SUCCESSIONI

Esercizi risolti

1. Calcolare i seguenti limiti:

$a)\ \lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{2n}$ $\qquad$ $b)\ \lim_{n\to\infty}\frac{\sqrt{n}-n+n^2}{2n^2-n^{3/2}+1}$

$c)\ \lim_{n\to\infty}\frac{2^n-3^n}{1+3^n}$ $\qquad$ $d)\ \lim_{n\to\infty}\frac{2^n+n^2}{3^n+n^3}$

$e)\ \lim_{n\to\infty}\frac{n\log n}{(n+1)(n+2)}$ $\qquad$ $f)\ \lim_{n\to\infty}\frac{1+\log n}{\sqrt{n}-\log n}$

$g)\ \lim_{n\to\infty}(-1)^n\frac{n}{n^2+1}$

$i)\ \lim_{n\to\infty}\sqrt[n]{2^n+3^n}$ $\qquad$ $\ell)\ \lim_{n\to\infty}\sqrt[n]{\frac{2n}{3n^2+1}}$

$m)\ \lim_{n\to\infty}\frac{n^2(3^n-3^{-n})}{4^n+n^2}$ $\qquad$ $n)\ \lim_{n\to\infty}\frac{n^6+\log n+3^n}{2^n+n^4+\log^5 n}$

$o)\ \lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^n$ $\qquad$ $p)\ \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n^2}$

$q)\ \lim_{n\to\infty}\frac{(n^2+1)^n}{n^{2n}}$

$s)\ \lim_{n\to\infty}\left(\sqrt[n]{3}-1\right)^n$

$u)\ \lim_{n\to\infty}n^2\,2^{-\sqrt{n}}$ $\qquad$ $v)\ \lim_{n\to\infty}\left(n^{\sqrt{n}}-2^n\right).$

2. Verificare che per $n\to\infty$

$$\frac{(n+3)!-n!}{(n+1)!}\sim n^2$$

3. Calcolare la parte principale per $n\to\infty$ di

$$\frac{\sqrt{n}-2n^3+n\log n}{5n+\log n}.$$

Soluzioni

1. a)

$$\lim_{n\to\infty}\left(1+\frac{1}{3n}\right)^{2n}=\lim_{n\to\infty}\left[\left(1+\frac{1}{3n}\right)^{3n}\right]^{2/3}=e^{2/3}.$$

Possiamo anche procedere utilizzando il limite fondamentale $\lim_{n\to\infty}(1+a/n)^n=e^a$,

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{3n}\right)^{n}\right]^{2}=(e^{1/3})^2=e^{2/3}.$$

b)

$$\lim_{n\to\infty}\frac{\sqrt{n}-n+n^2}{2n^2-n^{3/2}+n}=\lim_{n\to\infty}\frac{\not{n^2}\left(1-\frac{1}{n}-\frac{1}{n^{3/2}}\right)}{\not{n^2}\left(2-\frac{1}{\sqrt{n}}+\frac{1}{n}\right)}=\frac{1}{2}.$$

c)

$$\lim_{n\to\infty}\frac{2^n-3^n}{1+3^n}=\lim_{n\to\infty}\frac{\not{3^n}(-1+(\frac{2}{3})^n)}{\not{3^n}(1+\frac{1}{3^n})}=-1.$$

d)

$$\lim_{n\to\infty}\frac{2^n+n^2}{3^n+n^3}=\lim_{n\to\infty}\frac{2^n(1+\frac{n^2}{2^n})}{3^n(1+\frac{n^3}{3^n})}=\lim_{n\to\infty}\left(\frac{2}{3}\right)^n=0.$$

e)

$$\lim_{n\to\infty}\frac{n\,\log n}{(n+1)(n+2)}=\lim_{n\to\infty}\frac{\not{n}\log n}{\not{n^2}}\frac{1}{(1+\frac{1}{n})(1+\frac{2}{n})}=\lim_{n\to\infty}\frac{\log n}{n}=0.$$

f)

$$\lim_{n\to\infty}\frac{1+\log n}{\sqrt{n}-\log n}=\lim_{n\to\infty}\frac{\log n}{\sqrt{n}}\frac{1+\frac{1}{\log n}}{1-\frac{\log n}{\sqrt{n}}}=\lim_{n\to\infty}\frac{\log n}{\sqrt{n}}=0.$$

g) Il limite vale zero perche la successione $(-1)^n$ è limitata mentre $\frac{n}{n^2+1}$ è infinitesima.

i)

$$\lim_{n\to\infty}\sqrt[n]{2^n+3^n}=\lim_{n\to\infty}3\left(1+\left(\frac{2}{3}\right)^n\right)^{\frac{1}{n}}=3.$$

l)

$$\lim_{n\to\infty} \sqrt[n]{\frac{2n}{3n^2+1}} = \lim_{n\to\infty} \frac{\sqrt[n]{n}\,\sqrt[n]{2}}{(\sqrt[n]{n})^2\,\sqrt[n]{3+\frac{1}{n^2}}} = 1.$$

m)

$$\lim_{n\to\infty} \frac{n^2(3^n - 3^{-n})}{4^n + n^2} = \lim_{n\to\infty} \frac{n^2 3^n(1 - 3^{-2n})}{4^n(1+\frac{n^2}{4^n})} = \lim_{n\to\infty} \frac{n^2}{(4/3)^n} = 0.$$

n)

$$\lim_{n\to\infty} \frac{n^6 + \log n + 3^n}{2^n + n^4 + \log^5 n} = \lim_{n\to\infty} \frac{3^n(1+\frac{n^6}{3^n}+\frac{\log n}{3^n})}{2^n(1+\frac{n^4}{2^n}+\frac{\log^5 n}{2^n})} = \lim_{n\to\infty} \left(\frac{3}{2}\right)^n = +\infty.$$

o)

$$\lim_{n\to\infty} \left(\frac{n+3}{n+1}\right)^n = \lim_{n\to\infty} \left(1+\frac{2}{n+1}\right)^n = \lim_{n\to\infty} \left[\left(1+\frac{1}{\frac{n+1}{2}}\right)^{\frac{n+1}{2}}\right]^{\frac{2n}{n+1}} = e^2.$$

p)

$$\lim_{n\to\infty} \left(\frac{n-1}{n}\right)^{n^2} = \lim_{n\to\infty} \left[\left(1-\frac{1}{n}\right)^n\right]^n = (e^{-1})^{+\infty} = 0,$$

essendo $1/e < 1$.

q)

$$\lim_{n\to\infty} \frac{(n^2+1)^n}{n^{2n}} = \lim_{n\to\infty} \left(\frac{n^2+1}{n^2}\right)^n = \lim_{n\to\infty} \left[\left(1+\frac{1}{n^2}\right)^{n^2}\right]^{\frac{1}{n}} = e^0 = 1.$$

s)

$$\lim_{n\to\infty} \left(\sqrt[n]{3}-1\right)^n = 0^{+\infty} = 0.$$

u) Si ha

$$n^2\, 2^{-\sqrt{n}} = e^{2\log n - \sqrt{n}\log 2} = e^{\sqrt{n}(-\log 2 + 2\frac{\log n}{\sqrt{n}})} \to e^{-\infty} = 0.$$

v) Si ha

$$n^{\sqrt{n}} - 2^n = e^{\sqrt{n}\log n} - e^{n\log 2} = -e^{n\log 2}\left(1 - e^{\sqrt{n}\log n - n\log 2}\right).$$

Ora

$$e^{\sqrt{n}\log n - n\log 2} = e^{n\left(-\log 2 + \frac{\log n}{\sqrt{n}}\right)} \to e^{-\infty} = 0,$$

e quindi

$$\lim_{n\to\infty}\left(n^{\sqrt{n}} - 2^n\right) = -\lim_{n\to\infty} e^{n\log 2} = -\infty.$$

2.

$$\lim_{n\to\infty} \frac{(n+3)! - n!}{n^2(n+1)!} = \lim_{n\to\infty} \frac{(n+3)(n+2)(n+1) - 1}{n^2(n+1)} = 1.$$

3.

$$\frac{\sqrt{n} - 2n^3 + n\log n}{5n + \log n} = -\frac{2n^3}{5n}\,\frac{1 - \frac{\sqrt{n}}{2n^3} - \frac{\log n}{2n^2}}{1 + \frac{\log n}{5n}} \sim -\frac{2}{5}n^2.$$