ON FINITELY STABLE DOMAINS, I

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ABSTRACT. We prove that an integral domain R is stable and one-dimensional if and only if R is finitely stable and Mori. If R satisfies these two equivalent conditions, then each overring of R also satisfies these conditions and it is 2-v-generated. We also prove that if R is an Archimedean stable domain such that R' is local, then R is one-dimensional and so Mori.

1. Introduction. In this introduction we start with a short remainder of finitely stable and stable rings, recall the definitions of other classes of rings that we use here, as Mori, Archimedean, etc., and finally summarize our main results. By a ring we mean a commutative ring with unity. A *local ring* is a ring with a unique maximal ideal, not necessarily Noetherian. A *semilocal ring* is a ring with just finitely many maximal ideals.

Motivated by earlier work of H. Bass [**3**] and J. Lipman [**11**] on the number of generators of an ideal, in 1972 J. Sally and W. Vasconcelos defined an ideal I of a ring R to be *stable* if I is projective over its endomorphism ring; they called R a *stable ring* if each nonzero ideal of R is stable [**24, 25**]. Stability of rings is often determined by the stability of regular ideals, that is, ideals containing a nonzero divisor. D. Rush studied the rings such that each finitely generated regular ideal is stable, in particular in connection with properties of their integral closure and to the 2-generator property [**22, 23**]. These rings are now called *finitely stable*.

In a note of 1987, D.D. Anderson, J. Huckaba and I. Papick considered the notion of stability for integral domains [**1**]. If I is a nonzero ideal of a domain R, then the endomorphism ring of I coincides with the overring $E(I) = (I : I)$ of R; also, I is projective over $E(I)$ if and

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only if I is invertible as an ideal of $E(I)$. We use here notations like $(I : I)$ in a more general context: If R and T are domains with the same field of fractions K, I is an ideal of R and S is a subset of K, we set $(I :_T S) = \{t \in T \mid tS \subseteq I\}$ and $(I : S) = (I :_K S)$. The stability property of a nonzero ideal I does not depend on the domain containing I: more precisely, if I is a common nonzero ideal of two domains A and B, then I is stable as an ideal of A if and only if I is stable as an ideal of B since $\operatorname{Frac} A = \operatorname{Frac} B$.

Since 1998, finitely stable and stable domains have been thoroughly investigated by Bruce Olberding in a series of papers [**14**]-[**19**]. In [**20**], he also studied finitely stable rings in the spirit of Rush, extending several results known for stable domains. Our paper heavily relies on Olberding's work. We thank B. Olberding for his valuable help. Also, as he communicated to us, his articles [**15, 16, 17**] contain some errors.

Of course, when R is a Noetherian ring, stability and finite stability coincide, but in general these two classes of rings are distinct, even if R is an integrally closed domain: in this case R is finitely stable if and only if it is Prüfer, that is, each nonzero finitely generated ideal of R is invertible. Indeed, a domain R is integrally closed if and only if $R = E(I)$ for each nonzero finitely generated ideal I. However, a valuation domain is stable if and only if it is *strongly discrete*, that is, each nonzero prime ideal is not idempotent [**4,** Proposition 7.6]. Thus a valuation domain that is not strongly discrete is finitely stable, but not stable.

A domain R is finitely stable if and only if it is locally finitely stable [**6,** Proposition 7.3.4]. Actually, if I is a stable ideal of R, then I_S is a stable ideal of R_S for each multiplicative part $S \subseteq R$.

Recall that a domain R has *finite character* if each nonzero element of R is contained at most in finitely many maximal ideals. A finitely stable domain need not have finite character, since any Prüfer domain is finitely stable. On the other hand, a domain is stable if and only if it is locally stable and has finite character [**17,** Theorem 3.3].

We denote by R' the integral closure of a domain R.

Olberding characterized finitely stable domains as follows:

Theorem 1.1. [**20,** Corollary 5.11] *A domain R is finitely stable if and only if it satisfies the following conditions:*

(1) R' *is a quadratic extension of* R*;*
(2) R' *is a Prüfer domain;*
(3) *Each maximal ideal of* R *has at most* 2 *maximal ideals of* R' *lying over it.*

Recall that a domain D is a *quadratic extension* of a domain R if for each $x, y \in D$ we have $xy \in xR + yR + R$. Olberding also proved that, in the local one-dimensional case, stability and finite stability are equivalent provided the maximal ideal is stable:

Proposition 1.2. [**21,** Theorem 4.2] *Let* R *be a local one-dimensional domain. The following conditions are equivalent:*

(i) R *is stable;*
(ii) R *is finitely stable with stable maximal ideal;*
(iii) R' *is a quadratic extension of* R *and* R' *is a Dedekind domain with at most two maximal ideals.*

Stability is related to divisoriality and to the 2-generator property. Recall that an ideal I of a domain R is *divisorial* if $I \neq (0)$ and $I = I_v = (R : (R : I))$. A domain R is called *divisorial* if each nonzero ideal of R is divisorial, and it is called *totally divisorial* if each overring of R is divisorial. An ideal I of R is called 2*-generated* if I can be generated by two elements. The domain R is 2*-generated* if each finitely generated ideal of R is 2-generated.

A domain R is stable and divisorial if and only if it is totally divisorial [**18,** Theorem 3.12]. Also, any stable Noetherian domain is one-dimensional [**25,** Proposition 2.1], and a Noetherian domain is stable and divisorial (i.e., totally divisorial) if and only if it is 2-generated ([**15,** Theorem 3.1] and [**4,** Theorem 7.3]). The 2-generator property for Noetherian domains is strictly stronger than stability. The first example of a stable Noetherian domain that is not 2-generated (equivalently, it is not divisorial) was given in [**25,** Example 5.4]. Several other examples can be found in [**18,** Section 3].

A *Mori domain* is a domain with the ascending chain condition on divisorial ideals. This is equivalent to the property that each nonzero ideal I of R contains a finitely generated nonzero ideal J such that $(R : I) = (R : J)$, that is, $I_v = J_v$ [**2,** Theorem 2.1]. Clearly Noetherian domains are Mori. For the main properties of Mori domains, see the survey [**2**] and the references there. A nonzero ideal I of an integral domain R is 2-*v-generated* if I contains a 2-generated ideal J such that $(R : I) = (R : J)$, and R is 2-*v-generated* if each nonzero ideal of R is 2-v-generated. Of course, a 2-v-generated domain is Mori. However, if each divisorial ideal of R is principal (hence 2-v-generated), then R is not necessarily Mori (see [**12,** page 561]). Clearly, a Mori 2-generated domain is 2-v-generated.

A Mori domain R satisfies the ascending chain condition on principal ideals (for short, accp), and so it is *Archimedean*, that is, $\bigcap_{n\geq 0} r^n R = (0)$, for each nonunit $r \in R$. Indeed, a domain R satisfies accp if and only if $\bigcap_{n\geq 1}(\prod_{i=1}^{n} r_i R) = (0)$ for any nonunits $r_i \in R$, equivalently $\bigcap_{n\geq 1} a_n R = (0)$ if the sequence of principal ideals $a_n R$ is strictly decreasing. Besides accp domains, the class of Archimedean domains includes also one-dimensional domains [**13,** Corollary 1.4] and completely integrally closed domains [**10,** Corollary 13.4]. We recall that a domain R is completely integrally closed if and only if $R = E(I)$ for each nonzero ideal I. Hence completely integrally closed domains are integrally closed and the converse holds in the Noetherian case. A completely integrally closed stable domain is Dedekind.

Here are our main results:

(1) *If R is an Archimedean stable domain such that R' is local, then R is one-dimensional (Corollary 2.9).*
(2) *A domain R is stable and one-dimensional if and only if it is finitely stable and Mori (Theorem 4.8). If R satisfies these two equivalent conditions, then each overring of R also satisfies these conditions and it is 2-v-generated.*

If R' is not local, an Archimedean local stable domain R need not be one-dimensional. Indeed, in a forthcoming paper we will give examples of Archimedean local stable domains of dimension n, for each $n \geq 1$; see [**8,** Section 6].

A class of one-dimensional local domains that are stable and not Noetherian was constructed by Olberding in [**19,** Theorems 4.1 and 4.4] (see also [**18,** Theorem 3.10]). By our results, all these domains are new examples of one-dimensional Mori domains.

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versions of this paper.

2. The one-dimensional case. In the following, R is an integral domain that is not a field. By an *ideal* we mean an integral ideal.

The following construction, due to Olberding, is basic for our paper.

Construction 2.1. [**17,** Section 4] Let (R, M) be a local domain. Set $R_i = \{0\}$ for $i < 0$, $R_0 = R$ and $M_0 = M$. Define inductively for $n > 0$: $R_n = R_{n-1}$ if R_{n-1} is not local, and $R_n = E(M_{n-1}) = (M_{n-1} : M_{n-1})$ if R_{n-1} is local with maximal ideal denoted by M_{n-1}. Set $T = \bigcup_{n \geq 0} R_n$.

Thus we have:

- (a) If there exists an integer $k > 0$ such that R_k is not local, but R_i is local for $0 \leq i < k$, then $R_n = R_k$ for all $n \geq k$, and $T = R_k$.
- (b) If $R_n \subsetneq R_{n+1}$ for all $n \geq 0$, all the rings R_n are local.

We will use repeatedly the following theorem of Olberding.

Theorem 2.2. [**17,** Corollary 4.3, Theorem 4.8] and its proof, and [**20,** Theorem 5.4] *Let R be a finitely stable local domain with stable maximal ideal M. With the notation of 2.1 we have:*

- (1) *Each R_n is finitely stable with stable maximal ideals, and there exists an element $m \in M$ such that $M = mR_1$. Moreover, for $k \geq 1$, if R_k is local with maximal ideal M_k, then $M_k = mR_{k+1} = MR_{k+1}$, and if T is local, then its maximal ideal is $mT = MT$.*
- (2) *Each R_n is a finitely generated R-module, thus T is an integral extension of R.*

We also have:

- (a) *If $T = R_n$ for some $n \geq 0$, then T is a finitely generated R-module, and T has at most two maximal ideals.*
- (b) *If $T \neq R_n$ for all $n \geq 0$, then T is local.*
- (c) *The maximal ideals of T are principal, and the Jacobson radical of T is equal to $mT = MT$, where $mR_1 = M$.*

In addition, if R is a stable domain, then T is equal to the integral closure R' of R, and R' is a strongly discrete Prüfer domain.

In the one-dimensional case we have:

Corollary 2.3. *Let R be a one-dimensional finitely stable local domain with stable maximal ideal M. Then R is stable, and in the setting of Theorem 2.2, $T = R'$ is a principal ideal domain with at most two maximal ideals. Hence, if T is local, in particular, if $T \neq R_n$ for each $n \geq 0$, T is a DVR.*

Proof. R is stable by Proposition 1.2, so $T = R'$. Since R' is one-dimensional with principal maximal ideals, R' is a principal ideal domain by [**10,** Corollary 37.9]. □

Proposition 2.4. *In the setting of Theorem 2.2, $T \neq R_n$ for each $n \geq 0$ if and only if T is a finite R-extension (that is, T is a finitely generated R-module). Hence $T = R_n$ for some $n \geq 0$ if and only if T is not a finite R-extension. (Recall that if R is stable, then $T = R'$.)*

Proof. If $T = R_n$ for some $n \geq 0$, then T is a finitely generated R-module by Theorem 2.2 (a). Conversely, assume that T is generated as an R-module by a finite subset F of T. Then there exists an integer $n \geq 0$ such that $F \subseteq R_n$, implying that $T = R_n$. □

Denote by $\mathcal{U}(A)$ the set of units of a domain A.

Remark 2.5. *In the setting of Theorem 2.2, for any integer $n \geq 0$ we have $\mathcal{U}(T) \cap R_n = \mathcal{U}(R_n)$, since T is an integral extension of R_n.*

Lemma 2.6. *Let R be a finitely stable local domain with stable maximal ideal. In the setting of Theorem 2.2, if T is local, in particular, if condition (b) holds, we have:*

(1) *For each $n \geq 0$, $(R :_T m^n) = (R :_T M^n) = R_n$; equivalently, $Tm^n \cap R = R_n m^n$ (here $M^0 = R$).*
(2) *Let $r = um^n$ be a nonzero element of R, where $u \in \mathcal{U}(T)$, and $n \geq 0$. Then $(R :_T r) = R_n$.*

Proof. (1) We prove the equality $(R :_T m^n) = R_n$ by induction on n starting with $n = 0$. Let $n > 0$. Since $M = R_1 m$, by applying the induction assumption to R_1 replacing R we obtain that:

$$(R :_T m^n) = (M :_T m^n) = (R_1 m :_T m^n) = (R_1 :_T m^{n-1}) = R_n.$$

Also $M^n = (R_1 m)^n = R_1 m^n$. Since $R_n = (R :_T m^n)$ and $R_1 \subseteq R_n$, we obtain

$$R_n \subseteq (R :_T R_1 m^n) = (R :_T M^n) \subseteq (R :_T m^n) = R_n,$$

so $(R :_T M^n) = R_n$.

(2) By item (1) we have $u \in R_n$, and also:

$$(R :_T r) = (R :_T um^n) = ((R :_T m^n) : u) = (R_n :_K u) = R_n,$$

where $K = \operatorname{Frac} R$, since $u \in \mathcal{U}(R_n)$. □

Lemma 2.7. *Let (R, M) be a finitely stable local domain with stable maximal ideal. In the notation of 2.1 assume that T is local. Then*

$$\Big(\bigcap_{n\geq 0} m^n T\Big)^2 \subseteq \bigcap_{n\geq 0} m^n R.$$

Proof. By Lemma 2.6 (1), we have for all $n \geq 0$:

$$\Big(R \cap \bigcap_{k\geq 0} m^k T\Big)^2 \subseteq (R \cap m^n T)^2 = (m^n R_n)^2 = m^n (m^n R_n) \subseteq m^n R,$$

so $\Big(R \cap \bigcap_{k\geq 0} m^k T\Big)^2 \subseteq \bigcap_{n\geq 0} m^n R$.

Now let $s, t \in \bigcap_{n\geq 0} m^n T$. Again by Lemma 2.6 (1), we have $sm^e, tm^e \in R$ for a sufficiently large integer e. Thus $(sm^e)(tm^e) \in \Big(R \cap \bigcap_{n\geq 0} m^n T\Big)^2 \subseteq \bigcap_{n\geq 0} m^n R$. It follows that $st = \frac{(sm^e)(tm^e)}{m^{2e}} \in \bigcap_{n\geq 0} m^n R$. Hence $\Big(\bigcap_{n\geq 0} m^n T\Big)^2 \subseteq \bigcap_{n\geq 0} m^n R$. □

Theorem 2.8. *Let R be an Archimedean finitely stable local domain with stable maximal ideal. In the notation of 2.1, if T is local, in particular, if condition (b) of Theorem 2.2 holds, then R is one-dimensional.*

Proof. By Theorem 2.2, the maximal ideal of T is mT, $m \in M$. Let $Q = \bigcap_{n\geq 0} m^n T$. By Lemma 2.7, $Q^2 \subseteq \bigcap_{n\geq 0} m^n R = (0)$. Hence $Q = (0)$. By [**10,** Theorem 7.6 (a) and (c)], Q is the largest non-maximal prime contained in mT. Thus T is one-dimensional, and so is R, as T is an integral extension of R. $\square$

Corollary 2.9. *Let R be an Archimedean stable domain satisfying one of the following two conditions:*

(a) *R is local and R' is not a finitely generated R-module;*
(b) *R' is local.*

Then R is one-dimensional.

Proof. (a) Since R is local and stable, we have $T = R'$. By Proposition 2.4, condition (a) here is equivalent to condition (b) of Theorem 2.2. Thus $T = R'$ is local and so R is one-dimensional by Theorem 2.8.

(b) Since R' is local, also R is local. As in (a), since R is also stable, we have $T = R'$, and again R is one-dimensional by Theorem 2.8. $\square$

If R' is not local, an Archimedean local stable domain R need not be one-dimensional. Indeed, in a forthcoming paper we will give examples of Archimedean local stable domains of dimension n, for each $n \geq 1$; see [**8,** Section 6].

3. The 2-v-generator property. In the Noetherian case, the next theorem was proved by Sally and Vasconcelos [**24,** Theorem 2.4]. Olberding proved that the hypotheses of the theorem imply Noetherianity.

Theorem 3.1. [**16,** Proposition 4.5] *Let R be a one-dimensional stable domain. If R' is a finite R-extension, then each ideal of R is 2-generated.*

In Proposition 3.13 below, we state that a stable one-dimensional domain R is 2-v-*generated*, that is, for each nonzero ideal I there are two elements $x, y \in I$ such that $I_v = \langle x, y\rangle_v$; thus R is Mori. We start with the following notation:

Notation 3.2. In the setting of Theorem 2.2, assume that the domain R is one-dimensional and that T is local (in particular, T is local if condition (b) holds). As T is a DVR (Corollary 2.3) with maximal ideal mT, we denote by $\mathbf{v}$ the discrete valuation of T such that $\mathbf{v}(m) = 1$.

Lemma 3.3. *In the setting of Theorem 2.2, assume that the domain R is one-dimensional and that T is local. Then, by using Notation 3.2, we have:*

(1) *Let r be a nonzero element of R. Then:*

$$(R :_T r) = R_{\mathbf{v}(r)}.$$

(2) *Let I be a nonzero ideal of R, and let a be an element of minimal value $\mathbf{v}(a) = k$ in I. Then:*

$$(R :_T I) = R_k.$$

Proof. (1) This follows from Lemma 2.6 (2).

(2) By item (1), we have

$$(R :_T I) = \bigcap_{r \in I\setminus\{0\}} (R :_T r) = \bigcap_{r \in I\setminus\{0\}} R_{\mathbf{v}(r)} = R_k.$$

□

From Lemma 2.6 (1) we obtain:

Lemma 3.4. *In the setting of Theorem 2.2, assume that the domain R is one-dimensional and that T is local. Then, in the notation 3.2, we have for all $k \geq 0$:*

$$\{r \in R \mid \mathbf{v}(r) \geq k\} = R \cap m^k T = R_k m^k.$$

Proposition 3.5. *A one-dimensional local stable domain R is 2-v-generated; hence R is a Mori domain.*

Proof. In case (a) of Theorem 2.2, every ideal of R is 2-generated by Theorem 3.1.

Assume condition (b) of Theorem 2.2, and use Notation 3.2. Let $I \neq R$ be a nonzero ideal of R. Since T is a DVR, there exists a nonzero element $t \in T$ of maximal value $\mathbf{v}(t)$ such that $\frac{1}{t} I \subseteq R$. Let

$J = \frac{1}{t}I$, so $(R : J) \subseteq T$. Since $\frac{1}{m} \notin T$, there exists a nonzero element $a_1 \in J$ such that $\frac{a_1}{m} \notin R$. Let a_2 be an element of minimal value k in J. If $\frac{a_2}{m} \notin R$, set $a = a_2$. Assume that $\frac{a_2}{m} \in R$. If $\mathbf{v}(a_1) = \mathbf{v}(a_2)$, set $a = a_1$. Otherwise $\mathbf{v}(a_1) > \mathbf{v}(a_2)$, so $\mathbf{v}(a_1 + a_2) = \mathbf{v}(a_2)$ and $\frac{a_1+a_2}{m} \notin R$. In this case we set $a = a_1 + a_2$. In each case, a is a nonzero element of minimal value k in J such that $\frac{a}{m} \notin R$. Thus $a = um^k$, where $u \in \mathcal{U}(R_k) \setminus R_{k-1}$, by Lemma 2.6 (1).

Since $(R : J) \subseteq T$ and $\frac{1}{um} \notin T$, there exists an element $b \in J$ such that $\frac{b}{um} \notin R$. We show that $(R : \{a, b\}) \subseteq T$.

If x is an element in $(R : \{a, b\}) \setminus T$, we have $x = \frac{1}{vm^i}$, where $v \in \mathcal{U}(T)$ and $i > 0$. Thus $\frac{1}{vm}a, \frac{1}{vm}b \in R$. Since $\frac{1}{vm}a = \frac{u}{v}m^{k-1} \in R$, we have $\frac{u}{v} \in \mathcal{U}(R_{k-1})$ by Lemma 2.6 (1). Since $\mathbf{v}(b) \geq k$, we have $\mathbf{v}(\frac{b}{vm}) \geq k-1$. As $\frac{b}{vm} \in R$, we obtain by Lemma 3.4 that $\frac{b}{vm} \in R_{k-1}m^{k-1}$. Hence, $\frac{b}{um} = \frac{v}{u}\frac{b}{vm} \in R_{k-1}m^{k-1} \subseteq R$, a contradiction. It follows that $(R : \{a, b\}) \subseteq T$.

Since a is of minimal value in J, by Lemma 3.3 (1)-(2), we have $(R :_T J) = (R :_T a)$.

Hence $(R : J) \subseteq (R : \{a, b\}) = (R :_T \{a, b\}) = (R :_T J) \subseteq (R : J)$, so $(R : J) = (R : \{a, b\})$. Thus J is 2-v-generated and so is $I = tJ$. We conclude that R is 2-v-generated. □

Corollary 3.6. *Let R be an Archimedean stable domain such that R' is local (in particular, assume that condition (b) of Theorem 2.2 holds). Then R is a one-dimensional Mori domain.*

Proof. R is local and one-dimensional by Corollary 2.9. Hence R is Mori by Proposition 3.5. □

In Proposition 3.13 below we globalize Proposition 3.5.

Lemma 3.7. *Let S be a multiplicative subset of an integral domain R. If I is a 2-v-generated nonzero ideal of R, then the ideal IR_S of R_S is 2-v-generated. Hence, if R is 2-v-generated, also R_S is 2-v-generated.*

Proof. There exists a 2-generated subideal J of I such that $(R : J) = (R : I)$. Since $(R : J)R_S = (R_S : JR_S)$, we have $(R_S : IR_S) = (R_S : JR_S)$ and so the ideal IR_S of R_S is 2-v-generated. □

Lemma 3.8. *Let (R, M) be a local one-dimensional domain, and let $a, b \in M$ be two nonzero elements. Then each element in $a + Rb^k$ is associated with a for all sufficiently large integers k.*

Proof. Since R is local and one-dimensional, we have $M = \sqrt{aM}$, so $b^k \in aM$ for each sufficiently large integer k. Hence for all $r \in R$ we have $a + rb^k = a(1 + r(\frac{b^k}{a}))$, where $1 + r\frac{b^k}{a}$ is a unit in R since $\frac{b^k}{a} \in M$. □

Proposition 3.9. *Let R be a one-dimensional domain of finite character. The following conditions are equivalent:*

(i) *R is 2-v-generated;*
(ii) *R is locally 2-v-generated.*

Proof. (i) $\Rightarrow$ (ii) If R is 2-v-generated, then R is locally 2-v-generated by Lemma 3.7.

(ii) $\Rightarrow$ (i) Assume that R is locally 2-v-generated. We prove that each nonzero ideal $I \neq R$ of R is 2-v-generated. Since R has finite character there are just finitely many maximal ideals containing I, say $M_1, \dots, M_e$, which we assume to be distinct. For each $1 \leq i \leq e$, the domain R_{M_i} is 2-v-generated, so there exist nonzero elements a_i, b_i in I such that $(R_{M_i} : I) = (R_{M_i} : \{a_i, b_i\})$. There exist pairwise comaximal elements $m_i \in M_i$, for $1 \leq i \leq e$. By the Chinese Remainder Theorem, for each positive integer k there exists an element $a \in I$ such that we have in R:

$$a \equiv a_i \mod Im_i^k$$

for $1 \leq i \leq e$. By Lemma 3.8, we may choose k sufficiently large such that for each i the elements a and a_i are associated in R_{M_i}, so $(R_{M_i} : I) = (R_{M_i} : \{a_i, b_i\}) = (R_{M_i} : \{a, b_i\})$.

Let N_q $(q = 1, 2, \dots, f)$ be the maximal ideals containing a but not I. There exist pairwise comaximal elements $n_i \in M_i$, for $1 \leq i \leq e$ that belong to no maximal ideal N_q. Also there exists an element $c \in I$ that belongs to no ideal N_q. By the Chinese Remainder theorem, for each positive integer j there exists an element $b \in I$ such that $b \equiv b_i \mod In_i^j$ for each $1 \leq i \leq e$, and $b \equiv c \mod IN_q$ for each ideal N_q. Hence $b \notin N_q$ for all $1 \leq q \leq f$. By Lemma 3.8, for a sufficiently large integer j and for each i, the elements b and b_i are associated in R_{M_i}, so $(R_{M_i} : I) = (R_{M_i} : \{a, b\})$ for all $1 \leq i \leq e$.

Let M be a maximal ideal of R. If M contains I, thus $M = M_i$ for some integer $1 \leq i \leq e$, then $(R_{M_i} : I) = (R_{M_i} : \{a, b\})$. If M contains a but not I, then $M = N_q$ for some integer $1 \leq q \leq f$, so $b \notin M$. Thus $(R_M : I) = R_M = (R_M : \{a, b\})$. If M does not contain a, then again $(R_M : I) = R_M = (R_M : \{a, b\})$. Thus for each maximal ideal M of R we have $(R_M : I) = (R_M : \{a, b\})$. Hence

$$(R : I) = \bigcap_M (R_M : I) = \bigcap_M (R_M : \{a, b\}) = (R : \{a, b\}),$$

where M runs over all the maximal ideals of R. We conclude that I is 2-v-generated, so the domain R is 2-v-generated. □

Corollary 3.10. *A one-dimensional stable domain is* 2*-v-generated if and only if it is locally* 2*-v-generated*

Proof. Indeed, a stable domain has finite character. □

A locally 2-v-generated domain R is not necessarily 2-v-generated even if R is one-dimensional. For example, if R is an almost Dedekind domain that is not Dedekind, then R is locally a DVR, but R is not Mori since an almost Dedekind and Mori domain is Dedekind. For a positive result, see Proposition 3.12 below.

Lemma 3.11. *A one-dimensional Mori domain has finite character.*

Proof. If R is Mori and one-dimensional, every maximal ideal of R is divisorial [**2,** Theorem 3.1]. By [**2,** Theorem 3.3 (c)], a Mori domain is an intersection of finite character of the localizations at its maximal divisorial ideals. It follows that R has finite character. □

Proposition 3.12. *Let R be a one-dimensional domain. The following conditions are equivalent:*

(i) *R is* 2*-v-generated;*
(ii) *R is locally* 2*-v-generated and R has finite character.*

Proof. (i) $\Rightarrow$ (ii) R is locally 2-v-generated by Lemma 3.7 and has finite character by Lemma 3.11.

(ii) $\Rightarrow$ (i) See Proposition 3.9. $\square$

Proposition 3.13. *A one-dimensional stable domain R is 2-v-generated; hence R is Mori.*

Proof. Since R is locally stable, R is locally 2-v-generated by Proposition 3.5. Thus R is 2-v-generated by Corollary 3.10. $\square$

The stability assumption in Propositions 3.5 and 3.13 cannot be relaxed to finite stability. Indeed, let R be a one-dimensional valuation domain that is not a DVR. Thus R is finitely stable, but R is neither Mori, nor stable (the maximal ideal of R is not stable); see [**16,** Example 3.3]. On the other hand, we prove below that a one-dimensional finitely stable Mori domain is stable (Proposition 4.4).

4. The Mori case. In this section, we give a characterization of one-dimensional stable domains. We need a few preliminary results.

Proposition 4.1. *Let I be a stable ideal of an integral domain R. Then $I_v = I(I_v : I_v)$ is stable, and $(I_v)^2 \subseteq I$.*

Proof. Let $D = (I_v : I_v)$. Thus $(I : I) \subseteq (I_v : I) = (I_v : I_v) = D$. Since I is an invertible ideal of $(I : I)$ and $(I : I) \subseteq D$, it follows that ID is an invertible ideal of D. As D is a fractional divisorial ideal of R, we obtain that ID is a fractional divisorial ideal of R. Hence $I_v \subseteq ID$, so $I_v = ID$ since I_v is an ideal of D. Thus $I_v = ID$ is invertible in $D = (I_v : I_v)$, that is, I_v is a stable ideal of R. Also $(I_v)^2 = I_v(ID) = (I_v D)I = I_v I \subseteq I$. $\square$

Corollary 4.2. [**7,** Lemma 2.7] *In a finitely stable domain all the v-finite divisorial ideals are stable. In particular, all the divisorial ideals of a finitely stable Mori domain are stable.*

A nonzero ideal I of a domain is called a *t-ideal* if $I = \bigcup J_v$, where J runs over all finitely generated subideals of I. Divisorial ideals are t-ideals, and in a Mori domain each t-ideal is divisorial.

Corollary 4.3.

(1) *A stable radical ideal is divisorial.*
(Cf. **[17,** Corollary 4.13]. Here we do not assume that the domain R is stable).
(2) *If I is a radical ideal and each finitely generated subideal of I is stable, then I is a t-ideal.*
(3) *Each nonzero radical ideal of a finitely stable domain is a t-ideal.*
(4) *All the nonzero radical ideals of a finitely stable Mori domain are divisorial and stable.*

Proof. (1) Let I be a stable radical ideal of R. By Proposition 4.1, we have $(I_v)^2 \subseteq I$, so $I_v \subseteq I$ as the ideal I is radical. Hence $I = I_v$ is a divisorial ideal.
(2) If J is a nonzero finitely generated subideal of I, then $(J_v)^2 \subseteq J \subseteq I$ by Proposition 4.1. Since the ideal I is radical, we obtain $J_v \subseteq I$, so I is a t-ideal.
(3) follows from (2).
(4) All the radical ideals of a Mori domain are divisorial by item (2), so they are also stable by Corollary 4.2.

□

Proposition 4.4. *A one-dimensional finitely stable Mori domain is stable.*

Proof. For each maximal ideal M of R, R_M is Mori and finitely stable. Hence MR_M is divisorial (Corollary 4.3 (3)) and so stable (Corollary 4.2). By Proposition 1.2, R_M is stable. Since R has finite character (Lemma 3.11), R is stable by **[17,** Theorem 3.3]. □

Actually, as shown in Theorem 4.8 below, a finitely stable Mori domain is one-dimensional, so it is stable and 2-v-generated (Propositions 4.4 and 3.13).

The following lemma is known; we give a proof for lack of a reference.

Lemma 4.5. *Let I be a divisorial ideal of a Mori domain R. Then the domain $(I : I)$ is Mori.*

Proof. Let $J_1 \subseteq J_2 \subseteq \dots$ an infinite increasing sequence of divisorial ideals of the domain $(I : I)$. Since I is a divisorial ideal of R, the domain $(I : I)$ is a fractional divisorial ideal of R, so $J_1, J_2, \dots$ are fractional divisorial ideals of R. Let c be a nonzero element of I. Then $cJ_1 \subseteq cJ_2 \subseteq \dots$ is a an increasing sequence of divisorial ideals of R, so $cJ_n = cJ_{n+1}$ for $n \gg 0$. Thus the sequence $J_1 \subseteq J_2 \subseteq \dots$ stabilizes, implying that $(I : I)$ is Mori. □

Proposition 4.6. *Let (R, M) be a finitely stable local Mori domain. If T is a finite extension of R, then R is one-dimensional, stable and every ideal of R is* 2*-generated, thus the domain R is Noetherian. (see 2.1 for the definition of T).*

Proof. By Corollary 4.3 (4), the maximal ideal M of R is divisorial and stable. We use the setting of Theorem 2.2. By Proposition 2.4, $T = R_n$ for some integer $n \geq 0$. By Lemma 4.5, the domain $R_1 = (M : M)$ is Mori. By induction, R_k is Mori for all $k \geq 0$, so $T = R_n$ is a Mori domain. Since T has principal maximal ideals (Theorem 2.2 (c)), T is one-dimensional [**2,** Theorem 3.4]. So R is one-dimensional. By Proposition 4.4, R is stable. By Theorem 3.1, every ideal of R is 2-generated. □

Proposition 4.7. *Let (R, M) be a local domain. The following conditions are equivalent:*

(i) *R is one-dimensional and stable.*
(ii) *R is finitely stable and Mori.*

Proof. (i) ⇒ (ii) See Proposition 3.5.

(ii) ⇒ (i) By Corollary 4.3 (4), the maximal ideal M of R is stable. By Proposition 4.6, we have to consider just the case (b) of Theorem 2.2. In this case, by Theorem 2.8, R is one-dimensional. By Proposition 4.4, R is stable. □

In the next theorem we globalize Proposition 4.7:

Theorem 4.8. *Let R be an integral domain. The following two conditions are equivalent:*

(i) *R is one-dimensional and stable.*

(ii) *R is finitely stable and Mori.*

Moreover, if R satisfies these two equivalent conditions, then every overring of R also satisfies the two conditions, every overring of R is 2-v-generated, and R' is a Dedekind domain.

Proof. (i) $\Rightarrow$ (ii) Since R is locally stable, we obtain that R is locally Mori by Proposition 4.7. Since R has finite character, it follows that R is Mori [**2,** Theorem 2.4].

(ii) $\Rightarrow$ (i) Since R is locally finitely stable and locally Mori, it follows from Proposition 4.7 that R is one-dimensional and locally stable. Since R has finite character (Lemma 3.11), R is stable.

Assume that R satisfies the two conditions. Let D be an overring of R. Since R is one-dimensional and R' is Prüfer (as R is stable), it follows that each overring of R is is one-dimensional by [**9,** Theorem 6]. Since R is stable, each overring of R is stable. A one-dimensional stable domain is 2-v-generated by Proposition 3.13. Finally, R' is Prüfer and Mori, so it is Dedekind (alternatively, this follows from that a stable one-dimensional Prüfer domain is Dedekind). $\square$

In connection with Theorem 4.8, recall that an integral domain is Noetherian 2-generated if and only if it is one-dimensional, stable and divisorial ([**15,** Theorem 3.1] and [**4,** Theorem 7.3]).

However, if we assume just that R is a 2-v-generated domain, then R is not necessarily one-dimensional, and so also not finitely stable. Indeed, any Krull domain is 2-v-generated [**12,** Proposition 1.2]. In addition, it is not true that in a 2-v-generated domain each divisorial ideal is stable. In fact, if R is a Krull domain, stability coincides with invertibility. Thus each divisorial ideal of a Krull domain R is stable (i.e., invertible) if and only if R is locally factorial [**5,** Lemma 1.1]. On the other hand, a one-dimensional Krull domain is Dedekind and so each nonzero ideal is divisorial and stable.

In view of this example and of the 2-generated case, we ask:

Question 4.9. *Let R be a 2-v-generated domain R. Are the divisorial ideals of R v-stable? If R is one-dimensional, are the divisorial ideals of R stable?*

Recall that an ideal I of a domain R is *v-invertible* if $(I(R:I))_v = R$ and that a divisorial ideal I of R is *v-stable* if I is v-invertible in the ring $(I:I)$, that is $(I(I:I^2))_v = (I:I)$.

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