Tutorato 1
FM210 - Meccanica Analitica (CdL in Matematica)
Meccanica Analitica (CdL in Fisica)

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Equazioni differenziali ordinarie

Esercizio 1. Si consideri il sistema di equazioni differenziali lineari

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^3, \qquad A = \begin{pmatrix} 1 & 0 & 4 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

con condizioni iniziali $x(0) = (2, 1, -1)$. Se ne determini la soluzione.

Esercizio 2. Solve the system

$$\begin{cases} \frac{dx(t)}{dt} = 3x(t) - 4y(t), \\ \frac{dy(t)}{dt} = 4x(t) - 7y(t) \end{cases} \tag{1}$$

with initial condition $x(0) = y(0) = 1$.

Esercizio 3. Si consideri il sistema di equazioni differenziali lineari

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^2, \qquad A = \begin{pmatrix} 0 & 2\alpha - 1 \\ -1 & 2\alpha \end{pmatrix}, \qquad \alpha \in \mathbb{R}$$

con condizioni iniziali $x(0) = (2, 0)$. Se ne determini la soluzione al variare del parametro α.

Esercizio 4. Solve the linear system

$$\begin{cases} \dot{x}_1(t) = \frac{x_1(t)}{2} + \frac{x_3(t)}{2} \\ \dot{x}_2(t) = x_2(t) \\ \dot{x}_3(t) = \frac{x_1(t)}{2} + \frac{x_3(t)}{2} \\ \dot{x}_4(t) = \frac{x_4(t)}{2} + \frac{x_6(t)}{2} \\ \dot{x}_5(t) = x_5(t) \\ \dot{x}_6(t) = \frac{x_4(t)}{2} + \frac{x_6(t)}{2} \end{cases} \tag{2}$$

whit initial condition $x(0) := (x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0)) = (1, 2, 3, 4, 5, 6)$.

Esercizio 5. Si consideri il sistema di equazioni differenziali lineari

$$\dot{x} = Ax + B(t), \qquad x \in \mathbb{R}^2, \qquad A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \qquad B(t) = \begin{pmatrix} t \\ 2t+1 \end{pmatrix}$$

con condizioni inizali $x(0) = (0, 1)$. Se ne determini la soluzione.

Esercizio 6. Solve the following second order differential equation using matrix exponential

$$m\ddot{x} = -a\dot{x} \tag{3}$$

with initial conditions $x(0) = 1$ and $\dot{x}(0) = 2$.

Soluzioni

Esercizio 1. La soluzione generale del sistema è

$$x(t) = e^{At}x(0).$$

Si noti che la matrice A si può scrivere come $A = \mathbb{1} + N$ dove

$$N = \begin{pmatrix} 0 & 0 & 4 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

è tale che $N^3 = (0)_{i,j}$ e chiaramente $[\mathbb{1}, N] = 0$. Si ha quindi che

$$\begin{aligned} e^{At} &= e^{(\mathbb{1}+N)t} = e^{At}e^{Nt} = \\ &= \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \left(\mathbb{1} + Nt + \frac{N^2t^2}{2} \right) = \\ &= \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & 0 & 3t \\ t & 1 & 2t + 2t^2 \\ 0 & 0 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} e^t & 0 & 3te^t \\ te^t & e^t & (2t + 2t^2)\, e^t \\ 0 & 0 & e^t \end{pmatrix}. \end{aligned}$$

Pertanto, la soluzione con dato iniziale $x(0) = (2, 1, -1)$ è data da

$$x(t) = e^{At}x(0) = \begin{pmatrix} e^t & 0 & 3te^t \\ te^t & e^t & (2t + 2t^2)\, e^t \\ 0 & 0 & e^t \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} e^t\,(2 - 3t) \\ e^t\,(1 - 2t^2) \\ -e^t \end{pmatrix}.$$

Esercizio 2. The solution of an homogeneous first order matrix ODE is given by

$$\boldsymbol{x}(t) = \boldsymbol{x}(0)e^{At}. \tag{4}$$

The matrix exponential can be computed using Caley-Hamilton theorem: essentially every matrix is a root of its characteristic polynomial. Therefore for a matrix $n \times n$, we can always reduce its power grater than $n - 1$ to a sum of the first $n - 1$ powers; so every matrix function can be rewritten as

$$f(A) = \sum_{m=0}^{n-1} f_m A^m, \tag{5}$$

and the coefficients can be find requiring that on the diagonal form of the matrix we have

$$f(\lambda_j) = \sum_{m=0}^{n-1} f_m \lambda_j^m \tag{6}$$

where λ_j are the eigenvalues. When the eigenvalues are repeated, that is $\lambda_i = \lambda_j$ for some $i \neq j$, two or more equations are identical and the linear system cannot be solved uniquely. For such cases, given an eigenvalue λ with multiplicity q, the first $q-1$ derivatives of characteristic polynomial vanish at the eigenvalue. This leads to the extra $q-1$ linearly independent equations

$$\frac{d^k f(\lambda_i)}{d\lambda_i^k} = \frac{d^k}{d\lambda_i^k}\left(\sum_{m=0}^{n-1} f_m \lambda_i^m\right), \quad k = 1, q-1 \tag{7}$$

which, combined with others, yield the required n equations to solve for f_m.

In our case, we have

$$A = \begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix}, \tag{8}$$

the eigenvalues of A are

$$\lambda_1 = -5, \quad \lambda_2 = 1. \tag{9}$$

Therefore, imposing 6 we find a linear system for the coefficients f_j,

$$e^{-5t} = f_0 - 5tf_1, \quad e^t = f_0 + tf_1; \tag{10}$$

whose solution is given by

$$f_0 = \frac{e^{-5t}}{6} + \frac{5e^t}{6}, \quad f_1 = \frac{e^t}{6t} - \frac{e^{-5t}}{6t} \tag{11}$$

Plugging into 5 and 4 we have

$$\vec{x}(t) := (x(t), y(t)) = \frac{1}{6}\left(\left(e^{-5t} + 5e^t\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \left(\frac{e^t - e^{-5t}}{t}\right)\begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{12}$$

so

$$\begin{aligned}\vec{x}(t) &= \exp\left(\begin{pmatrix} 3 & -4 \\ 4 & -7 \end{pmatrix} t\right)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4e^t/3 - e^{-5t}/3 & 2e^{-5t}/3 - 2e^t/3 \\ 2e^t/3 - 2e^{-5t}/3 & 4e^{-5t}/3 - e^t/3 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \\ &= \begin{pmatrix} e^{-5t}/3 + 2e^t/3 \\ e^t/3 + 2e^{-5t}/3 \end{pmatrix}.\end{aligned} \tag{13}$$

Esercizio 3. Per prima cosa, calcoliamo il polinomio caratteristico di A:

$$p(\lambda) = \det(A - \lambda\mathbb{1}) = \det\begin{pmatrix} -\lambda & 2\alpha - 1 \\ -1 & 2\alpha - \lambda \end{pmatrix} = -\lambda(2\alpha - \lambda) + 2\alpha - 1 = \lambda^2 - 2\alpha\lambda + 2\alpha - 1,$$

da cui segue che $p(\lambda) = 0 \iff \lambda = 2\alpha - 1, 1$.

Lo spettro di A è pertanto $\Sigma(A) = \{2\alpha - 1, 1\}$. Dobbiamo quindi studiare separatamente i casi $\alpha \neq 1$ e $\alpha = 1$.

Consideriamo per primo il caso $\alpha \neq 1$ e calcoliamo gli autospazi generalizzati.

$E_{2\alpha-1} = Ker\,(A - (2\alpha - 1)\,\mathbb{1})$ è dato dalle equazioni

$$\begin{cases} (2\alpha - 1)\,x_2 - (2\alpha - 1)\,x_1 = 0 \\ -x_1 + 2\alpha x_2 - (2\alpha - 1)\,x_2 = 0 \end{cases}$$

da cui segue che

$$E_{2\alpha-1} = \left\{(t, t) \in \mathbb{R}^2 : t \in \mathbb{R}\right\}.$$

Allo stesso modo, $E_1 = Ker\,(A - \mathbb{1})$ è dato dalle equazioni

$$\begin{cases} (2\alpha - 1)\,x_2 - x_1 = 0 \\ -x_1 + 2\alpha x_2 - x_2 = 0 \end{cases}$$

da cui segue che

$$E_1 = \left\{((2\alpha-1)t, t) \in \mathbb{R}^2 : t \in \mathbb{R}\right\}.$$

Pertanto, una base di autovettori generalizzati è data da

$$v_1 = (1,1)\,, \qquad v_2 = (2\alpha-1, 1)$$

da cui

$$D = \begin{pmatrix} 2\alpha-1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 2\alpha-1 \\ 1 & 1 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} \frac{1}{2(1-\alpha)} & \frac{1-2\alpha}{2(1-\alpha)} \\ -\frac{1}{2(1-\alpha)} & \frac{1}{2(1-\alpha)} \end{pmatrix}.$$

Inoltre, $PDP^{-1} = A$ e quindi

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} = \\ &= \begin{pmatrix} 1 & 2\alpha-1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{(2\alpha-1)t} & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} \frac{1}{2(1-\alpha)} & \frac{1-2\alpha}{2(1-\alpha)} \\ -\frac{1}{2(1-\alpha)} & \frac{1}{2(1-\alpha)} \end{pmatrix} = \\ &= \begin{pmatrix} \frac{e^{(2\alpha-1)t}-(2\alpha-1)e^t}{2(1-\alpha)} & \frac{\left(e^{(2\alpha-1)t}-e^t\right)(1-2\alpha)}{2(1-\alpha)} \\ \frac{e^{(2\alpha-1)t}-e^t}{2(1-\alpha)} & \frac{e^{(2\alpha-1)t}(1-2\alpha)+e^t}{2(1-\alpha)} \end{pmatrix}. \end{aligned}$$

Pertanto, la soluzione con dato iniziale $x(0) = (2,0)$ è data da

$$x(t) = e^{At}x(0) = \begin{pmatrix} \frac{e^{(2\alpha-1)t}-(2\alpha-1)e^t}{2(1-\alpha)} & \frac{\left(e^{(2\alpha-1)t}-e^t\right)(1-2\alpha)}{2(1-\alpha)} \\ \frac{e^{(2\alpha-1)t}-e^t}{2(1-\alpha)} & \frac{e^{(2\alpha-1)t}(1-2\alpha)+e^t}{2(1-\alpha)} \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{e^{(2\alpha-1)t}-(2\alpha-1)e^t}{1-\alpha} \\ \frac{e^{(2\alpha-1)t}-e^t}{1-\alpha} \end{pmatrix}.$$

Studiamo ora il caso $\alpha = 1$. L'unico autospazio generalizzato è $E_1 = Ker\,(A-\mathbb{1})^2$ e si può facilmente verificare che $(A-\mathbb{1})^2 = (0)_{ij}$. Pertanto risulta $E_1 = \mathbb{R}^2$ e possiamo scegliere come autovettori generalizzati quelli della base canonica su $\mathbb{R}^2$

$$v_1 = (1,0)\,, \qquad v_2 = (0,1)\,,$$

da cui si ottiene che $D = P = P^{-1} = \mathbb{1}$.

Poniamo

$$N = A - D = A - \mathbb{1} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}.$$

Allora $N^2 = (0)_{ij}$ e ovviamente $[\mathbb{1}, N] = 0$, da cui segue che

$$e^{At} = e^{\mathbb{1}t}e^{Nt} = e^{\mathbb{1}t}\,(\mathbb{1} + Nt) = \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 1-t & t \\ -t & 1+t \end{pmatrix} = \begin{pmatrix} (1-t)\,e^t & te^t \\ -te^t & (1+t)\,e^t \end{pmatrix}.$$

Pertanto, la soluzione con dato iniziale $x(0) = (2,0)$ è data da

$$x(t) = e^{At}x(0) = \begin{pmatrix} (1-t)\,e^t & te^t \\ -te^t & (1+t)\,e^t \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2\,(1-t)\,e^t \\ -2te^t \end{pmatrix}.$$

Esercizio 4. The ODEs system can be written in matrix notation as

$$\dot{x}(t) = Ax(t) \tag{14}$$

where

$$A = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}. \tag{15}$$

The solution is again given by

$$x(t) = x(0)e^{At}. \tag{16}$$

We may note that the matrix A is a projection matrix. Remember that a projection matrix is a linear operator $P : V \to V$ on a vector space V such that $P^2 = P$ (idempotent); therefore A satisfies $A^n = A \; \forall n \in \mathbb{N}$. This simplifies the computation of the matrix exponential by the use of the definition itself. Indeed,

$$e^{At} := \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \mathcal{I} + \sum_{n=1}^{\infty} \frac{t^n}{n!} A =$$

$$= \begin{pmatrix} 1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!} & 0 & \frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!} & 0 & 0 & 0 \\ 0 & 1+1\sum_{n=1}^{\infty}\frac{t^n}{n!} & 0 & 0 & 0 & 0 \\ \frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!} & 0 & 1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!} & 0 & \frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!} \\ 0 & 0 & 0 & 0 & 1+1\sum_{n=1}^{\infty}\frac{t^n}{n!} & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!} & 0 & 1+\frac{1}{2}\sum_{n=1}^{\infty}\frac{t^n}{n!} \end{pmatrix} =$$

$$= \begin{pmatrix} \frac{e^t+1}{2} & 0 & \frac{e^t-1}{2} & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 & 0 & 0 \\ \frac{e^t-1}{2} & 0 & \frac{e^t+1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{e^t+1}{2} & 0 & \frac{e^t-1}{2} \\ 0 & 0 & 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & \frac{e^t-1}{2} & 0 & \frac{e^t+1}{2} \end{pmatrix}. \tag{17}$$

Therefore the solution is

$$x(t) = \begin{pmatrix} \frac{e^t+1}{2} & 0 & \frac{e^t-1}{2} & 0 & 0 & 0 \\ 0 & e^t & 0 & 0 & 0 & 0 \\ \frac{e^t-1}{2} & 0 & \frac{e^t+1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{e^t+1}{2} & 0 & \frac{e^t-1}{2} \\ 0 & 0 & 0 & 0 & e^t & 0 \\ 0 & 0 & 0 & \frac{e^t-1}{2} & 0 & \frac{e^t+1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(e^t+1) + \frac{3}{2}(e^t-1) \\ 2e^t \\ \frac{1}{2}(e^t-1) + \frac{3}{2}(e^t+1) \\ 2(e^t+1) + 3(e^t-1) \\ 5e^t \\ 2(e^t-1) + 3(e^t+1) \end{pmatrix}. \tag{18}$$

Esercizio 5. Per prima cosa, calcoliamo il polinomio caratteristico di A:

$$p(\lambda) = \det(A - \lambda\mathbb{1}) = \det\begin{pmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{pmatrix} = (1-\lambda)(-1-\lambda) = \lambda^2 - 1,$$

da cui segue che $p(\lambda) = 0 \iff \lambda = \pm 1$.

Lo spettro di A è pertanto $\Sigma(A) = \{1, -1\}$.

Calcoliamo gli autospazi generalizzati.

$E_1 = Ker\,(A - \mathbb{1})$ è dato dalle equazioni

$$\begin{cases} x_1 - x_1 = 0 \\ 2x_1 - x_2 - x_2 = 0 \end{cases}$$

da cui segue che

$$E_1 = \left\{(t,t) \in \mathbb{R}^2 : t \in \mathbb{R}\right\}.$$

Allo stesso modo, $E_{-1} = Ker\,(A + \mathbb{1})$ è dato dalle equazioni

$$\begin{cases} x_1 + x_1 = 0 \\ 2x_1 - x_2 + x_2 = 0 \end{cases}$$

da cui segue che

$$E_{-1} \left\{(0,t) \in \mathbb{R}^2 : t \in \mathbb{R}\right\}.$$

Pertanto, una base di autovettori generalizzati è data da

$$v_1 = (1,1)\,, \qquad v_2 = (0,1)$$

da cui

$$D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad P = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad P^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Inoltre, $PDP^{-1} = A$ e quindi

$$\begin{aligned} e^{At} &= Pe^{Dt}P^{-1} = \\ &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \\ &= \begin{pmatrix} e^t & 0 \\ e^t - e^{-t} & e^{-t} \end{pmatrix}. \end{aligned}$$

Pertanto, la soluzione con dato iniziale $x(0) = (0,1)$ è data da

$$\begin{aligned} x(t) &= e^{At}\left(x(0) + \int_0^t e^{-As} B(s)\, ds\right) = \\ &= \begin{pmatrix} e^t & 0 \\ e^t - e^{-t} & e^{-t} \end{pmatrix} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} e^{-s} & 0 \\ e^{-s} - e^s & e^s \end{pmatrix} \begin{pmatrix} s \\ 2s+1 \end{pmatrix} ds\right) = \\ &= \begin{pmatrix} e^t & 0 \\ e^t - e^{-t} & e^{-t} \end{pmatrix} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_0^t \begin{pmatrix} se^{-s} \\ se^{-s} + se^s + e^s \end{pmatrix} ds\right) = \\ &= \begin{pmatrix} e^t & 0 \\ e^t - e^{-t} & e^{-t} \end{pmatrix} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 - (1+t)\,e^{-t} \\ t\left(e^t - e^{-t}\right) - e^{-t} + 1 \end{pmatrix}\right) = \\ &= \begin{pmatrix} e^t & 0 \\ e^t - e^{-t} & e^{-t} \end{pmatrix} \begin{pmatrix} 1 - (1+t)\,e^{-t} \\ t\left(e^t - e^{-t}\right) - e^{-t} + 2 \end{pmatrix} = \\ &= \begin{pmatrix} e^t - t - 1 \\ e^t + e^{-2t} + e^{-t} - 2 \end{pmatrix}. \end{aligned}$$

Esercizio 6. First of all let us rewrite the equation as a system of first order ODEs. We define

$$z = \dot{x}, \tag{19}$$

and the system can be rewritten as

$$\begin{cases} \dot{x} &= z, \\ \dot{z} &= -\frac{a}{m} z, \end{cases} \tag{20}$$

while the initial condition becomes

$$\vec{x}(0) := (x(0), z(0)) = (1, 2). \tag{21}$$

The solution is given by

$$\vec{x}(t) = \vec{x}(0)e^{At} \tag{22}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & -\frac{a}{m} \end{pmatrix}. \tag{23}$$

Let us compute the characteristic polynomial:

$$p_{ch}(\lambda) = -\lambda\Big(- \frac{a}{m} - \lambda\Big) = 0; \tag{24}$$

the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = -\frac{a}{m}$. We now compute the exponential of A using Calay-Hamilton theorem, the system to solve in order to determine the coefficients is

$$\begin{cases} e^0 & = f_0, \\ e^{-\frac{at}{m}} & = f_0 - f_1\frac{at}{m} \end{cases}; \tag{25}$$

the solution is

$$f_0 = 1, \quad f_1 = \frac{m(1 - e^{-\frac{at}{m}})}{at}. \tag{26}$$

Therefore

$$\vec{x}(t) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{m}{at}(1 - e^{-\frac{at}{m}}) \begin{pmatrix} 0 & 1 \\ 0 & -\frac{a}{m} \end{pmatrix} t\right] \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{m}{a}(1 - e^{-\frac{at}{m}}) \\ 0 & e^{-\frac{at}{m}} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 + 2\frac{m}{a}(1 - e^{-\frac{at}{m}}) \\ 2e^{-\frac{at}{m}} \end{pmatrix}. \tag{27}$$