ON FINITELY STABLE DOMAINS, II

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ABSTRACT. Among other results, we prove the following:

(1) A locally Archimedean stable domain satisfies accp.
(2) A stable domain R is Archimedean if and only if every nonunit of R belongs to a height-one prime ideal of R' (this result is related to Ohm's Theorem for Prüfer domains).
(3) An Archimedean stable domain R is one-dimensional if and only if R' is equidimensional (generally, an Archimedean stable local domain is not necessarily one-dimensional).
(4) An Archimedean finitely stable semilocal domain with stable maximal ideals is locally Archimedean, but generally, neither Archimedean stable domains, nor Archimedean semilocal domains are necessarily locally Archimedean.

1. Introduction. In the following, R is an integral domain with quotient field K and $R \neq K$. An overring of R is a domain T such that $R \subseteq T \subseteq K$. We denote by R' the integral closure of R. By an *ideal* we mean an integral ideal.

This paper deals with Archimedean finitely stable domains and is a sequel of **[7]**.

We recall that a nonzero ideal I of R is called *stable* if I is invertible in its endomorphism ring $E(I) := (I : I)$. R is *finitely stable* if each nonzero finitely generated ideal is stable and is *stable* if each ideal is stable.

Since 1998, finitely stable and stable domains have been thoroughly investigated by Bruce Olberding in a series of papers **[16]**-**[21]**. Our paper heavily relies on Olberding's work. We thank B. Olberding for his

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valuable help. Also, as he communicated to us, his articles [**17, 18, 19**] contain some errors.

Of course, when R is a Noetherian domain, stability and finite stability coincide, but in general these two classes of rings are distinct, even if R is integrally closed: in this case R is finitely stable if and only if it is Prüfer, that is, each nonzero finitely generated ideal is invertible. Indeed, a domain R is integrally closed if and only if $R = E(I)$ for each nonzero finitely generated ideal I. However, a valuation domain is stable if and only if it is *strongly discrete*, that is, each nonzero prime ideal is not idempotent [**3,** Proposition 7.6]. Thus a valuation domain that is not strongly discrete is finitely stable, but not stable.

A domain R is finitely stable if and only if R_M is finitely stable, for each maximal ideal M [**4,** Proposition 7.3.4]. Actually, if I is a stable ideal of R, then I_S is a stable ideal of R_S for each multiplicative part $S \subseteq R$.

A domain has *finite character* if each nonzero element is contained at most in finitely many maximal ideals. A finitely stable domain need not have finite character, since any Prüfer domain is finitely stable. On the other hand, a domain is stable if and only if it is locally stable and has finite character [**19,** Theorem 3.3].

We recall that a domain R is called *Archimedean* if $\bigcap_{n\geq 0} r^n R = (0)$, for each nonunit $r \in R$. If R satisfies the ascending chain condition on principal ideals (for short, accp), then R is Archimedean. Indeed, the domain R satisfies accp if and only if $\bigcap_{n\geq 1}(\prod_{i=1}^{n} r_i R) = (0)$ for any nonunits $r_i \in R$, equivalently $\bigcap_{n\geq 1} a_n R = (0)$ if the sequence of principal ideals $a_n R$ is strictly decreasing. A *Mori domain* is a domain satisfying the ascending chain condition on divisorial ideals, so a Mori domain is Archimedean. Besides accp domains, the class of Archimedean domains includes also one-dimensional domains [**15,** Corollary 1.4] and completely integrally closed domains [**8,** Corollary 13.4]. We recall that a domain R is completely integrally closed if and only if $R = E(I)$ for each nonzero ideal I. Hence completely integrally closed domains are integrally closed and the converse holds in the Noetherian case. A completely integrally closed stable domain is Dedekind.

In [**7,** Theorem 4.8] we proved that a domain is stable and one-dimensional if and only if it is Mori and finitely stable. Here, among other results, we show that an Archimedean stable domain is one-dimensional if and only if R' is equidimensional (Proposition 4.1). The assumption that R' is equidimensional is essential, as shown in Example 5.17.

As usual, if $\mathcal{P}$ is a property of rings, then a ring R is *locally* $\mathcal{P}$ if R_M is $\mathcal{P}$ for each maximal ideal M of R. Generally, this does not imply that R_P is $\mathcal{P}$ for every prime ideal P even for a local domain (see Example 5.8 for the Archimedean property). The property $\mathcal{P}$ *localizes* if every ring satisfying $\mathcal{P}$ is locally $\mathcal{P}$. The following properties localize: stability, finite stability, Mori. However, as it is well-known, the Archimedean property, the accp and the c.i.c. property do not localize (see Section 5 below).

When studying the Archimedean property, we use Corollary 3.13: a stable domain R is Archimedean if and only if each nonunit of R belongs to a height-one prime ideal of R' (this result is related to Ohm's Theorem for Prüfer domains [**15,** Corollary 1.2]). We also prove that a stable domain is locally Archimedean if and only if $\bigcap_{n\geq 1} M^n = (0)$ for each maximal ideal M (Proposition 2.16); this condition implies accp (Proposition 2.18). So that a stable locally Archimedean domain satisfies accp (Corollary 2.19).

By Example 5.13, a stable Archimedean domain need not be locally Archimedean, and by Example 5.9 a semilocal Archimedean domain (even completely integrally closed) need not be locally Archimedean. On the positive side we show that an Archimedean finitely stable semilocal domain with stable maximal ideals is locally Archimedean (Proposition 3.14).

We thank T. Dumitrescu for pointing out some errors in previous versions of this paper.

The following results (1.1-1.4), due to Olberding, are basic for our paper.

Theorem 1.1. [**22,** Corollary 5.11] *A domain R is finitely stable if and only if it satisfies the following conditions:*

(1) *R' is a quadratic extension of R;*
(2) *R' is a Prüfer domain;*

(3) *Each maximal ideal of R has at most 2 maximal ideals of R' lying over it.*

Recall that a domain D is a *quadratic extension* of a domain R if for each $x, y \in D$ we have $xy \in xR + yR + R$. Olberding also proved that, in the local one-dimensional case, stability and finite stability are equivalent provided the maximal ideal is stable:

Proposition 1.2. [**23,** Theorem 4.2] *Let R be a local one-dimensional domain. The following conditions are equivalent:*

(i) *R is stable;*
(ii) *R is finitely stable with stable maximal ideal;*
(iii) *R' is a quadratic extension of R and R' is a Dedekind domain with at most two maximal ideals.*

Construction 1.3. [**19,** Section 4] Let (R, M) be a local domain. Set $R_i = \{0\}$ for $i < 0$, $R_0 = R$ and $M_0 = M$. Define inductively for $n > 0$: $R_n = R_{n-1}$ if R_{n-1} is not local, and $R_n = E(M_{n-1}) = (M_{n-1} : M_{n-1})$ if R_{n-1} is local with maximal ideal denoted by M_{n-1}. Set $T = \bigcup_{n \geq 0} R_n$.

Thus we have:

(a) If there exists an integer $k > 0$ such that R_k is not local, but R_i is local for $0 \leq i < k$, then $R_n = R_k$ for all $n \geq k$, and $T = R_k$.
(b) If $R_n \subsetneq R_{n+1}$ for all $n \geq 0$, all the rings R_n are local.

We will use repeatedly the following theorem of Olberding.

Theorem 1.4. [**19,** Corollary 4.3, Theorem 4.8] and its proof, and [**22,** Theorem 5.4] *Let R be a finitely stable local domain with stable maximal ideal M. With the notation of 1.3 we have:*

(1) *Each R_n is finitely stable with stable maximal ideals, and there exists an element $m \in M$ such that $M = mR_1$. Moreover, for $k \geq 1$, if R_k is local with maximal ideal M_k, then $M_k = mR_{k+1} = MR_{k+1}$, and if T is local, then its maximal ideal is $mT = MT$.*

(2) *Each R_n is a finitely generated R-module, thus T is an integral extension of R.*

We also have:

(a) *If $T = R_n$ for some $n \geq 0$, then T is a finitely generated R-module, and T has at most two maximal ideals.*
(b) *If $T \neq R_n$ for all $n \geq 0$, then T is local.*
(c) *The maximal ideals of T are principal, and the Jacobson radical of T is equal to $mT = MT$, where $mR_1 = M$.*

In addition, if R is a stable domain, then T is equal to the integral closure R' of R, and R' is a strongly discrete Prüfer domain.

2. On the Archimedean property. We start with some generalities on the Archimedean property. Then we prove that a finitely stable domain R with stable maximal ideals is locally Archimedean if and only if $\bigcap_{n\geq 1} M^n = (0)$ for each maximal ideal M of R (Proposition 2.16). We deduce from this result that a locally Archimedean stable domain satisfies accp (Corollary 2.19).

Many results in this section are related to the following theorem of J. Ohm, which will be extended in Theorem 3.12 below.

Theorem 2.1. [15, Corollary 1.6]. *Let R be a Prüfer domain. We have:*

(1) *If a is a nonunit of R belonging to just finitely many maximal ideals, then $\bigcap_{n\geq 1} a^n R = (0)$ if and only if a belongs to a height-one prime ideal.*

Hence:

(2) *If R has finite character, then R is Archimedean if and only if each nonunit of R belongs to a height-one prime ideal.*

Corollary 2.2. *An Archimedean Prüfer domain of finite character and with just finitely many height-one prime ideals is one-dimensional. In particular, an Archimedean Prüfer semilocal domain is one-dimensional.*

Proof. Let M be a maximal ideal of R. By Ohm's Theorem 2.1 (2), M is contained in the finite union of the height-one prime ideals of R.

Hence M has height one, so R is one-dimensional.

If R is Prüfer and semilocal, then R has just finitely many height-one prime ideals. Hence, if R is Archimedean, then R is one-dimensional. $\square$

Remark 2.3. *An integral domain R is Archimedean if and only if for each nonzero nonunit r of R there is an Archimedean domain D (depending on r) containing R such that r is a nonunit in D. Moreover, replacing D by $D \cap \mathrm{Frac}(R)$, we may assume that D is an overring of R.*

In particular, an intersection of Archimedean domains is Archimedean. Hence a locally Archimedean domain is Archimedean.

Corollary 2.4. *A domain R is Archimedean if and only if R has an Archimedean integral extension overring.*

Corollary 2.5. *Let $A \subseteq B$ be an extension of domains. If every nonzero nonunit of A belongs to a height-one prime ideal of B, then A is Archimedean.*

Proof. Let a be a nonzero nonunit of A. If Q is an height-one prime ideal of B containing a, then a is a nonunit in the one-dimensional (so Archimedean) domain B_Q. By Remark 2.3, A is Archimedean. $\square$

Corollary 2.6. *Let (R, M) be a local domain. If some integral extension of R has a height-one maximal ideal, then R is Archimedean.*

Proof. If Q is a height-one maximal ideal of an integral extension D of R, then $Q \cap R = M$. Hence R is Archimedean by Corollary 2.5. $\square$

Proposition 2.7. *Let R be an integral domain, and let a be a nonunit of R that belongs to just finitely many maximal ideals.*

Then $\bigcap_{n\geq 1} a^n R = (0)$ if and only if a belongs to a maximal ideal M such that $\bigcap_{n\geq 1} a^n R_M = (0)$.

Proof. Let $\mathfrak{F}$ be the set of maximal ideals containing a. We have

$$\bigcap_{n\geq 1} a^n R = R \cap \bigcap_{n\geq 1} (\bigcap_{M\in\mathfrak{F}} a^n R_M) = R \cap \bigcap_{M\in\mathfrak{F}} (\bigcap_{n\geq 1} a^n R_M) = \bigcap_{M\in\mathfrak{F}} R \cap (\bigcap_{n\geq 1} a^n R_M).$$

Since the set $\mathfrak{F}$ is finite it follows that $\bigcap_{n\geq 1} a^n R = (0)$ if and only if $R \cap \bigcap_{n\geq 1}(a^n R_M) = (0)$ for some $M \in \mathfrak{F}$, equivalently $\bigcap_{n\geq 1} a^n R_M = (0)$ for some $M \in \mathfrak{F}$ □

Proposition 2.8. *If R is an Archimedean domain and P is a principal prime ideal of R, then R_P is a DVR.*

Proof. If $P = rR$, then by [**8,** Theorem 7.6 (a) and (c)] $\bigcap_{n\geq 0} P^n = \bigcap_{n\geq 0} r^n R = (0)$ is the largest prime ideal of R properly contained in P. It follows that R_P is a one-dimensional local domain with principal maximal ideal, and so R_P is a DVR. □

Corollary 2.9. *Let R be an integral domain.*

(1) *If R is Archimedean with principal maximal ideals, then R is a principal ideal domain.*
(2) *If R is locally Archimedean with invertible maximal ideals, then R is a Dedekind domain.*

Proof. (1) By Proposition 2.8, R is one-dimensional. Since every nonzero prime ideal of R is principal, R is a principal ideal domain by [**8,** Corollary 37.9].

(2) By Proposition 2.8, R is locally a DVR (i.e., R is almost Dedekind); in particular R is one-dimensional. It follows that R is a Dedekind domain by [**8,** Theorem 37.8 (1) $\Leftrightarrow$ (4)]. □

However, an Archimedean domain R with invertible maximal ideals is not necessarily one-dimensional, even if R is Prüfer and stable: see Example 5.13 below.

Corollary 2.10. *An Archimedean Bézout domain R with stable maximal ideals is a principal ideal domain.*

Proof. As mentioned at the end of the proof of [**19,** Lemma 4.5], a stable maximal ideal M of a Prüfer domain R is invertible since

$(M:M)=R$. Thus the maximal ideals of R are finitely generated, so they are principal. Hence R is a principal ideal domain by Corollary 2.9. $\square$

None of the two conditions on the Bézout domain R in Corollary 2.10 to be a principal ideal domain can be omitted. Indeed, $R=\mathbb{Z}+X\mathbb{Q}[X]$ is a two-dimensional Bézout domain with principal maximal ideals. On the other hand, the ring of entire functions is an infinite-dimensional completely integrally closed Bézout domain. Thus R is Archimedean; see also Remark 2.11 below. Hence R has non-principal maximal ideals: these are the free maximal ideals: see [**4,** Ch. VIII, §8.1] and [**24,** Ch.6, §3].

Remark 2.11. *By* [**25,** Corollary 2.4]*, a GCD domain (in particular, a Bézout) domain is Archimedean if and only if it is completely integrally closed.*

Lemma 2.12. *Let I and J be two ideals of a ring R. If I contains a power of J, then*

$$\bigcap_{n\geq 1} J^n \subseteq \bigcap_{n\geq 1} I^n.$$

Hence, if $J\subseteq \sqrt{I}$ and the ideal J is finitely generated, then

$$\bigcap_{n\geq 1} J^n \subseteq \bigcap_{n\geq 1} I^n.$$

Proof. Let $J^k\subseteq I$ for some $k\geq 1$. Then $\bigcap_{n\geq 1} J^n=\bigcap_{n\geq 1}(J^k)^n\subseteq \bigcap_{n\geq 1} I^n$. If $J\subseteq\sqrt{I}$ and J is finitely generated, then I contains a power of J. $\square$

Corollary 2.13. *Let I be an ideal of an integral domain R. If $\bigcap_{n\geq 1} a^nR=(0)$ for all $a\in I$, then $\bigcap_{n\geq 1} a^nR=(0)$ for all $a\in\sqrt{I}$.*

Lemma 2.14. [**22,** Corollary 5.7] *Let R be a finitely stable local domain. Then a stable ideal I of R is principal in $(I:I)$. Moreover, if $I=x(I:I)$, then $I^2=xI$.*

Lemma 2.15. *Let R be a finitely stable local domain with stable maximal ideal M. Then M is the radical of a principal ideal and*

$$\bigcap_{n\geq 0} M^n = \bigcap_{n\geq 0} a^n R$$

for each element $a \in R$ such that $\sqrt{aR} = M$.

Proof. By Lemma 2.14, there exists an element $m \in M$ such that $M^2 = mM$. Clearly $\bigcap_{n\geq 0} M^n = \bigcap_{n\geq 0} m^n R$, and $M = \sqrt{mR}$. If $\sqrt{aR} = M$, then $\sqrt{aR} = \sqrt{mR}$, so

$$\bigcap_{n\geq 0} a^n R = \bigcap_{n\geq 0} m^n R = \bigcap_{n\geq 0} M^n,$$

by Lemma 2.12. □

Proposition 2.16. *Let R be a finitely stable domain with stable maximal ideals. Then R is locally Archimedean if and only if $\bigcap_{n\geq 1} M^n = (0)$ for each maximal ideal M.*

Proof. By Lemma 2.15, R is locally Archimedean if and only if $\bigcap_{n\geq 1} M^n R_M = (0)$ for every maximal ideal M. On the other hand, for every maximal ideal M we have

$$\bigcap_{n\geq 1} M^n = \bigcap_{n\geq 1} (M^n R_M \cap R) = \left(\bigcap_{n\geq 1} M^n R_M \right) \cap R,$$

so R_M is Archimedean if and only if $\bigcap_{n\geq 1} M^n = (0)$. The proposition follows. □

Remark 2.17. *For any integral domain R, the following two conditions are equivalent:*

(i) $\bigcap_{n\geq 1} I^n = (0)$ *for each ideal I;*
(ii) $\bigcap_{n\geq 1} M^n = (0)$ *for each maximal ideal M.*

If R satisfies these conditions, then R is locally Archimedean by Proposition 2.16.

Proposition 2.18. *Let R be an integral domain of finite character such that $\bigcap_{n\geq 1} M^n = (0)$ for each maximal ideal M of R. Then R satisfies accp.*

Proof. Assume that R does not satisfy accp. Then there exists an infinite sequence of nonunits r_n in R such that $\bigcap_{n\geq 1}(\prod_{i=1}^{n} r_i R) \neq (0)$. Let c be an element in this intersection. For all $n \geq 1$, each maximal ideal containing r_n contains also c, since $c \in r_n R$. As c belongs to just finitely many maximal ideals, there exists a maximal ideal M containing c such that $r_n \in M$ for infinitely many n's. Thus for each $n \geq 1$, there exist integers $1 \leq i_1 < i_2 < \ldots < i_n$ such that $r_{i_k} \in M$ for all $1 \leq k \leq n$. We have $c \in \prod_{j=1}^{i_n} r_j R \subseteq M^n$. Hence $c \in \bigcap_{n\geq 1} M^n$, a contradiction. $\square$

From Proposition 2.18 we obtain, by using Proposition 2.16:

Corollary 2.19. *A locally Archimedean finitely stable domain with stable maximal ideals and of finite character (in particular, a locally Archimedean stable domain) satisfies accp.*

However a domain R of finite character satisfying accp is not necesarily locally Archimedean, even if R is stable (see Example 5.13 below).

3. An extension of Ohm's Theorem to finitely stable domains. By using the fact that an integral extension overring of a finitely stable domain is quadratic (Theorem 1.1), so algebraically bounded, as defined in 3.1 below, we extend Ohm's Theorem from Prüfer domains to finitely stable domains (Theorem 3.12). We present a criterion for the locally Archimedean property of a stable domain in Proposition 3.8. As an application, we prove that a semilocal finitely stable Archimedean domain is locally Archimedean (Proposition 3.14).

Definition 3.1. Let $A \subseteq B$ be an extension of integral domains. The domain B is a *bounded algebraic* extension of A if there exist a nonzero element $d \in A$ and an integer $e \geq 1$ such that for each element $b \in B$ there exists a monic polynomial $f(X)$ of degree e in $A[X]$ satisfying $f(db) = 0$. The domain B is called a *bounded integral extension* of A if this property holds for $d = 1$.

Remark 3.2. *Let $A \subseteq B$ be an extension of integral domains. Then:*

(1) *B is a bounded algebraic extension of A if and only if there exists a nonzero element $d \in B$ such that $A + dB$ is a bounded integral extension of A.*
(2) *If $(A : B) \neq (0)$, then B is a bounded algebraic extension of A.*

Proposition 3.3. *Let A be an integral domain, let B be a bounded algebraic overring of A, and let a be an element of A. Then*

$$\bigcap_{n\geq 1} a^n A = (0) \Leftrightarrow \bigcap_{n\geq 1} a^n B = (0).$$

Hence, if B is Archimedean, also A is Archimedean.

Proof. Assume that $\bigcap_{n\geq 1} a^n A = (0)$. Let b be an element in $\bigcap_{n\geq 1} a^n B$. Since B is a bounded algebraic extension of A, there exist a nonzero element $d \in A$ and an integer $e \geq 1$ such that for each $x \in B$, the element dx is a root of a monic polynomial of degree e in $A[X]$. Thus, for each $b \in B$ and $n \geq 1$, by taking $x = \frac{b}{a^n}$, there exist elements $a_0, \ldots, a_{e-1} \in A$ (depending on b and on n) such that we have:

$$\left(\frac{db}{a^n}\right)^e + a_{e-1}\left(\frac{db}{a^n}\right)^{e-1} + \cdots + a_0 = 0. \tag{1}$$

Since B is an overring of A, there exists a nonzero element $c \in A$ (depending just on b) such that $c(db)^i \in A$ for all $1 \leq i \leq e$. Multiplying the equation (1) by $ca^{n(e-1)}$ we obtain that $\frac{c(db)^e}{a^n} \in A$ for all $n \geq 1$. Hence $b = 0$. We conclude that $\bigcap_{n\geq 1} a^n B = (0)$. The proposition follows. □

Corollary 3.4. *Let A be a finitely stable domain, let a be a nonzero element of A, and let B be an integral extension overring of A. Then:*

$$\bigcap_{n\geq 1} a^n A = (0) \Leftrightarrow \bigcap_{n\geq 1} a^n B = (0).$$

Proof. By Theorem 1.1, B is a quadratic extension of A, so B is a bounded integral extension of A. The corollary follows from Proposition 3.3. □

Proposition 3.5. *Let (R, M) be a finitely stable local domain with stable maximal ideal, and let D be an integral extension overring of R.*

Then R is Archimedean if and only if D has a maximal ideal N such that D_N is Archimedean.

Proof. Assume that R is Archimedean. Let $M = m(M : M)$, $m \in M$ (Lemma 2.14 or Theorem 1.4). By Corollary 3.4, $\bigcap_{n\geq 1} m^n D = (0)$. By Theorem 1.1, D has at most two maximal ideals. By Proposition 2.7, there exists a maximal ideal N of D such that $\bigcap_{n\geq 1} m^n D_N = (0)$. Since $M^2 = mM$, we see that $\bigcap_{n\geq 1} M^n D_N = (0)$. Since D is an integral extension of R and R is local, it follows that a prime ideal of D contains M if and only if it is a maximal ideal of D. Hence the only prime ideal of D_N containing MD_N is ND_N, so $ND_N = \sqrt{MD_N}$. By Corollary 2.13, D_N is Archimedean.

Conversely, if D_N is Archimedean, then R is Archimedean by Remark 2.3 since $R \subseteq D_N$ and $N \cap R = M$. □

Corollary 3.6. *Let (R, M) be a finitely stable local domain with stable maximal ideal, and let D be an integral extension overring of R. Assume that if N is a maximal ideal of D such that the domain D_N is Archimedean, then D_N is one-dimensional. Then R is Archimedean if and only D has a height-one maximal ideal.*

Proposition 3.7. *Let (R, M) be a local domain.*

(1) *If some integral extension of R has a height-one maximal ideal, then R is Archimedean.*
(2) *Conversely, we have:*
   (a) *If R is Archimedean and finitely stable, then R' has a height-one maximal ideal.*
   (b) *If R is Archimedean, finitely stable and the ideal M is stable, then T has a height-one maximal ideal (T is defined in Construction 1.3).*

Proof. (1) is Corollary 2.6.

(2, a) By Theorem 1.1, R' has at most two maximal ideals. Since R' is Prüfer, R' has at most two height-one prime ideals: Q_1 and Q_2 (not necessarily distinct). Let $P_i = Q_i \cap R$, $i = 1, 2$. Since R is Archimedean, by Corollary 3.4 we have $\bigcap_{n\geq 1} a^n R' = (0)$ for all $a \in R$. By Theorem 2.1, $M \subseteq P_1 \cup P_2$. We may assume that $M \subseteq P_1$, so $M = P_1 = Q_1 \cap R$. Hence Q_1 is a height-one maximal ideal of R'.

(2, b) By Proposition 3.5, T has a maximal ideal N such that the domain T_N is Archimedean. Hence T_N is a DVR by Proposition 2.8 as N is a principal ideal. Thus N is a height-one maximal ideal of T. □

Proposition 3.8. *Let R be a finitely stable domain. The following conditions are equivalent:*

(i) *R is locally Archimedean;*
(ii) *Each maximal ideal of R is contained in a height-one prime ideal of R' (which is necessarily maximal);*
(iii) *Each proper ideal of R is contained in a height-one maximal ideal of R'.*

Proof. (i) $\Rightarrow$ (ii) If R is local, then (ii) follows from Proposition 3.7(2)(a).

In the general case, let M be a maximal ideal of R. By the local case, the ideal MR_M of R_M is contained in a height-one prime Q of $(R_M)' = R'_M$, where R'_M is the localization of R' at the multiplicative subset $R \setminus M$. Thus $Q \cap R'$ is a height-one prime ideal of R' containing M.

(ii) $\Rightarrow$ (i) Let M be a maximal ideal of R. Let Q be a height-one prime ideal of R' containing M. Thus QR'_M is a height-one prime ideal of $R'_M = (R_M)'$ containing M. By Corollary 2.6, R_M is Archimedean, so R is locally Archimedean.

(ii) $\Leftrightarrow$ (iii) Clear. □

In the notation of 1.3, if R_k is one-dimensional for some $k \geq 0$, then all the rings R_n, as well as T, are one-dimensional since T is an integral extension of R_n, for all $n \geq 0$. For the Archimedean property we have:

Corollary 3.9. *Let (R, M) be a finitely stable local domain with stable maximal ideal. Set $R_\infty = T = \bigcup_{n\geq 0} R_n$ (see Construction 1.3). Assume that R_k is Archimedean for some $0 \leq k \leq \infty$. Then R_n is Archimedean for each n such that R_n is local. Thus R_n is Archimedean at least for each $R_n \neq T$.*

Proof. For all $0 \leq n \leq \infty$ we have $(R_n)' = R'$, so the corollary follows from Proposition 3.7. □

Corollary 3.9 might fail when T is not local, so $T = R_n$ for some integer n. Indeed, in Example 5.17, R is a stable local Archimedean domain, but $T = R' = R_1$ is not Archimedean. Moreover, we have:

Proposition 3.10. *Let R be a finitely stable local domain with stable maximal ideal. In the notation of 2.1, T is Archimedean if and only if R is one-dimensional.*

Proof. If R is one-dimensional, then T is one-dimensional, and so Archimedean, since T is an integral extension of R. Conversely, if T is Archimedean, then T, and so also R, is one-dimensional by Corollary 2.9, as the maximal ideals of T are principal. □

Corollary 3.11. [**7,** Theorem 2.8] *Let R be a finitely stable local domain with stable maximal ideal. If R is Archimedean and T is local, then R is one-dimensional.*

Proof. If T is local, T is Archimedean by Corollary 3.9 and so R is one-dimensional by Proposition 3.10. □

We now state the promised generalization of Ohm's Theorem 2.1.

Theorem 3.12. *Let R be a finitely stable domain, and let a be a nonzero nonunit of R belonging to just finitely many maximal ideals of R. The following conditions are equivalent:*

(i) $\bigcap_{n\geq 1} a^n R = (0)$;
(ii) *a belongs to a height-one prime ideal of R';*
(iii) *a belongs to a prime ideal P of R such that the domain R_P is Archimedean.*

Proof. (i) $\Rightarrow$ (ii) By Corollary 3.4, $\bigcap_{n\geq 1} a^n R' = (0)$. If N is a maximal ideal of R' containing a, then $N \cap R$ is a maximal ideal of R containing a. Since each maximal ideal of R is contained in at most two maximal ideals of R' (Theorem 1.1), it follows that a belongs to just finitely many maximal ideals of R'. Since R' is Prüfer, a belongs to a height-one prime ideal of R' (Theorem 2.1).

(ii) $\Rightarrow$ (iii) Let Q be a height-one prime ideal of R' containing a, and let $P = Q \cap R$. By Corollary 2.5 for $A = R_P$ and $B = R'_Q$, we obtain that R_P is Archimedean.

(iii) $\Rightarrow$ (i) follows from Remark 2.3. $\square$

Corollary 3.13. *Let R be a finitely stable domain of finite character (this holds, in particular, if R is a stable domain). Then R is Archimedean if and only if every nonzero nonunit in R belongs to a height-one prime ideal of R'.*

In the next proposition we extend Corollary 2.2 to finitely stable domains:

Proposition 3.14. *An Archimedean finitely stable domain of finite character such that its integral closure has just finitely many height-one prime ideals is locally Archimedean. In particular, an Archimedean finitely stable semilocal domain is locally Archimedean.*

Proof. Let M be a maximal ideal of R. As R is Archimedean, by Theorem 3.12, M is contained in the finite union of the height-one primes of R'. Thus the ideal MR' of R' is contained in one of these primes. By Proposition 3.8, R is locally Archimedean.

If R is an Archimedean finitely stable semilocal domain, then R' is Prüfer and semilocal. Thus R' has just finitely many height-one prime ideals. It follows that R is locally Archimedean. $\square$

In connection with Proposition 3.14, by Example 5.13, a stable Archimedean domain need not be locally Archimedean, and by Example 5.9 a semilocal Archimedean (even completely integrally closed) domain need not be locally Archimedean.

Question 3.15. *By Proposition 3.8, if a finitely stable domain R is locally Archimedean, then each nonzero nonunit of R belongs to a height-one maximal ideal of R'. Is the converse true? Cf. Corollary 3.13.*

4. One-dimensionality of Archimedean stable domains. In [**7,** Theorem 4.8], we proved that a finitely stable Mori domain is one-dimensional. In this section, we illustrate a general method for constructing local Archimedean stable domains of any dimension (Propositions 4.7 and 4.8); see also Example 5.17 below.

First we state a criterion for one-dimensionality of an Archimedean stable domain. We say that a domain R is *equidimensional* if $\dim R = \dim R_M$, for each maximal ideal M.

Proposition 4.1. *Let R be an Archimedean finitely stable domain of finite character (this includes the case that R is Archimedean and stable). The following conditions are equivalent:*

(i) *R is one-dimensional;*
(ii) *Every integral extension of R is equidimensional;*
(iii) *R' is equidimensional;*
(iv) *The pair (R, R') satisfies GD (the going down property) and R is equidimensional.*

Proof. (i) $\Rightarrow$ (ii) Every integral extension of R is one-dimensional, so also equidimensional.

(ii) $\Rightarrow$ (iii) Obvious.

(iii) $\Rightarrow$ (i) By Corollary 3.13, R' has a height-one maximal ideal. Thus R' is one-dimensional, and so is R.

(i) $\Rightarrow$ (iv) Clear.

(iv) $\Rightarrow$ (iii) Indeed, if B is any ring extension of an equidimensional (in particular, local) ring A such that the pair (A, B) satisfies GD, then B is equidimensional and $\dim B = \dim A$. □

Proposition 4.2. *Let R be an Archimedean finitely stable local domain with stable maximal ideal. The following conditions are equivalent:*

(i) *R is one-dimensional;*
(ii) *R is Mori;*
(iii) *T is Archimedean;*
(iv) *T is equidimensional;*
(v) *The pair (R, T) satifies GD.*

(See 1.3 for the notation T.)

Proof. (i) $\Rightarrow$ (ii) R is stable by Proposition 1.2. Thus R is Mori by [**7,** Proposition 3.13].

(ii) $\Rightarrow$ (i) by [**7,** Proposition 4.7].

(i) $\Leftrightarrow$ (iii) is Proposition 3.10.

(i) $\Rightarrow$ (iv) because T is one-dimensional.

(iv) $\Rightarrow$ (i) T has a height-one maximal ideal by Proposition 3.7 (2,b). Thus T is one-dimensional, and so is R.

(i) $\Rightarrow$ (v) This follows from that both R and T are one-dimensional.

(v) $\Rightarrow$ (iv) Since R is local, we may use the proof of the implication (iv) $\Rightarrow$ (iii) in Theorem 4.1. □

Proposition 4.3. *Let R be an Archimedean finitely stable semilocal domain. Then R is one-dimensional if and only if the pair (R, R') satisfies GD.*

Proof. Assume that the pair (R, R') satisfies GD. Let M be a maximal ideal of R. By Corollary 3.13, M is contained in the union of the height-one maximal ideals of R'. Since R' is semilocal (Theorem 1.1), it follows that M is contained in a height-one maximal ideal N of R'. As the pair (R, R') satisfies GD, this implies that M has height one, so R is one-dimensional. □

We now turn to the question how to obtain an Archimedean stable local domain (R, M) of dimension greater than one. Here we use again B. Olberding's work, and also a useful suggestion of W. Heinzer.

If R is such a domain, with the usual notation 1.3, by Corollary 3.11, T is not local and so R must satisfy condition (a) of Theorem 1.4, that is, $T = R_n$ for some $n \geq 0$. Since R is stable, $T = R'$ is a Prüfer domain and T has exactly 2 maximal ideals, which we denote by N_1 and N_2. Since R is Archimedean, T has a height-one maximal ideal by Proposition 3.7. We may assume that height $N_1 = 1$ and height $N_2 > 1$. Let $T = R_k$ with minimal $k \geq 0$, so $k > 0$ since T is not local. Thus R_{k-1} is local and, since any overring of a stable domain is stable [**19,** Theorem 5.1], R_{k-1} is stable. By Corollary 3.9, R_{k-1} is Archimedean.

Also, $\dim R_{k-1} = \dim R > 1$. Replacing R by R_{k-1}, we may assume that $R_1 = T$.

We have canonical isomorphisms $R/M \cong T/N_i$ for $i = 1, 2$, so $T = R+N_1 = R+N_2$. In Example 5.17, k is a subfield of R canonically identified with R/M, so $T = k + N_1 = k + N_2$ and $M = N_1 \cap N_2$.

Lemma 4.4. *Let D be an integral domain. The following conditions are equivalent:*

(i) *D is Prüfer and it has exactly two maximal ideals.*
(ii) *D is an intersection of two valuation domains.*

Proof. (i) $\Rightarrow$ (ii) Let Q_1 and Q_2 be the maximal ideals of D. Then $D = D_{Q_1} \cap D_{Q_2}$ is an intersection of two valuation domains.

(ii) $\Rightarrow$ (i) See [**14,** Theorem 12.2]. $\square$

Lemma 4.5. *Let $R \subseteq D$ be an extension of domains such that $D = R+xR$ for some element $x \in D$. Then D is a quadratic extension of R.*

Proof. Let $s = s_0 + s_1x, t = t_0 + t_1x$ be two elements in D, where $s_i, t_i \in R$ for $i = 1, 2$. Let I be the ideal $s_1R + t_1R$ of R. Thus $R + sR + tR = R + xI$. We have

$st \in (R + xI)^2 = (R + xI)R + (R + xI)xI \subseteq R + xI + (R + xR)I = R + xI$, implying that D is a quadratic extension of R. $\square$

We use the following lemma of Olberding:

Lemma 4.6. *Let R be a finitely stable domain. If I is a nonzero ideal of R such that IR' is principal, then I is principal in $(I : I)$, in particular I is stable.*

Proof. By Theorem 1.1, R' is a quadratic extension of R and has at most two maximal ideals. Hence we can apply [**22,** Proposition 3.6]. $\square$

For the next proposition cf. [**12,** Theorem 14]. (The statement in the proof of [**12,** Theorem 14], that $u-u^2 \in R$ for each nonunit $u \in A$, is false in general, but this error can be easily corrected.)

Proposition 4.7. *Let (V_1, Q_1) and (V_2, Q_2) be two valuation domains with no inclusion relation among them, with principal maximal ideals, with the same field of fractions L, containing a field k, and such that $V_i = k + Q_i$, for $i = 1, 2$. Let $D = V_1 \cap V_2$, $N_i = Q_i \cap D$, for $i = 1, 2$, $M = N_1 \cap N_2$, $R = k + M$, and $R_1 = (M : M)$. Then:*

(1) *N_1, N_2 are the only maximal ideals of D, $N_1 \neq N_2$, and N_1, N_2 are principal. We have $D_{N_i} = V_i$ for $i = 1, 2$, so D is Prüfer,* $\operatorname{Frac} D = L$, *and $D = k + N_i$ for $i = 1, 2$.*
(2) *If $N_1 = xD$, then $D = R + xR$, so D is a 2-generated R-module. Moreover, M is a principal ideal of D and a 2-generated ideal of R. Also D is a quadratic extension of R and $D = R' = R_1$.*
(3) *$R = k + M$ is a local domain with maximal ideal M.*
(4) *The domain R is finitely stable with stable maximal ideal M.*
(5) *R is stable if and only D is stable, equivalently D is strongly discrete.*
(6) *R is Archimedean if and only if one of the two valuation domains V_1, V_2 is one-dimensional, and so a DVR.*
(7) $\dim R > 1$ *if and only if* $\dim V_i > 1$ *for some $i = 1, 2$.*

Proof. (1) By [**14,** Theorem 12.2], N_1 and N_2 are the only maximal ideals of D, $N_1 \neq N_2$, and $D_{N_i} = V_i$ for $i = 1, 2$. For $i = 1, 2$, the maximal ideal N_i of D is locally principal, and so it is principal since D is semilocal.

For $i = 1, 2$ we have natural isomorphisms $D/N_i \cong D_{N_i}/N_i D_{N_i} = V_i/Q_i \cong k$, implying that $D = k + N_i$.

(2) Since the ideals N_1, N_2 of D are principal, we deduce that also $M = N_1 N_2$ is a principal ideal of D. Thus D is an overring of R, and $D = (M : M) = R_1$.

Since $xN_2 \subseteq N_1 N_2 = M$, we have:

$$D = k + N_1 = k + xD = k + x(k + N_2) = k + xN_2 + xk \subseteq R + kx.$$

Hence $D = R + xR$, implying by Lemma 4.5 that D is a quadratic extension of R. Since D is a Prüfer domain, and D is a quadratic, so integral, overring of R, it follows that $D = R'$.

As M is a principal ideal of D and $D = R + Rx$ is a 2-generated R-module, it follows that M is a 2-generated ideal of R.

(3) By definition, $R = k + M$, so M is a maximal ideal of R. If P is a maximal ideal of R then $P = N_i \cap R$ for some integer $i = 1, 2$, because $D = R'$ by (2). Thus $M = (N_1 \cap R) \cap (N_2 \cap R) \subseteq P$, implying that $M = P$. Thus (R, M) is a local domain.

(4) By item (1), $D = R'$ is Prüfer with two maximal ideals and by item (2), D is a quadratic extension of R. By Olberding's characterization 1.1, R is finitely stable. Since M is a principal, so stable, ideal of D and $\operatorname{Frac} R = \operatorname{Frac} D$, it follows that M is a stable ideal of R.

(5) If R is stable then D is stable, since each overring of a stable domain is stable.

Conversely, assume that D is stable. By item (2), we have $D = R' = R_1$ and $M = mR'$, where $m \in M$.

Let I be a nonzero ideal of R, and let $A = (I : I)$.

The domain D is Prüfer, and, as shown at the end of the proof of [**19,** Theorem 4.2], $D = R_1$ is a minimal overring of R. Hence by [**10,** Proposition 2.4 and Terminology on page 137], D is contained in every overring of R that is different from R. Hence, either $A = R$, or $D \subseteq A$. If $D \subseteq A$, then $A = (I : I)$ is a stable domain, so the ideal I of A is invertible in $(I : I)$, implying that I is a stable ideal of R.

Now assume that $A = R$. Since M is a principal ideal of R', it follows that $(IR' : IR') = (IM : IM)$. Also $(IM : IM)MI \subseteq IM \subseteq I$, so $(IM : IM)M \subseteq (I : I) = R$. Hence $(IR' : IR') \subseteq (R : M)$. If $(R : M) \neq (M : M)$, then the maximal ideal M of the local domain R is invertible, so principal, implying that $R = R_1 = (M : M)$, a contradiction. If $(R : M) = (M : M)$, then $(IR' : IR') = R'$. Hence IR' is an invertible, so principal, ideal of R' since R' is stable and semilocal. By Lemma 4.6, I is a stable ideal.

(6) Since R is local finitely stable, and $D = R'$, this follows from Proposition 3.7.

(7) Indeed, $\dim R = \dim D = \max(\dim V_1, \dim V_2)$. □

Corollary 4.8. *Let (V_1, Q_1) and (V_2, Q_2) be two strongly discrete val-*

uation domains with no inclusion relation among them, with principal maximal ideals, with the same field of fractions L, containing a field k, and such such that $V_i = k + Q_i$, for $i = 1, 2$. Let $\dim V_1 = 1$ and $\dim V_2 = n$, where $2 \leq n \leq \infty$. Let $D = V_1 \cap V_2$, $N_i = Q_i \cap D$, for $i = 1, 2$, $M = N_1 \cap N_2$, and $R = k + M$.

Then R is an n-dimensional Archimedean stable local domain.

Moreover, we have:

(1) *R satisfies accp, but R' is not Archimedean.*
(2) *The pair (R, D) does not satisfy GD (the going down property).*

Proof. By Proposition 4.7, R is an n-dimensional Archimedean stable local domain since D is a strongly discrete Prüfer domain.

(1) By Corollary 2.19, any Archimedean stable domain satisfies accp. By Corollary 2.2, R' is not Archimedean since R' is semilocal of dimension greater than 1.

(2) R does not satisfy GD by Proposition 4.3. □

5. Examples. It is well-known that the accp and the Archimedean properties do not localize. In [**11,** Example 2] Anne Grams constructs a one-dimensional Prüfer domain of finite character which satisfies accp (the ascending chain condition on principal ideals) and each of its localizations but one is a DVR, while the other one is a valuation domain that is not a DVR, so it does not satisfy accp (see comments and more examples in [**1**] and its references). Also, [**11**] (page 328) provides a general construction of an almost Dedekind domain A with accp whose Nagata ring $A(X)$ is not an accp domain (so that $A[X]$ is accp, while its localization $A(X)$ is not accp). This example as well as [**11,** Example 2] is one-dimensional, so it is locally Archimedean.

The ring of entire functions E is an infinite-dimensional completely integrally closed (hence Archimedean) Bézout domain [**4,** Section 8.1], but it is not locally Archimedean since the localizations at maximal ideals are valuation domains, and a valuation domain that is not a field is Archimedean if and only if it is one-dimensional. The ring E does not satisfy accp and it does not have finite character: for example, if f is a nonzero entire function with infinitely many zeros $c_1, c_2, \ldots$ (e.g., $\sin z$), then $f \in \bigcap_{n=1}^{\infty} \prod_{i=1}^{n} (Z - c_i)$, so the domain E does not satisfy accp, and E does not have finite character since f belongs to the maximal ideals $(Z - c_i)E$ for all i. .

We construct in Example 5.9 below a completely integrally closed (for short, c.i.c.) domain R satisfying accp with only two maximal ideals such that, for each maximal ideal M, R_M is not Archimedean; thus R_M does not satisfy accp. Of course, R is Archimedean and has finite character. We construct first a c.i.c. local domain A with accp such that A_P is not Archimedean for some prime ideal P (Example 5.8). Then we "double" this construction to obtain Example 5.9 (see Remark 5.10).

We also construct a stable Prüfer domain R with accp that is not locally Archimedean (Example 5.13), thus the converse of Corollary 2.19 is false.

In Example 5.14 we give an example of a local one-dimensional domain R such that R' is a finite extension of R, the ring R' is a PID, so stable, but R is not even finitely stable (cf. Proposition 4.7 (5) and Lemma 4.6).

Finally, following Olberding ([**16,** Proposition 5.4]), in Example 5.15 we construct a stable valuation domain with prime spectrum consisting of an infinite descending chain of prime ideals. We use this example in the last Example 5.17, where we present a stable Archimedean local domain of arbitrary dimension.

Recall that a set of subrings $\mathbf{S}$ of a ring R is *directed* if for each $A, B \in \mathbf{S}$ there exists $C \in \mathbf{S}$ such that both A and B are contained in C.

Lemma 5.1. *Let R be an integral domain that is a directed union of a set $\mathbf{S}$ of c.i.c. subrings. Assume $A = R \cap \mathrm{Frac}(A)$ for each $A \in \mathbf{S}$. Then R is c.i.c..*

Proof. Assume for $f \in R \setminus \{0\}$ and $g \in \mathrm{Frac}(R)$ that $fg^n \in R$ for all $n \geq 1$. Since the union of the subrings in $\mathbf{S}$ is directed, there exists a domain $A \in \mathbf{S}$ such that $f \in A$ and $g \in \mathrm{Frac}(A)$. Hence $fg^n \in R \cap \mathrm{Frac}(A) = A$, for all $n \geq 1$. Since A is c.i.c., we obtain that $g \in A \subseteq R$. Thus R is c.i.c. $\square$

Lemma 5.2. *Let R be an integral domain that is a directed union of a set $\mathbf{S}$ of accp subrings. Assume that for each $A \in \mathbf{S}$ there exists a retraction $\varphi_A : R \to A$ mapping nonunits of R to nonunits of A. Then R satisfies accp.*

Proof. Assume that R does not satisfy accp. Hence there exists a strictly increasing infinite sequence of nonzero principal ideals in R:

$$r_1 R \subsetneq r_2 R \subsetneq r_3 R \subsetneq \dots$$

We have $r_1 \in A$ for some domain $A \in \mathbf{S}$. Let $\varphi = \varphi_A$. Since $r_1 \neq 0$, there is an increasing sequence of nonzero principal ideals in the ring A:

$$r_1 A = \varphi(r_1)A \subseteq \varphi(r_2)A \subseteq \varphi(r_3)A \subseteq \dots$$

For each $n \geq 1$, we have $\frac{r_n}{r_{n+1}} \in R \setminus \mathcal{U}(R)$; hence $\varphi\left(\frac{r_n}{r_{n+1}}\right) = \frac{\varphi(r_n)}{\varphi(r_{n+1})} \in A \setminus \mathcal{U}(A)$. It follows that all the inclusions in the sequence

$$\varphi(r_1)A \subseteq \varphi(r_2)A \subseteq \varphi(r_3)A \subseteq \dots$$

are strict, contradicting the assumption that A satisfies accp. □

Proposition 5.3. *Let if $\varphi : A \to B$ be an homomorphism of rings. Consider the following two conditions:,*

1. *φ maps nonunits to nonunits.*
2. $\ker \varphi \subseteq \mathrm{Jac}(A)$.

Then (1) $\Rightarrow$ (2). If φ is surjective, then the two conditions are equivalent. In particular, if A is local, then any surjective homomorphism $\varphi : A \twoheadrightarrow B$ maps nonunits to nonunits.

Proof. (1) $\Rightarrow$ (2) Let $c \in \ker \varphi$. Assume that $c \notin \mathrm{Jac}(A)$. Since φ is surjective, there exists an element $a \in A$ such that $1 + ac$ is not a unit in A, although $\varphi(1 + ac) = 1$, a contradiction.

(2) $\Rightarrow$ (1) assuming that φ is surjective. Assume that for some nonunit $c \in A$, the element $\varphi(c)$ is invertible in B. Since φ is surjective, there exists an element $a \in A$ such that $\varphi(c)\varphi(a) = 1$. Hence $\varphi(1 - ca) = 0$, so $1 - ca \in J(A)$, implying that ca is invertible in A. Thus c is invertible in A, a contradiction. □

Proposition 5.4. *Let R be an integral domain that is a directed union of a set $\mathbf{S}$ of c.i.c. subrings satisfying accp. Assume that for every $A \in \mathbf{S}$ there exists a retraction $\varphi_A : R \to A$ mapping nonunits of R to nonunits of A. Then R is c.i.c. and it satisfies accp.*

Proof. The domain R satisfies accp by Lemma 5.2.

For $A \in \mathbf{S}$ we have $A = R \cap \mathrm{Frac}(A)$, since A is a retract of R. Thus R is c.i.c. by Lemma 5.1. $\square$

Corollary 5.5. *Let R be an integral domain that is a directed union of a set $\mathbf{S}$ of integrally closed Noetherian subrings. Assume that for every $A \in \mathbf{S}$ there exists a retraction $\varphi_A : R \to A$ mapping nonunits of R to nonunits of A. Then R is c.i.c. and it satisfies accp.*

Proof. Indeed, a Noetherian ring satisfies accp, and an integrally closed Noetherian domain is c.i.c. Hence the corollary follows from Proposition 5.4. $\square$

Lemma 5.6. *Let A be an integrally closed domain, let $n \geq 1$ and let X, Y, Z_i $(1 \leq i \leq n)$ be independent indeterminates over A. Then the domain*

$$D = A[X, Y, Z_i, \frac{XZ_i}{Y^i} \ (1 \leq i \leq n)]$$

is integrally closed.

Proof. Let S be the multiplicative monoid generated by $X, Y, Z_i, \frac{XZ_i}{Y^i}$ $(1 \leq i \leq n)$. We show that the monoid S is integrally closed. Let G be the group of fractions of S, that is, G is the multiplicative group generated by X, Y, Z_i $(1 \leq i \leq n)$. Let g be an element of G such that $g^k \in S$ for some integer $k \geq 1$. Since the monoid generated by $X, Y, \frac{1}{Y}, Z_i$ $(1 \leq i \leq n)$ is integrally closed, it follows that g belongs to this monoid. Thus

$$g = X^f Y^m \prod_{i=1}^{n} Z_i^{r_i},$$

where f, r_i are nonnegative integers for all i, and m is an integer. We have

$$g^k = X^{kf}Y^{km}\prod_{i=1}^{n} Z_i^{kr_i} = X^aY^b\prod_{i=1}^{n} Z_i^{c_i}\prod_{i=1}^{n}\left(\frac{XZ_i}{Y^i}\right)^{e_i}, \tag{2}$$

where a, b, c_i, e_i are nonnegative integers for all i. We may assume that the sum $a + \sum_{i=1}^n ic_i$ is minimal.

First assume that $c_i = 0$ for all i. Comparing exponents of the indeterminates Z_i on the two sides of (2), we obtain that $e_i = kr_i$ for all i, so a and b are divisible by k. It follows that $g \in S$.

Now assume that $c_{i_0} > 0$ for some index i_0. If $a > 0$, then

$$g^k = X^{a-1}Y^{b+i_0}\left(Z_{i_0}^{c_{i_0}-1}\prod_{i\neq i_0} Z_i^{c_i}\right)\left(\frac{XZ_{i_0}}{Y^{i_0}}\prod_{i=1}^{n}\frac{X^{e_i}Z_i^{e_i}}{Y^{ie_i}}\right),$$

contradicting the minimality of $a + \sum_{i=1}^n ic_i$. Thus $a = 0$.

Let j, q be integers such that $c_j > 0$ and $e_q > 0$. If $j > q$, we interchange Z_j and Z_q as follows:

$$g^k = Y^{b+j-q}\left(Z_qZ_j^{c_j-1}\prod_{i:i\neq j} Z_i^{c_i}\right)\left(\frac{XZ_j}{Y^j}\left(\frac{XZ_q}{Y^q}\right)^{e_q-1}\prod_{i:i\neq q}^{n}\left(\frac{XZ_i}{Y^i}\right)^{e_i}\right),$$

contradicting the minimality assumption on $a + \sum_{i=1}^n ic_i$. Hence $j \le q$ for all j and q such that c_j and e_q do not vanish. We have

$$g^k = X^{kf}Y^{km}\prod_{i=1}^{n} Z_i^{kr_i} = Y^b\prod_{i=1}^{n} Z_i^{c_i}\prod_{i=1}^{n}\left(\frac{XZ_i}{Y^i}\right)^{e_i}. \tag{3}$$

Let $1 \le q \le n$ be an integer such that $q \neq q_0 = \min_{e_i>0} i$. Since either $c_q = 0$ or $e_q = 0$, and since by (3) we have $c_q + e_q = kr_q$, it follows that both c_q and e_q are divisible by k. Comparing the exponents of X on both sides of (3), since all c_i, e_i for $i \neq q_0$ are divisible by k, we see that also e_{q_0} is divisible by k. Clearly, also c_{q_0} and b are divisible by k. Thus $g \in S$, so the monoid S is integrally closed. By **[9,** Corollary 12.11 (2)], the domain D is integrally closed. □

Remark 5.7. *The domain D in Lemma 5.6 is isomorphic to a subring of a polynomial ring over the domain A in $n+2$ indeterminates. Indeed, for $U_i = \frac{Z_i}{Y^i}$ $(0 \le i \le n)$ we have $D = A[X, Y, XU_i, Y^iU_i\ (1 \le i \le n)] \subseteq$*

$k[X, Y, U_i\ (0 \leq i \leq n)]$. *Similarly, the domains* D *of Example 5.8 and* A *of Example 5.9 below may be viewed as subrings of a polynomial ring over* k *in infinitely many indeterminates.*

Example 5.8. *A completely integrally closed local domain* R *with accp such that* R_P *is not Archimedean for some prime ideal* P*.*

Let k be a field and let

$$D = k[X, Y, Z_n, \frac{XZ_n}{Y^n}\ (n \geq 1)],$$

where $X, Y, Z_n\ (n \geq 1)$ are independent indeterminates over k. Let M be the maximal ideal of D generated by the elements $X, Y, Z_n, \frac{XZ_n}{Y^n}\ (n \geq 1)$. Set

$$R = D_M \text{ and } P = \langle X, Y, \frac{XZ_n}{Y^n}\ (n \geq 1)\rangle R.$$

For each $n \geq 1$, let $D_n = k[X, Y, Z_i, \frac{XZ_i}{Y^i}\ (1 \leq i \leq n)]$ and $R_n = (D_n)_{M_n}$, where M_n is the maximal ideal of D_n generated by $X, Y, Z_i, \frac{XZ_i}{Y^i}\ (1 \leq i \leq n)$, thus $M_n = M \cap D_n$.

Clearly $R_1 \subseteq R_2 \subseteq \dots$ and $R = \bigcup_n R_n$. For each n, there exists a retraction $\varphi_n : R \to R_n$ that maps to 0 each indeterminate Z_i, for $i > n$. Clearly $\varphi_n(MR) \subseteq M_n R_n$. By Lemma 5.6, the domains R_n are integrally closed. Since the domains R_n are Noetherian, from Corollary 5.5 it follows that R is c.i.c. and R satisfies accp.

The ideal P is prime since P is the set of all rational functions in R vanishing when plugging in first $X = 0$, and then $Y = 0$ (thus these rational functions are defined for $X = 0$, and *after* plugging in $X = 0$, we obtain a function defined for $Y = 0$). For all $n \geq 1$, the elements Z_n are invertible in R_P, so $\frac{X}{Y^n} \in R_P$. Since Y is not invertible in R_P, we see that the domain R_P is not Archimedean. □

Example 5.9. *A completely integrally closed domain* R *satisfying accp with just two maximal ideals such that, for each maximal ideal* M*, the domain* R_M *is not Archimedean.*

Let k be a field and let

$$A = k[X_1, Y_1, Z_{1,n}, \frac{X_1 Z_{1,n}}{Y_1^n};\ X_2, Y_2, Z_{2,n}, \frac{X_2 Z_{2,n}}{Y_2^n}\ (n \geq 1)],$$

where $X_i, Y_i, Z_{i,n} (i = 1, 2,\ n \geq 1)$ are independent indeterminates over k.

Let

$$P_1 = \langle X_1, Y_1, \frac{X_1 Z_{1,n}}{Y_1^n}, Z_{2,n}, \frac{X_2 Z_{2,n}}{Y_2^n}\ (n \geq 1)\rangle A \text{ and}$$
$$P_2 = \langle X_2, Y_2, \frac{X_2 Z_{2,n}}{Y_2^n}, Z_{1,n}, \frac{X_1 Z_{1,n}}{Y_1^n}\ (n \geq 1)\rangle A.$$

The ideal P_1 is prime since it is the set of all rational functions in A vanishing when plugging in first $X_1 = Z_{2,n} = 0$ for all n, and then $Y_1 = 0$. Similarly, the ideal P_2 is prime.

For all $n \geq 1$, the elements $Z_{1,n}$ are invertible in A_{P_1}, so $\frac{X_1}{Y_1^n} \in A_{P_1}$. Since Y_1 is not invertible in A_{P_1}, we see that the domain A_{P_1} is not Archimedean. Similarly, the domain A_{P_2} is not Archimedean.

Let $S = A \setminus (P_1 \cup P_2)$, and $R = A_S$, thus $R = A_{P_1} \cap A_{P_2}$. Hence R has just two maximal ideals, namely $M_1 = P_1 A_{P_1} \cap R$ and $M_2 = P_2 A_{P_2} \cap R$. We have $R_{M_i} = A_{P_i}$ for $i = 1, 2$, so the domains R_{M_1} and R_{M_2} are not Archimedean.

For each $n \geq 1$, let

$$A_n = k[X_1, Y_1, Z_{1,j}, \frac{X_1 Z_{1,j}}{Y_1^j};\ X_2, Y_2, Z_{2,j}, \frac{X_2 Z_{2,j}}{Y_2^j}\ (1 \leq j \leq n)]$$

and $R_n = (A_n)_{S_n}$, where $S_n = S \cap A_n$.

By Lemma 5.6, the domains

$$D_n = k[X_1, Y_1, Z_{1,j}, \frac{Z_{1,j} X_1}{Y_1^j}\ (1 \leq j \leq n)]$$

and $A_n = D_n[X_2, Y_2, Z_{2,j}, \frac{X_2 Z_{2,j}}{Y_2^j}\ (1 \leq j \leq n)]$ are integrally closed. Hence R_n is integrally closed.

Clearly $R_1 \subseteq R_2 \subseteq \dots$ and $R = \bigcup_n R_n$. For each $n \geq 1$ we have a retraction $\varphi_n : R = A_S \to R_n$ that maps to 0 each indeterminate $Z_{i,j}$ for $i = 1, 2$ and $j > n$ since the elements $Z_{1,j}$ and $Z_{2,j}$ do not belong to S. Clearly the elements in $\varphi_n(M_1 \cup M_2)$ are nonunits in R_n. Since the domains R_n are Noetherian and integrally closed, it follows from Corollary 5.5 that R is c.i.c. and R satisfies accp. □

Of course, the domain R in Example 5.9 is not Mori, since any localization of a Mori domain is Mori and so Archimedean.

Remark 5.10. *If D and A are the domains defined in Examples 5.8 and 5.9, respectively, then $A \cong D \otimes_k D$.*

The next example 5.13 shows that a stable Archimedean domain may not be locally Archimedean. We will use below the following well-known facts:

Lemma 5.11. *(see* [**4,** Lemma 1.1.4 and Proposition 5.3.3]*) Let U be a valuation domain (possibly a field), let $K = \mathrm{Frac}(U)$, and let X be an indeterminate over U. Then $V = U + XK[X]_{\langle X \rangle}$ is a valuation domain. If U is strongly discrete, then also V is strongly discrete. The prime ideals of V are all the ideals $P + XK[X]_{\langle X \rangle}$, where P is a prime ideal of U. Moreover, if P is nonzero, then $P + XK[X]_{\langle X \rangle} = PV$ and $(P + XK[X]_{\langle X \rangle}) \cap U = P$. For $P = (0)$ the ideal $XK[X]_{\langle X \rangle}$ is the least nonzero prime ideal of V. Thus, if U is finite dimensional, then* $\dim V = \dim U + 1$.

Corollary 5.12. *Let X and Y be two independent indeterminates over a field k, let $C = k[Y, \frac{X}{Y^n}\ (n \geq 1)]$, and let P be the maximal ideal $YC = \langle X, \frac{X}{Y^n}\ (n \geq 1) \rangle$ of C. Then $V = C_P$ is a strongly discrete 2-dimensional valuation domain.*

Proof. Clearly, $V = k[Y]_{\langle Y \rangle} + Xk(Y)[X]_{\langle X \rangle}$. By Lemma 5.11, V is a strongly valuation domain of dimension 2. □

Example 5.13. *A stable 2-dimensional Prüfer domain R satisfying accp with just two maximal ideals of height 2. Thus for each maximal ideal M of R, except the two maximal ideals of height 2, the domain R_M is a DVR. Also R is Archimedean, but not locally Archimedean: R_M is not Archimedean if M is a maximal ideal of R of height 2.*

Let X and Y be two independent indeterminates over a field k. Set

$$R = k[X, Y, \frac{X(1-X)^n}{Y^n}, \ \frac{Y^{n+1}}{(1-X)^n} \ (n \geq 1)]_S,$$

where $S = k[Y] \setminus Yk[Y]$.

Let $T = \frac{1-X}{Y}$. We have $X = 1 - YT$, so

$$R = k[Y, YT, (1-YT)T^n, \frac{Y}{T^n} \ (n \geq 1)]_S$$

(as shown in item (1) below, R satisfies accp, thus R is Archimedean, although $\frac{Y}{T^n} \in R$ for all $n \geq 1$. This is not a contradiction since $T \notin R$).

(1) *R satisfies accp.*

Let f and g_n $(n \geq 1)$ be nonzero elements of R such that $\frac{f}{\prod_{i=1}^n g_i} \in R$ for all $n \geq 1$. To prove that g_i is a unit for $i \gg 0$, we may assume that $g_i \in k[Y, YT, (1-YT)T^n, \frac{Y}{T^n} \ (n \geq 1)]$, for all $i \geq 1$.

Since the elements Y and T are algebraically independent over k, we may view the ring $k[Y, YT, (1-YT)T^n, \frac{Y}{T^n} \ (n \geq 1)]$ as a subring of the polynomial ring $k(T)[Y]$. Thus for $i \gg 0$ we have $\deg_Y(g_i) = 0$, that is, $g_i \in k(T)$.

For $i \gg 0$, since $g_i \in k[Y, YT, (1-YT)T^n, \frac{Y}{T^n} \ (n \geq 1)]$, by plugging in $Y = 0$, we obtain that $g_i \in k[T]$; by plugging in $Y = \frac{1}{T}$, we obtain that $g_i \in k[\frac{1}{T}]$, so $g_i \in k[T] \cap k[\frac{1}{T}] = k$. We conclude that R satisfies accp.

(2) *R is a stable 2-dimensional Prüfer domain with just two maximal ideals of height 2.*

Let M be a maximal ideal of R.

(3) *Assume that $Y \notin M$. Then:*
- *R_M is a DVR, so* height $M = 1$.
- *Each nonzero element of R belongs to just finitely many maximal ideals of R not containing Y.*

Clearly $R \subseteq D = k(Y)[X, \frac{1}{1-X}] \subseteq R_M$, and R_M is a ring of fractions of D. Hence R_M is a local PID, that is, a DVR.

For each maximal ideal M of R not containing Y we have $MR_M = PD_P$ for $P = M \cap D$, and since D is a PID, each nonzero element of R belongs to just finitely many prime ideals of D, and so it belongs to just finitely many maximal ideals of R not containing Y.

(4) *Assume that* $Y \in M$. *Then* R_M *is a stable* 2*-dimensional valuation domain, in particular* height $M = 2$.

Since $X(1-X) \in RY \subseteq M$ it follows that either $X \in M$ or $1 - X \in M$.

(a) *Assume that* $Y, X \in M$.
Clearly, $C = k[Y, \frac{X}{Y^n}\ (n \geq 1)] \subseteq R_M$. Since the maximal ideal $P = \langle Y, \frac{X}{Y^n}\ (n \geq 1)\rangle$ of C is contained in MR_M, it follows that $P = MR_M \cap C$. Since $R \subseteq C[\frac{1}{1-X}] \subseteq C_P \subseteq R_M$, it follows that $C_P = R_M$. By Corollary 5.12, $R_M = C_P$ is a 2-dimensional strongly discrete, and so stable, valuation domain. Also M is uniquely determined by the requirement $Y, X \in M$, namely $M = PC_P \cap R$.

(b) *Assume that* $Y, 1 - X \in M$.
Recall that $T = \frac{1-X}{Y}$. Since $XT = \frac{X(1-X)}{Y} \in R$ and X is a unit in R_M, we see that $T \in R_M$. Hence

$$\widetilde{C} = k[T, \frac{Y}{T^n}\ (n \geq 1)] \subseteq R_M,$$

the maximal ideal $\widetilde{P} = \langle T, \frac{Y}{T^n}\ (n \geq 1)\rangle$ of the ring $\widetilde{C}$ is contained in MR_M, and $R \subseteq \widetilde{C}_{\widetilde{P}} \subseteq R_M$. As in item (b) (i), we conclude that $R_M = \widetilde{C}_{\widetilde{P}}$ is a 2-dimensional strongly discrete, and so stable, valuation domain and that M is uniquely determined by the requirement $Y, 1 - X \in M$.

Thus R has finite character and each localization of R at a maximal ideal is a stable valuation domain. Hence R is a stable Prüfer domain [**19,** Theorem 3.3].

We have also proved that R is 2-dimensional with exactly 2 maximal ideals of height 2. The localizations at these two maximal ideals are not Archimedean, as seen directly from the above proof. Actually, as it is well-known, a valuation domain is Archimedean if and only if it is one-dimensional. The maximal ideals of R are invertible since R is stable and Prüfer. Thus in Corollary 2.9 (2) we may not assume just that R is Archimedean rather than locally Archimedean.

Example 5.13 shows that the converse of Corollary 2.19 is false: a stable domain R which satisfies accp need not be locally Archimedean, even if R is Prüfer and 2-dimensional.

Example 5.14. *A local integral domain (R, M) with the following properties:*

(1) *R is one-dimensional, Noetherian, not (finitely) stable, but with stable maximal ideal.*
(2) *$R' = (M : M)$ is a finitely generated R-module.*
(3) *R' is a principal ideal local domain, so R' is stable and Prüfer.*

Let $K = \mathbb{Q}(\sqrt[3]{2})$ and $R = \mathbb{Q} + XK[[X]]$. Thus $R' = K[[X]]$ is a principal ideal local domain with maximal ideal $M = XK[[X]]$; so R is one-dimensional, and R' is a 3-generated R-module. By the Eakin-Nagata Theorem, R is Noetherian. Clearly, R' is not a quadratic extension of R, so R is not finitely stable. Explicitly, the fractional ideal $I = \langle 1, \sqrt[3]{2} \rangle$ of R is not stable (equivalently, the ideal $\langle X, X\sqrt[3]{2} \rangle$ of R is not stable). Indeed, $I^2 = \langle 1, \sqrt[3]{2}, \sqrt[3]{4} \rangle$ and $(I : I^2) = XR$, so $I(I : I^2) = XR \neq R$. It follows that I is not stable. The maximal ideal M of R is stable, since M is an ideal of the stable domain R' which is an overring of R. $\square$

In the next example we present a well-known construction which is related to the construction in the proof of the Kaplansky-Jaffard-Ohm Theorem [**5,** Ch.III, Theorem 5.3]. This example illustrates explicitly a particular case of Olberding's Theorem [**16,** Proposition 5.4], and will be also used for Example 5.17.

Example 5.15. *For each $1 \leq n \leq \infty$ and for a field k, a strongly discrete, so stable, n-dimensional valuation domain V containing k. In particular, if $n = \infty$, the nonzero prime ideals of V form a descending infinite sequence, so the height of every nonzero prime ideal of V is infinite. Moreover, for all n,* $\mathrm{Frac}\, V$ *is a purely transcendental extension of k of transcendence degree $\aleph_0$.*

First let $n = \infty$. Let $V = A_Q$, where

$$A = k[X_n, \frac{X_{n+1}}{X_n^i}\ (n \geq 1, i \geq 1)],$$

k is a field, X_n $(n \geq 1)$ are independent indeterminates over k, and $Q = X_1 A = \langle X_n, \frac{X_{n+1}}{X_n^i}\ (n \geq 1)\rangle A$ is a maximal ideal of A.

It is easy to show that $V = \bigcup_{n=1}^{\infty} V_n$ (an ascending union), where V_n are subrings of V defined inductively as follows: $V_0 = k$, and for $n \geq 1$, we let $V_n = V_{n-1} + X_n\,(k(X_1, \ldots, X_{n-1}[X_n])_{\langle X_n\rangle}$.

By induction, $\mathrm{Frac}(V_n) = k(X_1, \ldots, X_n)$ for $n \geq 1$. Thus $V_n = V_{n-1} + X_n\left(\mathrm{Frac}(V_{n-1})[X_n]_{\langle X_n\rangle}\right)$ for $n \geq 1$. Hence by Corollary 5.12, we obtain inductively that V_n is a strongly discrete valuation domain of dimension n, with maximal ideal $M_n = X_1 V_n$, and that the nonzero prime ideals of V_n form a descending chain

$$M_n = P_{n,n} \supsetneqq P_{n,n-1} \supsetneqq \cdots \supsetneqq P_{n,1}.$$

It follows that the domain $V = \bigcup_{n=1}^{\infty} V_n$ is a strongly discrete, so stable, valuation domain with maximal ideal $M = X_1 V$. Let P be a nonzero prime ideal of V. Since $P = \bigcup_{n=1}^{\infty}(P \cap V_n)$, we have $P \cap V_n \neq (0)$ for some integer $n \geq 1$. By Lemma 5.11, $P = (P \cap V_n)V = P_{n,i}V$ for an integer $1 \leq i \leq n$. If n is minimal, then $P_{n,i}$ is the least nonzero prime ideal of V_n, so $i = 1$. Hence the nonzero prime ideals of V form an infinite descending chain $M = P_1 \supsetneqq P_2 \supsetneqq \ldots$, where $P_n = P_{n,1}V$ for all $n \geq 1$.

Thus for all $n \geq 1$, P_n is the ideal of V generated by the one-dimensional subspace $X_n k(X_1, X_2, \ldots, X_{n-1})$ of V over the field $k(X_1, X_2, \ldots, X_{n-1})$.

Explicitly, for all $n \geq 1$ we have

$$P_n = \sum_{i=n}^{\infty} X_i \left(k(X_1, \ldots, X_{i-1})[X_i]_{\langle X_i\rangle}\right).$$

If n is finite, similarly to the definition of V above, we define $V_n = A_{Q_n}$, where

$$A = k[X_j, \frac{X_{j+1}}{X_j^i}\ (1 \leq 1 < n, i \geq 1)],$$

and $Q = X_1 A$ is a maximal ideal of A. (if $n = 1$, then $A = k[X_1]$). $\square$

In the last example we exhibit an n-dimensional Archimedean stable local domain, for each $n \geq 2$; thus answering in the negative the question posed in [**6,** Problem 7.1]. (For details concerning this example, see Propositions 4.7 and 4.8 above.)

We need the following lemma:

Lemma 5.16. *Let k be a field, and let $L \neq k$ be a purely transcendental field extension of k with* $\operatorname{tr.d.} L/k \leq \aleph_0$. *Then there exists a DVR* (V, N) *such that* $\operatorname{Frac} V = L$ *and* $V/N = k$.

Proof. Let $L = k(B)$, where B is a set of algebraically independent elements over k. Since $\operatorname{tr.d.} L/k \leq \aleph_0 \leq \operatorname{tr.d.} k((X))/k$ [**13,** Lemma 1, Section 3], there exists a subset B_0 of $k((X))$ containing X such that $|B_0| = |B|$. Thus there exists an isomorphism over k of the fields L and $k(B_0)$ mapping B onto B_0. Hence we may assume that $L = k(B) \subseteq k((X))$ and that $X \in B$. Define $V = k[[X]] \cap L$. Thus V is a DVR with maximal ideal XV, and $V/XV \cong k$. Since $k[B] \subseteq V \subseteq L = k(B)$, it follows that $\operatorname{Frac}(V) = L$. $\square$

Example 5.17. *For $1 \leq n \leq \infty$, a stable n-dimensional Archimedean local domain* (R, M).

By Example 5.15, for any field k, there exists a stable n-dimensional valuation domain (V_2, Q_2) containing k such that $\operatorname{Frac} V_2 = L$ is a purely transcendental extension of k and $V_2/Q_2 = k$. By Lemma 5.16, there exists a DVR (V_1, Q_1) containing k such that $\operatorname{Frac}(V_1) = L$, and $V_1/Q_1 = k$. By Proposition 4.7, there exists a local Archimedean finitely stable domain R such that $R' = V_1 \cap V_2$ and by Proposition 4.8 such a domain is stable. $\square$

By Example 5.17 and by Proposition 4.8 (1), the integral closure of an Archimedean domain, or even an accp stable domain, is not necessarily Archimedean. The domain $\mathbb{Z} + X\overline{\mathbb{Z}}[X]$, where $\overline{\mathbb{Z}}$ is the ring of all algebraic integers, satisfies accp while $R' = \overline{\mathbb{Z}}[X]$ does not, although R' is Archimedean [**2,** Example 5.1].

REFERENCES

1. D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Factorization in integral domains*, J. Pure Appl. Algebra **69** (1990), 1–19.

2. D. D. Anderson, D. F. Anderson, and M. Zafrullah, *Rings between* $D[X]$ *and* $K[X]$, Houston J. Math. **17** (1991), 109–129.

3. S. Bazzoni and L. Salce, *Warfield domains*, J. Algebra **185** (1996), 836–868.

4. M. Fontana, J.A. Huckaba and I.J. Papick, *Prüfer domains*, Monographs and Textbooks in Pure and Applied Mathematics **203**, M. Dekker, New York, 1997.

5. L. Fuchs and L. Salce, *Modules over Non-Noetherian Domains*, Mathematical Surveys and Monographs **84**, American Mathematical Society, 2001.

6. S. Gabelli, *Ten problems on stability of domains*, Commutative Algebra - Recent Advances in Commutative Rings, Integer-valued Polynomials, and Polynomial functions, 175–193, Springer, New York, 2014.

7. S. Gabelli and M. Roitman, *On finitely stable domains, I*, submitted to J. Comm. Alg.

8. R. Gilmer, *Multiplicative Ideal Theory*, Dekker, New York, 1972.

9. R. Gilmer, *Commutative semigroup rings*, University of Chicago, 1984.

10. R.Gilmer and W. Heinzer, *Intersections of quotients rings of an integral domain*, J. Math. Kyoto Univ. **7** (1967), 133–149.

11. A. Grams, *Atomic rings and the ascending chain condition for principal ideals*, Proc. Camb. Phil. Soc., **75** (1974), 321–329.

12. A. Jaballah, *Maximal non-Prüfer and maximal non-integrally closed subrings of a field*, J. Algebra Appl., **11** (2012), 877–895.

13. S. MacLane and O. F. G. Schilling, *Zero-Dimensional Branches of Rank One on Algebraic Varieties*, Ann. of Math., **40**1 (1939), 507–520.

14. H. Matsumura, *Commutative ring theory*, Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989.

15. J. Ohm, *Some Counterexamples Related to Integral Closure in* $D[[x]]$, Trans. Amer. Math. Soc., **122** Issue 2 (1966), 321–333.

16. B. Olberding, *Globalizing local properties of Prüfer domains*, J. Algebra **205** (1998), 480–504.

17. B. Olberding, *Stability, duality and 2-generated ideals, and a canonical decomposition of modules*, Rend. Semin. Mat. Univ. Padova **106** (2001), 261–290.

18. B. Olberding, *On the classification of stable domains*, J. Algebra **243** (2001), 177–197.

19. B. Olberding, *On the structure of stable domains*, Comm. Algebra **30** (2002), 877–895.

20. B. Olberding, *Noetherian rings without finite normalizations*, Progress in commutative algebra 2, 171–203, W. de Gruyter, Berlin, 2012.

21. B. Olberding, *One-dimensional bad Noetherian domains*, Trans. Amer. Math. Soc. **366** (2014), 4067–4095.

22. B. Olberding, *Finitely stable rings*, Commutative Algebra - Recent Advances in Commutative Rings, Integer-valued Polynomials, and Polynomial functions, 269–291, Springer, New York, 2014.

23. B. Olberding, *One-dimensional stable rings*, J. Algebra **456** (2016), 93–122.

24. R. Remmert, *Classical topics in complex function theory*, Springer-Verlag, New York, 1998.

25. M. Zafrullah, *Completely integrally closed Prüfer v-multiplication domains*, preprint.

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