KAM theory, Lindstedt series and the stability of the upside-down pendulum

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Abstract. *We consider the planar pendulum with support point oscillating in the vertical direction; the upside-down position of the pendulum corresponds to an equilibrium point for the projection of the motion on the pendulum phase space. By using the Lindstedt series method recently developed in literature starting from the pioneering work by Eliasson, we show that such an equilibrium point is stable for a full measure subset of the stability region of the linearized system inside the two-dimensional space of parameters, by proving the persistence of invariant KAM tori for the two-dimensional Hamiltonian system describing the model.*

1. Introduction

1.1. *The state of the art.* The upside-down pendulum with the support point oscillating with a frequency ω large enough has been extensively studied in literature as a simple model exhibiting a quite nontrivial behaviour; see [5] and the references quoted therein.

The stability of the upside-down position can be proven by the averaging method (see for instance [19], Ch. 9): the result is that if the support point oscillates fast enough then the upward equilibrium position becomes stable. However such a kind of analysis is not completely rigorous both because no explicit control on the corrections can be obtained and because it can lead to incorrect results, as already pointed out in [8] and [6]. In fact the averaging method approach can be followed also for studying the stability of the downward position, and the result one finds in doing so is that such a position is always stable – provided that ω is large enough to make it possible to apply the averaging, say $\omega > \omega_1$, for some ω_1 –, a result obviously unacceptable as by varying ω above ω_1 one can lose even the linear stability (as it follows from Mathieu's equation theory). A rather complete review on the averaging method can be found in [2].

A rigorous proof has been given for the linearized system in [1], where also the case of several pendula has been considered. In the latter case the linearized system can be written (by a diagonalization procedure) as a system of several uncoupled Mathieu's equations. So both for a single pendulum and for more than one pendulum, the theory of Mathieu's equation applies. In particular for a single pendulum the physical parameters describing the system have to be such that the parameters (a, q) appearing in Mathieu's equation $\ddot{x} + (a + 2q\cos 2\tau)x = 0$ lay inside the stability region corresponding to negative values of a (see [1]): if the amplitude of the oscillation of the support point is small enough (with respect to the length of the pendulum), then one has stability if the frequency ω is above a threshold value ω_0 (see below), and the same value predicted by the averaging method is found for ω_0.

1.2. *Stability and KAM theory.* What is missing in the literature is a rigorous discussion of the full system (not just the linearized one). In this paper we achieve such a task, by studying the full system by means of perturbation theory techniques.

In [2] it is suggested that KAM methods are necessary, but there is no explicit discussion; we think that it can be of interest to discuss in detail the problem by means of the Lindstedt series method recently developed in literature (see [4] for a list of references) both as an application of the general theory presented in [4] and as an occasion to improve the bounds in a case in which additional properties come into play. The method, based on a tree representation of the quasiperiodic solutions describing the KAM invariant tori, was originally introduced in [12], hence revisited in [14] and [11], and adapted to the case of isochronous systems in [4]: it is on the latter that we shall rely.

Since the linearized system can be written as a two-dimensional integrable Hamiltonian system, the full system becomes a perturbation of an integrable one: so KAM theory applies. Then we can prove that a large quantity of invariant tori persist under the perturbations, and the nearer the initial data $(\theta(0), \dot\theta(0))$ are to the upside-down position $(\pi, 0)$ the nearer the curves obtained by projection of the tori on the pendulum phase space $(\theta, \dot\theta)$ are to such a position: this proves the stability of the upside-down position for the pendulum.

Note that in applying the above argument it is a fundamental fact that the system is a two-dimensional one, so that the existence of the tori yield a topological obstruction for trajectories starting inside a torus to cross it and move far from it. So such a result cannot be used in order to study the stability of the upside-down position in the case of more than one pendulum: as a matter of fact, in the latter case, it is even likely that Arnol'd diffusion can occur and, as a consequence, the position is not stable at all.

1.3. *Contents.* As far as the proof is concerned, in such a note we simply show that the system can be written in terms of action-angle variables, and gives rise to a perturbed isochronous system; thus we can refer to [4] for the proof of persistence of KAM invariant tori within the framework of the Lindstedt series approach.

Note that, provided that for our system the conditions under which the KAM theorem for perturbations of isochronous systems applies, one could proceed also by means of the usual KAM techniques. We prefer to use the Lindstedt series method of [4] for the reasons illustrated above: from a technical point of view this means that the theorem 1.4 of [4] briefly recalled in Section 2.2 could be proved also through other approaches to KAM theory.

In Section 2 we introduce the model which we are interested in, and we show how the analysis of [4] can be applied to it; this yields some Diophantine-type condition on the oscillation frequency of the support point In Appendix A3 we explicitly check that all the hypotheses under which the results in [4] are proved are verified by our model. Finally in Appendix A4 we show how the condition on the oscillation frequency of the support point of the pendulum can be relaxed into a weaker non-resonance condition. Here we use explicitly the techniques introduced in [4]: from a technical point of view Appendix A4 represents the main interest of the present note. [Appendix A1 reviews the basic properties of Mathieus equation that we use, while Appendix A2 briefly describes the canonical change of coordinates casting the Hamiltonian into action-angle variables.]

2. The vertically driven pendulum

2.1. *The upside-down pendulum.* Consider a pendulum with support point $P = (x_P, y_P)$ oscillating in the vertical direction with the law $y_P(t) = b\cos\omega t$ (and $x_P(t) \equiv 0$). The system is described by the Lagrangian (see, for instance [18] or [19])

$$\mathcal{L}_0 = \frac{1}{2} m\ell^2\dot\theta^2 + mg\ell\,(\cos\theta + 1) - m\ell b\omega^2\,(\cos\theta + 1)\cos\omega t, \qquad \dot\theta = \frac{\mathrm{d}\theta}{\mathrm{d}t}, \tag{2.1}$$

which, by the change of variable

$$\theta(t) = \pi + x(\tau), \qquad \tau = \omega t/2, \tag{2.2}$$

can be written as

$$\mathcal{L}_0 = \frac{1}{8} m\ell^2\omega^2\dot x^2 - mg\ell\,(\cos x - 1) + m\ell b\omega^2\,(\cos x - 1)\cos 2\tau, \qquad \dot x = \frac{\mathrm{d}x}{\mathrm{d}\tau}. \tag{2.3}$$

The corresponding (non-autonomous) Hamiltonian is

$$H_0 = \frac{2y_1^2}{m\ell^2\omega^2} + \left(mg\ell - m\ell b\omega^2\cos 2\tau\right)(\cos x_1 - 1)\,, \tag{2.4}$$

where $x_1 = x$ and y_1 is the momentum conjugated to x_1. We are interested in the stability of the position $(\theta,\dot\theta) = (\pi, 0)$, *i.e.* $(x,\dot x) = (0,0)$, for ω large enough (see [19] and [1]), so that we can define

$$\begin{cases} y_1 = \delta p_1, \\ x_1 = \delta q_1, \end{cases} \tag{2.5}$$

and write (2.4), divided by δ^2, as

$$\begin{aligned} H = {} & \frac{2p_1^2}{m\ell^2\omega^2} - \left(mg\ell - m\ell b\omega^2\cos 2\tau\right)\frac{q_1^2}{2} \\ & + \left(mg\ell - m\ell b\omega^2\cos 2\tau\right)\sum_{p=2}^{\infty}(-1)^p\frac{1}{(2p)!}q_1^{2p}\delta^{2(p-1)}, \end{aligned} \tag{2.6}$$

where the last sum is obtained by Taylor expanding the cosine function and disappears for $\delta = 0$.

We can consider the autonomous Hamiltonian

$$\begin{aligned} H = {} & \frac{2p_1^2}{m\ell^2\omega^2} + I_2 - \left(mg\ell - m\ell b\omega^2\cos 2\alpha_2\right)\frac{q_1^2}{2} \\ & + \left(mg\ell - m\ell b\omega^2\cos 2\alpha_2\right)\sum_{p=2}^{\infty}(-1)^p\frac{1}{(2p)!}q_1^{2p}\delta^{2(p-1)}, \end{aligned} \tag{2.7}$$

with (I_2, α_2) conjugated variables.

By a (canonical) rescaling

$$p_1 \to \frac{2}{\sqrt{m\ell^2\omega^2}}p_1, \qquad q_1 \to \frac{\sqrt{m\ell^2\omega^2}}{2}q_1, \tag{2.8}$$

we can put (2.7) into the form

$$\begin{aligned} H = {} & \frac{p_1^2}{2} + I_2 + \left(-\frac{4g}{\ell\omega^2} + \frac{4b}{\ell}\cos 2\alpha_2\right)\frac{q_1^2}{2} \\ & - \left(-\frac{4g}{\ell\omega^2} + \frac{4b}{\ell}\cos 2\alpha_2\right)\sum_{p=2}^{\infty}(-1)^p\frac{1}{(2p)!}\left(\frac{4}{m\ell^2\omega^2}\right)^{p-1} q_1^{2p}\delta^{2(p-1)}, \end{aligned} \tag{2.9}$$

where, for notational simplicity, we still denote by (p_1, q_1) the new variables.

Let us consider the Hamiltonian obtained from (2.9) by putting $\delta = 0$: one easily realizes that the corresponding equation for q_1 is Mathieu's equation

$$\ddot q_1 + (a + 2q\cos 2\tau)\, q_1 = 0 \qquad a = -\frac{4g}{\ell\omega^2}, \qquad q = \frac{2b}{\ell}, \tag{2.10}$$

so that the solution is of the form

$$q_1(\tau) = \rho\,\mathrm{Re}\,u_1(\tau), \qquad u_1(\tau) = e^{i\mu\tau}p_0(\tau), \tag{2.11}$$

for some ρ (and for a particular choice of the initial phase) and with $p_0(\tau)$ a periodic function of period π; μ is a real number in $(0,1)$, for ω large enough (see [3], [17] and [9]; see also Appendix A1 for the notations and for a review of some basic properties of Mathieu's equation which will be used below). In (2.11) we can choose ρ so that $q_1^2(0) + p_1^2(0) = 1$ (*i.e.* $x_1^2(0) + y_1^2(0) = O(\delta^2)$).

It is possible to pass to action-angle variables through a canonical transformation

$$(p_1, I_2, q_1, \alpha_2) \to (\mathbf{A}, \boldsymbol{\alpha}) \equiv (A_1, A_2, \alpha_1, \alpha_2), \tag{2.12}$$

see [13] and [10] (see also Appendix A2 below), with α_2 the same in both sides. Then the Hamiltonian (2.9) at $\delta = 0$ becomes

$$H = \boldsymbol{\omega} \cdot \mathbf{A}, \qquad \boldsymbol{\omega} = (\mu, 1), \tag{2.13}$$

which is an isochronous Hamiltonian, while the perturbation

$$(a + 2q\cos 2\alpha_2) \sum_{p=2}^{\infty} (-1)^p \frac{c^{p-1}}{(2p)!} q_1^{2p} \delta^{2(p-1)}, \qquad c = \frac{4}{m\ell^2\omega^2}, \tag{2.14}$$

in (2.9) can be written in terms of the variables $(\mathbf{A}, \boldsymbol{\alpha})$, and gives a function analytic in its arguments (in a suitable domain). By setting $\delta^2 = \varepsilon$ and defining

$$g(\alpha_2) \equiv (a + 2q\cos 2\alpha_2) = \sum_{|\nu_2| \le 1} g_{\nu_2}\, e^{i2\nu_2\alpha_2}, \tag{2.15}$$

the Hamiltonian (2.9) becomes

$$H = \boldsymbol{\omega} \cdot \mathbf{A} + f(\boldsymbol{\alpha}, A_1, \varepsilon), \tag{2.16}$$

where (see Appendix A2)

$$\begin{aligned} f(\boldsymbol{\alpha}, A_1, \varepsilon) &= \sum_{k=1}^{\infty} \varepsilon^k f^{(k)}(\boldsymbol{\alpha}, A_1) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\boldsymbol{\nu} \in \mathbb{Z}^2} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} f_{\boldsymbol{\nu}}^{(k)}(A_1) \\ &= \sum_{p=2}^{\infty} \varepsilon^{p-1} \Phi_p(\boldsymbol{\alpha}, A_1) = \sum_{p=2}^{\infty} \varepsilon^{p-1} \sum_{|\nu_1| \le 2p} \sum_{\nu_2 \in \mathbb{Z}} e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} \Phi_{p,\boldsymbol{\nu}}(A_1), \end{aligned} \tag{2.17}$$

with

$$\Phi_p(\boldsymbol{\alpha}, A_1) = g(\alpha_2) F_p(\boldsymbol{\alpha}, A_1), \tag{2.18}$$

where the functions $F_p(\boldsymbol{\alpha}, A_1)$ are analytic in $(\boldsymbol{\alpha}, A_1)$ uniformly for all p. Call $\mathcal{A}$ the domain of analyticity in A_1, and, for any $A \in \mathcal{A}$, let us denote by $\mathcal{B}_\rho(A)$ the ball of radius ρ and center A; the analyticity in $\boldsymbol{\alpha}$ is in a strip $|\mathrm{Im}\,\alpha_j| < \kappa$, $j = 1, 2$.

The system (2.16) represents two harmonic oscillators interacting through a potential depending only on the angles and on the action variable A_1: the latter condition, in particular, implies that one oscillator is simply a clock, *i.e.* it rotates with fixed frequency.

The corresponding equations of motion are

$$\begin{cases} \dot{\alpha}_1 = \mu + \partial_{A_1} f, \\ \dot{\alpha}_2 = 1, \\ \dot{A}_1 = -\partial_{\alpha_1} f, \\ \dot{A}_2 = -\partial_{\alpha_2} f, \end{cases} \tag{2.19}$$

so that one sees that $\alpha_2(t) = t$.

2.2. *Stability of the upside-down pendulum.* We want to study the persistence of tori near the origin. Note that, for any value of $\mathbf{A}$, the origin is recovered by setting $\varepsilon = \delta^2 = 0$ (see (2.5)). On the other hand for $\varepsilon = 0$ the scaled Hamiltonian (2.16) reduces to $\boldsymbol{\omega} \cdot \mathbf{A}$, so that it admits invariant tori for any value $\mathbf{A}_0 = (A_{01}, A_{02})$ of the action variable $\mathbf{A}$, all run with the same rotation vector $\boldsymbol{\omega}$: such tori are defined by

$$\mathcal{T} = \{\boldsymbol{\alpha}(t) = \boldsymbol{\omega} t, \mathbf{A}(t) = \mathbf{A}_0\}, \tag{2.20}$$

where $\boldsymbol{\omega}$ is fixed and $\mathbf{A}_0 \in \mathbb{R}^2$.

Note that the Hamiltonian (2.16) is of the form considered in [4], so that we can apply the theorem 1.4 of [4]; for simplicity we write here the complete statement (adapted to the notations used above).

THEOREM. [4]. *Fix $\mathbf{A}_0 = (A_{01}, A_{02})$, with $A_{01} \in \mathcal{A}$, and $\rho > 0$ such that $\mathcal{B}_\rho(A_{01}) \subset \mathcal{A}$. Consider the equations of motions (2.19), corresponding to the Hamiltonian (2.16), with $\boldsymbol{\omega} = (\mu, 1)$ satisfying the Diophantine condition*

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| > C\,|\boldsymbol{\nu}|^{-\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}, \tag{2.21}$$

with $C > 0$ and $\tau > 1$, and suppose that

$$\partial_{A_1} f^{(1)}_{\mathbf{0}}(A_1) \equiv \int_{\mathbb{T}^2} \mathrm{d}\boldsymbol{\alpha}\, \partial_{A_1} f^{(1)}(\boldsymbol{\alpha}, A_1) \neq 0 \qquad \forall A_1 \in \mathcal{B}_\rho(A_{01}). \tag{2.22}$$

There is a universal constant $b \in (0,1)$ and three b-dependent constants $a > 0$, $\rho' \in (0,\rho)$ and $\kappa' \in (0,\kappa)$ such that for all $\mu_0 \in (\mu - aC, \mu + aC)$, with $\boldsymbol{\omega}_0 = (\mu_0, 1)$ satisfying the Diophantine condition

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| > C_0 |\boldsymbol{\nu}|^{-\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\}, \qquad C_0 = bC, \tag{2.23}$$

and for all $A_1 \in \mathcal{B}_{\rho'}(A_{01})$, there is a value $\varepsilon = \varepsilon(\mu_0, A_1)$, depending analytically on $A_1 \in \mathcal{B}_{\rho'}(A_{01})$, and two functions $\overline{\mathbf{h}}(\boldsymbol{\psi}, A_1, \mu_0)$ and $\overline{\mathbf{H}}(\boldsymbol{\psi}, A_1, \mu_0)$, analytic in $(\boldsymbol{\psi}, A_1) \in \Sigma_{\kappa'} \times \mathcal{B}_{\rho'}(A_{01})$ and with zero $\boldsymbol{\psi}$-average, such that

$$\begin{cases} \boldsymbol{\alpha}(t) = \boldsymbol{\omega}_0 t + \overline{\mathbf{h}}(\boldsymbol{\omega}_0 t, A_1, \mu_0), \\ \mathbf{A}(t) = \mathbf{A} + \overline{\mathbf{H}}(\boldsymbol{\omega}_0 t, A_1, \mu_0) \end{cases} \tag{2.24}$$

with $\mathbf{A} = (A_1, A_{02})$, is a solution of (2.19). The constant a depends on b, but it is independent of C.

In Appendix A3 we shall verify that the condition (2.22) of the theorem is satisfied for the Hamiltonian (2.16) corresponding to the veryically driven pendulum.

Then fix any value $\mathbf{A}_0$ for which $\mathcal{B}_\rho(A_{01}) \subset \mathcal{A}$ for some $\rho > 0$, and suppose $\boldsymbol{\omega} = (\mu, 1)$ to satisfy the Diophantine condition (2.21): by the above theorem the following scenario arises. By the lemma 2.2 of [4], there exist two functions $\mathbf{h}(\boldsymbol{\psi}, A_1, \varepsilon, \mu_0)$ and $\mathbf{H}(\boldsymbol{\psi}, A_1, \varepsilon, \mu_0)$, called the *conjugating functions*, analytic in their first three arguments in a domain

$$\mathcal{D}_{\rho',\kappa',\varepsilon_0} = \{(\mathbf{A}, \boldsymbol{\alpha}, \varepsilon) \,:\, |A_1 - A_{01}| < \rho', \quad |\mathrm{Im}\,\alpha_j| < \kappa', j = 1,2, \quad |\varepsilon| < \varepsilon_0\}, \tag{2.25}$$

for some $\rho' \in (0,\rho)$, $\kappa' \in (0,\kappa)$ and for $\varepsilon_0 = EC_0$, for a C_0-independent constant E small enough, and vanishing for $\varepsilon = 0$, such that the following holds: it is possible to find (arbitrarily) near to μ a value μ_0 with $\boldsymbol{\omega}_0 = (\mu_0, 1)$ Diophantine with Diophantine constants $C_0 = bC$ and τ, for some constant b, and, correspondingly, a value $\varepsilon = \varepsilon^*$ with $|\varepsilon^*| < \varepsilon_0$, such that

$$\begin{cases} \boldsymbol{\alpha}(t) = \boldsymbol{\omega}_0 t + \mathbf{h}(\boldsymbol{\omega}_0 t, A_1, \varepsilon^*, \mu_0), \\ \mathbf{A}(t) = \mathbf{A} + \mathbf{H}(\boldsymbol{\omega}_0 t, A_1, \varepsilon^*, \mu_0) \end{cases} \tag{2.26}$$

is a solution of the equations of motion (2.19). As the value of $\varepsilon = \varepsilon^*$ is such that difference between μ and μ_0 is $O(\varepsilon)$, see [4], the closer μ_0 is to μ, the smaller is the value of ε, *i.e.* , by (2.5), the closer is the motion to the origin. With the notations (2.24) this means that one has $\overline{\mathbf{h}}(\boldsymbol{\omega}_0 t, A_1, \mu_0) = \mathbf{h}(\boldsymbol{\omega}_0 t, A_1, \varepsilon^*, \mu_0)$ and $\overline{\mathbf{H}}(\boldsymbol{\omega}_0 t, A_1, \mu_0) = \mathbf{H}(\boldsymbol{\omega}_0 t, A_1, \varepsilon^*, \mu_0)$. The quantity $\eta = \mu - \mu_0$ is called counterterm for reasons that are explained in [4]; the analysis in [4] also shows that $\eta = \varepsilon \partial_{A_1} f^{(1)}_{\mathbf{0}}(A_1)$ up to corrections.

Note that (2.26) defines an invariant torus $\mathcal{T}$ run with rotation vector $\boldsymbol{\omega}_0$. The projection of such a torus on the plane (x_1, y_1) gives a closed curve $\mathcal{C}$ with distance of order $O(\varepsilon)$ from the origin: in fact in (2.26) one has $|A_1(t) - A_{01}| = O(\varepsilon)$, by analyticity, while we can take $A_{01} = O(1)$ (in ε).

In the original variables, along the motion corresponding to the considered torus $\mathcal{T}$, one has

$$x_1^2(t) + y_1^2(t) = O(\delta^2), \qquad \delta^2 = \varepsilon, \tag{2.27}$$

in the sense that, for initial data $(x_1(0), y_1(0))$ with $x_1^2(0) + y_1^2(0) = \delta^2$, one has $|x_1^2(t) + y_1^2(t) - \delta^2| < C\delta^4$, for some constant C; in particular for ε small enough along the motion one has $A_1(t) \neq 0$, so that the trajectory never crosses the origin (which can be outside the analyticity domain of the conjugating functions). So we have a closed curve surrounding the origin and which can be made arbitrarily near to it. As all trajectories starting from initial data contained inside the torus described by (2.26) have to remain inside, we have that their projections onto the plane (x_1, y_1)

have to remain inside the closed curve $\mathcal{C}$. Therefore we obtain the stability of the origin for the motion of the variables (x_1, y_1).

2.3. *Extension to vectors satisfying only a non-resonance condition.* Note that the theorem 1.4 of [4] requires $\boldsymbol{\omega}$ to be Diophantine. In the study of stability of elliptic equilbrium points it is well known (see for instance [16]) that a Diophantine condition on the unperturbed frequencies is not necessary, and in fact it can be relaxed (by using that the perturbation depends on $\mathbf{A}$ and ε in a precise way). A mechanism of this kind works also in the present case, and, by using the special form of the perturbation (see (2.15)÷(2.18)), one can show that also for the vertically driven pendulum the conditions on the rotation vectors can be weakened. This could certainly be done within the formalism of the classical approaches to KAM theory, but here we prefer to discuss such an extension of the proof within the tree formalism introduced in [4]. The analysis is performed in Appendix A4, and gives the following result.

Given the Hamiltonian (2.16), if $\boldsymbol{\omega} = (\mu, 1)$ verifies the non-resonance condition

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \neq 0 \qquad \forall \boldsymbol{\nu} \in \{\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \;:\; |\nu_1| \leq 4\}, \tag{2.28}$$

then the equilibrium position $(0, 0)$ for the motion of the variables (x_1, y_1) is stable, *i.e.* for any $\varepsilon > 0$ there exists $\delta > 0$ such that for all initial data $(x_1(0), y_1(0))$ verifying

$$x_1^2(0) + y_1^2(0) \leq \delta^2, \tag{2.29}$$

one has

$$x_1^2(t) + y_1^2(t) \leq \varepsilon^2, \tag{2.30}$$

for all $t \in \mathbb{R}$. The stability follows from the existence of KAM invariant tori corresponding to rotation vectors $\boldsymbol{\omega}_0 = (\mu_0, 1)$ which can be made arbitrarily close to $\boldsymbol{\omega}$ and verify some suitable Diophantine condition (see Appendix A4 for details).

2.4. *Remark.* Note that the Diophantine condition imposed on $\boldsymbol{\omega}$ in the theorem 1.4 of [4] (which allows a full measure set of rotation vectors) can in fact be improved: for weaker conditions holding in general we refer to [20], while a case in which optimal conditions can be explicitly worked out can be found in [7]. Nevertheless *in general*, that is without making any assumptions on the perturbation, no condition of the kind of (2.28), or whatever else, can be obtained: the possibility of imposing only a non-resonance condition like (2.28) can arise only if the perturbation is of some special form, as it is the case for the elliptic equilibrium points and for the vertically driven pendulum studied in the present paper. In this sense, the results formulated in [4] are as general as in any other KAM approach to the study of isochronous systems.

Appendix A1. Some basic properties of Mathieu's equation

A1.1. *Standard form of Mathieu's equation.* Consider the equation (2.10), that we write again as

$$\ddot{x} + (a + 2q\cos 2\tau)\, x = 0, \tag{A1.1}$$

where $a < 0$ and $q \in \mathbb{R}$. The solutions of (A1.1) are of the form

$$u(\tau) = e^{i\mu_0\tau} p_0(\tau), \tag{A1.2}$$

where $\mu_0 \in \mathbb{C}$ and p_0 is π-periodic. The regions of the plane (a, q) such that $\mu_0 \in \mathbb{R}$ are called *stability regions*: the corresponding solutions are quasiperiodic, hence (in particular) bounded.

A1.2. *Stable solutions.* The curves delimiting the stability regions are called *characteristic values* of Mathieu's equation. The only characteristic values for which one can have $a < 0$ are the first two, which are usually denoted by $a_0(q)$ and $b_1(q)$ (see [3], Ch. III); for a, q such that

$a_0(q) < a < \min\{b_1(q), 0\}$ the solution of (A1.1) is stable, (see [3], Ch. VI). More precisely there exist two independent (Floquet) solutions

$$u_1(\tau) = e^{i\mu_0\tau} p_0(\tau), \qquad u_2(\tau) = e^{-i\mu_0\tau} \bar{p}_0(\tau), \tag{A1.3}$$

where $0 < \mu_0 < 1$ (see [3] and [9]) and

$$p_0(\tau) = \sum_{\nu\in\mathbb{Z}} e^{2i\nu\tau} p_{0\nu}, \qquad p_0(0) = 1, \qquad p_0(-\tau) = \bar{p}_0(\tau); \tag{A1.4}$$

here and henceforth, given $z \in \mathbb{C}$, $\bar{z}$ denotes the complex conjugate of z. Both μ_0 and $p_0(\tau)$ depend on a and q. Note that

$$|p_{0\nu}| \le P\, e^{-\kappa|\nu|}, \qquad P = \max_{\tau\in[0,\pi]} |p_0(\tau)|\,, \tag{A1.5}$$

for some $\kappa \in \mathbb{R}_+$.

One can construct two real independent solutions as

$$w_1(\tau) = \frac{1}{2}\left[e^{i\mu_0\tau} p_0(\tau) + e^{-i\mu_0\tau} \bar{p}_0(\tau)\right], \qquad w_2(\tau) = \frac{1}{2ic}\left[e^{i\mu_0\tau} p_0(\tau) - e^{-i\mu_0\tau} \bar{p}_0(\tau)\right], \tag{A1.6}$$

where $c \in \mathbb{R}$ is such that $\dot{w}_2(0) = 1$. Note that w_1 is even and w_2 is odd.

Appendix A2. Action-angle variables for the upside-down pendulum

A2.1. *First change of coordinates.* Define $F_0(\varphi, \psi) = e^{i\varphi} p_0(\psi)$, so that $u_1(t) = F_0(\mu t, t)$. Consider the change of coordinates $\mathcal{C}_1 : \mathbb{R}^3 \times \mathbb{T} \setminus \{(0,0) \times \mathbb{R} \times \mathbb{T}\} \to \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R} \times \mathbb{T}$ defined as

$$\begin{aligned}
&(q_1, p_1, I_2, \alpha_2) \to (\rho, \varphi, I_2, \psi),\\
&q_1 = q_1(\rho, \varphi, \psi) = \rho\,\mathrm{Re}\,(F_0(\varphi, \psi)) \equiv \rho\, Q(\varphi, \psi),\\
&p_1 = p_1(\rho, \varphi, \psi) = \rho\,\mathrm{Re}\,(D\, F_0(\varphi, \psi)) \equiv \rho\, P(\varphi, \psi),\\
&\alpha_2 = \psi,
\end{aligned} \tag{A2.1}$$

where D is the differential operator

$$D = \mu\,\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}. \tag{A2.2}$$

Define also

$$K = \frac{i}{2}\left(u_1(t)\,\frac{\mathrm{d}\overline{u}_1(t)}{\mathrm{d}t} - \frac{\mathrm{d}u_1(t)}{\mathrm{d}t}\,\overline{u}_1(t)\right); \tag{A2.3}$$

with the notations of Appendix A1 one has $K = c \ne 0$; see (A1.6). One can easily show that the change of coordinates (A2.1) is analytic (see [10] and [13]).

A2.2. *Second change of coordinates.* Consider the change of coordinates $\mathcal{C}_2 : \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+ \times \mathbb{T} \times \mathbb{R} \times \mathbb{T}$ defined as

$$\begin{aligned}
&(\rho, \varphi, I_2, \psi) \to (A_1, \alpha_1, A_2, \alpha_2),\\
&A_1 = \frac{1}{2} K \rho^2,\\
&A_2 = I_2 + \frac{1}{2}\rho^2 \left(\frac{\partial Q}{\partial\psi}\, P - Q\,\frac{\partial P}{\partial\psi}\right),\\
&\alpha_1 = \varphi,\\
&\alpha_2 = \psi.
\end{aligned} \tag{A2.4}$$

Then (A2.4) is an analytic change of coordinates, such that the change of coordinates $\mathcal{C}_2 \circ \mathcal{C}_1$ is analytic and canonical, and, in the new variables, the Hamiltonian (2.9) for $\delta = 0$ becomes

$$\boldsymbol{\omega} \cdot \mathbf{A} \equiv \mu A_1 + A_2. \tag{A2.5}$$

For the proof see [10] and [13], in particular the theorem 2 in Section 1.3 of [10].

Moreover the second line of (2.9) can be written as a function

$$f(\boldsymbol{\alpha}, A_1, \delta^2) = \sum_{p=2}^{\infty} \varepsilon^p \Phi_p(\boldsymbol{\alpha}, A_1), \qquad \varepsilon = \delta^2, \tag{A2.6}$$

which does not depend on A_2, and such that the dependence of $\Phi_p(\boldsymbol{\alpha}, A_1)$ on α_1 involves only harmonics with $|\nu_1| \le 2p$ (simply combine the definitions of the two changes of coordinates $\mathcal{C}_1$ and $\mathcal{C}_2$).

Appendix A3. Nonvanishing of the counterterm

A3.1. *First order contribution of the counterterm.* For the Hamiltonian model (2.16), by setting

$$g(\tau) = a + 2q \cos 2\tau, \tag{A3.1}$$

and writing (see (2.14) and (2.17))

$$\begin{aligned} f(\boldsymbol{\alpha}, A_1, \varepsilon) &= g(\alpha_2) \sum_{k=1}^{\infty} \varepsilon^k \left[(-1)^{k+1} \frac{c^k}{(2k+2)!} q_1^{2(k+1)} \right] \\ &= \sum_{k=1}^{\infty} \sum_{\boldsymbol{\nu} \in \mathbb{Z}} \varepsilon^k e^{i\boldsymbol{\nu}\cdot\boldsymbol{\alpha}} f_{\boldsymbol{\nu}}^{(k)}(\mathbf{A}), \qquad f_{\boldsymbol{\nu}}^{(k)}(\mathbf{A}) = \Phi_{k+1,\boldsymbol{\nu}}(A_1), \end{aligned} \tag{A3.2}$$

where for fixed k one has $|\nu_1| \le 2k+2$, the counterterm to first order, *i.e.* $\eta^{(1)}(\mathbf{A}) \equiv \partial_{A_1} f_{\mathbf{0}}^{(1)}(\mathbf{A})$ (see §2.2, and [4] for further details), is obtained by considering the quantity

$$f^{(1)}(\boldsymbol{\alpha}, \mathbf{A}) = g(\alpha_2) \frac{c}{4!} q_1^4, \tag{A3.3}$$

by expressing q_1 in terms of the variables $(\boldsymbol{\alpha}, A_1)$ hence by deriving it with respect to A_1 and computing the Fourier component $\boldsymbol{\nu} = \mathbf{0}$.

By using the change of coordinates (A2.4) we see that one has

$$\partial_{A_1} f_{\mathbf{0}}^{(1)}(\mathbf{A}) = \frac{A_1}{8K^2} \left\langle g\, p_0^2 \overline{p}_0^2 \right\rangle, \tag{A3.4}$$

where, given any quasiperiodic function $f(\tau)$, we are denoting by $\langle f \rangle$ its Fourier component $\boldsymbol{\nu} = \mathbf{0}$.

A3.2. *Nonvanishing of the counterterm to first order.* The function u_1 in (2.11) solves Mathieu's equation, so that

$$\ddot{u}_1 + g\, u_1 = 0; \tag{A3.5}$$

of course also $\overline{u}_1$ solves the same Mathieu's equation. Define

$$w = |u_1|^2 = u_1 \overline{u}_1, \qquad s = w\dot{w}; \tag{A3.6}$$

Then

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}\tau} &= \dot{w}\dot{w} + w\ddot{w} = |\dot{w}|^2 + w\left(\ddot{u}_1 \overline{u}_1 + 2\dot{u}_1 \dot{\overline{u}}_1 + u_1 \ddot{\overline{u}}_1\right) \\ &= |\dot{w}|^2 + 2w\left(\dot{u}_1 \dot{\overline{u}}_1 - g u_1 \overline{u}_1\right), \end{aligned} \tag{A3.7}$$

where

$$\left\langle w \dot{u}_1 \dot{\overline{u}}_1 \right\rangle = \frac{1}{4} \left\langle \left[du_1^2/d\tau\right] \left[d\overline{u}_1^2/d\tau\right] \right\rangle = \frac{1}{4} \left\langle \left| du_1^2/d\tau \right|^2 \right\rangle , \tag{A3.8}$$

so that

$$\begin{aligned} \left\langle g p_0^2 \overline{p}_0^2 \right\rangle &= \left\langle g u_1^2 \overline{u}_1^2 \right\rangle = \left\langle g w u_1 \overline{u}_1 \right\rangle \\ &= \left(\frac{1}{2} \left\langle |\dot{w}|^2 \right\rangle + \left\langle w \dot{u}_1 \dot{\overline{u}}_1 \right\rangle \right) - \frac{1}{2} \left\langle \frac{\mathrm{d}s}{\mathrm{d}\tau} \right\rangle \\ &= \frac{1}{4} \left(2 \left\langle \left| d|u_1|^2/d\tau \right|^2 \right\rangle + \left\langle \left| du_1^2/d\tau \right|^2 \right\rangle \right) > 0, \end{aligned} \tag{A3.9}$$

which yields that the counterterm to first order is nonvanishing for all $A_1 \neq 0$.

Appendix A4. Non-resonance conditions on the rotation vectors

A4.1. *Non-resonant condition on the rotation vector.* The result in Section 2 can be improved in the following way.

Let $\boldsymbol{\omega}$ be such that

$$\boldsymbol{\omega} \cdot \boldsymbol{\nu} \neq 0 \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \text{ such that } |\nu_1| \leq 4. \tag{A4.1}$$

Then it is still possible to perform the analysis of [4] with some minor changes. We refer to the formalism introduced in [4]: we use the same notations used there, by assuming a full knowledge of that paper by the reader, with no further comments henceforth.

Assume that the rotation vector $\boldsymbol{\omega}$ satisfies the non-resonance condition (A4.1). This means that there exists a constant α such that

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| > \alpha \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \text{ such that } |\nu_1| \leq 4, \tag{A4.2}$$

so that, for ε small enough

$$|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}| > \alpha/2 \qquad \forall |\nu_1| \leq 4, \tag{A4.3}$$

provided that $|\boldsymbol{\omega} - \boldsymbol{\omega}_0| < b\varepsilon$ for some constant b. Defining

$$N_\alpha(\theta) = \left| \{ \ell \in \Lambda(\theta) \ : \ |\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_\ell| \leq \alpha/8 \} \right|, \tag{A4.4}$$

then one can prove the following bound.

A4.2. Lemma. *For any tree $\theta \in \mathcal{T}_{k,\boldsymbol{\nu}}$ one has*

$$N_\alpha(\theta) \leq \frac{3k}{5}, \tag{A4.5}$$

where k is the order of θ.

A4.3. *Proof of the lemma A4.2.* We prove by induction on the number of nodes $V = |V(\theta)|$ that

$$N_\alpha(\theta) \leq \max\{(3k-1)/5, 0\}. \tag{A4.6}$$

For $V = 1$ one has $N_\alpha(\theta) = 1$ if $N_\alpha(\theta) \neq 0$. On the other hand, if k is the order of the tree, one has $p = k + 1$ in (2.17) so that $|\nu_1| \leq 2k + 2$ by (2.17). Then by (A4.3) one can have $N_\alpha(\theta) \neq 0$ only for $k \geq 2$: as $3k - 1 \geq 5$ for $k \geq 2$ the bound (A4.6) follows for $V = 1$.

In general note that $k = 1$ implies $V = 1$ so that the above argument yields $N_\alpha(\theta) = 0$ for θ of order 1.

Given $V > 1$, assuming (A4.6) to hold for $V' < V$ we can show that it holds also for V. Let θ be a tree of order k with V nodes and let v_0 be the node which the root line $\ell_0 \equiv \ell_{v_0}$ of θ exits from. Call $\theta_1, \ldots, \theta_m$ the subtrees of order ≥ 2 entering the node v_0 and denote by $k_1, \ldots, k_m$ and $V_1, \ldots, V_m$, respectively, the orders and the numbers of nodes of such subtrees. Call also k' and V' the sum of the orders and of the numbers of nodes, respectively of the subtrees of order 1 entering v_0 (of course $k' = V'$). One has $m \geq 0$, $k_1 + \ldots + k_m + k_{v_0} + k' = k$ and $V_1 + \ldots + V_m + 1 + V' = V$.

If either $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| > \alpha/8$ or $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| \leq \alpha/8$ and $k_{v_0} + k' \geq 2$, then the bound (A4.6) follows by the inductive hypothesis. In fact one has in the first case ($|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| > \alpha/8$)

$$\begin{aligned} N_\alpha(\theta) &= N_\alpha(\theta_1) + \ldots + N_\alpha(\theta_m) \\ &\leq \frac{3k_1 - 1}{5} + \ldots \frac{3k_m - 1}{5} \leq \frac{3(k-1) - m}{5} \leq \frac{3k-1}{5}, \end{aligned} \tag{A4.7}$$

while in the second case ($|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| \le \alpha/8$ and $k_{v_0} + k' \ge 2$)

$$
\begin{aligned}
N_\alpha(\theta) &= 1 + N_\alpha(\theta_1) + \ldots + N_\alpha(\theta_m) \\
&\le 1 + \frac{3k_1 - 1}{5} + \ldots \frac{3k_m - 1}{5} \le 1 + \frac{3(k-2) - m}{5} \\
&\le \frac{3k-1}{5} + \left(1 + \frac{1 - 6 - m}{5}\right) \le \frac{3k-1}{5}.
\end{aligned} \tag{A4.8}
$$

So consider the case $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_0}| \le \alpha/8$ and $k_{v_0} + k' = 1$ (*i.e.* $k_{v_0} = 1$ so that $k' = 0$ and $m \ge 1$). One has

$$
\begin{aligned}
N_\alpha(\theta) &= 1 + N_\alpha(\theta_1) + \ldots + N_\alpha(\theta_m) \\
&\le 1 + \frac{3k_1 - 1}{5} + \ldots + \frac{3k_m - 1}{5} \le \frac{3k-1}{5} + \left(1 + \frac{1 - 3 - m}{5}\right),
\end{aligned} \tag{A4.9}
$$

so that, for $m \ge 3$, (A4.6) follows also for V.

If $m = 2$ one has

$$
N_\alpha(\theta) = 1 + N_\alpha(\theta_1) + N_\alpha(\theta_2). \tag{A4.10}
$$

In such a case the lines ℓ_1 and ℓ_2 entering v_0 can not be such that, simultaneously, $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_1}| \le \alpha/8$ and $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_2}| \le \alpha/8$: this would imply $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{v_0}| < \alpha/2$, which is not possible by (A4.3), as $|\nu_{v_0 1}| \le 4$ for $k_{v_0} = 1$ (see (2.17)). Therefore at least one line entering v_0, say ℓ_1, does not contribute to $N_\alpha(\theta)$, so that, if $\theta'_1, \ldots, \theta'_{m'}$ are the subtrees of orders ≥ 2 entering the node v_1 which the line ℓ_1 exits from, one has in (A4.10)

$$
N_\alpha(\theta_1) = N_\alpha(\theta'_1) + \ldots + N_\alpha(\theta'_{m'}), \tag{A4.11}
$$

and by the inductive hypothesis, by denoting by k'' the sum of the subtrees of order 1 entering v_1 and noting that $k'_1 + \ldots + k'_{m'} = k_1 - k_{v_1} - k'' \le k_1 - 1$ and $k_1 + k_2 = k - 1$, one has

$$
\begin{aligned}
N_\alpha(\theta) &= 1 + N_\alpha(\theta'_1) + \ldots + N_\alpha(\theta'_{m'}) + N_\alpha(\theta_2) \\
&\le 1 + \frac{3(k_1 - 1) - m'}{5} + \frac{3k_2 - 1}{5} \\
&\le \frac{3k-1}{5} + \left(1 - \frac{3 + 3 + m'}{5}\right) \le \frac{3k-1}{5}.
\end{aligned} \tag{A4.12}
$$

Therefore (A4.6) is proved also in such a case.

If $m = 1$ one has

$$
N_\alpha(\theta) = 1 + N_\alpha(\theta_1), \tag{A4.13}
$$

and the line ℓ_1 entering v_0 must have a momentum $\boldsymbol{\nu}_{\ell_1}$ such that $|\boldsymbol{\omega}_0 \cdot \boldsymbol{\nu}_{\ell_1}| > \alpha/8$ (simply reason as above by using that $|\nu_{v_0 1}| \le 4$), so that

$$
N_\alpha(\theta) = N_\alpha(\theta'_1) + \ldots + N_\alpha(\theta'_{m'}), \tag{A4.14}
$$

if $\theta'_1, \ldots, \theta'_{m'}$ are the subtrees of order ≥ 2 entering the node v_1 which the line ℓ_1 exits from; if $m' \ge 2$ the bound follows again by the inductive hypothesis. If $m' = 1$ then one has only one subtree θ'_1 entering v_1, such that its order is $k'_1 \le k - 1 - k_{v_1} \le k - 2$ and

$$
N_\alpha(\theta) = 1 + N_\alpha(\theta'_1), \tag{A4.15}
$$

so that the inductive hypothesis gives

$$
N_\alpha(\theta) \le 1 + \frac{3k'_1 - 1}{5} \le 1 + \frac{3(k-2) - 1}{5} \le \frac{3k-1}{5} + \left(1 - \frac{6}{5}\right) \le \frac{3k-1}{5}, \tag{A4.16}
$$

so that (A4.6) follows once more. If $m' = 0$ the discussion becomes trivial: one has $V \ge 2$, hence $k \ge 2$, while $N_\alpha(\theta) = 1$, so that (A4.6) follows.

Finally if $m = 0$ then again one has $V \geq 2$, hence $k \geq 2$, while $N_\alpha(\theta) = 1$, so that (A4.6) trivially holds. ■

A4.4. *Improvement of the bound on the radius of convergence.* With respect to [4] we can modify the multiscale decomposition by defining the scales starting from n such that $C_0 2^n < \alpha/8 \leq C_0 2^{n+1}$.

Then one can reason as in [15] in order to show that, by using the lemma A4.2, instead of a factor C_0^{-k} one has a factor $(8\alpha^{-1})^k C_0^{-3k/5}$ (here the exponent k for $8\alpha^{-1}$ is a bound on the number of lines which do not contribute to $N_\alpha(\theta)$).

This implies that the radius of convergence ε_0 of the series defining the functions $\mathbf{h}, \mathbf{H}, \eta$ can be bounded by $\varepsilon_0 = E' C_0^{3/5}$ (instead of $\varepsilon_0 = EC_0$ as in Section 2), for some constant E' depending on α but not on C_0.

A4.5. *Persistence of KAM tori.* Under the condition (A4.1) on $\boldsymbol{\omega}$ the analysis of [4] can be repeated for the Hamiltonian (2.16) with f as in (2.17). By the discussion in §A4.4 one finds $\varepsilon_0 \equiv E' C_0^{3/5}$, so that one can choose $\varepsilon_* \in [0, \varepsilon_0]$ and repeat the same argument as in Section 5 of [4]: the main difference is that one can fix the interval I of size $|I| = aC_0^{3/5}$ and, as $C_0^{3/5} \gg C_0$ for C_0 small enough (if C is not small one can choose the constant b in the theorem 1.4 of [4] so that the same conclusions hold), one can easily prove that there are infinitely many $\mu_0 \in I$ such that the corresponding $\boldsymbol{\omega}_0$ verify the Diophantine condition (1.11) of [4].

A4.6. *Remark.* The above argument could be adapted also to the study of existence of KAM tori around an elliptic equilibrium point: in that case one finds that, if $\boldsymbol{\omega}$ are the unperturbed frequencies, the Diophantine condition (2.21) can be weakened into

$$\boldsymbol{\omega} \cdot \boldsymbol{\nu} \neq 0 \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \text{ such that } |\boldsymbol{\nu}| \leq 3, \tag{A4.17}$$

or even into

$$\boldsymbol{\omega} \cdot \boldsymbol{\nu} \neq 0 \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} \text{ such that } |\boldsymbol{\nu}| \leq 4, \tag{A4.18}$$

if one performs a preliminary canonical transformation casting the perturbation to degree 4. Of course such results are known also from the usual KAM theory [16].

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