Explicit estimates on the torus for the sup-norm and the crest factor of solutions of the Modified Kuramoto-Sivashinky Equation in one and two space dimensions

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Abstract

We consider the Modified Kuramoto-Sivashinky Equation (MKSE) in one and two space dimensions and we obtain explicit and accurate estimates of various Sobolev norms of the solutions. In particular, by using the sharp constants which appear in the functional interpolation inequalities used in the analysis of partial differential equations, we evaluate explicitly the sup-norm of the solutions of the MKSE. Furthermore we introduce and then compute the so-called crest factor associated with the above solutions. The crest factor provides information on the distortion of the solution away from its space average and therefore, if it is large, gives evidence of strong turbulence. Here we find that the time average of the crest factor scales like $\lambda^{(2d-1)/8}$ for λ large, where λ is the bifurcation parameter of the source term and $d = 1, 2$ is the space dimension. This shows that strong turbulence cannot be attained unless the bifurcation parameter is large enough.

Short title: On the Crest Factor for the Modified Kuramoto-Sivashinsky Equation.

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1 Introduction

Accurate analysis of solutions of partial differential equations (PDEs) is an essential part in our understanding of many of the features of physical and biological phenomena. There are various approaches that strive to obtain detailed information on the behaviour of solutions of PDEs. In this work we use functional analysis methods and we employ the latest explicit and sharp estimates for the embedding constants appearing in the functional inequalities widely used in the study of any PDE. More precisely we have computed, as accurately as we possibly can, the estimates for some of the classical Sobolev norms of solutions of a model very close to some classical PDEs to which it reduces in particular cases. In the following we will refer to our model as the Modified Kuramoto-Sivashinky Equation (MKSE); in two space dimensions it reads

$$u_t = -\Delta^2 u - \Delta u + \lambda u - u^3 - u(u_x + u_y), \tag{1}$$

where Δ is the Laplacian, $u = u(x, y, t)$ for $(x, y) \in \Omega = [0, L]^2$, with $L > 0$ and $t > 0$, subject to the initial condition $u(x, y, 0) = u_0(x, y)$ and periodic boundary conditions on the boundary of Ω. The real constant λ is called the *bifurcation parameter*; since we are mainly interested in the behaviour of the system for large λ, for simplicity we take $\lambda > 0$. In this work we will obtain accurate estimates of some Sobolev norms of the MKSE such as the the L^∞ norm of its solutions. Furthermore we have introduced an important concept in the analysis of the behaviour of solutions of dissipative PDEs, namely the so-called "crest factor", which is defined as the ratio between the L^∞ and the L^2 norm of solutions. It has therefore the dimension of the square root of the inverse of the "volume" of the torus in d spatial dimensions, and hence it can be made dimensionless by multiplying it by $L^{\frac{d}{2}}$. The crest factor contains important informations on the "distortions" between the amplitude and the L^2 norm of the solution. It is in fact a standard measurement used in turbulence experiments in fluid dynamics. Effectively what it says is that if it is of order one then the dynamics is relatively "mild", in the sense that the solution does not have major excursions in space-time. However, when the maximum amplitude of the solution becomes much larger with respect to its spatial average, then the solution does have strong deviations in space and time; these strong intermittent fluctuations away from the averages are one of the hallmarks of hard turbulence. This phenomenon is now well established in many physical contexts such as, for example, in fluid convection. Thus the main aim of this work is to estimate in an explicit and accurate manner both some classical Sobolev norms of the solutions of the MKSE and the associated crest factor of these solutions.

Going back to our model first note that in the one space dimensional case the (1) naturally reduces to the classical Kuramoto-Sivashinsky equation in the presence of a source term and a saturation term, namely one has

$$u_t = -u_{xxxx} - u_{xx} + \lambda u - u^3 - u u_x, \tag{2}$$

for $x \in \Omega = [0, L]$, with $L > 0$, and $\lambda > 0$.

Note also that by neglecting the last term in (2) it reduces to another classical dissipative PDE, namely the Swift-Hohenberg equation. Both the Kuramoto-Sivashinsky equation and the Swift-Hohenberg equation have been extensively investigated because of their fundamental importance in many mathematical, physical, biological and other contexts. So the literature on these two PDEs is huge and here we are forced to cite only a few of the relevant papers devoted to them: see for example [34, 13, 23, 32, 22, 17, 35, 11, 21, 10, 30].

The layout and main results of the paper are as follows: in Section 2 we state some standard functional setting and the notation used in this work. In Section 3 we obtain explicit and accurate estimates for the sup-norm of the solutions of the MKSE in one and two spatial dimensions. These estimates are stated after proving the Lemmas 1,2,3 and Theorem 1. In Section 4 we compute the time averaged dissipative length scale also in one and two spatial dimensions. Finally in Section 5 we obtain the "crest factor" of the solutions of the MKSE and we express the conclusion and open problems.

2 Functional Settings and Notation

Let us first give a brief standard preliminary functional setting and notation [1, 31, 24, 36]. Denote by $\Omega = [0, L]^d$ the $d-$dimensional torus; for any scalar function $\phi(x)$ with $x \in \Omega$ let $\|\phi\|_p^p = \int_\Omega |\phi(x)|^p \,\mathrm{d}x$ be the norm associated with the Banach space of $\Omega-$periodic functions; we also define the L^∞ norm as

$$\|\phi\|_\infty = \sup_{x\in\Omega} |\phi(x)| \,.$$

For $p = 2$ we denote by $L^2(\Omega)$ the Hilbert space of $\Omega-$periodic functions ϕ with $\|\phi\|_2 < +\infty$. Given a multi-index $\vec{n} = (n_1, n_2, \ldots, n_d)$, with all the n_i non-negative integers, let $|\vec{n}| = n_1 + \ldots + n_d$ and

$$D^{\vec{n}} := \frac{\partial^{|\vec{n}|}}{\partial x_1^{n_1} \partial x_2^{n_2} \cdots \partial x_d^{n_d}},$$

and let

$$H^n := \Big\{\phi : \int_\Omega (D^{\vec{n}}\phi)^2 \mathrm{d}x < +\infty \text{ for all } \vec{n} \text{ such that } |\vec{n}| = n\Big\},$$

together with

$$\|\phi\|_{H^n}^2 := \sum_{\substack{n_1,\ldots,n_d \geq 0 \\ n_1+\ldots+n_d = n}} \frac{n!}{n_1! \cdot \cdot n_d!} \|D^{\vec{n}}\phi\|_2^2, \tag{3}$$

be the Sobolev space of $\Omega-$periodic functions with up to $n-$derivatives in $L^2(\Omega)$. We also set $Du := (\partial u/\partial x_1, \ldots, \partial u/\partial x_d)$. In (3), we naturally identify the functions having the same "mixed" partial derivatives, because it is well known that the solutions of the MKSE are sufficiently smooth [2, 36, 31]; for example we identify the differential operators

$$\frac{\partial^{n_1+n_2+\cdots+n_d}}{\partial x_1^{n_1} \ldots \partial x_i^{n_i} \ldots \partial x_j^{n_j} \ldots \partial x_d^{n_d}} \equiv \frac{\partial^{n_1+n_2+\cdots+n_d}}{\partial x_1^{n_1} \ldots \partial x_j^{n_j} \ldots \partial x_i^{n_i} \ldots \partial x_d^{n_d}}, \tag{4}$$

and of course any other possible combination of the indices. Also from Parseval's identity we have that

$$\sum_{\substack{n_1,\ldots,n_d\geq 0\\ n_1+\ldots+n_d=n}} \frac{n!}{n_1!\ldots n_d!}\|D^{\vec{n}}\phi\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^{2n} \sum_{\vec{k}\in\mathbb{Z}^d} |\vec{k}|^{2n}|\phi_{\vec{k}}|^2 \,. \tag{5}$$

In (5) the Fourier series expansion has been used,

$$\phi = \sum_{\vec{k}\in\mathbb{Z}^d} \phi_{\vec{k}}\, e^{2\pi i\vec{k}\cdot\vec{x}/L}\,,$$

and

$$|\vec{k}|^2 = \vec{k}\cdot\vec{k} = k_1^2 + k_2^2 + \ldots + k_d^2.$$

By the same token the definition of Sobolev space can be extended to any *real* number s as

$$H^s \;= \Big\{\phi = \sum_{\vec{k}\in\mathbb{Z}^d} \phi_{\vec{k}}\, e^{2\pi i\vec{k}\cdot\vec{x}/L} : \overline{\phi}_{\vec{k}} = \phi_{-\vec{k}} \text{ and } \sum_{\vec{k}\in\mathbb{Z}^d} |\vec{k}|^{2s}|\phi_{\vec{k}}|^2 < +\infty\Big\}, \tag{6}$$

and the corresponding norm is given by

$$\|\phi\|_{H^s}^2 := L^d \left(\frac{2\pi}{L}\right)^{2s} \sum_{\vec{k}\in\mathbb{Z}^d} |\vec{k}|^{2s}|\phi_{\vec{k}}|^2 \,.$$

These Sobolev spaces, defined on the $d-$dimensional torus, are used below as we need to deal with the negative Laplacian $A := -\Delta$ (as a self-adjoint unbounded operator) and its fractional powers. More precisely, the eigenvalues of A are given by the numbers $(2\pi/L)^2|\vec{k}|^2$, so the domain of its powers A^s is the set of functions such that

$$L^d \left(\frac{2\pi}{L}\right)^{4s} \sum_{\vec{k}\in\mathbb{Z}^d} |\vec{k}|^{4s}|\phi_{\vec{k}}|^2 = \|A^s\phi\|_2^2 < +\infty\,. \tag{7}$$

Thus in this paper, for any $s > 0$, we make the *formal* identification

$$\|A^{\frac{s}{2}}\phi\|_2^2 = \|(-\Delta)^{\frac{s}{2}}\phi\|_2^2 = L^d \left(\frac{2\pi}{L}\right)^{2s} \sum_{\vec{k}\in\mathbb{Z}^d} |\vec{k}|^{2s}|\phi_{\vec{k}}|^2,$$

provided it is understood that these operators are being used as differential operators "acting" on functions in H^s, according to (6) and (7).

3 Explicit Estimates of Sobolev norms of the MKSE

In this section we wish to obtain explicit (and as accurately as we possibly can) estimates for various norm of solutions of the MKSE. We then use such estimates to compute the corresponding crest factor associated to these solutions. In the light of this we then define

$$J_n := \|u\|^2_{H^n} = \sum_{\substack{n_1,\dots,n_d \geq 0 \\ n_1+\dots+n_d=n}} \frac{n!}{n_1!\dots n_d!}\|D^{\vec{n}}u\|^2_2. \tag{8}$$

The MKSE has been defined in Section 2 and it is given by (2) in $d=1$ and by (1) in $d=2$, in the domain $\Omega=[0,L]^d$, $d=1,2$, with d being the spatial dimension. The MKSE is known to have a unique solution for every initial datum $u_0 \in L^2(\Omega)$; the solution $u \in C([0,T];H)$, where $H=L^2(\Omega)$, for any $T>0$; in addition the corresponding semigroup $S_t u_0 = u(t)$ has a global attractor $\mathcal{A} \subset\subset H$ (for details see [2, 36, 31]). Therefore all the calculations and estimates obtained below are *not formal,* but they reflect the actual behaviour of the solutions of the MKSE. Hence in the following we wish to find as accurately as possible estimates for the J_n and then use them to obtain the corresponding estimates for the L^∞ norm of the solutions by using the sharp estimate found in [7, 6, 3] (see also [37, 38, 16, 25].

First note that one can show that the time-dependent functionals J_n introduced above satisfy a so-called *ladder differential inequality* [5, 15, 8], namely for any $n > d/2$, where d is the spatial dimension, we have that

$$\frac{1}{2}\dot{J}_n \leq -J_{n+2} + J_{n+1} + \lambda J_n + \left(c_n\|u\|^2_\infty + \tilde{c}_n\|Du\|_\infty\right) J_n,$$

where the constants c_n and $\tilde{c}_n$ do not depend upon the solution $u=u(x,t)$. Because we need to know explicitly all the constants appearing in our analysis, we are somehow forced to restrict ourselves to the lower values of the non-negative integer n. In particular in the one-dimensional case we can restrict ourselves to the analysis of J_0 and J_1, which in $d=1$ are sufficient for having an upper bound on the $\|u\|_\infty$ norm of the solution of any PDE. On the other hand for the $d=2,3$ case we will have to analyze J_2 also.

Before starting our formal analysis let us make clear what we mean by the time-asymptotic behavior of a given function of time $F(t)$. *From now on with an overbar over a given function of time $F(t)$, namely $\overline{F(t)}$, we mean the limit superior, taken over all the initial conditions, as time goes to plus infinity. More formally we mean that we are using the classical Gronwall inequality, hence we take the limit superior as time goes to infinity and thence we consider the supremum over all the initial conditions. Occasionally the set of initial conditions may be restricted to the global attractor of the PDE under investigation, but this will be clear from the context if not explicitly stated [2, 36, 31].*

3.1 Analysis in the one spatial-dimension case

We can now start our analysis of our PDE on the torus in one spatial dimension, namely we study

$$u_t = -u_{xxxx} - u_{xx} + \lambda u - u^3 - uu_x, \tag{9}$$

with periodic boundary conditions on $\Omega = [0, L]$.

In space dimension one it is sufficient to have control on the J_0 and the J_1 in order to have control on the sup norm of any solution of any PDE. Thus we start with the analysis of $J_0(t)$.

Lemma 1. *The time-asymptotic behaviour of $J_0(t)$, namely $\overline{J}_0$, is given by*

$$\overline{J}_0 = \limsup_{t\to\infty} J_0(t) \le L\Big(\lambda + \frac{1}{4}\Big). \tag{10}$$

Proof: By taking the time-dependent quantity $J_0(t) = \int_\Omega u^2(x,t)\,\mathrm{d}x$ and differentiating it with respect to time one finds

$$\frac{1}{2}\dot{J}_0 = -J_2 + J_1 + \lambda J_0 - \int_\Omega (u)^4\,\mathrm{d}x. \tag{11}$$

Note that the contribution from the last term in (9) is zero on periodic boundary conditions. Also note that, for non-trivial behaviour one can see that we must have a restriction on the values of the parameter λ; in fact, after splitting the J_1 term by using first a Cauchy-Schwarz inequality and then a Young inequality, namely

$$J_1 \le (J_2)^{\frac{1}{2}}(J_0)^{\frac{1}{2}} = (2J_2)^{\frac{1}{2}}\Big(\frac{J_0}{2}\Big)^{\frac{1}{2}} \le J_2 + \frac{1}{4}J_0,$$

and also noting that $-\displaystyle\int_\Omega (u)^4\,\mathrm{d}x \le -\frac{J_0^2}{L}$, it follows that (11) becomes

$$\frac{1}{2}\dot{J}_0 \le \Big(\lambda + \frac{1}{4}\Big)J_0 - \frac{J_0^2}{L}. \tag{12}$$

Hence one can see that if $\lambda \le -1/4$ the zero solution becomes a global attractor. Since we have taken $\lambda > 0$ we are excluding such a situation. Thus going back to our analysis of J_0 we have to study (12). By standard analysis one can see that the fixed points of the corresponding nonlinear ordinary differential equation are given by $J_0 = 0, L(\lambda + \frac{1}{4})$ with 0 being unstable and $L(\lambda + \frac{1}{4})$ being stable. Thus the long-time asymptotic behaviour of J_0 (denoted with $\overline{J}_0$) satisfies (10). In particular it is independent of the initial condition $u(x, t = 0) = u_0(x)$. ■

We now turn our analysis to the estimate of J_1.

Lemma 2. *The time-asymptotic behaviour of* $J_1(t)$*, namely* $\overline{J}_1$*, is given by*

$$\overline{J}_1 := \limsup_{t\to\infty} J_1(t) \le \sqrt{\frac{24\lambda+13}{11}}\, L\Big(\lambda+\frac{1}{4}\Big). \tag{13}$$

Proof: Here we take the time-dependent quantity $J_1(t) = \int_\Omega (u_x(x,t))^2 \mathrm{d}x$ and differentiating it with respect to time we find

$$\frac{1}{2}\dot{J}_1 = -J_3 + J_2 + \lambda J_1 - 3\int_\Omega u^2(u_x)^2\,\mathrm{d}x - \int_\Omega (u_x)^3\,\mathrm{d}x - \int_\Omega (u)(u_x)(u_{xx})\,\mathrm{d}x.$$

An integration by parts on the last term gives

$$\frac{1}{2}\dot{J}_1 = -J_3 + J_2 + \lambda J_1 - 3\int_\Omega u^2(u_x)^2\,\mathrm{d}x - \int_\Omega (u_x)^3\,\mathrm{d}x + \int_\Omega (u_x)^3\,\mathrm{d}x + \int_\Omega (u)(u_x)(u_{xx})\,\mathrm{d}x;$$

hence two terms cancel out and then by performing first a Cauchy-Schwarz inequality and then a judicious Young inequality so as to generate the terms $3\int u^2(u_x)^2 + \frac{J_2}{12}$ one obtains

$$\frac{1}{2}\dot{J}_1 \le -J_3 + J_2 + \lambda J_1 + \frac{1}{12}J_2.$$

By using a Young inequality on the term J_2 and simplifying we arrive at

$$\dot{J}_1 \;\le\; -\frac{11}{12}J_3 + \left(2\lambda + \frac{13}{12}\right) J_1. \tag{14}$$

We now use the inequality [5, 15, 8]

$$J_p \le J_{p+r}^{\frac{q}{r+q}} J_{p-q}^{\frac{r}{r+q}}, \qquad p \ge q, \quad r \ge 0, \tag{15}$$

with $p = 1$, $r = 2$ and $q = 1$ to obtains $-J_3 \le -J_1^3/J_0^2$. Hence inserting this into (14), so as to obtain

$$\dot{J}_1 \le -\frac{J_1^3}{J_0^2} + \left(\frac{24\lambda+13}{11}\right) J_1. \tag{16}$$

and performing a similar analysis to that used in obtaining the estimate (10), one finds

$$\overline{J}_1 := \limsup_{t\to\infty} J_1(t) \le \sqrt{\frac{24\lambda+13}{11}}\,\overline{J}_0,$$

which, together with (10), yields the result. ■

By using the estimates above it is interesting to obtain the corresponding estimate for the $\|u\|_\infty$ of the solution in the $d = 1$ case. Here we can apply the sharp results found in [7, 6, 3]: for any function $u \in H^{1+\epsilon}$ one has

$$\|u\|_\infty \;\le\; \left(\frac{\zeta(1+\epsilon)}{\pi}\right)^{\frac{1}{2}} \|(-\Delta)^{\frac{1+\epsilon}{4}} u\|_2 \,+ L^{-\frac{1}{2}} J_0^{\frac{1}{2}}, \tag{17}$$

where $\epsilon > 0$ and

$$\zeta(1+\epsilon) = \sum_{n\geq 1} \frac{1}{n^{1+\epsilon}} \tag{18}$$

is the Riemann zeta function. The last term in (17) takes into account the mean of u. By taking the value $\epsilon = 1$ we therefore obtain

$$\|u\|_\infty \leq \sqrt{\frac{\pi}{6}}\, \|Du\|_2 + L^{-\frac{1}{2}} J_0^{\frac{1}{2}} = \sqrt{\frac{\pi}{6}}\, J_1^{\frac{1}{2}} + L^{-\frac{1}{2}} J_0^{\frac{1}{2}}; \tag{19}$$

thus by using (10) and (13) we obtain

$$\overline{\|u\|}_\infty \leq \left(\frac{L\pi}{24}(4\lambda+1)\sqrt{\frac{24\lambda+13}{11}}\right)^{\frac{1}{2}} + \frac{\sqrt{4\lambda+1}}{2}. \tag{20}$$

3.2 Analysis in the two spatial-dimensions case

We can now turn our attention to the two-dimensional case having domain $[0, L]^2$; as it is well known in this case having control on the J_1 norm alone is not sufficient, but it is necessary to have control on the J_2 norm as well. Before actually computing the time-asymptotic behaviour of J_2 we note that the estimates for $\overline{J}_0$ and $\overline{J}_1$ in two spatial dimension are different because of the nonlinear terms; indeed all we have to do is estimating the nonlinear part as best as we can. We start with the estimate of J_0. Here the only difference with respect to the $d = 1$ case comes from the term $-\int (u)^4 \,\mathrm{d}x\,\mathrm{d}y \leq -J_0^2/L^2$; it follows that the differential inequality for $J_0(t)$ becomes

$$\frac{1}{2}\dot{J}_0 \leq \left(\lambda + \frac{1}{4}\right) J_0 - \frac{J_0^2}{L^2}.$$

Therefore one obtains for the time-asymptotic behaviour of $J_0(t)$ the estimate

$$\overline{J}_0 := \limsup_{t\to\infty} J_0(t) \leq L^2\left(\lambda + \frac{1}{4}\right). \tag{21}$$

Similarly for the time-asymptotic behaviour of J_1 one finds that

$$\frac{1}{2}\dot{J}_1 = -J_3 + J_2 + \lambda J_1 - \sum_{|\vec{n}|=1} \int_\Omega \left[(D^{\vec{n}}u)D^{\vec{n}}(u^3)\,\mathrm{d}x\,\mathrm{d}y + (D^{\vec{n}}u)[D^{\vec{n}}(uu_x + uu_y)]\,\mathrm{d}x\,\mathrm{d}y\right].$$

Hence by neglecting the negative definite term given by the first summation and by expanding all the derivatives present in the second summation one arrives at

$$\frac{1}{2}\dot{J}_1 \leq -J_3 + J_2 + \lambda J_1 + \sqrt{\frac{24}{\pi}} J_1 J_2^{\frac{1}{2}}. \tag{22}$$

Hence a similar analysis to the one done for obtaining the time-asymptotic behaviour of J_0 gives the estimate

$$\overline{J}_1 := \limsup_{t\to\infty} J_1(t) \leq \left[\frac{5}{3}+\frac{5}{6}(4)^{\frac{14}{5}}\left(\frac{6}{\pi}\right)^{\frac{3}{5}}\right]^{\frac{1}{3}} \overline{J}_0 \ \leq \ \left[\frac{5}{3}+\frac{5}{6}(4)^{\frac{14}{5}}\left(\frac{6}{\pi}\right)^{\frac{3}{5}}\right]^{\frac{1}{3}} L^2\Big(\lambda+\frac{1}{4}\Big). \quad (23)$$

We now turn our attention to the analysis of $J_2(t)$; the corresponding first order non-linear differential equation is given by

$$\frac{1}{2}\dot{J}_2 = -J_4 + J_3 + \lambda J_2 - \sum_{|\vec{n}|=2} \int_\Omega \Big((D^{\vec{n}}u)D^{\vec{n}}(u^3) + \ (D^{\vec{n}}u)[D^{\vec{n}}(uu_x+uu_y)]\Big)\mathrm{d}x\,\mathrm{d}y. \quad (24)$$

where the terms in the summations represent the non-linear terms. Their accurate estimates is given by the following result.

Lemma 3. *The nonlinear terms above obeys the estimate*

$$-\sum_{|\vec{n}|=2} \int_\Omega \Big((D^{\vec{n}}u)D^{\vec{n}}(u^3) + (D^{\vec{n}}u)[D^{\vec{n}}(uu_x+uu_y)]\Big)\mathrm{d}x\,\mathrm{d}y \leq \frac{78}{\pi}J_1J_2 + 5\|Du\|_\infty J_2.$$

Proof: We first analyse the terms

$$-\sum_{|\vec{n}|=2} \int_\Omega (D^{\vec{n}}u)D^{\vec{n}}(u^3)\,\mathrm{d}x\,\mathrm{d}y.$$

One starts by making the explicit differentiation, thereby obtaining

$$\begin{aligned}&-\sum_{|\vec{n}|=2} \int_\Omega (D^{\vec{n}}u)D^{\vec{n}}(u^3)\,\mathrm{d}x\,\mathrm{d}y = -6\int_\Omega u(u_x)^2u_{xx}\\ &-3\int_\Omega u^2(u_{xx})^2\,\mathrm{d}x\,\mathrm{d}y - 6\int_\Omega u(u_y)^2u_{yy}\,\mathrm{d}x\,\mathrm{d}y - 3\int_\Omega u^2(u_{yy})^2\,\mathrm{d}x\,\mathrm{d}y\\ &-6\int_\Omega u^2(u_{xy})^2\,\mathrm{d}x\,\mathrm{d}y - 12\int_\Omega uu_xu_yu_{xy}\,\mathrm{d}x\,\mathrm{d}y;\end{aligned}$$

integrating by parts the first, the third and the last terms and then rearranging we obtain

$$\begin{aligned}&-\sum_{|\vec{n}|=2} \int_\Omega (D^{\vec{n}}u)D^{\vec{n}}(u^3)\,\mathrm{d}x\,\mathrm{d}y = 2\int_\Omega (u_x)^4\,\mathrm{d}x\,\mathrm{d}y\\ &-3\int_\Omega u^2(u_{xx})^2\,\mathrm{d}x\,\mathrm{d}y + 2\int_\Omega (u_y)^4\,\mathrm{d}x\,\mathrm{d}y - 3\int_\Omega u^2(u_{yy})^2\,\mathrm{d}x\,\mathrm{d}y\\ &-6\int_\Omega u^2(u_{xy})^2\,\mathrm{d}x\,\mathrm{d}y + 6\int_\Omega (u_x)^2(u_y)^2\,\mathrm{d}x\,\mathrm{d}y + 6\int_\Omega uu_{xx}(u_y)^2\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

By splitting the last two terms by applying first a Cauchy-Schwarz inequality and then a Young inequality we get

$$
\begin{aligned}
&-\sum_{|\vec{n}|=2}\int_\Omega (D^{\vec{n}}u)D^{\vec{n}}(u^3)\,\mathrm{d}x\,\mathrm{d}y = 2\int_\Omega (u_x)^4\,\mathrm{d}x\,\mathrm{d}y\\
&-3\int_\Omega u^2(u_{xx})^2\,\mathrm{d}x\,\mathrm{d}y + 2\int_\Omega (u_y)^4\,\mathrm{d}x\,\mathrm{d}y - 3\int_\Omega u^2(u_{yy})^2\,\mathrm{d}x\,\mathrm{d}y\\
&-6\int_\Omega u^2(u_{xy})^2\,\mathrm{d}x\,\mathrm{d}y + 3\int_\Omega (u_x)^4\,\mathrm{d}x\,\mathrm{d}y + 3\int_\Omega (u_y)^4\,\mathrm{d}x\,\mathrm{d}y\\
&+3\int_\Omega u^2(u_{xx})^2\,\mathrm{d}x\,\mathrm{d}y + 3\int_\Omega (u_y)^4\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Simplifying we finally obtain that the nonlinear term can be estimated as

$$
-\sum_{|\vec{n}|=2}\int_\Omega (D^{\vec{n}}u)D^{\vec{n}}(u^3)\,\mathrm{d}x\,\mathrm{d}y \le\ 5\int_\Omega (u_x)^4\,\mathrm{d}x\,\mathrm{d}y + 8\int_\Omega (u_y)^4\,\mathrm{d}x\,\mathrm{d}y. \tag{25}
$$

In the two-dimensional case we can use an improved version of the Ladyzhenskaya inequality [19], namely for any mean zero function $\phi(x,y)$ on the $2d$ torus we have the inequality

$$
\int_\Omega (\phi(x,y))^4\,\mathrm{d}x\,\mathrm{d}y \le \frac{6}{\pi}\int_\Omega (\phi(x,y))^2\,\mathrm{d}x\,\mathrm{d}y\int_\Omega |\nabla\phi|^2\,\mathrm{d}x\,\mathrm{d}y.
$$

Hence we can estimate the term $5\int_\Omega (u_x)^4\,\mathrm{d}x\,\mathrm{d}y$ in (25) as

$$
5\int_\Omega (u_x)^4\,\mathrm{d}x\,\mathrm{d}y\ \le\ \frac{30}{\pi}\left(\int_\Omega (u_x)^2\,\mathrm{d}x\,\mathrm{d}y\right)\left(\int_\Omega (u_{xx}^2+u_{xy}^2)\,\mathrm{d}x\,\mathrm{d}y\right)
$$

and similarly

$$
8\int_\Omega (u_y)^4\,\mathrm{d}x\,\mathrm{d}y\ \le\ \frac{48}{\pi}\left(\int_\Omega (u_y)^2\,\mathrm{d}x\,\mathrm{d}y\right)\left(\int_\Omega (u_{yy}^2+u_{xy}^2)\,\mathrm{d}x\,\mathrm{d}y\right).
$$

By noting that $\int_\Omega (u_x)^2\,\mathrm{d}x\,\mathrm{d}y\ \le\ J_1$, $\int_\Omega (u_y)^2\,\mathrm{d}x\,\mathrm{d}y\ \le\ J_1$, $\int_\Omega (u_{xx}^2+u_{xy}^2)\,\mathrm{d}x\,\mathrm{d}y\ \le J_2$ and $\int_\Omega (u_{yy}^2+u_{xy}^2)\,\mathrm{d}x\,\mathrm{d}y\ \le J_2$, we therefore obtain

$$
-\sum_{|\vec{n}|=2}\int_\Omega (D^{\vec{n}}u)D^{\vec{n}}(u^3)\,\mathrm{d}x\,\mathrm{d}y \le \frac{78}{\pi}J_1J_2. \tag{26}
$$

We now turn to the other remaining nonlinear terms; again we start be expressing them explicitly, namely

$$
\begin{aligned}
&-\sum_{|\vec{n}|=2}\int_\Omega (D^{\vec{n}}u)D^{\vec{n}}(uu_x+uu_y)\,\mathrm{d}x\,\mathrm{d}y = \\
&-\int_\Omega u_{xx}[uu_x+uu_y]_{xx} - \int_\Omega 2u_{xy}[uu_x+uu_y]_{xy} - \int_\Omega u_{yy}[uu_x+uu_y]_{yy} = \\
&-\frac{5}{2}\int_\Omega u_x(u_{xx})^2\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_\Omega u_y(u_{xx})^2\,\mathrm{d}x\,\mathrm{d}y - 2\int_\Omega u_xu_{xx}u_{xy}\,\mathrm{d}x\,\mathrm{d}y \\
&-3\int_\Omega u_x(u_{xy})^2\,\mathrm{d}x\,\mathrm{d}y - 2\int_\Omega u_yu_{xx}u_{xy}\,\mathrm{d}x\,\mathrm{d}y - 3\int_\Omega u_y(u_{xy})^2\,\mathrm{d}x\,\mathrm{d}y \\
&-2\int_\Omega u_xu_{xy}u_{yy}\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2}\int_\Omega u_x(u_{yy})^2\,\mathrm{d}x\,\mathrm{d}y - 2\int_\Omega u_yu_{xy}u_{yy}\,\mathrm{d}x\,\mathrm{d}y \\
&-\frac{5}{2}\int_\Omega u_y(u_{yy})^2\,\mathrm{d}x\,\mathrm{d}y,
\end{aligned}
$$

where any term with three derivatives has first been integrated by parts to move one derivative away to the remaining terms in the integral. All integrals are of the form

$$
\int_\Omega u_x(u_{xx})^2\,\mathrm{d}x\,\mathrm{d}y, \quad \int_\Omega u_xu_{xx}u_{xy}\,\mathrm{d}x\,\mathrm{d}y, \quad \int_\Omega u_x(u_{xy})^2\,\mathrm{d}x\,\mathrm{d}y,
$$

$$
\int_\Omega u_xu_{xy}u_{yy}\,\mathrm{d}x\,\mathrm{d}y, \quad \int_\Omega u_x(u_{yy})^2\,\mathrm{d}x\,\mathrm{d}y,
$$

or with the variables x and y exchanged. We pull the terms u_x or u_y in the L^∞ norm thereby obtaining, for instance, $\int u_x(u_{xx}^2) \le \|u_x\|_\infty J_{2,x}$, where with $J_{2,x}$ we mean the "component of J_2 along the x coordinate"; the other similar terms such as $\int u_y(u_{xx}^2)$, $\int u_x(u_{yy}^2)$, etc. are handled in the same way. Other terms of the form, say, $\int u_xu_{xy}u_{yy}$ are dealt with by first pulling out the u_x term in L^∞, then applying a Cauchy-Scharwz to the two remaining terms and then splitting the two terms with a Young inequality. We collect all the terms together thereby finally obtaining

$$
-\sum_{|\vec{n}|=2}\int_\Omega (D^{\vec{n}}u)D^{\vec{n}}(uu_x+uu_y)\,\mathrm{d}x\,\mathrm{d}y \le \|Du\|_\infty(5J_{2,xx}+2\cdot 5J_{2,xy}+5J_{2,yy}) \le 5\|Du\|_\infty J_2,
$$

where we have used that $\|u_x\|_\infty, \|u_y\|_\infty \le \|Du\|_\infty$. The last estimate, together with (26) implies the result. ■

By using the results obtained above we can now prove the following result.

Theorem 1. *The time-asymptotic behaviour of $J_2(t)$, namely $\overline{J}_2$, satisfies*

$$\overline{J}_2 \leq \overline{J}_0^{\frac{3}{2}} \left[108 + 4\lambda^2 + 108\left(\frac{5}{\sqrt{\pi}}\right)^4 \overline{J}_0^2 + 108\left(\frac{78}{\pi}\right)^4 \overline{J}_0^4\right]^{\frac{1}{2}}$$
$$\leq \left[L^3\Big(\frac{4\lambda+1}{4}\Big)^3\Big(108 + 4\lambda^2 + 108L^2\left(\frac{5}{\sqrt{\pi}}\right)^4\Big(\frac{4\lambda+1}{4}\Big)^2 + 108L^4\left(\frac{78}{\pi}\right)^4\Big(\frac{4\lambda+1}{4}\Big)^4\right]^{\frac{1}{2}}.$$

Proof: First we write the estimate for the time derivative of J_2, as obtained from (24) and Lemma 3, namely

$$\frac{1}{2}\dot{J}_2 \;\leq\; -J_4 + J_3 + \lambda J_2 + \frac{78}{\pi}J_1 J_2 + 5\|Du\|_\infty J_2. \tag{27}$$

To handle the last term we use the (almost sharp) estimate $\|Du\|_\infty \leq \frac{1}{\sqrt{\pi}}J_3^{\frac{1}{4}}J_1^{\frac{1}{4}}$ [20]. To absorb the (27) term we split it as follows:

$$\begin{aligned}
&J_3 \leq J_4^{\frac{3}{4}}J_0^{\frac{1}{4}} \leq \frac{1}{8}J_4 + 54J_0,\\
&\lambda J_2 \leq \lambda J_4^{\frac{1}{4}}J_0^{\frac{1}{2}} \leq \frac{J_4}{8} + 2\lambda^2 J_0,\\
&\frac{78}{\pi}J_1J_2 \leq \frac{78}{\pi}\left(J_4^{\frac{1}{4}}J_0^{\frac{3}{4}}\right)\left(J_4^{\frac{1}{2}}J_0^{\frac{1}{2}}\right) \leq \frac{78}{\pi}J_4^{\frac{3}{4}}J_0^{\frac{5}{4}} \leq \frac{J_4}{8} + \frac{1}{4}\left(216J_0^5\Big(\frac{78}{\pi}\Big)^4\right),\\
&5\|Du\|_\infty J_2 \leq \frac{5}{\sqrt{\pi}}J_3^{\frac{1}{4}}J_1^{\frac{1}{4}}J_2 \leq \frac{5}{\sqrt{\pi}}J_4^{\frac{3}{4}}J_0^{\frac{3}{4}} \leq \frac{J_4}{8} + \frac{1}{4}\left(216\Big(\frac{5}{\sqrt{\pi}}\Big)^4 J_0^3\right),
\end{aligned}$$

where (15) has bee used repeatedly. Using all of this one arrives at

$$\frac{\dot{J}_2}{2} \leq -\frac{J_4}{2} + \frac{1}{2}J_0\left[108 + 4\lambda^2 + 108\left(\frac{5}{\sqrt{\pi}}\right)^4 J_0^2 + 108\left(\frac{78}{\pi}\right)^4 J_0^4\right].$$

Therefore the time-asymptotic behaviour of J_2 is given by

$$\overline{J}_2 \;\leq\; \overline{J}_0^{\frac{3}{2}}\left[108 + 4\lambda^2 + 108\left(\frac{5}{\sqrt{\pi}}\right)^4 \overline{J}_0^2 + 108\left(\frac{78}{\pi}\right)^4 \overline{J}_0^4\right]^{\frac{1}{2}}. \tag{28}$$

By substituting the estimate for $\overline{J}_0$ we finally obtain the result. ∎

Thus for the estimate of $\|u\|_\infty$ we use the result proved in [7], where it is shown that on the two-dimensional torus $\Omega = [0,L]^2$, for every $\epsilon > 0$, the L^∞ norm of a mean zero scalar function $u \in H^{1+\epsilon}$ satisfies the estimate

$$\|\phi\|_\infty \;\leq\; [4\zeta(1+\epsilon)\beta(1+\epsilon)]^{\frac{1}{2}}\, L^{-1}\left(\frac{L}{2\pi}\right)^{(1+\epsilon)} \|(-\Delta)^{\frac{1+\epsilon}{2}}\phi\|_2\,, \tag{29}$$

where the coefficient $4\zeta(1+\epsilon)\beta(1+\epsilon)$ is sharp, and where

$$\zeta(1+\epsilon)=\sum_{n\geq 1}\frac{1}{n^{1+\epsilon}}\,,\qquad \beta(1+\epsilon)=\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)^{1+\epsilon}}\,,$$

are the Riemann zeta-function and Dirichlet series respectively. Thus for the estimate of $\overline{\|u\|_\infty}$ we use (29) with $\epsilon=1$, namely

$$\overline{\|u\|_\infty}\leq\frac{L}{2\pi^2}(\zeta(2)\beta(2))^{\frac{1}{2}}\,\overline{\|\Delta u\|_2}+L^{-1}\overline{J}_0^{\frac{1}{2}}\leq\frac{L}{2\pi^2}(\zeta(2)\beta(2))^{\frac{1}{2}}\,\overline{J}_2^{\frac{1}{2}}+L^{-1}\overline{J}_0^{\frac{1}{2}}.$$

By using the values for $\zeta(2)\beta(2)=6\pi^{-2}K$ with $K=0.915965594...$ we obtain

$$\overline{\|u\|_\infty}\leq\frac{L}{2\pi^3}\sqrt{6K}\overline{J}_2^{\frac{1}{2}}+L^{-1}\overline{J}_0^{\frac{1}{2}},$$

where the estimate for $\overline{J}_2^{\frac{1}{2}}$ is provided by (28) and that for $\overline{J}_0^{\frac{1}{2}}$ is provided by (21).

4 The Crest Factor of Solutions of Dissipative PDEs

So far we have obtained various Sobolev norms estimates of solutions of our equation, such us the estimates for $J_0,\ J_1,\ J_2$ and the corresponding estimate for the sup-norm. An important question which naturally arise from our analysis is to investigate the so-called *crest factor* (also known as the *peak to average ratio*), namely the ratio between the L^∞ norm and the L^2 norm of the solution:

$$C_f\;:=L^{\frac{d}{2}}\frac{\|u\|_\infty}{J_0^{\frac{1}{2}}}. \tag{30}$$

It is therefore by definition dimensionless and it contains important information on the "distortions" between the sup-norm (the amplitude) and the L^2 norm of the solution. It is in fact a standard measurement used in turbulence experiments in fluid dynamics. The ideal result would be to have a time-pointwise estimate of C_f. However this is very difficult due essentially to the non-linearity of the equation. Alternatively one could try to estimate the time-asymptotic behaviour of C_f, but this also proves to be very hard to handle and it is essentially due to the lack of knowledge of a "decent" lower bound on the quantity J_0, namely an estimate of the form $J_0(t)\geq\alpha>0$. The problem of estimating the lower bound appears in many contexts in the theory of nonlinear dissipative PDEs, such as for example in the theory of the Navier-Stokes equations where it is notoriously very hard to find a "proper" lower bound for the energy even on the torus [14]. So in this work we will compute the time-average of the quotient between the L^∞ norm and the L^2 norm of the solution, namely $\left\langle\|u\|_\infty/J_0^{\frac{1}{2}}\right\rangle$. First of all let us derive sharp estimates for the $\|u\|_\infty$

of typical solutions $u(x,t)$. Note that in general we cannot assume that the solutions of our equation have zero-mean. Hence we have to "carry along" the mean value of our solutions. Thus define $u^*(t) := \fint_\Omega u(x,t)\,\mathrm{d}x$ and write $u(x,t) = u^*(t) + u'(x,t)$, where $\int_\Omega u'(x,t)\,\mathrm{d}x = 0$. Then using the inequality

$$|u^*| = L^{-d} \left| \int_\Omega u(x)\,\mathrm{d}x \right| \leq L^{-\frac{d}{2}} J_0^{\frac{1}{2}} \tag{31}$$

and defining $J_0' := \|u'\|_2^2$, we obtain [9]

$$\|u\|_\infty \leq |u^*| + \|u'\|_\infty \leq L^{-\frac{d}{2}} J_0^{\frac{1}{2}} + c(n)(J_0')^{\frac{2n-d}{4n}} J_n^{\frac{d}{4n}} \,. \tag{32}$$

with $n > 1/2$ and $c(n)$ a suitable constant, where we have used a Gagliardo-Nirenberg inequality to obtain the estimate on $\|u'\|_\infty$. By substituting $u = 1$ in (31) we see that the constant $L^{-\frac{d}{2}}$ is sharp. Therefore we obtain the following estimate

$$\frac{\|u\|_\infty}{J_0^{\frac{1}{2}}} \leq \frac{|u^*| + \|u'\|_\infty}{J_0^{\frac{1}{2}}} \leq L^{-\frac{d}{2}} + \frac{\|u'\|_\infty}{J_0^{\frac{1}{2}}}.$$

Hence by using (32) we obtain

$$\frac{\|u\|_\infty}{J_0^{\frac{1}{2}}} \leq L^{-\frac{d}{2}} + c(n) \left(\frac{J_n}{J_0}\right)^{\frac{d}{4n}} \left(\frac{J_0'}{J_0}\right)^{\frac{2n-d}{4n}} .$$

Thus our estimate for the crest factor is obtained by taking the time-average

$$\widetilde{C}_f := \left\langle L^{\frac{d}{2}} \frac{\|u\|_\infty}{J_0^{\frac{1}{2}}} \right\rangle .$$

It is useful to concentrate on the "pure" distortion between the sup-norm and the L^2 norm for non-constant solutions (note that of course constant functions have crest factor equal to 1). Bearing this in mind one obtains

$$\widetilde{C}_f = 1 + \overline{C}_f, \qquad \overline{C}_f := \left\langle L^{\frac{d}{2}} \frac{\|u'\|_\infty}{J_0^{\frac{1}{2}}} \right\rangle \leq c(n) L^{\frac{d}{2}} \left\langle \left(\frac{J_n}{J_0}\right)^{\frac{d}{4n}} \right\rangle , \tag{33}$$

where the last bound follows noting that $J_0' \leq J_0$. Note that, since one has trivially $\overline{C}_f = 0$ if $u(x,t)$ does not depend on x, in order to estimate the crest factor we may assume in the following that $u' \neq 0$. Hence $J_n > 0$ for all $n \geq 0$.

4.1 Time-averaged crest factor in one spatial dimension

In one spatial dimension it is sufficient to take $n = 1$ in (33) and so one has

$$\overline{C}_f \leq c(1) \left\langle L^{\frac{1}{2}} \left(\frac{J_1}{J_0}\right)^{\frac{1}{4}} \right\rangle. \tag{34}$$

From [18] or Appendix A in [9] we have that $c(1) = 1$. Thus one needs to derive as best as possible the time average of the quantity $(J_1/J_0)^{\frac{1}{4}}$. This is achieved as follows. First take the differential inequality (16) and divide throughout by J_1. This leads to

$$\frac{\dot{J}_1}{J_1} \leq -\left(\frac{J_1}{J_0}\right)^2 + \left(\frac{24\lambda + 13}{11}\right).$$

Then we take the time average of both sides of the inequality thereby getting

$$\left\langle \left(\frac{J_1}{J_0}\right)^2 \right\rangle \leq \left(\frac{24\lambda + 13}{11}\right),$$

where we have used that J_1 is bounded both from below and from above by two positive constants. Going back to (34) one obtains (with $c^2(1) = 1$),

$$\overline{C}_f \leq L^{\frac{1}{2}} \left\langle \left(\frac{J_1}{J_0}\right)^{\frac{1}{4}} \right\rangle \leq L^{\frac{1}{2}} \left\langle \left(\frac{J_1}{J_0}\right)^2 \right\rangle^{\frac{1}{8}} \leq L^{\frac{1}{2}} \left(\frac{24\lambda + 13}{11}\right)^{\frac{1}{8}}, \tag{35}$$

which shows that $C_f = O(\lambda^{\frac{1}{8}})$ for large λ.

4.2 Time-averaged crest factor in two spatial dimensions

The strategy for obtaining the time-averaged crest factor in two spatial dimensions is similar to the one-dimensional case with the corresponding changes, namely here $d = 2$ and also one has to insert the explicit values of the constants $c(2)$ in (33). Also it is well known that in two spatial dimensions it is sufficient to take $n = 2$, and so we need to estimate the quantity

$$\overline{C}_f \leq c(2) L \left\langle \left(\frac{J_2}{J_0}\right)^{\frac{1}{4}} \right\rangle \leq c(2) L \left\langle \frac{J_2}{J_0} \right\rangle^{\frac{1}{4}}. \tag{36}$$

So we start from the differential inequality (see (27))

$$\frac{1}{2}\dot{J}_2 \leq -\frac{1}{2}J_4 + \frac{1}{2}\left(2\lambda + 1 + \frac{156}{\pi}J_1 + 10\|Du\|_\infty\right) J_2.$$

We now use again the inequality $\|Du\|_\infty \leq \frac{1}{\sqrt{\pi}}(J_3 J_1)^{\frac{1}{4}}$ and also (15), with $p = q = r = 2$, and so we obtain

$$\dot{J}_2 \leq -\frac{J_2^2}{J_0} + \left(2\lambda + 1 + \frac{156}{\pi} J_1 + \frac{10}{\sqrt{\pi}}(J_3 J_1)^{\frac{1}{4}}\right) J_2.$$

Similarly to the one-dimensional case we divide throughout by J_2 and then we take the time average of both sides of the inequality obtaining

$$\left\langle \frac{J_2}{J_0} \right\rangle \leq (2\lambda + 1) + \frac{156}{\pi}\langle J_1 \rangle + \frac{10}{\sqrt{\pi}} \langle J_3 \rangle^{\frac{1}{4}} \langle J_1 \rangle^{\frac{1}{4}},$$

where we have used the properties of the time average in order to obtain the last term. In order to estimate $\langle J_1 \rangle$ we use (11), that we re-write here:

$$\frac{1}{2}\dot{J}_0 = -J_2 + J_1 + \lambda J_0 - \int_\Omega (u)^4 \,\mathrm{d}x.$$

By using $J_1 \leq J_2^{\frac{1}{2}} J_0^{\frac{1}{2}}$ and then splitting the right hand side with the Young inequality we obtain

$$\frac{1}{2}\dot{J}_0 = -\frac{J_2}{2} + \left(\lambda + \frac{1}{2}\right) J_0,$$

where here we have neglected the last term. Time averaging both sides we finally get

$$\langle J_2 \rangle \leq (2\lambda + 1)\langle J_0 \rangle \leq (2\lambda + 1)\overline{J}_0 \leq (2\lambda + 1) L^2 \left(\lambda + \frac{1}{4}\right). \tag{37}$$

Thus by first time averaging the inequality $J_1 \leq J_2^{\frac{1}{2}} J_0^{\frac{1}{2}}$ and then splitting the time average of the product on the right hand side one obtains

$$\langle J_1 \rangle \leq \langle J_2 \rangle^{\frac{1}{2}} \langle J_0 \rangle^{\frac{1}{2}} \leq (2\lambda + 1)^{\frac{1}{2}} \langle J_0 \rangle^{\frac{1}{2}} \langle J_0 \rangle^{\frac{1}{2}} \leq (2\lambda + 1)^{\frac{1}{2}} \langle J_0 \rangle \leq (2\lambda + 1)^{\frac{1}{2}} \overline{J}_0. \tag{38}$$

We now estimate the other term, namely $\langle J_3 \rangle$. Here we use the time-average of the formula (22), obtaining

$$\langle J_3 \rangle \leq \langle J_2 \rangle + \lambda \langle J_1 \rangle + \frac{24}{\pi} \langle J_1 J_2^{\frac{1}{2}} \rangle \leq \langle J_2 \rangle + \lambda \langle J_1 \rangle + \frac{24}{\pi} \langle J_0^{\frac{1}{2}} J_2 \rangle \leq \langle J_2 \rangle + \lambda \langle J_1 \rangle + \frac{24}{\pi} \langle J_2 \rangle \overline{J}_0^{\frac{1}{2}}.$$

Therefore by inserting the estimates for $\overline{J}_0$, $\langle J_1 \rangle$ and $\langle J_2 \rangle$ given by (21), (38) and (37), respectively, one finds

$$\langle J_3 \rangle \leq (2\lambda + 1)^{\frac{1}{2}} L^2 \left(\lambda + \frac{1}{4}\right) \left[\lambda + (2\lambda + 1)^{\frac{1}{2}} \left(1 + \sqrt{\frac{24}{\pi}} L \left(\lambda + \frac{1}{4}\right)^{\frac{1}{2}}\right)\right]. \tag{39}$$

So going back to the computation of the crest factor in the space two dimensional case, in (36) we have to insert the value of the constant $c(2)$, which is $c(2) = \sqrt{1/\pi}$ [20], and bound $\langle (J_2/J_0)^{\frac{1}{4}} \rangle$ by using the estimates (38) and (39) found above for $\langle J_1 \rangle$ and $\langle J_3 \rangle$. In particular, one finds $C_f = O(\lambda^{\frac{3}{8}})$ for λ large.

5 Conclusions and Open Problems

In this work we have analysed various Sobolev norms of solutions of a modified version of KSE, with the aim to estimate as accurately as possible both the sup-norm of solutions and then the corresponding *crest factor*. More specifically, by using the *best* available explicit estimates for the coefficients which appear in the Sobolev norms used, we have first derived explicit estimates for the $\overline{J}_0, \overline{J}_1, \overline{J}_2$, namely their time-asymptotic behaviour, and then we have used these estimates to compute the time-asymptotic behaviour of the L^∞ norm of the solution, namely the $\overline{\|u\|_\infty}$ in one and two space dimension. We then addressed another very important indicator of the dynamics of solutions of dissipative PDEs, namely the accurate estimate of the so-called *crest factor*. This is defined as the ratio between the L^∞ norm of the solution and the L^2 norm of the solution:

$$C_f \;:= L^{\frac{d}{2}} \frac{\|u\|_\infty}{J_0^{\frac{1}{2}}}, \tag{40}$$

where d is the spatial dimension. It is therefore a dimensionless pure number and it contains important information on the “distortion” between the “amplitude“ and the L^2 norm of the solution. It is in fact a standard measurement used in turbulence experiments in fluid dynamics.

Let us now discuss the implications of the estimates we have found in both one and two space dimensions. In space dimension one we found that th time-average of C_f is

$$\widetilde{C}_f = 1 + O(\lambda^{\frac{1}{8}}), \tag{41}$$

while in space dimension two we found

$$\widetilde{C}_f = 1 + O(\lambda^{\frac{3}{8}}). \tag{42}$$

The two formulas above reveal some of the features related to the dynamics of the solutions of our PDE. In fact in one space dimension the time average of the ratio between the “peak to the root mean square” (the crest factor) scales like (41) as a function of the positive parameter λ. So for small λ the distortion $\overline{C}_f = \widetilde{C}_f - 1$ is small (as it should), but what it really says is that our PDE cannot have *major* excursions in space-time as λ increases because the crest factor goes like $\lambda^{\frac{1}{8}}$, and $\frac{1}{8}$ is “pretty small“. On the other hand as a function of the parameter L (the length of the torus) it scales like $\sqrt{L}$; this shows that the crest factor are more sensitive to the length of the torus than to the parameter λ.

In the two space dimension case the crest factor shows (of course) stronger potential fluctuations. Indeed it scales like the $\lambda^{\frac{3}{8}}$ for large λ, which is naturally much larger then in the one space dimension case. As a function of L, for large L it scales like $L^{\frac{3}{2}}$,which again naturally is much larger than in the one space dimension case where it goes like $\sqrt{L}$ for large L.

It would be interesting to compute the crest factor for other important PDEs, such as the Complex Ginzburg-Landau equation and the Navier-Stokes equations. As one can infer from our analysis above, the crest factor sheds some light on the nature of the solutions of any PDE. In particular, as a function of the parameters and the length of the torus, it gives important indication on the fluctuations of solutions away from their spatial average. Thus it can provide insight on regimes of "soft" and "hard" turbulent behaviour of the solutions of any dissipative PDE.

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