SELECTION RULES FOR PERIODIC ORBITS AND SCALING LAWS FOR A DRIVEN DAMPED QUARTIC OSCILLATOR

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Abstract. In this paper we investigate the conditions under which periodic solutions of the nonlinear oscillator $\ddot{x} + x^3 = 0$ persist when the differential equation is perturbed so as to become $\ddot{x} + x^3 + \varepsilon x^3 \cos t + \gamma \dot{x} = 0$. For any frequency ω, there exists a threshold for the damping coefficient γ, above which there is no periodic orbit with period $2\pi/\omega$. We conjecture that this threshold is infinitesimal in the perturbation parameter, with integer order depending on the frequency ω. Some rigorous analytical results toward the proof of this conjecture are given: these results would provide a complete proof if we could rule out the possibility that other periodic solutions arise besides subharmonic solutions. Moreover the relative size and shape of the basins of attraction of the existing stable periodic orbits are investigated numerically, showing that all attractors different from the origin are subharmonic solutions and hence giving further support to the validity of the conjecture. The method we use is different from those usually applied in bifurcation theory, such as Mel′nikov's method or that of Chow and Hale's, and allows us to investigate situations in which the non-degeneracy assumptions on the perturbation are violated.

1. Introduction

We study the existence of periodic solutions of the ordinary differential equation

$$\ddot{x} + x^3 + \varepsilon f(t) x^3 + \gamma \dot{x} = 0, \qquad f(t) = \cos t, \tag{1.1}$$

where $\gamma > 0$ is the friction coefficient and ε is a real parameter characterising the strength of the external driving force. As the driving function $f(t)$ has fixed period 2π, this means that the frequency ω of the periodic orbit must be such that $2\pi/\omega$ is commensurable with 2π, that is $\omega = p/q$, where p and q are relatively prime integers. We refer to such a case as a $p\,{:}\,q$ *resonance*. The corresponding solution has period $2\pi q$. If a periodic solution of the system (1.1) for $\varepsilon = \gamma = 0$, with frequency $\omega = p/q$, persists when $\varepsilon, \gamma \neq 0$, then it is called a *subharmonic solution of order* q/p. Both ε and γ are assumed to be small. For ε fixed and small, assuming that γ is also small is natural, in order to avoid the origin becoming a global attractor [3].

Our techniques could be applied to more general differential equations, in particular to the class considered in Chapter 11 of the book by Chow and Hale on bifurcation theory [13], as shown in [17]. We prefer to concentrate on a specific model, such as (1.1), in order to deal with a case in which all analytical calculations can be performed explicitly. Moreover excessive generality has the disadvantage of hiding that some assumptions on the perturbation, although generically true, can fail

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to be satisfied in concrete problems; see below for details. Hence existence of certain attractos can not be explained for some values of the parameters by relying only on the analysis performed in [13]. Finally, equation (1.1) is an equation which has its own interest in physics; see Appendix A for some background and physical applications.

With respect to [13], we are interested not only in the problem of investigating bifurcations from given periodic orbits, but also in the problem of understanding, for fixed values of the parameters, which periodic orbits coexist, and of studying the relative sizes of the corresponding basins of attraction. Moreover we shall consider situations in which some hypotheses of Theorem 2.1 of [13] are not satisfied. This will be essential in order to interpret the numerical results we shall present later, because, typically, not all the periodic orbits which are detected in numerical simulations satisfy all the non-degeneracy hypotheses assumed in [13]. Strictly speaking [13] deals only with periodic solutions whose frequencies $\omega = p/q$ have $p = 1$, but the method can be applied to the case $p \neq 1$ as well. An alternative method was previoulsy developed by Mel′nikov [32] (see also [23]), and it requires essentially the same assumptions, which, expressed in terms of the so-called Mel′nikov function, mean that this has simple zeroes.

Our method is essentially based on perturbation theory, but rigorously implemented (so as to control the perturbation series to all orders and check convergence). It turns out to be a very efficient tool, despite the comments made in [25], as it allows removal of restrictive conditions on the perturbation. Hence our appoach is along the same lines of the papers by Cesari, Gambill and Hale on existence of subharmonic and ultraharmonic solutions for weakly nonlinear systems (see in particular [11, 24, 16, 12]), but deals with strongly nonlinear systems, i.e. with the case in which the unperturbed system is itself nonlinear. An important difference between the linear and nonlinear cases is that in the first one only one periodic solution can be expected to exist, depending on the parameters of the unperturbed linear system, whereas in the nonlinear cases a full range of periods is possible. In both cases one finds that a periodic solution, if *a priori* possible, can really exist only if the friction coefficient is small enough. Hence we show that perturbation theory is a very efficient method to investigate problems of this kind, and it can be naturally extended to the nonlinear setting. Moreover one can show (see for instance [15, 18] and references therein) that the formalism can be extended to even more interesting – and technically more difficult – problems, where also small divisor issues arise.

One motivation to consider equation (1.1) in particular was to explain analytically some numerical results that we discussed in [3]. There, the following scenario emerged. Besides the origin, the only other attractors are periodic orbits with frequencies which are submultiples of that of the driving force, that is of the form $\omega = 1/q$, with $q \in \mathbb{N}$. Furthermore, only the lowest values of q are really relevant, and even values turn out to be preferred with respect to odd values. Here we study equation (1.1) in a regime where perturbation theory can be applied (small ε), and we find in such a regime a theoretical explanation of those results. A direct application of the bifurcation theory developed in [13] is in no way possible, because of the problem of detecting the periodic orbits that one can expect to be continued under perturbation, and, as remarked above, we must also consider periodic orbits for which some non-degeneracy assumptions of [13] are violated (which are for instance those with frequency $\omega = 1/q$, q even). We will come back to this later on in this section.

Another motivation for this work arises from celestial mechanics, where problems of this type appear when studying the resonance locking between the orbital and rotational periods of satellites via the mechanism of spin-orbit coupling [8, 19, 20]. It is well known that the two periods are rationally dependent. Not only is this so: in almost all cases the two periods are equal, so that there is a 1:1 resonance; only in the case of the Sun-Mercury is there a 3:2 resonance. This can be considered as an effect of friction in planetary motions. By deriving a suitable model to describe the problem (spin-orbit model) and introducing some friction terms, one can show that for fixed values of the parameters only a finite number of periodic orbits are actually possible, and each of them appears only when the friction is below some suitable threshold value. For a more detailed discussion see [8, 19, 20]. One can imagine that, in the evolution of the solar system, the friction term decreased over astronomical timescales and the satellite motion eventually became locked into some resonance. In other words, the corresponding trajectory was captured by some periodic attracting orbit. Of course, within such a picture, it becomes essential to analyse the relative sizes of the basins of attraction in order to decide if the existence of a certain periodic orbit can really have some impact on the dynamics – if there exists an attracting periodic orbit but its basin of attraction is negligible, then that orbit will have no physical relevance. This can be difficult to assess from an analytical point of view, but it is essentially routine to perform the numerical simulations.

What we have in mind is an analogous description for the model equation (1.1). It is clear that a given periodic orbit may possibly exist only if the friction coefficient γ is not too large: we have already remarked that if γ is large enough then any trajectory is attracted to the origin. Numerically one finds that for values of γ relatively small, only a few periodic orbits appear and they are asymptotically stable. Moreover the union of (the closure of) the corresponding basins of attraction and of the basin of attraction of the origin appears to fill the entire phase space.

The system described by the equation (1.1) can be considered as a Hamiltonian quasi-integrable system with a friction term. In the absence of friction most of phase space is filled by KAM invariant tori. When friction (however small) is present, not only are all tori destroyed, but it appears that all motions are attracted to a few surviving periodic orbits. Therefore it is of interest to investigate the selection criteria with determine the surviving periodic orbits. By decreasing the value of γ, new periodic orbits can arise. They are less relevant than the old ones, in the sense that their basins of attraction are very small compared with those of the orbits already appearing for larger values of γ. In practice it is very difficult to see the orbits arising for very small values of γ; as we shall see, the corresponding basins of attraction have very small area and the convergence to the orbits is obviously very slow.

A periodic orbit of given rational frequency ω, if it exists at all, exists only if γ is less than a suitable threshold $\gamma_0(\omega, \varepsilon)$. We now make and investigate the following conjecture.

- *The threshold $\gamma_0(\omega, \varepsilon)$ is infinitesimal in ε. The order of magnitude in ε of $\gamma_0(\omega, \varepsilon)$ is given by an integer exponent $m(\omega)$, that is $\gamma_0(\omega, \varepsilon) = O(\varepsilon^{m(\omega)})$.*

Therefore the thresholds $\gamma_0(\omega, \varepsilon)$ are characterised by a scaling law in terms of the perturbation parameter ε, with a scaling exponent which depends on the frequency ω. If we could prove that no other periodic solutions exist besides subharmonic solutions, then the analysis performed here would provide a complete proof of the conjecture. But of course we cannot exclude *a priori* that other periodic solutions exist, possibly with different regularity properties.

To investigate the conjecture, we cannot rely on the analysis performed in [13], as the discussion requires dealing with degenerate cases in which Chow and Hale's hypotheses are not satisfied. Indeed, in [13] existence of subharmonic solutions is proved under two kinds of assumptions, one involving the unperturbed system – hypothesis (H_1) – and one involving the perturbation – hypotheses (H_2) and (H_3), which become (H_4) for the forced systems studied in [25]. In our case the first assumption is satisfied, whereas it can be the case that once the values of the parameters ε and γ have been fixed then the second one holds only for some periods. The result of this is that the other periodic orbits cannot be studied by simply applying the results of [25] and [13].

In this paper we provide results, both analytical and numerical, which support the conjecture. For fixed ε and γ we can write

$$\gamma = \varepsilon^m C_m \tag{1.2}$$

(or more generally $\varepsilon^m C_m$ + higher order terms, which are irrelevant anyway), and we find that subharmonic solutions with given rational frequency ω either are impossible or appear only if C_m is less than some threshold value depending on ω, say $C_m < C_{m,0}(\omega)$.

We study the system (1.1) by writing γ as in (1.2), so that, for fixed m and γ given by (1.2), we can consider ε and C_m as independent parameters. We shall discuss in full detail the cases $m = 1$ and $m = 2$. The other cases can be treated in the same way, but they add nothing new from a qualitative point of view (once the general strategy is understood). Nevertheless, we stress once more that dealing with the cases $m > 1$ allow us to extend the results of Chow and Hale to situations in which their assumptions do not apply, and in particular to study bifurcations of periodic orbits with periods which are excluded under the hypotheses described in [13]. An advantage of our method is that it allows a unified approach to any subharmonic solutions, without making any restrictive assumption on the perturbation. In particular we can study bifurcations from unperturbed periodic orbits of any given frequency ω: when the perturbation is switched on, either an orbit of this kind becomes possible for some m in (1.2) and $C < C_{m,0}(\omega)$ or it never arises (unlikely). Of course one could also envisage generalizations of the bifurcation method described in [13] to problems in which the first assumption is not satisfied. Some results in this direction were provided in [26].

For fixed C_m we prove that only a finite number of periodic orbits are present, as an upper bound on the values of p and q arises. Once a periodic orbit with frequency ω has appeared, it remains for all values of C_m less than the threshold value $C_{m,0}(\omega)$ – so, as the value of m in (1.2) increases, the periodic orbits found for smaller m still survive, while at the same time, new ones appear. We call the resonances appearing for $m = 1$ *primary* (or *first order*) resonances, those appearing for $m = 2$ *secondary* (or *second order*) resonances, and so on.

Note that in this way we more or less reverse the point of view explained initially: in fact, given m, we determine the periodic orbits which are possible under the parameterisation (1.2) of the friction coefficient. Some of the periodic orbits can exist for more than one value of m, and therefore the lowest m for a given frequency $\omega = p/q$ defines $m(\omega)$.

2. Perturbation theory of periodic solutions for primary resonances

We start by considering $\ddot{x}+x^3+\varepsilon f(t)x^3+\varepsilon C\dot{x} = 0$, with $f(t) = \cos t$, for small $\varepsilon \in \mathbb{R}$: this corresponds to $m = 1$ in (1.2). Here C is a real parameter; we shall be interested in the case $\varepsilon C > 0$, so that $\varepsilon C\dot{x}$ can be interpreted as a dissipative term.

For $\varepsilon = 0$ the system reduces to

$$\ddot{x} + x^3 = 0, \tag{2.1}$$

and the solution $x^{(0)}(t)$ can be written in terms of Jacobi elliptic functions. If we set $\alpha = (4E)^{1/4}$, where the energy

$$E = H(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{4}x^4, \tag{2.2}$$

is a constant of motion for (2.1), we can write the solution as $x^{(0)}(t) = \alpha\,\mathrm{cn}(\alpha(t - t_0))$, where $\mathrm{cn}\ t \equiv \mathrm{cn}(t, 1/\sqrt{2})$ is the cosine-amplitude function with elliptic modulus $k = 1/\sqrt{2}$, and $-\alpha t_0$ is the initial phase. In Appendix B we recall some basic properties of elliptic functions.

We can also write $f(t + t_0)$ instead of $f(t)$, so obtaining for $\gamma = \varepsilon C$

$$\ddot{x} + x^3 + \varepsilon f(t + t_0)x^3 + \varepsilon C\dot{x} = 0, \qquad f(t) = \cos t. \tag{2.3}$$

The advantage of introducing t_0 is that we can suitably choose t_0 in order to fix as zero the phase of the solution, which becomes

$$x^{(0)}(t) = \alpha\,\mathrm{cn}(\alpha t). \tag{2.4}$$

In other words, as we have an infinite number of unperturbed solutions of energy E, all differing just by a phase, we prefer to fix this phase as zero by implicitly using an initial condition $x^{(0)}(0) = \alpha$, and move the freedom of choice in the initial condition to the phase of the driving force (the only one which is time-dependent). In this way no generality is lost in fixing the initial condition as in (2.4).

If we denote by $K(k)$ the complete elliptic integral of the first kind

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2\sin^2\theta}}, \tag{2.5}$$

which for $k = 1/\sqrt{2}$ gives $K(1/\sqrt{2}) = [\Gamma(1/4)]^2/(4\sqrt{\pi}) \approx 1.85407$, we have that the solution $x^{(0)}(t)$ is periodic with period $T_0 = 4K/\alpha$, with $K = K(1/\sqrt{2})$. Its derivative is given by

$$y^{(0)}(t) \equiv \dot{x}^{(0)}(t) = -\alpha^2\,\mathrm{sn}(\alpha t)\ \mathrm{dn}(\alpha t), \tag{2.6}$$

with $\mathrm{sn}\ t \equiv \mathrm{sn}(t, 1/\sqrt{2})$ and $\mathrm{dn}\ t \equiv \mathrm{dn}(t, 1/\sqrt{2})$. The constant α is determined by the initial conditions $(\bar{x}^{(0)}, \bar{y}^{(0)}) \equiv (x^{(0)}(0), y^{(0)}(0)) = (\alpha, 0)$, while the initial phase is such that $t_0 \in [0, 4K/\alpha]$.

It is more convenient to work with action-angle variables. A straightforward calculation gives (see Appendix C)

$$\begin{cases} x = (3I)^{1/3} \, \mathrm{cn}\, \varphi, \\ y = -(3I)^{2/3} \, \mathrm{sn}\, \varphi \, \mathrm{dn}\, \varphi, \end{cases} \tag{2.7}$$

where $(\varphi, I) \in \mathbb{R}/4K\mathbb{Z} \times \mathbb{R}_+$ are conjugate variables. Then (2.2) becomes

$$E = \frac{1}{4}(3I)^{4/3} \equiv \mathcal{H}_0(I), \tag{2.8}$$

which yields $3I = (4E)^{3/4}$, and the equations of motion (2.1) can be written as (see Appendix C)

$$\begin{cases} \dot\varphi = (3I)^{1/3} + \varepsilon\,(3I)^{1/3} f(t+t_0) \, \mathrm{cn}^4 \varphi - \varepsilon\, C \, \mathrm{cn}\, \varphi \, \mathrm{sn}\, \varphi \, \mathrm{dn}\, \varphi, \\ \dot I = \varepsilon\,(3I)^{4/3} f(t+t_0) \, \mathrm{cn}^3 \varphi \, \mathrm{sn}\, \varphi \, \mathrm{dn}\, \varphi - \varepsilon\, C\,(3I) \, \mathrm{sn}^2 \varphi \, \mathrm{dn}^2 \varphi. \end{cases} \tag{2.9}$$

In terms of the variables (φ, I) the unperturbed solution ($\varepsilon = 0$) becomes $(\varphi^{(0)}(t), I^{(0)}(t)) = (\alpha t, I^{(0)}) = (\alpha t, \alpha^3/3)$, so that one has $\alpha = (3I^{(0)})^{1/3}$.

The Wronskian matrix $W(t)$ is a solution of the unperturbed linear system

$$\dot W = M(t)\, W, \qquad M(t) = \begin{pmatrix} 0 & (3I^{(0)})^{-2/3} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha^{-2} \\ 0 & 0 \end{pmatrix}, \tag{2.10}$$

so that the matrix $W(t)$ can be taken as

$$W(t) = \begin{pmatrix} 1 & \alpha^{-2} t \\ 0 & 1 \end{pmatrix} \tag{2.11}$$

and one can check that $W(0) = \mathbb{1}$ and $\det W(t) \equiv 1$.

We look for periodic solutions $z(t) = (\varphi(t), I(t))$ with period $2\pi/\omega$, $\omega = p/q \in \mathbb{Q}$, of the form

$$z(t) = \sum_{n=0}^{\infty} \varepsilon^n z^{(n)}(t) \tag{2.12}$$

where the coefficients $z^{(n)}(t)$ are given by

$$z^{(n)}(t) = \begin{pmatrix} \varphi^{(n)}(t) \\ I^{(n)}(t) \end{pmatrix} = W(t) \begin{pmatrix} \bar\varphi^{(n)} \\ \bar I^{(n)} \end{pmatrix} + W(t) \int_0^t \mathrm{d}\tau \, W^{-1}(\tau) \begin{pmatrix} F_1^{(n)}(\tau) \\ F_2^{(n)}(\tau) \end{pmatrix}, \tag{2.13}$$

with

$$\begin{aligned} F_1^{(n)}(t) &= \left[(3I(t))^{1/3} - \left(3I^{(0)}\right)^{-2/3} I(t) \right]^{(n)} \\ &\quad + \left[(3I(t))^{1/3} f(t+t_0) \, \mathrm{cn}^4 \varphi(t) - C \, \mathrm{cn}\, \varphi(t) \, \mathrm{sn}\, \varphi(t) \, \mathrm{dn}\, \varphi(t) \right]^{(n-1)}, \\ F_2^{(n)}(t) &= \left[(3I(t))^{4/3} f(t+t_0) \, \mathrm{cn}^3 \varphi(t) \, \mathrm{sn}\, \varphi(t) \, \mathrm{dn}\, \varphi(t) - C\,(3I(t)) \, \mathrm{sn}^2 \varphi(t) \, \mathrm{dn}^2 \varphi(t) \right]^{(n-1)}, \end{aligned} \tag{2.14}$$

where by $[\ldots]^{(n)}$ we denote the terms of order n in ε within $[\ldots]$.

If $\varepsilon = 0$ one has $\omega = 2\pi/T_0 = 2\pi\alpha/4K$, so that one has a periodic solution with period commensurate with 2π if $2\pi\alpha/4K \in \mathbb{Q}$, which imposes a condition on the energy (2.8); the value of t_0 can be arbitrarily chosen in $[0, 4K/\alpha]$.

If $\varepsilon \neq 0$ we study the conditions under which the unperturbed free solution is preserved, i.e. we look for a solution of the form (2.12), with

$$\omega = \frac{2\pi\alpha}{4K} = \frac{p}{q} \in \mathbb{Q}, \tag{2.15}$$

which reduces to the unperturbed case for $\varepsilon = 0$, only for a suitable choice of the initial time t_0. Note that if ω satisfies (2.15), i.e. if $\alpha = \alpha(p,q) \equiv 4Kp/2\pi q$, then $t_0 \in [0, 2\pi q/p]$ and the solution $z(t)$ is periodic with period $T = 2\pi q$.

In (2.13) we have

$$W(t)W^{-1}(\tau)\begin{pmatrix} F_1^{(n)}(\tau) \\ F_2^{(n)}(\tau)\end{pmatrix} = \begin{pmatrix} 1 & \alpha^{-2}(t-\tau) \\ 0 & 1\end{pmatrix}\begin{pmatrix} F_1^{(n)}(\tau) \\ F_2^{(n)}(\tau)\end{pmatrix} = \begin{pmatrix} F_1^{(n)}(\tau) + \alpha^{-2}(t-\tau)\,F_2^{(n)}(\tau) \\ F_2^{(n)}(\tau)\end{pmatrix}, \tag{2.16}$$

so that, by taking into account that one has

$$\int_0^t \mathrm{d}\tau\,(t-\tau)\,F_2^{(n)}(\tau) = \int_0^t \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau'\,F_2^{(n)}(\tau'), \tag{2.17}$$

we can write the first component $\varphi^{(n)}(t)$ of $z^{(n)}(t)$ in (2.13) as

$$\varphi^{(n)}(t) = \bar\varphi^{(n)} + \alpha^{-2}t\,\bar I^{(n)} + \int_0^t \mathrm{d}\tau\,F_1^{(n)}(\tau) + \alpha^{-2}\int_0^t \mathrm{d}\tau\int_0^\tau \mathrm{d}\tau'\,F_2^{(n)}(\tau'), \tag{2.18}$$

while the second component $I^{(n)}(t)$ is given by

$$I^{(n)}(t) = \bar I^{(n)} + \int_0^t \mathrm{d}\tau\,F_2^{(n)}(\tau). \tag{2.19}$$

Given a periodic function g, let us denote with $g_0 = \langle g\rangle$ the average of g and with $\tilde g$ the zero average function $g - \langle g\rangle$. Suppose that one has

$$\left\langle F_2^{(n)}\right\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\,F_2^{(n)}(t) = 0, \tag{2.20}$$

so that we can write

$$\begin{aligned} \mathcal{F}_1^{(n)}(t) &= \int_0^t \mathrm{d}\tau\,F_1^{(n)}(\tau) = \left\langle F_1^{(n)}\right\rangle t + \int_0^t \mathrm{d}\tau\,\tilde F_1^{(n)}(\tau), \\ \mathcal{F}_2^{(n)}(t) &= \int_0^t \mathrm{d}\tau\,F_2^{(n)}(\tau) = \int_0^t \mathrm{d}\tau\,\tilde F_2^{(n)}(\tau). \end{aligned} \tag{2.21}$$

Then in (2.18) and (2.19) we can write

$$\begin{cases} \varphi^{(n)}(t) = \bar\varphi^{(n)} + \alpha^{-2}t\,\bar I^{(n)} + \left\langle F_1^{(n)}\right\rangle t + \displaystyle\int_0^t \mathrm{d}\tau\,\tilde F_1^{(n)}(\tau) + \alpha^{-2}\left\langle \mathcal{F}_2^{(n)}\right\rangle t + \alpha^{-2}\int_0^t \mathrm{d}\tau\,\tilde{\mathcal{F}}_2^{(n)}(\tau), \\ I^{(n)}(t) = \bar I^{(n)} + \mathcal{F}_2^{(n)}(t) = \bar I^{(n)} + \displaystyle\int_0^t \mathrm{d}\tau \tilde F_2^{(n)}(\tau), \end{cases} \tag{2.22}$$

where all terms which are not linear in time are periodic. By choosing the initial conditions $\bar{I}^{(n)}$ such that one has

$$\bar{I}^{(n)} = -\left(\alpha^2 \left\langle F_1^{(n)} \right\rangle + \left\langle \mathcal{F}_2^{(n)} \right\rangle\right), \tag{2.23}$$

we obtain

$$\begin{cases} \varphi^{(n)}(t) = \bar{\varphi}^{(n)} + \int_0^t \mathrm{d}\tau\, \tilde{F}_1^{(n)}(\tau) + \alpha^{-2} \int_0^t \mathrm{d}\tau\, \tilde{\mathcal{F}}_2^{(n)}(\tau), \\ I^{(n)}(t) = \bar{I}^{(n)} + \int_0^t \mathrm{d}\tau \tilde{F}_2^{(n)}(\tau), \end{cases} \tag{2.24}$$

so that $z^{(n)}(t)$ is a periodic function with period T. So we are left with the problem of proving (2.20). We shall see that this will require fixing also the initial phase t_0 and the initial conditions $\bar{\varphi}^{(n)}$ (which give corrections to t_0).

We shall prove first that the series expansion (2.12) is formally defined, that is that the coefficients $z^{(n)}(t)$ are well defined to all perturbation orders. Then we shall show that the coefficients admit a bound $|z^{(n)}(t)| < Z^n$ for some constant Z, so that, by taking ε and C small enough, the series (2.12) converges to a $2\pi/\omega$-periodic function, analytic in t, ε and C.

3. Existence of periodic solutions for primary resonances

The validity of assumption (2.20) is guaranteed by the following result.

Lemma. *Consider the formal series* (2.12). *If* $p/q = 1/2n$, $n \in \mathbb{N}$, *and* C *is small enough, it is possible to fix the initial conditions* $(\bar{\varphi}^{(n)}, \bar{I}^{(n)})$ *in such a way that* (2.20) *holds for all* $n \geq 1$. *If* $p/q \neq 1/2n$ *for all* n *then* (2.21) *can be satisfied only for* $C = 0$.

Proof. For $n = 1$ one has

$$\begin{aligned} F_2^{(1)}(t) &= \left(3I^{(0)}\right)^{4/3} f(t+t_0)\, \mathrm{sn}\, \varphi^{(0)}(t)\, \mathrm{dn}\, \varphi^{(0)}(t)\, \mathrm{cn}^3\, \varphi(t) - C\left(3I^{(0)}(t)\right) \mathrm{sn}^2\, \varphi^{(0)}(t)\, \mathrm{dn}^2\, \varphi^{(0)}(t) \\ &= \alpha^4 f(t+t_0)\, \mathrm{sn}(\alpha t)\, \mathrm{dn}(\alpha t)\, \mathrm{cn}^3(\alpha t) - C\alpha^3\, \mathrm{sn}^2(\alpha t)\, \mathrm{dn}^2(\alpha t) \end{aligned} \tag{3.1}$$

Define

$$\Delta = \frac{1}{T_0} \int_0^{T_0} \mathrm{d}t\, \mathrm{sn}^2(\alpha t)\, \mathrm{dn}^2(\alpha t) = \frac{1}{4K} \int_0^{4K} \mathrm{d}t\, \mathrm{sn}^2\, t\, \mathrm{dn}^2\, t = \frac{1}{3}, \tag{3.2}$$

(see Appendix D) and

$$\begin{aligned} \Gamma_1(t_0; p, q) &= \frac{1}{T} \int_0^T \mathrm{d}t\, \mathrm{sn}(\alpha t)\, \mathrm{dn}(\alpha t)\, \mathrm{cn}^3(\alpha t)\, f(t+t_0) \qquad (3.3) \\ &= \cos t_0 \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \mathrm{sn}\, t\, \mathrm{dn}\, t\, \mathrm{cn}^3\, t\, \cos(t/\alpha) - \sin t_0 \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \mathrm{sn}\, t\, \mathrm{dn}\, t\, \mathrm{cn}^3\, t\, \sin(t/\alpha) \\ &= -\sin t_0 \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t\, \mathrm{sn}\, t\, \mathrm{dn}\, t\, \mathrm{cn}^3\, t\, \sin(t/\alpha), \end{aligned}$$

(as the integral which multiplies $\cos t_0$ is 0 because of parity) which we rewrite as

$$\Gamma_1(t_0; p, q) = -\sin t_0\, G_1(p, q), \tag{3.4}$$

where

$$G_1(p,q) \equiv \frac{1}{4Kp} \int_0^{4Kp} \mathrm{d}t \, \mathrm{sn}\, t \, \mathrm{dn}\, t \, \mathrm{cn}^3 \, t \, \sin(t/\alpha). \tag{3.5}$$

Then we obtain that, by choosing (if possible) t_0 such that

$$C = \mathcal{G}_1(t_0;p,q) \equiv -\alpha \sin t_0 \frac{G_1(p,q)}{\Delta} \equiv -\sin t_0 \frac{4K}{2\pi\Delta} \left(\frac{p}{q} G_1(p,q) \right), \tag{3.6}$$

one has $\left\langle F_2^{(1)} \right\rangle = 0$.

The function $G_1(p,q)$ is identically vanishing for odd q, while for even q one has $G_1(p,q) = 0$ for all $p \neq 1$ and $G_1(1,q) \neq 0$; see Appendix C for a proof of this assertion. Moreover the function $G_1(p,q)$ is decreasing in q, so that, for a fixed value of C, there will be $q_0 = q_0(C)$ such that (3.5) can be satisfied only for $q \leq q_0$: in other words only a finite number of periodic orbits will exist. Again we refer to Appendix C for details. A list of nontrivial values for $G_1(p,q)$ up to $q = 10$ is given in table 1.

Table 1: Values of $G_1(1,q)$ for $q = 2,4,6,8,10$. All the other values of $G_1(p,q)$, $q \leq 10$, are vanishing. The corresponding threshold values $C_0(p/q)$ for C are computed according to (3.8).

q	$G_1(1,q)$	$\alpha(1/q)$	$C_0(1/q)$
2	0.100773	0.590170	0.178442
4	0.069555	0.295085	0.061574
6	0.015217	0.196723	0.008980
8	0.002078	0.147543	0.000920
10	0.000220	0.118034	0.000078

Note that the existence of a value t_0 satisfying (3.6) is possible only if

$$\min_{t_0 \in [0, 2\pi q/p]} \mathcal{G}_1(t_0;p,q) \leq C \leq \max_{t_0 \in [0, 2\pi q/p]} \mathcal{G}_1(t_0;p,q), \tag{3.7}$$

that is only if

$$|C| \leq C_0(p/q) \equiv \frac{4K}{2\pi\Delta} \left(\frac{p}{q} G_1(p,q) \right) \approx 3.54102 \left(\frac{p}{q} G_1(p,q) \right), \tag{3.8}$$

which represents a smallness condition on C. See table 1 for some threshold values $C_0(p/q)$.

Once t_0 has been set according to (3.6) one has to fix $\bar{I}^{(1)}$ as prescribed by (2.23) for $n = 1$.

To obtain (2.20) for $n \geq 2$ one has to fix the initial conditions $\bar{\varphi}^{(n)}$ for $n \geq 1$: notice that in order to eliminate the terms diverging in time we had to fix only the initial conditions $\bar{I}^{(n)}$. For all $n \geq 2$ one has

$$\begin{aligned} F_2^{(n)}(t) = {} & \left(3I^{(0)}\right)^{4/3} f(t+t_0) \frac{\partial}{\partial \varphi} \left(\mathrm{sn}\, \varphi \, \mathrm{dn}\, \varphi \, \mathrm{cn}^3 \, \varphi \right) \Big|_{\varphi = \alpha t} \bar{\varphi}^{(n-1)} \\ & - C \left(3I^{(0)}\right) \frac{\partial}{\partial \varphi} \left(\mathrm{sn}^2 \, \varphi \, \mathrm{dn}^2 \, \varphi \right) \Big|_{\varphi = \alpha t} \bar{\varphi}^{(n-1)} + R_2^{(n)}(t), \end{aligned} \tag{3.9}$$

where $R_2^{(n)}(t)$ is a suitable function which does not depend on $\bar{\varphi}^{(n-1)}$. One has $\left\langle F_2^{(n)} \right\rangle = 0$ if and only if one has

$$\begin{aligned} \left\langle R_2^{(n)} \right\rangle = -\alpha^3 \Bigg(& \frac{\alpha}{T} \int_0^T \mathrm{d}t \, f(t+t_0) \, \frac{1}{\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{sn}\,(\alpha t) \; \mathrm{dn}(\alpha t) \, \mathrm{cn}^3(\alpha t) \right) \\ & - \frac{C}{T} \int_0^T \mathrm{d}t \, \frac{1}{\alpha} \frac{\mathrm{d}}{\mathrm{d}t} \left(\mathrm{sn}^2(\alpha t) \; \mathrm{dn}^2(\alpha t) \right) \Bigg) \bar{\varphi}^{(n-1)}. \end{aligned} \tag{3.10}$$

An easy computation shows that one has

$$\left\langle R_2^{(n)} \right\rangle = \mathcal{M}_1(t_0; p, q) \, \bar{\varphi}^{(n-1)}, \qquad \mathcal{M}_1(t_0; p, q) = \alpha^3 \cos t_0 \, G_1(p, q), \tag{3.11}$$

so that $\mathcal{M}_1(t_0; p, q)$ is non-vanishing for t_0 such that (3.6) is satisfied; for a proof of (3.11) see Appendix D. Therefore we can fix the initial conditions $\bar{\varphi}^{(n)}$ in such a way that one has $\langle F_2^{(n)} \rangle$ to all orders. This completes the proof of the lemma. □

The analysis performed so far shows that the coefficients $z^{(n)}(t)$ of the series expansions in (2.12) are well defined to all orders. To complete the proof of existence of periodic solutions for primary resonances one has still to show that the series expansions converge. This is assured by the following result.

Theorem 1. *Fix $\omega = p/q = 1/2n$, $n \in \mathbb{N}$. Then there exists $C_0 = C_0(1/q)$, decreasing to zero in q, and, for all $|C| < C_0$, a value $\varepsilon_0 > 0$ such that for all $|\varepsilon| < \varepsilon_0$ the system* (2.3) *admits $2q$ $2\pi/\omega$-periodic solutions $x_1(t, \varepsilon, C)$ analytic in (t, ε, C). The analyticity domain in (ε, C) contains the region*

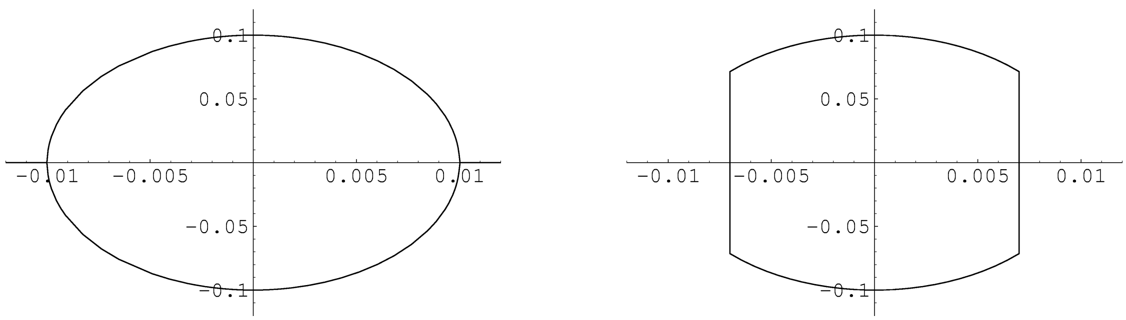
$$\left\{ (\varepsilon, C) \in \mathbb{R}^2 \, : \, \left(\frac{\varepsilon}{a} \right)^2 + \left(\frac{C}{C_0} \right)^2 < 1 \text{ and } |\varepsilon| < b \right\}, \tag{3.12}$$

for two suitable positive constant a and b. Furthermore there are no periodic solutions with frequency $\omega \neq 1/2n$.

Proof. As we are looking for periodic solutions there are no small divisors, and one easily shows that the periodic functions $z^{(n)}(t)$ are analytic in t and admit a bound like Z^k for some positive constant Z. For instance one could use the tree expansion as in [15, 18, 8]; we note that the analysis turns out to be a particular case of that developed in [15], and it is rather trivial as no small divisors appear. Otherwise one could also reason as in [34], Appendix A, to deduce the analyticity of the periodic solutions, hence the convergence of the series, from general considerations.

Then it is sufficient to take ε small enough, say less than some value ε_0, and analyticity in ε for $|\varepsilon| < \varepsilon_0$ follows. The constant ε_0 depends on C; to make explicit such a dependence first of all we express t_0 in terms of C, as $\sin t_0 = -C/C_0$, with $C_0 = \alpha \, G_1(p,q)/\Delta$ (see (3.6)), and hence $\cos t_0 = \sqrt{1 - (C/C_0)^2}$. The study of the perturbation series (again we refer to [15] for details) shows that, for $\omega = 1/2n$ and $|C| < C_0(1/2n)$, as far as the dependence on C is concerned, to all orders n the functions $z^{(n)}(t)$ are just polynomials of order n in C and $\bar{\varphi}^{(n')}$, $n' < n$, and the initial conditions $\bar{\varphi}^{(n)}$ can be written as $(1/\cos t_0)$ times a quantity which again is a polynomial in C and $\bar{\varphi}^{(n')}$, $n' < n$. In the end we find that $z^{(n)}(t)$ can be written as a polynomial of degree n in C and $\sqrt{1 - (C/C_0)^2}$, so that for $|C| < C_0$ analyticity in C follows.

The condition of smallness on ε is of the form $\varepsilon_0 < \min\{b, a|\cos t_0|\}$, for suitable positive constants a and b (again see [15] for details), so that we can write $\varepsilon_0 < \min\left\{b, a\sqrt{1-(C/C_0)^2}\right\}$, and we find that for $\varepsilon < \min\{b, a\}$ we can choose any value of C such that $|C| < C_0\sqrt{1-(\varepsilon/a)^2}$; see figure 1. Then the relation (3.12) between ε and C follows.



(a) Case $a \leq b$. (b) Case $a > b$.

Figure 1: Projection on the real plane of the (estimated) analyticity domain in (ε, C) of the periodic solutions $x_1(t, \varepsilon, C)$. Case (b) is more realistic, as discussed in the text.

Finally, from the proof of the lemma (see comments after (3.6)), we obtain that only periodic solutions with frequency $\omega = p/q$ with $p = 1$ and q even, are possible, and the larger the value of q is the smaller is $C_0(1/q)$. □

If one performs explicitly the calculations of the constants a and b appearing in the estimates, one finds $a > b$, so that figure 1b gives a shape of the estimated analyticity domain which is more realistic. The existence of the formal series (that is the fact that the coefficients are well defined to all orders), which follows from the Lemma, requires a threshold value which is independent of ε, whereas the analysis of the convergence of that series gives a dependence on ε. An approximately square form of the analyticity domain would imply that with good approximation one can take the same threshold value for all ε in the domain.

As the condition (3.7) shows for $|C| < C_0(1/2n)$ there are $4n$ periodic solutions of (2.12) with frequency $\omega = 1/2n$: $2n$ of them are asymptotically stable, hence attracting, and $2n$ unstable [23], so that, when performing numerical analysis, only the first $2n$ can actually be detected: this means that numerically, by letting the system evolve in the future for a long time, only $2n$ periodic orbits with a frequency ω can be found.

We conclude this section by making some connection with the literature. Equation (1.1) belongs to the class of planar equations considered in [13], Chapter 11 (cf. in particular equation (2.2)), hence Theorem 2.1 of [13] applies. As equation (1.1) describes a parametrically driven system, not a forced one, strictly speaking it is not of the form of equation (2.3) considered in [13] (cf. also [25]). However, the analsysis performed in [13] can be easily adapted to the case of driven systems, and a result analogous to Corollary 2.3 of [13] can be derived. Hence a function which represents the analogue of the function $G_k(\alpha)$ of equation 2.9 in [13] can be introduced. It would be, essentially, the same as function $\Gamma_1(t_0; p, q)$ in our case, with $p = 1$ and $q = k$, and with t_0 playing the role of the parameter α. Then, assumption (H_4) of [13] is satisfied in our case only for values of p, q such that $G_1(p, q) \neq 0$ (otherwise the function is identically vanishing): for $p = 1$ – as assumed in [13] – this requires q even. For such values the results of [13] (equivalently of [23]) can be applied. In particular the domain we found in the proof of Theorem 1, in the plane (ε, γ), near the origin, becomes a sector, and it corresponds to the set S_2^k described in [13]. So, for such orbits, our result provides an alternative method to study the bifurcation of surviving periodic orbits, and it can be extended to more general equations as in [13], provided that the associated vector fields are analytic. To deal with all the other periodic orbits we need to go to high orders of perturbation theory. This will be done in next section. We shall see that this will allow us to study situations in which the curves delimiting the sector S_2^k are tangent, even with very high order of tangency.

4. Secondary and higher order resonances

Consider now the equation

$$\ddot{x} + x^3 + \varepsilon f(t + t_0)x^3 + \varepsilon^2 D\dot{x} = 0, \qquad f(t) = \cos t, \tag{4.1}$$

for which we seek for periodic solutions analytic in ε.

We can repeat the analysis performed in the previous section: the only difference is that now the contributions due to the dissipative terms start arising from the second order on.

To first order we find

$$\left\langle F_2^{(1)} \right\rangle = -\sin t_0 \, \alpha^4 \, G_1(p, q), \tag{4.2}$$

as immediately follows from (3.1) by taking into account definitions (3.4) to (3.5) and the fact that there is no term proportional to C. Existence of the formal solution requires $\langle F_2^{(1)} \rangle = 0$. Therefore if $G_1(p, q) = 0$, that is if $p/q \neq 1/2n$, we have that (4.2) is identically satisfied and t_0 is arbitrary, while if $G_1(p, q) \neq 0$, that is if $p/q = 1/2n$, we have to fix $\sin t_0 = 0$.

The latter solutions are dealt with through the following result.

Theorem 2. *Fix $\omega = p/q = 1/2n$. There are $2\pi/\omega$-periodic solutions $x_2(t, \varepsilon, D)$ of (4.1), analytic in (t, ε, D), and they coincide with the solutions $x_1(t, \varepsilon, \varepsilon D)$ of (2.3) given by Theorem 1.*

Proof. Fix $\omega = p/q = 1/2n$. By reasoning as for the proof of Theorem 1, we find that there exist $2\pi/\omega$-periodic solutions of (4.1) for ε small enough and D not too large. The equation (4.2) requires $\sin t_0 = 0$. A second order computation gives

$$\left\langle F_2^{(2)} \right\rangle = \alpha^3 \cos t_0 \, G_1(p, q) \, \bar{\varphi}^{(1)} + \alpha^4 \, \Gamma_2(t_0; p, q) - \frac{1}{3}\alpha^3 D, \tag{4.3}$$

where the first term is that obtained in the previous case, the last one is due to the dissipative term, and the second one takes into account all the contributions which depend neither on $\bar{\varphi}^{(1)}$ nor on D.

It is easy to see that, just by parity properties, one has $\Gamma_2(t_0;p,q)=0$ for $\sin t_0=0$, so that (4.3) requires

$$\cos t_0\, G_1(p,q)\,\bar{\varphi}^{(1)}-\frac{D}{3}=0, \qquad \cos t_0=\pm 1, \tag{4.4}$$

which imposes a constraint on the two parameters D and $\bar{\varphi}^{(1)}$. From higher order computations (as in [15] and [8]) one finds also that, in order to have convergence of the perturbation series, one has to require

$$\varepsilon\bar{\varphi}^{(1)}c_1<1, \tag{4.5}$$

for some positive constant c_1. In the end one obtains that there exists $\varepsilon_1>0$ such that for $|\varepsilon|<\varepsilon_1$ and $D<D_0(1/2n)=O(1/\varepsilon)$ the equation (4.1) admits $2q$ solutions $x_2(t,\varepsilon,D)$ analytic in t, ε and D. For ε small enough choose D so small that, by fixing $C=\varepsilon D$, one has $|C|<C_0$ and (3.12) is satisfied. Then any function $x_1(t,\varepsilon,\varepsilon D)$ is analytic in ε, as the composition of two analytic functions; note also that for $C=\varepsilon D$ the equation (2.3) reduces to (4.1). The analysis of Section 3 shows that there are $2q$ periodic solutions analytic in ε, as there are $2q$ possible values of t_0 such that (3.6) is satisfied in $[0,2\pi q]$. On the other hand there are also $2q$ periodic solutions of (4.1), corresponding to the values $t_0\in[0,2\pi q]$ such that $t_0=k\pi$ with k an integer, so that the solutions have to be pairwise equal to each other. By the uniqueness of analytic continuation, such solutions have to be equal pairwise as long as they are defined. □

For the new periodic solutions (the ones corresponding to frequencies $\omega=p/q\neq 1/2n$) the following result applies.

Theorem 3. *Fix $\omega=p/q=1/(2n+1)$, $n\in\mathbb{Z}_+$. There exists $D_0=D_0(1/q)$, decreasing to zero in q, and, for all $|D|<D_0$, a value $\varepsilon_0>0$ such that for all $|\varepsilon|<\varepsilon_0$ the system (4.1) admits $4q$ $2\pi/\omega$-periodic solutions analytic in (t,ε,D). There are no periodic solutions with frequency $\omega=p/q$, for $p\neq 1$.*

Proof. Fix $\omega=p/q\neq 1/2n$. As explained in the remark after (4.2) in this case $G_1(p,q)=0$, hence t_0 is arbitrary.

To second order a tedious computation (see Appendix D) gives

$$\Gamma_2(t_0;p,q)=\sin 2t_0\, G_2(p,q), \tag{4.6}$$

where

$$
\begin{aligned}
G_2(p,q) = -\frac{1}{8Kp}\int_0^{4Kp} \mathrm{d}t \Big\{ & \sin(t/\alpha)\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{cn}^3\, t\ \mathrm{sn}\, t\ \mathrm{dn}\, t\right)\int_0^t \mathrm{d}t'\ \cos(t'/\alpha)\ \mathrm{cn}^4\, t' \\
& + \cos(t/\alpha)\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{cn}^3\, t\ \mathrm{sn}\, t\ \mathrm{dn}\, t\right)\int_0^t \mathrm{d}t'\ \sin(t'/\alpha)\ \mathrm{cn}^4\, t' \\
& + \sin(t/\alpha)\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{cn}^3\, t\ \mathrm{sn}\, t\ \mathrm{dn}\, t\right)\int_0^t \mathrm{d}t'\ \Bigg[\int_0^{t'} \mathrm{d}t''\ \cos(t''/\alpha)\ \mathrm{cn}^3\, t''\ \mathrm{sn}\, t''\ \mathrm{dn}\, t'' \\
& \qquad - \frac{1}{4Kp}\int_0^{4Kp}\mathrm{d}t\int_0^t \mathrm{d}t'\ \cos(t'/\alpha)\ \mathrm{cn}^3\, t'\ \mathrm{sn}\, t'\ \mathrm{dn}\, t'\Bigg] \\
& + \cos(t/\alpha)\,\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{cn}^3\, t\ \mathrm{sn}\, t\ \mathrm{dn}\, t\right)\int_0^t \mathrm{d}t'\int_0^{t'} \mathrm{d}t''\ \sin(t''/\alpha)\ \mathrm{cn}^3\, t''\ \mathrm{sn}\, t''\ \mathrm{dn}\, t'' \\
& + 4\Big(\sin(t/\alpha)\ \mathrm{cn}^3\, t\ \mathrm{sn}\, t\ \mathrm{dn}\, t\int_0^t \mathrm{d}t'\ \cos(t'/\alpha)\ \mathrm{cn}^3\, t'\ \mathrm{sn}\, t'\ \mathrm{dn}\, t' \\
& + \cos(t/\alpha)\ \mathrm{cn}^3\, t\ \mathrm{sn}\, t\ \mathrm{dn}\, t\int_0^t \mathrm{d}t'\ \sin(t'/\alpha)\ \mathrm{cn}^3\, t'\ \mathrm{sn}\, t'\ \mathrm{dn}\, t'\Big)\Big\},
\end{aligned} \tag{4.7}
$$

where the sum of the last two terms gives a vanishing contribution, as it is the average of a total derivative, so that (4.6) can be satisfied only if

$$
|D| \le D_0(p/q) \equiv \frac{4K}{2\pi\Delta}\left(\frac{p}{q}G_2(p,q)\right) \approx 3.54102\left(\frac{p}{q}G_2(p,q)\right), \tag{4.8}
$$

which defines the threshold values $D_0(p/q)$.

By analogous reasoning to that used for for $G_1(p,q)$ (see Appendix C), one can show that one can have $G_2(p,q) \neq 0$ only if $p/q = 1/n$; see Appendix D. As we are excluding $p/q = 1/2n$, we obtain the result that p has to be 1, and q has to be odd. See table 2 for the quantities $G_2(1,q)$ and the corresponding threshold values $D_0(1/q)$, for the first odd values of q.

Table 2: Values of $G_2(1,q)$ for $q = 1,3,5,7,9$. All the other values of $G_2(p,q)$, with odd $q \le 10$, are vanishing. The corresponding threshold values $D_0(p,q)$ for D are computed according to (4.7).

q	$G_2(1,q)$	$\alpha(1,q)$	$D_0(1/q)$
1	0.041322	1.180341	0.146322
3	0.055069	0.393447	0.065001
5	0.009161	0.236068	0.006488
7	0.000351	0.168620	0.000177
9	0.000006	0.131149	0.000002

So far we have seen that the first order gives no condition, while the second order fixes the value of the initial phase t_0. Then one can show that the higher order contributions fix the values of the corrections $\bar{\varphi}^{(n)}$: contrary to what happens in the case discussed in Section 3 now the condition on $\bar{\varphi}^{(n)}$, $n \geq 1$, is found at step $n+2$ instead than at step $n+1$. We omit the details of the proof, which are rather cumbersome. □

An important difference with respect to the primary resonances discussed in Section 3 is that, as implied by the condition (4.7) for t_0, for $|D| < D_0(p/q)$ we now have $4q$ periodic orbits with frequency $\omega = p/q$: $2q$ of them will be asymptotically stable and $2q$ will be unstable, so that, numerically, by considering only evolution in forward time, only $2q$ periodic orbits can be detected. This is in agreement with the numerical results presented in Section 5.

We can also consider models (1.1) with γ given by (1.2), with other values of m. The general scenario is that, by increasing the value of m, one still finds all the periodic orbits found for the previous values of m, and then new periodic orbits appear, with a threshold $C_{m,0}(\omega)$ of order 1 in ε. For fixed m and C_m, only a finite number of periodic orbits exist, as the following result shows.

Theorem 4. *Consider the system* (1.1)*, with γ given by* (1.2)*. For fixed C_m only a finite number of periodic orbits exist, and the corresponding frequencies $\omega = p/q$ are such that $|p| \leq m$ and $1 \leq q \leq q_m(C_m)$, where $q_m(C_m)$ goes to infinity when C_m goes to zero.*

Proof. We give only a sketch of the proof, which can be performed by induction. For $m = 1$ the statement follows from theorem 1.

Then assume that the statement is true for all $m' < m$: one has to check that only a finite number of new periodic orbits appear. By using the perturbation expansion envisaged in the previous sections one looks for periodic orbits with frequencies ω which were not possible for any previous value m'. Therefore, up to order m, no condition has arisen for such new periodic orbits. To order m one obtains a condition of the form

$$\alpha^4 \, \Gamma_m(t_0; p, q) - \frac{1}{3} \alpha^3 C_m = 0, \tag{4.9}$$

where $\Gamma_m(t_0; p, q)$ is the average (in t) of a function which depends on t_0 as a trigonometric polynomial of order at most m:

$$\Gamma_m(t_0; p, q) = \sum_{r_1=-m}^{m} \sum_{r_2=-m}^{m} \sum_{n \in \mathbb{Z}} P_{r_1,r_2,n} e^{i r_1 t_0} \frac{1}{4Kp} \int_0^{4kp} \mathrm{d}t \, e^{i\pi q r_2 t/2Kp} e^{i\pi n t/2K}, \tag{4.10}$$

so that one can prove (see Appendix D) that (4.10) can be different from zero only if $q/n = n/r_2$, which yields $|p| \leq r_2 \leq m$. For fixed p only the component with $r_2 = p$, hence with $n = q$, can contribute to the average in (4.10). On the other hand the coefficients $P_{r_1,r_2,n}$ tend to zero exponentially for $n \to \infty$, by the analyticity of the elliptic functions, so that, for a fixed value of C_m, there is a value $q = q_m(C_m)$ such that the corresponding $\alpha\gamma(t_0; p, q)$ is less than $C_m/3$, so that (4.9) cannot be satisfied. □

What emerges numerically is that the size of the basins of attraction of the attracting periodic orbits increases when γ is decreased: we can interpret such a phenomenon by saying that if we let γ decrease, when it crosses some value $\gamma_0(p/q, \varepsilon)$ an attracting periodic orbit with frequency p/q appears, and its basin of attraction enlarges as γ continues to decrease. The periodic orbits obtained for $m = 1$ in (1.2), which were the first to appear, have the largest basins of attraction, the orbits appeared at $m = 2$ have smaller basins of attraction, and so on, until the orbits which appear for the largest values of n will be the less relevant ones, that is the ones with the smallest basins of attraction. Such a scenario is well accounted by the numerical results given in the next Section.

5. Numerical results

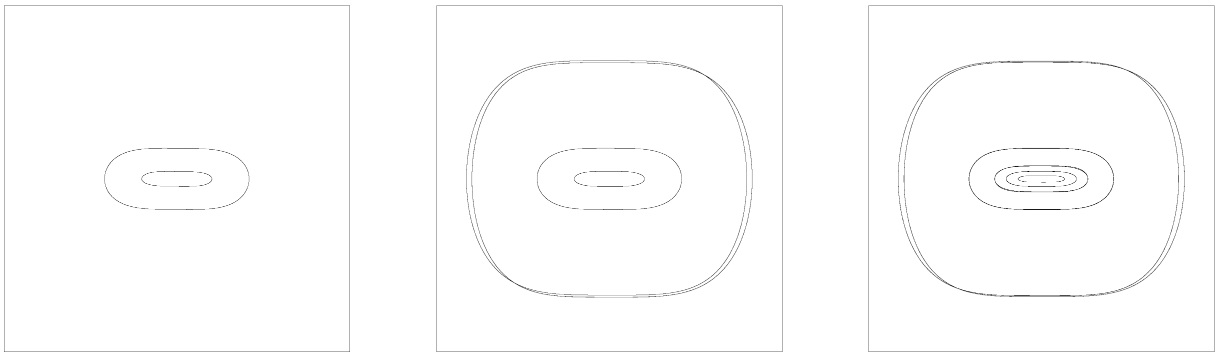
In this Section we give some numerical results for the model (1.1), with $\varepsilon = 0.1$. For numerical purposes it is more convenient to have the same initial phase for all solutions: so, if there is some periodic solution $x(t)$ with frequency ω, then t_0 will be defined by the condition that, denoting the corresponding unperturbed solution by $x^{(0)}(t)$, one has $x^{(0)}(0) = \alpha \, \mathrm{cn}(-\alpha t_0)$.

If one tries to obtain explicit bounds for the constants a and b appearing in the statement of theorem 1, by proceeding as outlined in the proof without looking for optimal estimates, the value of ε_0 as found in the proof of Theorem 1 turns out to be very small. However one expects that such bounds could be greatly improved through a more careful analysis, so that we assume that the analytical results found in the previous sections still apply to the chosen value of ε. Problems of this kind are unavoidable in perturbation theory. For instance a similar situation arises in KAM theory, where analytical bounds on the radius of convergence of the perturbation series are in general very bad, and to obtain bounds compatible with realistic values of the perturbation parameters can be very difficult in simple models as well (such as the standard map, the periodically forced pendulum and the circular restricted three-body problem), even if in such cases better (even optimal) bounds can be worked out with some effort [10, 9]. On the other hand, taking ε extremely small would make any numerical analysis difficult as the system would become a very small perturbation of the free system (for $\varepsilon = 0$) and the corresponding value of the damping coefficient γ would also be small. Decay of the transient, and hence attraction to the periodic orbit, would become very slow and hence difficult to detect without increasing the numerical precision and the running times of the programs. A reasonable compromise turns out to be taking $\varepsilon = 0.1$.

We recall that if the value of γ is large enough all trajectories tend to the origin, which is a global attractor according to the analysis performed in [3]. By decreasing γ new attractors appear.

For instance if we fix $\gamma = 0.005$, hence $C = 0.05$ in (2.3) and $D = 0.5$ in (4.1), according to table 1 we can have only periodic orbits with frequency 1/2 and 1/4, while no periodic orbit among the ones described in Section 4 is possible, as the corresponding value of D is above the threshold value (cf. table 2). Moreover for given frequency $\omega = 1/2n$ there are $4n$ periodic orbits, and $2n$ of them are asymptotically stable. All these orbits are obtained from each other by a shift of 2π in the time direction, so that if they are projected into the $(x, \dot{x})$ plane they give the same closed curve. As we are interested in understanding which periods arise, we do not want to distinguish between orbits with the same periods. This will allow us to study the basins of attraction of all orbits which are identified when projected into the $(x, \dot{x})$ plane.

By taking a grid of 1024×1024 initial conditions in the square $\mathcal{Q} = [-1, 1] \times [-1, 1]$ around the origin, we indeed find that all trajectories are captured either by the origin or by one of the six periodic orbits represented in figure 2a: two orbits with period 4π and four orbits with period 8π.



(a) $\varepsilon = 0.1$ and $\gamma = 0.005$.

(b) $\varepsilon = 0.1$ and $\gamma = 0.001$

(c) $\varepsilon = 0.1$ and $\gamma = 0.0005$.

Figure 2: Attracting periodic orbits in the square $[-1.4, 1.4] \times [-1.4, 1.4]$. Starting from the outer to the inner ones, the frequencies of the orbits are: (a) 1/2 and 1/4, (b) 1 (I and II), 1/2 and 1/4, (c) 1 (I and II), 1/2, 1/3 (I and II), 1/4 and 1/6.

The parts of the basins of attraction of the origin and the periodic orbits with frequencies $\omega = 1/2$ and $\omega = 1/4$ contained in the square $\mathcal{Q}$, are represented in figure 3. As is typically the case [22, 5, 7] there is a thick basin core surrounded by thin layers. Basins of curves with the same projection into the plane $(x, \dot{x})$ are represented together. Note that the basins of attractions are invariant under the transformation $(x, \dot{x}) \to (-x, -\dot{x})$.

Of course there are faster and more sophisticated methods one could use to study the basins of attraction, such as the *straddle orbit method* or its variants [5, 33, 2]. However we are mostly interested in the relative sizes of the basins, so the method we use, which consists in just following the evolution of the initial data point by point [22, 35, 6], even if very simple and slow, is better suited for our purposes.

If we fix $\gamma = 0.001$, hence $C = 0.01$ in (2.3) and $D = 0.1$ in (4.1), the analysis in Section 3 predicts that only the periodic orbits with frequency 1/2 and 1/4 appear, according to table 1, while by using the threshold values in table 2 we see that only the periodic orbit with frequency 1 has to be added to the previous one. We expect that other models (1.1) with γ given by (1.2) for $m \geq 3$ do not imply other periodic orbits than the ones considered, for fixed γ, so that, in the end, we see that the only attractors which are possible for $\varepsilon = 0.1$ and $\gamma = 0.001$ are, besides the origin, the periodic orbits with frequencies 1/2, 1/4 and 1. This is in agreement with the numerical results. Indeed if we take as before a grid of 1024×1024 initial conditions in the square $\mathcal{Q} = [-1, 1] \times [-1, 1]$ around the origin we find that all trajectories are captured either by the origin or by one of the periodic orbits represented in figure 2b, which have exactly the frequencies predicted by the theory. As anticipated there are two attracting periodic orbits with frequency $\omega = 1$ (with different projections into the plane $(x, \dot{x})$), while, for $\omega = 1/2$ and $\omega = 1/4$, there are, respectively, two and four attracting orbits (which reduce, when projected into the plane $(x, \dot{x})$ to two curves, one for the orbits with frequency 1/2 and one for the orbits with frequency 1/4).

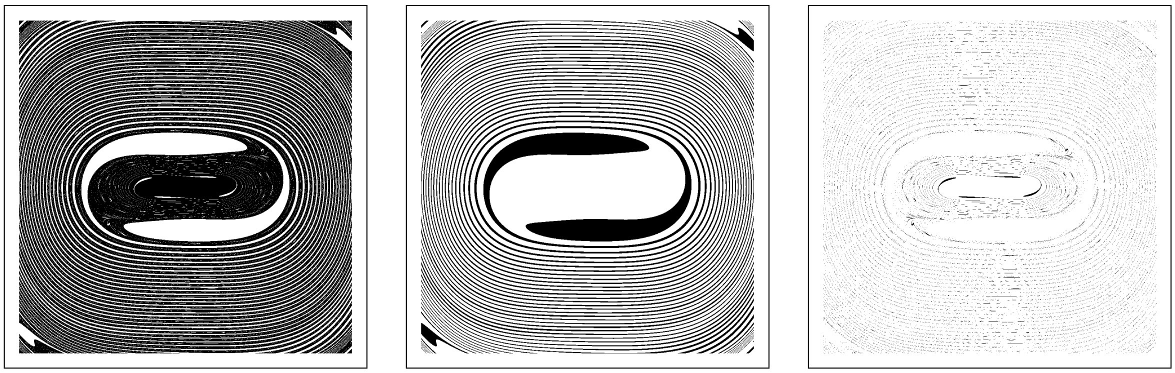


Figure 3: Basins of attraction of the origin (left), the periodic orbits with frequency $\omega = 1/2$ (middle) and with frequency $\omega = 1/4$ (right) for $\varepsilon = 0.1$, $\gamma = 0.005$ in (1.1). Basins of orbits with the same projections into the plane $(x, \dot{x})$ are represented together.

The parts of the basins of attraction of the origin and of the eight periodic orbits contained in the square $\mathcal{Q}$, for $\varepsilon = 0.1$ and $\gamma = 0.001$ are represented in figure 4.

A natural criterion to measure the relative size of the basins of attraction is provided by the respective areas (as given by the number of points of the grid of initial data evolving toward the corresponding attractor). For instance, for $\gamma = 0.001$, one finds that the basin of attraction of the origin is still the largest one as it covers 44.39% of the square $\mathcal{Q}$. On the other hand the size of the basin of attraction of the periodic orbits with frequency 1/2 is comparable, as it covers 40.94% of $\mathcal{Q}$. The relative measures of the basins of attraction of the periodic orbits with frequency 1/2 and of the two periodic orbits with frequency 1 are, respectively, given by 13.32%, 0.67% and 0.67% of the overall area of the square $\mathcal{Q}$.

Analogous numerical analysis performed for higher and lower values of γ shows that, when γ is decreasing, the basin of attraction of the origin, which at the beginning (that is for γ such that $C = \gamma/\varepsilon$ is above the critical threshold $C_0(1/2)$) filled the entire phase space (see also [3]), begins to decrease, to the advantage of the basins of attraction of the newly appearing attractors.

For $\gamma = 0.0005$ there are several attracting periodic orbits: when projected into the plane $(x, \dot{x})$ those with frequency 1 and 1/3 appear in pairs, while only one curve is obtained by projecting the periodic orbits for each frequency of the form $\omega = 1/q$, with $q = 2, 4, 6$; see figure 2c. Of course each projection corresponds to q phase-shifted periodic trajectories. The two projected curves corresponding to the periodic orbits with frequencies 1/3, which appear indistinguishable in figure 2c, can be seen to be different if the figure is enlarged.

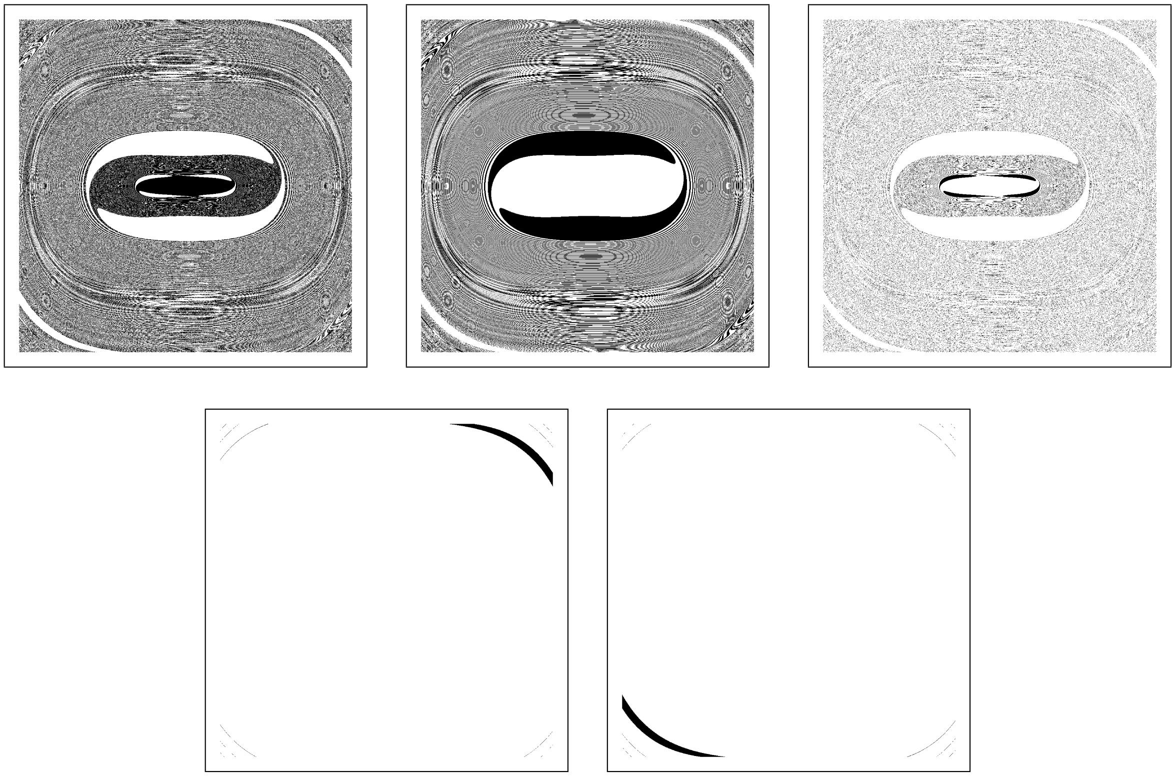


Figure 4: Basin of attraction of the origin (top left) and the periodic orbits with frequency $\omega = 1/2$ (top middle), $\omega = 1/4$ (top right) and $\omega = 1$ (bottom) for $\varepsilon = 0.1$ and $\gamma = 0.001$ in (1.1). Basins of orbits with the same projections into the plane $(x, \dot{x})$ are represented together.

The parts of the basins of attraction of the origin and of the coexisting periodic orbits contained in the square $\mathcal{Q}$, are represented in figure 5.

The relative sizes of the (parts contained in $\mathcal{Q}$) of the basins of attraction of the origin and of the attracting periodic orbits for some values of γ are given in table 3.

It emerges from table 3 that as γ decreases new attracting orbits appear and, once either $C = \gamma/\varepsilon$ or $D = \gamma/\varepsilon^2$ (or whatever else) have became smaller than the corresponding critical threshold, their basins of attraction get larger at the expense of the basin of attraction of the origin. For example the relative measure of the basin of attraction of the periodic orbits with frequency 1/2 increases: for instance for $\gamma = 0.001$ (that is for $C = 0.01$ and $D = 0.1$) it becomes almost equal to the relative measure of the basin of attraction of the origin, and for $\gamma = 0.0005$ it actually becomes larger. Analogous considerations hold for the other attracting orbits.

In general to see orbits with frequency p/q, where either q or p or both of them are large, one needs a very small value for the friction coefficient γ: in the limit that $\gamma = 0$ all periodic orbits appear, as a byproduct of the analysis of the previous sections.

The appearance of the moiré-like patterns in some of the figures can be explained by the use of a regular grid overlaid on the basins of attraction.

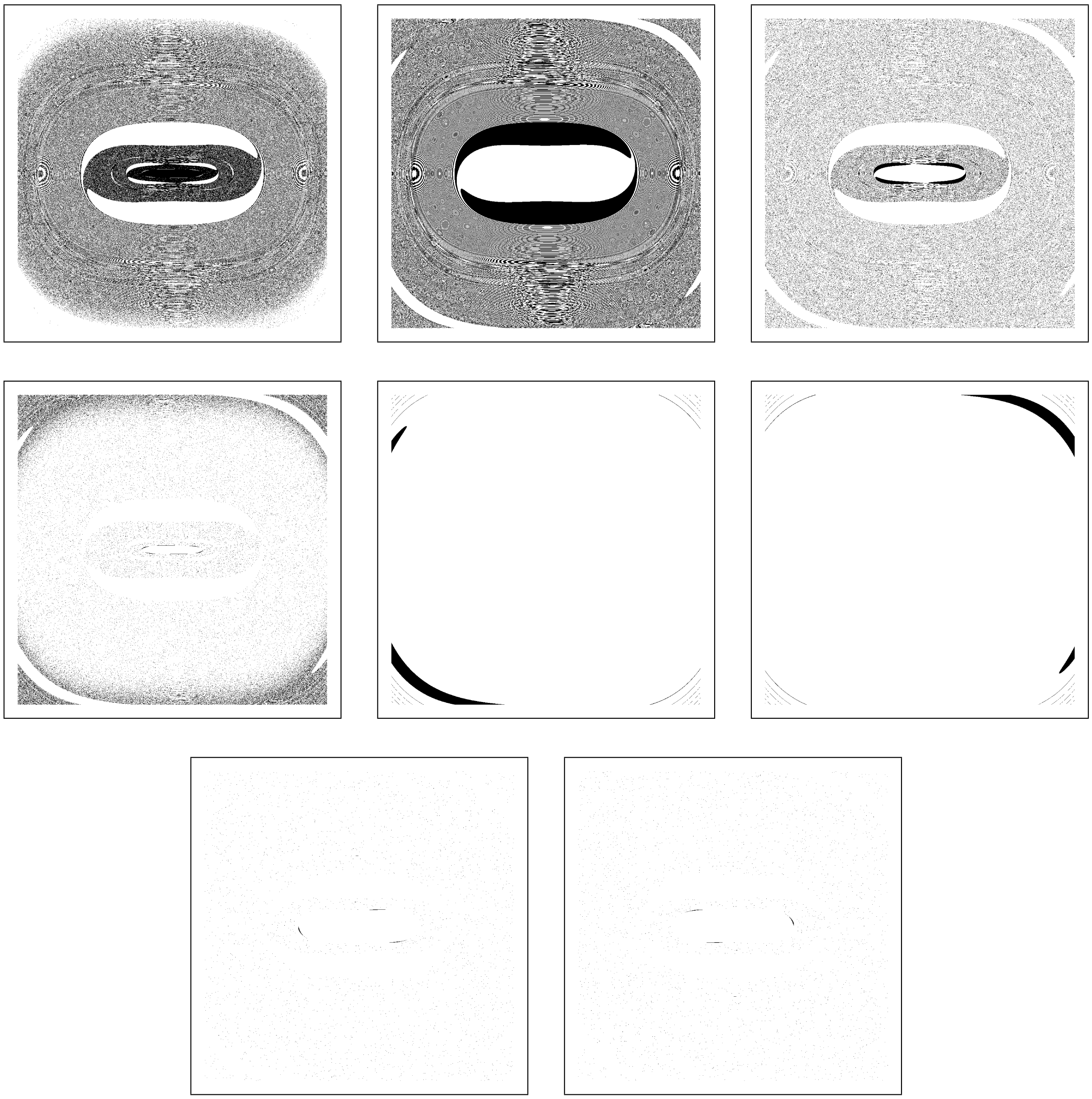


Figure 5: Basin of attraction of the origin (top left) and the periodic orbits with frequency $\omega = 1/2$ (top middle), $\omega = 1/4$ (top right), $\omega = 1/6$ (middle left), $\omega = 1$ (middle middle and right) and $\omega = 1/3$ (bottom) for $\varepsilon = 0.1$, $\gamma = 0.0005$ in (1.1). Basins of orbits with the same projections into the plane $(x, \dot{x})$ are represented together.

Numerically we find that each initial condition in the grid belongs to the basin of attraction of one of the coexisting periodic orbits. This suggests that the union of the closure of all basins of attraction fills the entire phase space: of course approximation errors in the numerical integration of the ordinary differential equation makes it impossible to study the forward evolution of the boundaries of the basins of attraction.

Table 3: Relative sizes of the parts of the basins of attraction contained inside the square $\mathcal{Q} = [-1, 1] \times [-1, 1]$ for some values of γ; the value of ε is fixed at $\varepsilon = 0.1$. The periodic orbits are labelled by the corresponding frequencies, and 0 denotes the origin. I and II refer to curves with different projections into the plane $(x, \dot{x})$. Vanishing percentages mean that there is no corresponding attracting orbit.

γ	0	1/2	1/4	1/6	1-I	1-II	1/3-I	1/3-II
0.0200	100.00%	00.00%	00.00%	00.00%	00.00%	00.00%	00.00%	00.00%
0.0150	91.08%	08.92%	00.00%	00.00%	00.00%	00.00%	00.00%	00.00%
0.0100	79.12%	20.88%	00.00%	00.00%	00.00%	00.00%	00.00%	00.00%
0.0050	64.83%	31.84%	03.34%	00.00%	00.00%	00.00%	00.00%	00.00%
0.0010	44.39%	40.94%	13.32%	00.00%	00.67%	00.67%	00.00%	00.00%
0.0005	34.03%	41.83%	14.56%	06.44%	01.29%	01.29%	00.29%	00.29%

6. Open problems

The analysis performed in Sections 3 and 4 deals with periodic orbits which are obtained for (1.1) in a very particular way: they are subharmonic solutions, that is solutions bifurcating from periodic solutions of the unperturbed system. Numerically all attracting periodic orbits which have been detected are of this kind: it would be interesting to find a mathematical justification for this phenomenon.

As a byproduct we have found that all attractors are, in the cases investigated, periodic orbits. This is unlikely to be an accident. In general, introducing a dissipation term into Hamiltonian equations can produce other kinds of attractors, as for instance in [4]. In our case the system, in the absence of friction, is a quasi-integrable system (that is a perturbation of an integrable system), whereas in [4], if we look at the model considered there as a perturbation of an integrable system, strange attractors appear when the values of the perturbation parameters are large enough, beyond the perturbation regime. Perhaps it is natural to expect that only periodic orbits appear when adding a dissipative term to a quasi-integrable system and confining ourselves to small values of the parameters: such an issue would deserve further study. Note also that in our case, the unperturbed system has a very simple structure as there is only a stable equilibrium point, while in the model studied in [4] there is also a saddle-point, with the corresponding stable and unstable manifolds, which can be responsible for strange attractors appearing for large values of the parameters.

It would be also interesting to see what happens in our case when increasing the value of ε, in particular when we are definitely beyond the range of validity of perturbation theory. Of course an analytic study in such a case is more difficult (even if we can rely on KAM-type results in the absence of friction [30]), but numerically the problem can be easily tackled.

In this paper we have limited ourselves to perturbations of the quartic oscillator. Of course one could consider more general systems, for instance any analytic anharmonic potential (diverging at

$\pm\infty$). Analogous results can be expected in such cases, even if analytically the unperturbed solutions would no longer have the nice properties of the Jacobi elliptic functions which we have used to perform the calculations of perturbation theory.

Finally, a more detailed study of the basins of attraction, for instance with the techniques quoted in Section 5, could be a further topic of investigation. Of course such a study would not shed any further light on the relative sizes of the basins of attraction, in order to understand the relevance that a given attracting periodic orbit can have for the dynamics, but we think that it could be interesting for its own sake.

Acknowledgements. All calculations have been done on Compaq Alpha computers using Fortran 77 and Mathematica. We thank the Department of Physics of the University of Rome "La Sapienza" for providing us with some of the computing resources.

Appendix A. Background and motivations

Differential equations like (1.1) arise in various branches of engineering and mathematical physics. Engineering applications include the description of nonlinear electronic circuits known as parametric amplifiers [29], whose differential equation is typically of the form (1.1). In a more direct application, an equation which reduces exactly to (1.1) for certain values of the parameters is explicitly considered in [37, 38], where the dynamic response of a micro-electromechanical sensor (MEMS) is investigated. According to the means by which the voltage signal is applied, the behaviour of the device can be described either by the Duffing equation, which has been extensively studied in literature (including applications to MEMS devices [1, 14]), or by a nonlinear Mathieu equation. In the latter case the equation becomes $\ddot{x} + \gamma\dot{x} + (\beta + \delta\cos t)x + (\delta_3 + \delta_3' \cos t)x^3 = 0$, and the cubic stiffness of the oscillator (due to both mechanical and electrical effects) affects strongly the dynamic response of the device: by varying the voltage amplitude of the applied electrical signal the frequency response of the first order parametric resonance changes drastically [37, 38]. Note that the presence of the nonlinear terms in the Mathieu equation globally modifies the dynamics. For instance, for $\delta_3 > |\delta_3'|$, unbounded motions are no longer possible. For $\gamma = 0$ this is a consequence of KAM theorem; it also occurs when the driving force is not small, but still such that $1 + \varepsilon f(t)$ in (1.1) is positive [30]. In fact there are plenty of invariant tori which provide a topological obstruction to the drifting of solutions. *A fortiori* boundedness of all motions follows also when there is a friction term – i.e. for $\gamma > 0$.

In [37, 38], β is typically of order 1, while the other parameters are small (say of order ε). In this paper we are interested in the case in which the nonlinear effects are more relevant, so that both β and δ_3 can be supposed to be of comparable magnitude. If they are both positive, then for $\varepsilon = 0$ the equations of motion describe a one-dimensional system with potential $\beta x^2/2 + \delta_3 x^4/4$, so that an explicit solution in terms of Jacobi elliptic functions exists. For simplicity we shall take $\beta = \delta = 0$. Of course this eliminates resonance effects in the linear regime, but we can assume that the same scenario we are investigating appears for $(\beta, \delta) \neq (0, 0)$ well inside some stability regions in the (β, δ) plane – that is outside the resonance tongues. In this case the origin is always a stable equilibrium point. Hence, we take $\beta = \delta = 0$, $\delta_3 = 1$ and $\delta_3' = \varepsilon$, which leads to (1.1).

One might think that confining ourselves to the case of linear stability is too restrictive. However, we are in fact interested not in the problem of stability of the origin and related issues (such as the study of the resonance tongues), but instead in that of studying which periodic orbits are preserved when the perturbation is switched on. This is an important question in the presence of friction, because in such a case all invariant KAM tori break up as an effect of the dissipation, and it is found numerically that all solutions are attracted by periodic orbits, so that the asymptotic behaviour of the system is completely governed by the periodic solutions. Hence our aim is not that of determining for which values of β and δ the origin is stable; this explains why we simplify the system by choosing $\beta = \delta = 0$, so producing (1.1). For the same reason, the role of the friction is very different in our case relative to what happens when studying the characteristic values, that is the boundaries of the resonance tongues, of the linear (and also nonlinear) Mathieu equation. There, a small friction term slightly changes the characteristic values [27, 31, 28], while for the nonlinear equation (1.1) the friction causes all motions to converge either to the origin or to one among a few attracting periodic orbits.

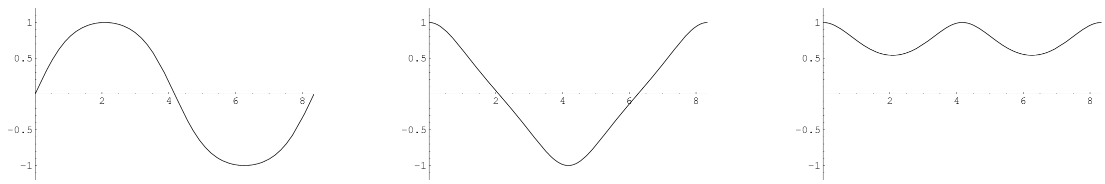
We stress that choosing $\beta = 0$ simply fixes to $k = 1/\sqrt{2}$ the elliptic modulus of the corresponding Jacobi elliptic functions which solve the unperturbed equation ($\varepsilon = \gamma = 0$). Of course the analysis could be extended to any choice of β (but still such that β and δ remain within a stability region). However this would introduce further technical intricacies without shedding any further light on the problem. Also generalisations to other super-quadratic potentials would be possible. Nevertheless, if on the one hand the qualitative features (conservation of only a finite number of periodic orbits) were the same, on the other hand all quantitative results (determination of the threshold values) would of course change, and would be much more difficult to work out analytically. Hence we preferred to consider an explicit model instead of setting up a general scheme which would in the end give more general, but only qualitative, results.

Appendix B. Basic properties of the elliptic functions

Let us denote by $\mathrm{sn}(u, k)$, $\mathrm{cn}(u, k)$ and $\mathrm{dn}(u, k)$ the Jacobi elliptic functions sine-amplitude, cosine-amplitude and delta-amplitude, respectively; see for instance [21] and [36]. Here $k \in (0, 1)$ is the elliptic modulus, and $k' = \sqrt{1 - k^2}$ is the complementary modulus. See figure 6.

The Jacobi elliptic functions are doubly periodic functions with periods $4K(k)$ and $4K'(k)i$, where

$$K(k) = \int_0^{\pi/2} \frac{\mathrm{d}\theta}{\sqrt{1 - k^2 \sin^2 \theta}}, \qquad K'(k) = K(k'), \tag{B.1}$$



(a) Sine-amplitude. (b) Cosine-amplitude. (c) Delta-amplitude.

Figure 6: Jacobi elliptic functions with elliptic modulus $k = 1/\sqrt{2}$.

are the complete elliptic integrals of the first kind. More precisely one has

$$\begin{aligned} \operatorname{sn}(u + 2mK + 2niK', k) &= (-1)^m \operatorname{sn}(u, k), \\ \operatorname{cn}(u + 2mK + 2niK', k) &= (-1)^{m+n} \operatorname{cn}(u, k), \\ \operatorname{dn}(u + 2mK + 2niK', k) &= (-1)^n \operatorname{dn}(u, k), \end{aligned} \tag{B.2}$$

where $K = K(k)$ and $K' = K'(k)$, so that, for real values of the arguments, one has

$$\begin{aligned} \operatorname{sn}(u + 2mK, k) &= (-1)^m \operatorname{sn}(u, k), \\ \operatorname{cn}(u + 2mK, k) &= (-1)^m \operatorname{cn}(u, k), \\ \operatorname{dn}(u + 2mK, k) &= \operatorname{dn}(u, k), \end{aligned} \tag{B.3}$$

which means that $\operatorname{cn}(u, k)$ and $\operatorname{sn}(u, k)$ are periodic functions with period $4K$, while $\operatorname{dn}(u, k)$ is periodic with period $2K$.

One has

$$\operatorname{cn}(-u, k) = \operatorname{cn}(u, k), \qquad \operatorname{sn}(-u, k) = -\operatorname{sn}(u, k), \qquad \operatorname{dn}(-u, k) = \operatorname{dn}(u, k), \tag{B.4}$$

and

$$\begin{aligned} \frac{\partial}{\partial u} \operatorname{cn}(u, k) &= -\operatorname{sn}(u, k)\, \operatorname{dn}(u, k), \\ \frac{\partial}{\partial u} \operatorname{sn}(u, k) &= \operatorname{cn}(u, k)\, \operatorname{dn}(u, k), \\ \frac{\partial}{\partial u} \operatorname{dn}(u, k) &= -k^2 \operatorname{sn}(u, k)\, \operatorname{cn}(u, k). \end{aligned} \tag{B.5}$$

Moreover the following identities hold

$$\begin{aligned} &\operatorname{cn}^2(u, k) + \operatorname{sn}^2(u, k) = 1, \\ &\operatorname{dn}^2(u, k) + k^2 \operatorname{sn}^2(u, k) = 1, \\ &\operatorname{dn}^2(u, k) - k^2 \operatorname{cn}^2(u, k) = 1 - k^2. \end{aligned} \tag{B.6}$$

If $k = 1/\sqrt{2}$ is fixed, we can write, for simplicity, $\operatorname{cn}(u) = \operatorname{cn}(u, 1/\sqrt{2})$, $\operatorname{sn}(u) = \operatorname{sn}(u, 1/\sqrt{2})$ and $\operatorname{dn}(u) = \operatorname{dn}(u, 1/\sqrt{2})$.

Appendix C. Action-angle variables for the quartic potential

To show that the coordinates (φ, I) given by (2.7) are canonical it is sufficient to show that one has

$$\{x, y\} \equiv \frac{\partial x}{\partial \varphi}\frac{\partial y}{\partial I} - \frac{\partial x}{\partial I}\frac{\partial y}{\partial \varphi} = 1, \tag{C.1}$$

and this is an easy computation.

To see that the coordinates (φ, I) can be interpreted as action-angle variables, just note that, by defining

$$A \equiv \frac{1}{4K}\oint y\,\mathrm{d}x = \frac{1}{2\sqrt{2}K}(4E)^{3/4}\int_{-1}^{1}\mathrm{d}x\,\sqrt{1-x^4} = \frac{1}{2\sqrt{2}K}(4E)^{3/4}\frac{2\sqrt{2}K}{3}, \tag{C.2}$$

one obtains $I = A$ (we defer the computation of the integral to Appendix D).

Actually (φ, I) are not strictly speaking action-angle variables as φ is not an angle (it is defined modulo $4K$ instead than 2π); formulae are slightly simpler with our choice of φ.

In the new variables the Hamiltonian for $\varepsilon = 0$ is given by (2.8). If we neglect the dissipative term, then the equation of motion can be derived by the Hamiltonian

$$\mathcal{H}(\varphi, I) = \mathcal{H}_0(I) + \frac{1}{4}\varepsilon\,(3I)^{4/3}\,f(t+t_0)\,\mathrm{cn}^4\,\varphi, \tag{C.3}$$

which yields (2.9) for $C = 0$.

To take into account the dissipative term can just write $\dot{\varphi} = (\partial\varphi/\partial x)\,\dot{x} + (\partial\varphi/\partial y)\,\dot{y}$, and $\dot{I} = (\partial I/\partial x)\,\dot{x} + (\partial I/\partial y)\,\dot{y}$, where the partial derivatives can be computed in terms of the entries of the Jacobian matrix of the inverse transformation:

$$\begin{pmatrix} \dfrac{\partial\varphi}{\partial x} & \dfrac{\partial\varphi}{\partial y} \\ \\ \dfrac{\partial I}{\partial x} & \dfrac{\partial I}{\partial y} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x}{\partial\varphi} & \dfrac{\partial x}{\partial I} \\ \\ \dfrac{\partial y}{\partial\varphi} & \dfrac{\partial y}{\partial I} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{\partial y}{\partial I} & -\dfrac{\partial x}{\partial I} \\ \\ -\dfrac{\partial y}{\partial\varphi} & \dfrac{\partial x}{\partial\varphi} \end{pmatrix}, \tag{C.4}$$

as the determinant of a symplectic matrix is 1, so that (2.9) is immediately obtained.

Appendix D. Some useful integrals

Given Δ as defined in (3.2), using obvious notational shorthands, one has

$$\begin{aligned} \Delta &= \left\langle \mathrm{sn}^2\,\mathrm{dn}^2\right\rangle = -\left\langle (\mathrm{cn})'\,(\mathrm{sn}\ \mathrm{dn})\right\rangle = \left\langle \mathrm{cn}\,(\mathrm{sn}\,\mathrm{dn})'\right\rangle \\ &= \left\langle \mathrm{cn}\left(\mathrm{cn}\,\mathrm{dn}^2 - \frac{1}{2}\,\mathrm{cn}\,\mathrm{sn}^2\right)\right\rangle = \left\langle \mathrm{cn}^2\,\mathrm{dn}^2 - \frac{1}{2}\,\mathrm{cn}^2\,\mathrm{sn}^2\right\rangle \\ &= \left\langle \mathrm{cn}^2\left(1 - \frac{1}{2}\,\mathrm{sn}^2\right) - \frac{1}{2}\,\mathrm{cn}^2\,\mathrm{sn}^2\right\rangle = \left\langle \mathrm{cn}^2 - \mathrm{cn}^2\,\mathrm{sn}^2\right\rangle = \left\langle \mathrm{cn}^2 - \left(2\,\mathrm{dn}^2 - 1\right)\mathrm{sn}^2\right\rangle \\ &= \left\langle \mathrm{cn}^2 + \mathrm{sn}^2\right\rangle - 2\left\langle \mathrm{sn}^2\,\mathrm{dn}^2\right\rangle = 1 - 2\left\langle \mathrm{sn}^2\,\mathrm{dn}^2\right\rangle = 1 - 2\Delta, \end{aligned} \tag{D.1}$$

where the prime denotes the derivative, and (B.6) have been repeatedly used; hence $\Delta = 1/3$.

One has

$$\begin{aligned}\int_{-1}^{1} \mathrm{d}x\,\sqrt{1-x^4} &= \int_0^{2K} \mathrm{d}t\,\mathrm{sn}\,t\,\mathrm{dn}\,t\,\sqrt{1-\mathrm{cn}^4\,t} = \int_0^{2K} \mathrm{d}t\,\mathrm{sn}\,t\,\mathrm{dn}\,t\,\sqrt{2\,\mathrm{sn}^2\,t\,\mathrm{dn}^2\,t} \\ &= 2\sqrt{2}K\left(\frac{1}{4K}\int_0^{4K}\mathrm{d}t\,\mathrm{sn}^2\,\mathrm{dn}^2\,t\right) = 2\sqrt{2}K\Delta = \frac{2\sqrt{2}K}{3},\end{aligned} \tag{D.2}$$

which implies the last identity in (C.2).

The Fourier series of the Jacobi elliptic functions considered in Appendix B are given by

$$\begin{aligned}\mathrm{sn}(u,k) &= \frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{\mathfrak{q}^{n-1/2}}{1-\mathfrak{q}^{2n-1}}\sin\frac{(2n-1)\pi u}{2K(k)}, \\ \mathrm{cn}(u,k) &= \frac{2\pi}{kK(k)}\sum_{n=1}^{\infty}\frac{\mathfrak{q}^{n-1/2}}{1+\mathfrak{q}^{2n-1}}\cos\frac{(2n-1)\pi u}{2K(k)}, \\ \mathrm{dn}(u,k) &= \frac{\pi}{2K(k)} + \frac{2\pi}{K(k)}\sum_{n=1}^{\infty}\frac{\mathfrak{q}^{n}}{1-\mathfrak{q}^{2n}}\cos\frac{2n\pi u}{2K(k)},\end{aligned} \tag{D.3}$$

where $\mathfrak{q} = \exp(-\pi K'(k)/K(k))$, so that $\mathfrak{q} = e^{-\pi}$ for $k = 1/\sqrt{2}$, while we can write

$$\begin{aligned}G_1(p,q) &= \frac{1}{4Kp}\int_0^{4Kp}\mathrm{d}t\,\mathrm{sn}\,t\,\mathrm{dn}\,t\,\mathrm{cn}^3\,t\,\sin(t/\alpha) = \frac{1}{4Kp}\int_0^{4Kp}\mathrm{d}t\left(-\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{cn}^4\,t\right)\sin(t/\alpha) \\ &= \frac{1}{4\alpha}\left(\frac{1}{4Kp}\int_0^{4Kp}\mathrm{d}t\,\mathrm{cn}^4\,t\,\cos(t/\alpha)\right),\end{aligned} \tag{D.4}$$

with

$$\cos(t/\alpha) = \cos\frac{\pi t}{2K(k)}\frac{q}{p}, \tag{D.5}$$

so that one can have $G_1(p,q) \neq 0$ only if, for suitable integers n_j one has

$$p\left(\pm(2n_1-1)\pm(2n_2-1)\pm(2n_3-1)\pm(2n_4-1)\right)\pm q = 0, \tag{D.6}$$

which requires for q to be of the form

$$q = 2np, \qquad n\in\mathbb{Z}. \tag{D.7}$$

Therefore first of all q has to be even. Moreover, for fixed q, one must have $p = q/2n$ for some $n\in\mathbb{Z}$. If we impose that (q,p) are relatively prime then the identity $p/q = 1/2n$ imposes $q = 2n$ and $p = 1$. Finally (D.4) also implies that for $p/q = 1/2n$ one has $G_1(1/2n) > 0$, as $G_1(p,q)$ is equal to the qth Fourier label of the function $\mathrm{cn}^4(t)$, which is strictly positive by the second of (D.3).

Note moreover that if we choose q to be large enough in (D.7) then also n has to be large, so that large Fourier labels of the elliptic functions have to be involved in order that the integral $G_1(p,q)$ be non-vanishing. This implies that the corresponding value of $G_1(p,q)$ has to be small enough (use the fact that in the expansions (D.3) one has $0 < \mathfrak{q} < 1$).

Now let us show how the condition (3.10) implies (3.11). One can write (3.10) as

$$\left\langle R_2^{(n)}\right\rangle = \alpha^3\left(\alpha\,U(p,q) - C\,V(p,q)\right)\bar{\varphi}^{(n-1)}, \tag{D.8}$$

with

$$U(p,q) \equiv \frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t\, \cos(t/\alpha)\, \frac{1}{\alpha}\, \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{sn}\, t\ \mathrm{dn}\, t\ \mathrm{cn}^3\, t\right)\right)\cos t_0 - \frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t\, \sin(t/\alpha)\, \frac{1}{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{sn}\, t\ \mathrm{dn}\, t\ \mathrm{cn}^3\, t\right)\right)\sin t_0 = \frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t\, \cos(t/\alpha)\frac{1}{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{sn}\, t\ \mathrm{dn}\, t\ \mathrm{cn}^3\, t\right)\right)\cos t_0, \tag{D.9}$$

and

$$V(p,q) \equiv \frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t\, \frac{1}{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{sn}^2\, t\ \mathrm{dn}^2\, t\right)\right) = 0. \tag{D.10}$$

In (D.9), by integrating twice by parts, we obtain

$$U(p,q) = -\frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t\left(\cos(t/\alpha)\frac{1}{\alpha}\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{1}{4}\ \mathrm{cn}^4\, t\right)\right)\right)\cos t_0 = -\frac{1}{4\alpha}\left(\frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t\, \sin(t/\alpha)\frac{1}{\alpha}\left(\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{cn}^4\, t\right)\right)\cos t_0 = \frac{1}{4\alpha^2}\left(\frac{1}{4Kp}\int_0^{4Kp} \mathrm{d}t\, \cos(t/\alpha)\ \mathrm{cn}^4\, t\right)\cos t_0 = \frac{1}{\alpha}\, G_1(p,q)\, \cos t_0, \tag{D.11}$$

which implies (3.11).

Now let us consider the case $\gamma = \varepsilon^2 D$ and $p/q \neq 1/2n$: first of all we want to prove (4.7). By shortening $O(t) = \mathrm{cn}^3(\alpha t)\ \mathrm{sn}\,(\alpha t)\ \mathrm{dn}\,(\alpha t)$, $E(t) = \dot{O}(t)$, $c(t) = \cos t$ and $s(t) = \sin t$, and by denoting with $I[F](t)$ the integral of F between 0 and t, we can write

$$\left\langle F_2^{(2)}\right\rangle = \alpha^4\left\{\cos t_0\, \langle cE\rangle\, \bar{\varphi}^{(1)} + \cos t_0\, \sin t_0\left(\left\langle sEI[cE^{(1)}]\right\rangle + \left\langle cEI[sE^{(1)}]\right\rangle\right) + \cos t_0\, \sin t_0\left(\left\langle sEI[I[cO^{(1)}] - \langle I[cO^{(1)}]\rangle]\right\rangle + \left\langle cEI[I[sO^{(1)}]]\right\rangle\right)\right\} + 4\,\alpha\left\{-\sin t_0\, \langle sO\rangle\, \bar{I}^{(1)} + \alpha^2\cos t_0\, \sin t_0\left(\left\langle sOI[cO^{(1)}]\right\rangle + \left\langle cOI[sO^{(1)}]\right\rangle\right)\right\} - \frac{1}{3}\alpha^3 D, \tag{D.12}$$

for suitable functions $E^{(1)}$ (even) and $O^{(1)}$ (odd); an explicit computation gives

$$E^{(1)}(t) = -\alpha\ \mathrm{cn}^4(\alpha t), \qquad O^{(1)}(t) = -\alpha^2\, \mathrm{cn}^3(\alpha t)\ \mathrm{sn}\,(\alpha t)\ \mathrm{dn}\,(\alpha t) = -\alpha^2 O(t). \tag{D.13}$$

This simply follows from the parity properties of the unperturbed solution (2.4), from the remark that if F is an even function then $I[F]$ is odd and from the the Fourier expansions (D.3) of the Jacobi elliptic functions (which imply that the averages $\langle cE^{(1)}\rangle$ and $\langle I[sO]\rangle$ are vanishing for $p/q \neq 1/2n$).

In (D.12) the averages $\langle cE\rangle$ and $\langle sO\rangle$ are vanishing because of (D.6) and (D.7); note also that the first one is simply $\alpha^{-1}G_1(p,q)$ (see (D.9) and (D.11), so that (4.6) and (4.7) are proved, with $\gamma_1(p,q)$ defined according to (4.8).

Imposing $\langle F_2^{(2)}\rangle = 0$ gives

$$\frac{1}{3}\alpha^3\, D = \frac{1}{2}\alpha^3\, \sin 2t_0\left\{\alpha\left(\left\langle sEI[cE^{(1)}]\right\rangle + \left\langle cEI[sE^{(1)}]\right\rangle\right) + \alpha\left(\left\langle sEI[I[cO^{(1)}]]\right\rangle + \left\langle cEI[I[sO^{(1)}]]\right\rangle\right) + 4\left(\left\langle sOI[cO^{(1)}]\right\rangle + \left\langle cOI[sO^{(1)}]\right\rangle\right)\right\}, \tag{D.14}$$

By writing

$$c(t) = \frac{1}{2} \sum_{\sigma=\pm 1} e^{i\sigma t}, \qquad s(t) = \frac{1}{2i} \sum_{\mu=\pm 1} \mu e^{i\mu t}, \qquad F(t) = \sum_{n\in\mathbb{Z}} e^{in\omega t} F_n, \tag{D.15}$$

for $F = E, O, E^{(1)}, O^{(1)}$, we can rewrite (D.13) as

$$\begin{aligned} &\frac{1}{3}\alpha^3 \, D = \alpha^3 \, \sin 2t_0 \, G_2(p,q), \\ &G_2(p,q) = \frac{1}{4i} \sum_{\omega(n+n')+\mu+\sigma=0} \left\{ \frac{\alpha}{i(\mu+\omega n')} \left(E_n E^{(1)}_{n'} \right) (\mu+\sigma) \right. \\ &\qquad \left. + \frac{\alpha}{i^2(\mu+\omega n')^2} \left(E_n O^{(1)}_{n'} \right) (\mu+\sigma) + \frac{4}{i(\mu+\omega n')} \left(E_n O^{(1)}_{n'} \right) (\mu+\sigma) \right\}, \end{aligned} \tag{D.16}$$

so that we immediately see that the contributions with $\mu+\sigma=0$ disappear. As $\mu+\sigma \in \{-2,0,2\}$ then we have to retain in (D.16) only the contributions with n and n' such that $n+n' = \pm 2q/p$. As n and n' have to be even, as it is easy to check from the expressions (D.13) by relying on the Fourier expansions (D.3), we are left with $2n \pm 2q/p = 0$, which requires $p/q = 1/n$. As we are excluding values of p,q such that $p/q = 1/2n$ we have finally shown that the quantity $G_2(p,q)$ in (4.7) can be different from zero only for $p/q = 1/n$, with n odd.

Finally we want to prove that for $\gamma = \varepsilon^m C_m$ to order m the equation $\langle F_2^{(m)} \rangle = 0$ takes the form (4.9). First note that one has a function $f(t+t_0)$ for each perturbation order, so that $\langle F_2^{(m)} \rangle$ is a polynomial of order m in t_0: for the same reason it has to be a polynomial in t/α. Because of the presence of the Jacobi elliptic functions any further dependence on t has to be analytic and $4K$-periodic: hence the expansion (4.10) follows. In order to have $\gamma_m(t_0;p,q) \neq 0$ one has to require (by the same reasoning used to obtain (D.6) for $m=1$) $qr_2/p = n \in \mathbb{Z}$: as p and q are relatively prime integers and $|r_2| \leq m$, then one must have $|p| \leq m$. The exponential decay in n of the coefficients $P_{r_1,r_2,n}$ can be proved as in the previous case $m=1$, by using the analyticity of the elliptic functions (which in turn implies the exponential decay of the Fourier coefficients appearing in the expansions (D.3)).

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