Tutorato 2
FM210 - Meccanica Analitica (CdL in Matematica)
Meccanica Analitica (CdL in Fisica)

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Sistemi conservativi, flussso e periodo, punti di equilibrio e relazioni utili

Esercizio 1. Si consideri un sistema autonomo del primo ordine con flusso $\varphi(t, x)$ di periodo T.

1. Dimostrare che la definizione di periodo non dipende dal punto sulla traiettoria, ovvero che ogni punto dell'orbita ha lo stesso periodo.
2. Dimostrare che $\varphi(t + nT, x) = \varphi(t, x)$ $\forall n \in \mathbb{N}$.

Esercizio 2. Given the following point particle conservative systems

1. $\begin{cases} \ddot{q}_1 + 2\ddot{q}_2 = F_1 = -kq_1 - 3\eta q_1^2 \\ 2\ddot{q}_1 + 3\ddot{q}_2 = F_2 = -kq_2 - 3\eta q_2^2 \end{cases} \quad , \quad F : \mathbb{R}^2 \to \mathbb{R}^2;$
2. $3\ddot{r} = F = -2aD_e(1 - e^{-a(r-r_e)})e^{-a(r-r_e)}, \quad F : \mathbb{R} \to \mathbb{R};$
3. $\begin{cases} \ddot{q}_1 + 2\ddot{q}_2 = F_1 = -2q_2 sin(q_1q_2) + 6q_2q_3 + 3q_1^2 \\ 2\ddot{q}_1 + 4\ddot{q}_2 + \ddot{q}_3 = F_2 = -2q_1 sin(q_1q_2) + q_3e^{q_2q_3} + 6q_1q_3 \\ \ddot{q}_2 + \ddot{q}_3 = F_3 = q_2e^{q_2q_3} + 6q_1q_2 \end{cases} \quad , \quad F : \mathbb{R}^3 \to \mathbb{R}^3,$

say how many degrees of freedom each system is composed of and compute the total energy.

Esercizio 3. Dimostrare che data una qualsiasi matrice $A \in M_n(\mathbb{R})$ si ha

$$\det(e^A) = e^{tr(A)}.$$

Esercizio 4. Given the following forces and their domains

1. $F : \mathbb{R}^3 \to \mathbb{R}^3$, $\;F(q_1, q_2, q_3) = (q_2^2 q_3^3, 2q_1 q_2 q_3^3, 3q_1 q_2^2 q_3^2)$;
2. $F : \mathbb{R}^3 \setminus \{0, 0, 0\} \to \mathbb{R}^3$, $\;F(q_1, q_2, q_3) = (3q_1^2 q_3, q_3^2, q_1^3 + 2q_2 q_3)$;
3. $F : \mathbb{R}^2 \setminus \{0, 0\} \to \mathbb{R}^2$, $\;F(q_1, q_2) = \frac{1}{q_1^2+q_2^2}(-q_2, q_1)$;
4. $F : \mathbb{R}^2 \setminus \{0, 0\} \to \mathbb{R}^2$, $\;F(q_1, q_2) = \frac{1}{(q_1^2+q_2^2)^2}(q_2^2 - q_1^2, -2q_1 q_2)$;
5. $F : \mathcal{S} \to \mathbb{R}^3$, $\;F(q_1, q_2, q_3) = (q_2 + e^{q_3}, q_1, q_1 e^{q_3})$, where $\mathcal{S} := \{ta | a \in A \subset \mathbb{R}^3, t \in [0, 1]\}$,

say if the mechanical systems associated are conservative or not.

Esercizio 5. Si consideri il sistema di equazioni differenziali lineari

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x} = (x, y, z) \in \mathbb{R}^3, \qquad A = \begin{pmatrix} -2 & 2 & 0 \\ -3 & 1 & -1 \\ 3 & -2 & 0 \end{pmatrix}.$$

1. Si determinino i punti di equilibrio e se ne discuta la stabilità.
2. Si determinino i dati iniziali per cui esiste $\lim_{t\to\infty} \mathbf{x}(t)$ e si calcoli il valore di tale limite.

Esercizio 6. Given the following dynamical systems

1. $\ddot{y} + y - \varepsilon \left(\frac{\dot{y}^3}{3} - \dot{y} \right) = 0, \quad \varepsilon > 0$;
2. $\begin{cases} \frac{dx}{dt} = \alpha x - \beta xy \\ \frac{dy}{dt} = \delta xy - \gamma y \end{cases}, \quad \alpha, \beta, \gamma, \delta \in \mathbb{R}^+.$

study the stability of the equilibrium points.

Soluzioni

Esercizio 1. 1. Sia $y = \varphi(\bar{t}, x)$ per un qualche tempo $\bar{t}$.

Allora

$$\varphi(t+T, y) = \varphi(t+T, \varphi(\bar{t}, x)) = \varphi(t+T+\bar{t}, x) = \varphi(t+\bar{t}, x) = \varphi(t, \varphi(\bar{t}, x)) = \varphi(t, y)$$

dove nella seconda e nella quarta uguaglianza si è usata la proprietà di gruppo della soluzione $\varphi(t, x)$.

2. Procediamo per induzione su $n > 1$, usando ancora una volta la proprietà di gruppo:
   - **$n = 2$:** $\varphi(t + 2T, x) = \varphi(t + T + T, x) = \varphi(t + T, x) = \varphi(t, x)$.
   - **$n \to n + 1$:** $\varphi(t + (n + 1)T, x = \varphi(t + T + nT, x) = \varphi(t + T, x) = \varphi(t, x)$.

Esercizio 2. The number of degrees of freedom is given by the dimension of the space to which the vector of generalized coordinates belong: if $q \in \mathbb{R}^l$, the system has l degrees of freedom. To compute the total energy

$$H(q, \dot{q}) = \frac{1}{2}\langle \dot{q}, A\dot{q}\rangle + V(q) \tag{1}$$

we need to compute the potential energy and to extrapolate the mass matrix. The mass matrix can be read directly form the ODE system while to compute the potential energy we have to solve $F(q) = -\nabla V(q)$.

1. The system has 2 degrees of freedom. The mass matrix is

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}, \tag{2}$$

therefore the kinetic energy is

$$K(\dot{q}) = \frac{1}{2}(\dot{q}_1 \ \dot{q}_2)\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} = \frac{1}{2}(\dot{q}_1 \ \dot{q}_2)\begin{pmatrix} \dot{q}_1 + 2\dot{q}_2 \\ 2\dot{q}_1 + 3\dot{q}_2 \end{pmatrix} = \frac{1}{2}(\dot{q}_1^2 + 4\dot{q}_1\dot{q}_2 + 3\dot{q}_2^2). \tag{3}$$

The energy potential is given by the following system of equation

$$\begin{cases} F_1 & = -\frac{\partial V}{\partial q_1} = kq_1 + 3\eta q_1^2 \\ F_2 & = -\frac{\partial V}{\partial q_2} = kq_2 + 3\eta q_2^2 \end{cases}; \tag{4}$$

integrating the first equation we get

$$V(q) = \frac{k}{2}q_1^2 + \eta q_1^3 + f(q_2) \tag{5}$$

where $f(q_2)$ is an arbitrary function of q_2. If we now insert 5 in the second of 4 we get

$$kq_2 + 3\eta q_2^2 = f'(q_2) \Rightarrow f(q_2) = \frac{k}{2}q_2^2 + \eta q_2^3. \tag{6}$$

In the end the total energy is

$$H(q, \dot{q}) = \frac{1}{2}(\dot{q}_1^2 + 4\dot{q}_1\dot{q}_2 + 3\dot{q}_2^2) + \frac{k}{2}(q_1^2 + q_2^2) + \eta(q_1^3 + q_2^3). \tag{7}$$

2. The system has 1 one degree of freedom. The kinetic energy is simply $K(\dot{r}) = \frac{3}{2}\dot{r}^2$ while the potential energy can be found from

$$V'(r) = 2aD_e(1 - e^{-a(r-r_e)})e^{-a(r-r_e)} = D_e(1 - e^{-a(r-r_e)})^2. \tag{8}$$

The total energy is

$$H(r, \dot{r}) = \frac{3}{2}\dot{r}^2 + D_e(1 - e^{-a(r-r_e)})^2. \tag{9}$$

3. The system has tree degrees of freedom. The mass matrix is given by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \tag{10}$$

so the kinetic energy is

$$K(\dot{q}) = \frac{1}{2}(\dot{q}_1\ \dot{q}_2\ \dot{q}_3)\begin{pmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{pmatrix} = \frac{1}{2}(\dot{q}_1\ \dot{q}_2\ \dot{q}_3)\begin{pmatrix} \dot{q}_1 + 2\dot{q}_2 \\ 2\dot{q}_1 + 4\dot{q}_2 + \dot{q}_3 \\ \dot{q}_2 + \dot{q}_3 \end{pmatrix} = \frac{1}{2}(\dot{q}_1^2 + 4\dot{q}_1\dot{q}_2 + 4\dot{q}_2^2 + 2\dot{q}_2\dot{q}_3 + \dot{q}_3^2). \tag{11}$$

The potential energy is given solving the equations

$$\begin{cases} F_1 & = -\frac{\partial V}{\partial q_1} = -2q_2 sin(q_1q_2) + 6q_2q_3 + 3q_1^2 \\ F_2 & = -\frac{\partial V}{\partial q_2} = -2q_1 sin(q_1q_2) + q_3e^{q_2q_3} + 6q_1q_3 \\ F_3 & = -\frac{\partial V}{\partial q_3} = q_2e^{q_1q_2} + 6q_2q_3 \end{cases}. \tag{12}$$

Integrating the first equation we get

$$V(q) = -2cos(q_1q_2) - 6q_1q_2q_3 - q_1^3 + f(q_2, q_3) \tag{13}$$

where $f(q_2, q_3)$ is an arbitrary function of q_2 and q_3; putting in the second equation we get

$$-2q_1 sin(q_1q_2) + q_3e^{q_2q_3} + 6q_1q_3 = -2q_1 sin(q_1q_2) + 6q_1q_3 - \frac{\partial f}{\partial q_2} \Rightarrow f(q_1, q_2) = -e^{q_2q_3} + h(q_3), \tag{14}$$

where $h(q_3)$ is an arbitrary function of q_3. For the moment we have

$$V(q) = -2cos(q_1q_2) - 6q_1q_2q_3 - q_1^3 - e^{q_2q_3} + h(q_3); \tag{15}$$

putting into the third equation we have

$$q_2e^{q_1q_2} + 6q_1q_2 = 6q_1q_2 + q_2e^{q_2q_3} + h'(q_3) \Rightarrow h(q_3) = c = 0. \tag{16}$$

The total energy is therefore

$$H(q, \dot{q}) = \frac{1}{2}(\dot{q}_1^2 + 4\dot{q}_1\dot{q}_2 + 4\dot{q}_2^2 + 2\dot{q}_2\dot{q}_3 + \dot{q}_3^2) - 2cos(q_1q_2) - 6q_1q_2q_3 - q_1^3 - e^{q_2q_3}. \tag{17}$$

Esercizio 3. Sia $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ l'operatore lineare rappresentato dalla matrice A. Allora è possibile trovare una base in cui T sia rappresentato da una matrice in forma canonica di Jordan. Si può quindi scrivere, presa Q la matrice del cambiamento di base, $A = Q^{-1}JQ$ dove J è una matrice in forma canonica, i.e.

$$J = \begin{pmatrix} \Lambda_{n_1} & 0 & \dots & 0 \\ 0 & \Lambda_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \Lambda_{n_r} \end{pmatrix}$$

con $n_1 + \ldots + n_r = n$ e Λ_{n_i} blocchi elementari di Jordan.

In particolare, si ha $J = N + \Lambda$ con N nilpotente e Λ diagonale tali che $[N, \Lambda] = 0$. Allora, $e^J = e^N e^\Lambda$. Ma

$$e^N = \mathbb{1} + \sum_{k\geq 1} \frac{N^k}{k!}$$

e $\det(e^N) = \det \mathbb{1} = 1$ perché le matrici N^k hanno elementi non nulli solo sulla k-esima sovradiagonale.

Si ha quindi

$$\det(e^J) = \det(e^N)\det(e^\Lambda) = \det(e^\Lambda).$$

Allora

$$\begin{aligned} \det(e^A) &= \det(Q^{-1}e^J Q) = \det(Q^{-1})\det(e^J)\det(Q) = \\ &= \det(e^J) = \det(e^\Lambda) = \prod_{i=1}^{n} e^{\lambda_i} = e^{\lambda_1 + \ldots + \lambda_n} = \\ &= e^{tr(J)} = e^{tr(Q^{-1}JQ)} = e^{tr(A)} \end{aligned}$$

e quindi

$$\det(e^A) = e^{tr(A)}.$$

Esercizio 4. A mechanical system is said to be conservative if the vector field representing the force is conservative. A vector field is conservative if its line integral over an arbitrary curve C is path-independent (so its value depends only on the endpoints of the curve) or, equivalently, if its circuitation is vanishing. If we are in a simply connected set a sufficient condition to be conservative is that the curl of the vector field F

$$\nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ F_{q_1} & F_{q_2} & F_{q_3} \end{vmatrix} = \mathbf{i}\left(\frac{\partial F_{q_3}}{\partial q_2} - \frac{\partial F_{q_2}}{\partial q_3}\right) + \mathbf{j}\left(\frac{\partial F_{q_1}}{\partial q_3} - \frac{\partial F_{q_3}}{\partial q_1}\right) + \mathbf{k}\left(\frac{\partial F_{q_2}}{\partial q_1} - \frac{\partial F_{q_1}}{\partial q_2}\right) \tag{18}$$

is zero. Indeed, using Stoke theorem we have

$$\iint_\Sigma (\nabla \times F) \cdot d\Sigma = \oint_{\partial\Sigma = C} F \cdot ds = 0 \Rightarrow \nabla \times F = 0. \tag{19}$$

To use Stokes' theorem, we just need to find a surface whose boundary is C If the domain of F is simply connected, so if it is path-connected and if every loop can be contracted to a point, we can always find such a surface. The surface can just go around any hole that is in the middle of the domain. With such a surface along which $\nabla \times F = 0$, we can use Stokes' theorem to show that the circulation around C is zero. Since we can do this for any closed curve, we can conclude that F is conservative. On the other hand, if the set is not simply connected, but has a hole going all the way through it, then $\nabla \times F = 0$ is not a sufficient condition for pat-independence. In this case, if C is a curve that goes around the hole, then we cannot find a surface that stays inside that domain whose boundary is C. Without such a surface, we cannot use Stokes' theorem to conclude that the circulation around C is zero. In that cases we need to compute directly the scalar function such that $F = -\nabla V$: if exist than F is conservative thanks to gradient theorem otherwise it is not.

1. The domain of F is simply connected and
$$\nabla \times F = 0. \tag{20}$$
The vector field is conservative and so the system.

2. The domain of F is simply connected and
$$\nabla \times F = 0. \tag{21}$$
The vector field is conservative and so the system.

3. The domain is not simply connected so the curl gives no information. We have to indicate that exist a scalar function V such that $F = -\nabla V$ to show the field is conservative or to find a curve along which the circuitation is not zero to show the field is not conservative. If we take every loop around the hole we can see that the circuitation is not zero. Indeed, let be $\gamma(t) = (Rcos(t), Rsin(t))$ with $t \in [0, 2\pi]$ and $R \in \mathbb{R}$ a parameterization of C , therefore
$$\begin{aligned} \oint_C F \cdot ds &= \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t) dt = \\ &= \int_0^{2\pi} \frac{1}{R^2}(-Rsin(t), Rcos(t)) \cdot (-Rsin(t), Rcos(t)) dt = \\ &= \int_0^{2\pi} dt = 2\pi. \end{aligned} \tag{22}$$
The vector field is not conservative and so the system.

4. The domain is not simply connected so the curl gives no information. Taken the same loop as before we get
$$\begin{aligned} \oint_C F \cdot ds &= \int_0^{2\pi} F(\gamma(t)) \cdot \gamma'(t) dt = \\ &\int_0^{2\pi} \frac{1}{R}(sin^2(t) - cos^2(t), -2sincos(t)) \cdot (-sin(t), cos(t)) dt = \\ &= \frac{1}{R}\int_0^{2\pi} sin^3(t) - sin(t)cos^2(t) = 0. \end{aligned} \tag{23}$$
So or we try again with other loops or we try to construct a scalar function such that $F = -\nabla V$, namely
$$\begin{cases} \frac{q_2^2 - q_1^2}{(q_1^2+q_2^2)^2} &= -\frac{\partial V}{\partial q_1} \\ \frac{-2q_1q_2}{(q_1^2+q_2^2)^2} &= -\frac{\partial V}{\partial q_2}. \end{cases} \tag{24}$$
Let us search for a scalar function $V(q_1, q_2)$; the first equation gives us
$$V(q_1, q_2) = -\frac{q_1}{q_1^2 + q_2^2} + f(q_2) \tag{25}$$
where $f(q_2)$ is an arbitrary function of q_2. From the second equation we have
$$\frac{-2q_1q_2}{(q_1^2 + q_2^2)^2} = \frac{-2q_1q_2}{(q_1^2 + q_2^2)^2} + f'(q_2) \Rightarrow f(q_2) = c; \tag{26}$$
so there exist a scalar function $V(q_1, q_2)$ and the system is conservative.

5. The set is a star one, therefore it is simply connected; the curl is

$$\nabla \times F = 0. \tag{27}$$

The vector field is conservative and so the system.

Esercizio 5. 1. L'unico punto di equilibrio è $\mathbf{x} = \mathbf{0}$.

Per studiarne la stabilità cerchiamo gli autovalori della matrice A calcolandone il polinomio caratteristico:

$$p(\lambda) = \det \begin{pmatrix} -2 & 2 & 0 \\ -3 & 1 & -1 \\ 3 & -2 & 0 \end{pmatrix} = \lambda(2+\lambda)(1-\lambda) - 6 - 6\lambda + 2(2+\lambda) = -(\lambda+1)(\lambda^2+2),$$

per cui $p(\lambda) = 0 \iff \lambda_1 = -1,\ \lambda_{2,3} = \pm i\sqrt{2}$.

Ma allora $\mathfrak{Re}(\lambda_1) < 0,\ \mathfrak{Re}(\lambda_{2,3}) = 0$, cioè $\mathbf{x} = \mathbf{0}$ è un punto di equilibrio stabile.

2. Siano ora $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ autovettori associati rispettivamente agli autovalori $\lambda_1, \lambda_2, \lambda_3$, con $\mathbf{v}_2$ e $\mathbf{v}_3$ necessariamente a coefficienti complessi.

La soluzione generale è data da

$$\mathbf{x}(t) = \alpha_1 \mathbf{v}_1 e^{-t} + \alpha_2 \mathbf{v}_2 e^{-i\sqrt{2}t} + \alpha_3 \mathbf{v}_3 e^{i\sqrt{2}t}$$

con $\alpha_1, \alpha_2, \alpha_3$ tali che

$$\mathbf{x}(t) = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3 = (x_0, y_0, z_0).$$

Allora $\lim_{t\to\infty} \mathbf{x}(t)$ esiste $\iff \alpha_2 = \alpha_3 = 0$.

In questo caso, $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{0}$.

Ma il vettore $\mathbf{v}_1$ è l'autovettore associato all'autovalore λ_1, cioè è tale che $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1$:

$$\begin{cases} -2x + 2y = -x \\ -3x + y - z = -y \\ 3x - 2y = -z \end{cases}$$

da cui segue che $\mathbf{v}_1 = (2, 1, -4)$.

Ma allora i dati iniziali corrispondenti sono

$$(x_0, y_0, z_0) = \alpha_1 \mathbf{v}_1 = \alpha_1 (2, 1, -4).$$

Esercizio 6. Given a dynamical system $\dot{x} = f(x)$, the equilibrium points are defined as those point x_{eq} where $f(x_{eq}) = 0$. Their stability can be studied using linearization or Lyapunov direct method. The linearized version of the system $\dot{x} = f(x)$ around an equilibrium point x_{eq} is given by

$$\dot{x} = J(x_{eq})(x - x_{eq}), \tag{28}$$

where $J(x_{eq})$ is the jacobian matrix computed at the equilibrium point. If all the eigenvalues have negative real part, x_{eq} is an asymptotically stable point while if there exist at least one eigenvalue with positive real part, x_{eq} is unstable. All other cases give no information of the stability of x_{eq} indeed in these cases can happen that the full non linear system behave completely differently.
A second possibility is to use a Lyapunov function $L(x)$: let be x_{eq} an equilibrium point for the full system and let be $L(x) \in C^1(\mathbb{R}) : B(x_{eq}) \to \mathbb{R}$ such that

- $L(x_{eq}) = 0$,
- $L(x) > 0 \;\; \forall x \in B(x_{eq}) \setminus \{x_{eq}\}$.

If

- $\dot{W}(x) \leq 0 \;\; \forall x \in B(x_{eq})$ than x_{eq} is a stable equilibrium point;
- $\dot{W}(x) < 0 \;\; \forall x \in B(x_{eq})$ than x_{eq} is an asymptotically stable equilibrium point
- $\dot{W}(x) > 0 \;\; \forall x \in B(x_{eq})$ than x_{eq} is an unstable equilibrium point.

When we use the Lyapunov function method, it is advisable to use as the first candidate of Lyapunov function the quadratic function $L(x) = \frac{1}{2}\langle x - x_{eq}, x - x_{eq}\rangle$ which is positive definite in any neighborhood $B(x_{eq})$ of x_{eq}. In the case we want to prove stability, we can try using constants of motion C since for them it is guaranteed that $\dot{C} = 0$.

1. Let $x_1 = y, x_2 = \dot{y}$ so that the corresponding system is given by

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -x_1 + \varepsilon \left(\frac{x_2^3}{3} - x_2 \right) \end{cases} \tag{29}$$

The origin $x_1 = 0,\; x_2 = 0$ is the only equilibrium point $x_{eq} = (0, 0)$; the jacobian matrix of the linearized system is given by

$$J(x_1, x_2) = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon \end{pmatrix}, \tag{30}$$

therefore

$$J(x_{eq}) = \begin{pmatrix} 0 & 1 \\ -1 & -\varepsilon \end{pmatrix}, \tag{31}$$

whose eigenvalues are $\lambda_{1,2} = \frac{-\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$, since $\mathcal{Re}(\lambda_{1,2}) < 0$ the equilibrium point is asymptotically stable.

2. This system is known as Lokta-Volterra or pray-predator system; it is used in biology as the simplest model for predation. The equilibrium point are given by the system

$$\begin{cases} x(\alpha - \beta y) = 0 \\ -y(\gamma - \delta x) = 0 \end{cases} \tag{32}$$

whose solution are $x_{eq1} = (0, 0)$ and $x_{eq2} = (\frac{\alpha}{\beta}, \frac{\gamma}{\delta})$. The jacobian matrix is

$$J(x, y) = \begin{pmatrix} \alpha - \beta y & -\beta x \\ \delta y & \delta x - \gamma \end{pmatrix} \tag{33}$$

and computed at the equilibrium points take the values

$$J(0, 0) = \begin{pmatrix} \alpha & 0 \\ 0 & -\gamma \end{pmatrix}, \quad J\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \begin{pmatrix} 0 & -\frac{\beta\gamma}{\delta} \\ \frac{\alpha\delta}{\beta} & 0 \end{pmatrix}. \tag{34}$$

In the case of x_{eq1} the jacobian matrix has eigenvalues $\lambda_1 = \alpha$ and $\lambda_2 = -\gamma$, therefore x_{eq1} is an unstable equilibrium point. In the case of x_{eq2} the jacobian matrix has eigenvalues $\lambda_1 = i\sqrt{\alpha\gamma}$ and

$\lambda_2 = -i\sqrt{\alpha\gamma}$; the linearization method gives us no information. Let us try to find a Lyapunov function. We can find a constant of motion eliminating time from the system to get one equation

$$\frac{dy}{dx} = -\frac{y}{x}\frac{\delta x - \gamma}{\beta y - \alpha} \Rightarrow \frac{\beta y - \alpha}{y}\,dy + \frac{\delta x - \gamma}{x}\,dx = 0, \tag{35}$$

the solution can be found by separation of variables

$$V = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y),. \tag{36}$$

where V is a constant quantity depending on the initial conditions. Let us define the function

$$l(x,y) = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y) - V; \tag{37}$$

it is clear that $\dot{l}(x,y) = 0$, moreover

$$l\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = -V + \gamma\left[1 - ln\left(\frac{\gamma}{\delta}\right)\right] + \alpha\left[1 - ln\left(\frac{\alpha}{\beta}\right)\right], \tag{38}$$

so we can define the Lyapunov function

$$L(x,y) = \delta x - \gamma \ln(x) + \beta y - \alpha \ln(y) - \gamma\left[1 - ln\left(\frac{\gamma}{\delta}\right)\right] - \alpha\left[1 - ln\left(\frac{\alpha}{\beta}\right)\right] \tag{39}$$

that satisfy $L\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = 0$ and $L(x,y) > 0\ \forall x, y \in B(x_{eq})$, indeed computing the hessian matrix

$$H_{L(x,y)}(x,y) = \begin{pmatrix} \frac{\gamma}{x^2} & 0 \\ 0 & \frac{\alpha}{y^2} \end{pmatrix} \Rightarrow \det(H_{L(x,y)}(x,y)) = \frac{\gamma}{x^2}\frac{\alpha}{y^2} \tag{40}$$

therefore we have

$$\det\left(H_{L(x,y)}\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right)\right) = \frac{\delta^2}{\gamma}\frac{\beta^2}{\alpha} > 0, \quad H_{L(x,y)11}\left(\frac{\gamma}{\delta}, \frac{\alpha}{\beta}\right) = \frac{\gamma}{\delta^2} > 0 \tag{41}$$

so x_{eq} is a minimum for the Lyapunov function. Moreover, by construction $\dot{L}(x,y) = 0$ and so x_{eq} is a stable equilibrium point.