in S$ is called *von Neuman regular* (for short, *vN-regular*), if there exists an element $a \in S$ such that $x = x^2a$. Idempotent and invertible elements are vN-regular. By a well-known theorem of Clifford, S is a disjoint union of groups if and only if all its elements are vN-regular: in this case, S is called a *Clifford semigroup*.

The set $\mathscr{F}(R)$ of nonzero fractional ideals of a domain R form a multiplicative semigroup, with unity R. The *class semigroup* of R is defined as the quotient semigroup of $\mathscr{F}(R)$ by the subgroup of nonzero principal ideals. A domain R is called a *Clifford regular domain* if its class semigroup is Clifford regular; this is equivalent to say that each nonzero fractional ideal is vN-regular in the semigroup $\mathscr{F}(R)$.

Dedekind domains are trivial examples of Clifford regular domains. S. Bazzoni and L. Salce showed that all valuation domains are Clifford regular and gave a complete description of the structure of the class semigroup in that case [11]. P. Zanardo and U. Zannier investigated the class semigroups of orders in number fields and showed that all orders in quadratic fields are Clifford regular domains [55]. The study of Clifford regular domains was then carried on by S. Bazzoni [6, 8, 9, 10].

Clifford regular domains are between stable and finitely stable domains.

Proposition 6.1. [9, Proposition 2.3] *A stable domain is Clifford regular and a Clifford regular domain is finitely stable.*

Hence, Clifford regularity and (finite) stability are equivalent for Mori domains (Theorem 2.17). Also, an integrally closed Clifford regular domain is Prüfer (Proposition 2.5(1)).

S. Bazzoni proved that a Clifford regular domain has finite character [10, Theorem 4.7]. This property characterizes Clifford regularity inside Prüfer domains and allows to show that the integral closure of a Clifford regular domain is still Clifford regular.

Theorem 6.2. [9, Theorem 4.5] *An integrally closed domain is Clifford regular if and only if it is a Prüfer domain with finite character.*

Proposition 6.3. [10, Corollary 4.8] *If R is a Clifford regular domain, its integral closure is Clifford regular.*

Even in the local case, Clifford regularity may not coincide with stability or finite stability. In fact, any valuation domain is Clifford regular [11, Theorem 3] but need not be stable (Proposition 2.5(3)). A local finitely stable domain that is not Clifford regular is exhibited in [9, Example 6.6].

The following result puts in relation stability and Clifford regularity.

Proposition 6.4. [53, Theorem 2.6] *Let R be a domain. The following conditions are equivalent:*

(i) R is stable;

(ii) R is Clifford regular and each nonzero idempotent fractional ideal of R is a ring.

Clifford regularity of overrings was investigated by L. Sega in [53]. Since ideals extended from vN-regular ideals are still vN-regular, the situation is similar to the one of quasi-stability.

Proposition 6.5. [53, Proposition 4.1] *Let S be an overring of R and assume that each ideal of S is extended from a fractional ideal of R. If R is Clifford regular, then S is Clifford regular.*

Proposition 6.6. [53, Theorem 4.6] *Let R be a Clifford regular domain with integral closure* R'. *If* $(R:R') \neq (0)$, *each overring of R is Clifford regular.*

7 Problems

As we summarize in the two tables below, all the stability conditions introduced in this paper are well understood when R is integrally closed or Noetherian. But in general there are several questions still unanswered; in this last section we illustrate some of them.

Table 1 The integrally closed case

Stable	$\Leftrightarrow$	Prüfer strongly discrete with finite character
$\Downarrow$		$\Downarrow$
Clifford regular	$\Leftrightarrow$	Prüfer with finite character
$\Downarrow$		$\Downarrow$
Quasi-stable	$\Leftrightarrow$	Prüfer
$\Updownarrow$		$\Updownarrow$
Finitely stable	$\Leftrightarrow$	Prüfer

Stable	$\Leftrightarrow$	Prüfer strongly discrete with finite character
$\Downarrow$		$\Downarrow$
Ratliff-Rush	$\Leftrightarrow$	Prüfer strongly discrete
$\Downarrow$		$\Downarrow$
Finitely stable	$\Leftrightarrow$	Prüfer
$\Downarrow$		
L-stable		

Problem 7.1 A domain R is *Archimedean*, if $\bigcap_{n\geq 0} r^n R = (0)$, for each nonunit $r \in R$. Since Mori domains satisfy the ascending chain condition on principal ideals, they are Archimedean. The class of Archimedean domains includes also completely integrally closed domains and one-dimensional domains.

Table 2 The Noetherian case

Stable	⇔	Clifford regular	⇔	Quasi-stable	⇔	Finitely stable
	⇔	Ratliff-Rush	⇔	L-stable one-dimensional		

Question. *Is a stable Archimedean domain one-dimensional?*

The answer is positive in the semilocal case [24], so that a semilocal stable Archimedean domain is Mori (Theorem 2.17). However, in general the Archimedean property does not pass to localizations. For example, the ring of entire functions is an infinite-dimensional completely integrally closed (hence Archimedean) Bezout domain [14, Section 8.1] which is not locally Archimedean, because an Archimedean valuation domain is one-dimensional. Hence, a way of approaching this problem is trying to understand if the Archimedean property localizes under the hypothesis of stability.

Problem 7.2 A (one-dimensional) Mori domain whose nonzero finitely generated ideals are L-stable is not necessarily (finitely) stable. In fact if R is a local one-dimensional integrally closed Mori domain, each nonzero finitely generated ideal I of R is L-stable (Proposition 2.3(2)); but if R is (finitely) stable it must be a discrete valuation domain (Proposition 2.5(1)). Example of local one-dimensional integrally closed Mori domains that are not valuation domains can be constructed by means of pullbacks [4, Theorem 2.2]. Does Proposition 2.4 extend to Mori domains? That is

Question. *Is a one-dimensional L-stable Mori domain stable?*

Problem 7.3 For Noetherian domains, the Ratliff-Rush property is equivalent to (finite) stability and in the one-dimensional case also to L-stability (Propositions 2.4 and 4.2). Since stable domains are Ratliff-Rush, a Mori Ratliff-Rush domain need not be Noetherian (Section 1.2).

Question. *Is a Mori Ratliff-Rush domain one-dimensional?*

Apart from the Noetherian case, the answer is positive if either $(R : R') \neq (0)$ or R is seminormal [33, Corollary 2.10].

Problem 7.4 If R is a Mori stable domain (equivalently, a one-dimensional stable domain) and I is a nonzero ideal of R, we have $I_v = \langle x, y \rangle_v$, for some $x, y \in I$ (Theorem 2.17); thus we can say that in a stable Mori domain each divisorial ideal is *2-v-generated*. Since a divisorial Mori domain is Noetherian, this result generalizes (i) $\Rightarrow$ (iii) of Theorem 3.6.

Question. *Assume that R is a one-dimensional Mori domain such that each divisorial ideal is 2-v-generated. Is it true that each divisorial ideal is stable?*

Note that the answer to this question is negative if R has dimension greater than one. For example, let R be a Krull domain. Then each divisorial ideal of R is 2-v-generated [37, Proposition 1.2] and stability coincides with invertibility (Proposition 2.3(1)). Hence each divisorial ideal of R is stable (i.e., invertible) if and only if R is locally factorial [13, Lemma 1.1].

Recall that a nonzero ideal I of a domain R is *v-invertible* if $(I(R:I))_v = R$ and that I is called *v-stable* if I_v is v-invertible as an ideal of $E(I_v)$, that is $(I_v(E(I_v) : I_v))_v = E(I_v)$. Clearly, v-invertibility implies v-stability. If each nonzero ideal of R is v-stable, we say that R is *v-stable* [21].

Each nonzero ideal of a Krull domain is v-invertible, thus a Krull domain is v-stable. However, a Krull domain is stable if and only if it is a Dedekind domain, that is it has dimension one (Proposition 2.3(1)). An example of a one-dimensional Mori domain that is v-stable but not stable is given in [22, Example 2.6].

Question. *Assume that R is a Mori domain such that each divisorial ideal is 2-v-generated. Is it true that R is v-stable?*

Problem 7.5 The *t-closure* of a nonzero ideal I is defined by setting

$$I_t := \bigcup\{J_v;\ J \text{ finitely generated and } J \subseteq I\},$$

for each nonzero ideal I of R. If $I = I_t$, I is called a *t-ideal*. Invertible ideals are divisorial and divisorial ideals are t-ideals.

Olberding proved that R is a stable domain if and only if each nonzero ideal I is divisorial in $E(I)$ [42, Theorem 3.5]. When R is finitely stable, each finitely generated nonzero ideal I is a t-ideal of $E(I)$, being invertible. Does the converse hold?

Question. [46, Question 2.5] *Assume that each (finitely generated) nonzero ideal I is a t-ideal of $E(I)$. Is it true that R is finitely stable?*

The answer is positive when, for each finitely generated nonzero ideal I, the ideal $(E(I):I)$ is finitely generated in $E(I)$ [46, Proposition 2.4].

Problem 7.6 It is well known that if R has finite character, a locally invertible ideal is invertible. Conversely, if each locally invertible ideal is invertible R need not have finite character (for example, a Noetherian domain need not have finite character). However, a Prüfer domain such that each locally invertible ideal is invertible does have finite character. This fact was conjectured by S. Bazzoni [6, p. 630] and proved by W. Holland, J. Martinez, W. McGovern and M. Tesemma in [29]. (A simplified proof is in [36]). F. Halter-Koch gave independently another proof, in the more general context of ideal systems [26]. Other contributions were given by M. Zafrullah in [54] and by C.A. Finocchiaro, G. Picozza and F. Tartarone in [16].

Following D.D. Anderson and M. Zafrullah, for short we call R an *LPI-domain* if each locally principal nonzero ideal of R is invertible [2].

Since a Prüfer domain is precisely a finitely stable integrally closed domain, one is lead to ask the following more general question.

Question. [10, Question 4.6] *Assume that R is a finitely stable LPI-domain. Is it true that R has finite character?*

The answer is positive if and only if the LPI-property extends to fractional overrings [18, Corollary 15], in particular when R is Mori or integrally closed. For an exhaustive discussion of this problem see [18].

Problem 7.7 By Proposition 5.3, a quasi-stable domain is locally quasi-stable. What about the converse? Note that a locally quasi-stable domain, being locally finitely stable, is finitely stable.

Question. [46, Section 3] *Is it true that a domain which is locally quasi-stable is quasi-stable?*

The answer is positive for integrally closed or Mori domains. Other than that, also when R is h-local [46, Corollary 3.13].

Problem 7.8 A similar question can be addressed for Clifford regular domains. Bazzoni proved that any localization of a Clifford regular domain is Clifford regular [9, Proposition 2.8] (Proposition 6.5) and that a Clifford regular domain has finite character [10, Theorem 4.7]. But it is not known if the converse is true in general.

Question. [9, Question 6.8] *Is it true that a domain which is locally Clifford regular and has finite character is Clifford regular?*

We know that the answer is positive in the following cases: (a) When R is integrally closed. (This follows from Theorem 6.2.) (b) When R is Mori. In this case, each localization of R is Mori and Clifford regular, hence (finitely) stable. Thus R is locally stable and the finite character implies that R is stable. (c) When each nonzero prime ideal of R is contained in a unique maximal ideal, for example if R is one-dimensional [23, Proposition 5.5].

Problem 7.9 A quasi-stable domain need not have finite character; thus a quasi-stable domain need not be Clifford regular.

Question. *Is a Clifford regular domain quasi-stable?*

The answer is positive when R is integrally closed or Mori.

Problem 7.10 It is easy to see that a vN-regular ideal is L-stable [9, Lemma 2.6], so that Clifford regular domains are L-stable. However, an L-stable domain need not be finitely stable; thus neither quasi-stable nor Clifford regular. For example, a Noetherian integrally closed domain is always L-stable, but it is stable if and only if it is a Dedekind domain (Proposition 2.3(2)).

Question. *Is a finitely stable domain, or a quasi-stable domain, L-stable?*

Again, the answer is positive when R is integrally closed or Mori.

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