# Globally and locally attractive solutions for quasi-periodically forced systems

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**Abstract**

We consider a class of differential equations, $\ddot{x} + \gamma\dot{x} + g(x) = f(\boldsymbol{\omega}t)$, with $\boldsymbol{\omega} \in \mathbb{R}^d$, describing one-dimensional dissipative systems subject to a periodic or quasi-periodic (Diophantine) forcing. We study existence and properties of trajectories with the same quasi-periodicity as the forcing. For $g(x) = x^{2p+1}$, $p \in \mathbb{N}$, we show that, when the dissipation coefficient is large enough, there is only one such trajectory and that it describes a global attractor. In the case of more general nonlinearities, including $g(x) = x^2$ (describing the varactor equation), we find that there is at least one trajectory which describes a local attractor.

## 1 Introduction

Consider the ordinary differential equation

$$\ddot{x} + \gamma\dot{x} + x^{2p+1} = f(\boldsymbol{\omega}t), \tag{1.1}$$

where $p \in \mathbb{N}$, $\boldsymbol{\omega} \in \mathbb{R}^d$ is the frequency vector, $f(\boldsymbol{\psi})$ is an analytic quasi-periodic function,

$$f(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d} \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} f_{\boldsymbol{\nu}}, \tag{1.2}$$

with average $\langle f \rangle \equiv f_{\mathbf{0}} \neq 0$, and $\gamma > 0$ is a real parameter (dissipation coefficient). Here and henceforth we denote with $\cdot$ the scalar product in $\mathbb{R}^d$. By the analyticity assumption on $f$ there are two strictly positive constants $F$ and $\xi$ such that one has $|f_{\boldsymbol{\nu}}| \leq F\,\mathrm{e}^{-\xi|\boldsymbol{\nu}|}$ for all $\boldsymbol{\nu} \in \mathbb{Z}^d$.

If $d > 1$ we shall assume a Diophantine condition on the frequency vector $\boldsymbol{\omega}$, that is

$$|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq C_0 |\boldsymbol{\nu}|^{-\tau} \qquad \forall \boldsymbol{\nu} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}, \tag{1.3}$$

where $|\boldsymbol{\nu}| = |\boldsymbol{\nu}|_1 \equiv |\nu_1| + \ldots + |\nu_d|$, and $C_0$ and $\tau$ are positive constants, with $\tau > d - 1$ and $C_0$ small enough. Note that for $d = 1$ the condition (1.3) is automatically satisfied for all $\boldsymbol{\omega} \neq \mathbf{0}$.

In this paper we want to show that for $\gamma$ large enough the system (1.1) admits a global attractor which is a quasi-periodic solution with the same frequency vector $\boldsymbol{\omega}$ as the forcing $f$. This will be done in two steps: first we prove that for $\gamma$ large enough there is a quasi-periodic solution $x_0(t)$ with frequency vector $\boldsymbol{\omega}$ (cf. Theorem 1 in Section 2); second we prove that, again for $\gamma$ large enough, any trajectory is attracted by $x_0(t)$ (cf. Theorem 2 in Section 3).

In particular, this solves for the system (1.1) a problem left as open in [12]. Indeed in [12] we considered a class of ordinary differential equations, including (1.1), and proved existence of a quasi-periodic solution with the same quasi-periodicity as the forcing, but we couldn't conclude, not even locally, that this was the only solution with such a property. The result stated above gives an affirmative answer to this problem for the system (1.1), by showing that the quasi-periodic solution $x_0(t)$ is unique; cf. Theorem 3 in Section 4.

This uniqueness result holds for the more general systems studied in [12], including the resistor-inductor-varactor circuit, or simply varactor equation, studied in [12, 4]. This is a simple electronic circuit described by the equation $\ddot{x} + \gamma\dot{x} + x^\mu = f(\boldsymbol{\omega}t)$, for $x > 0$, where $R = \gamma$ is the resistance, $L = 1$ is the (normalised) inductance, $f(\boldsymbol{\omega}t)$ is the electromotive force, $v(t) = x(t)$ is the varactor voltage, and $i(t) = \dot{x}(t)$ is the current. The varactor is a particular type of diode, and it is described by the nonlinear term $x^\mu$, where typically $\mu \in [1.5, 2.5]$. In [12, 4] the case $\mu = 2$ was explicitly considered, for the sake of simplicity and concreteness. In these more general cases, the solution $x_0(t)$ is not a global attractor, but it turns out to be the only attractor in a neighbourhood of the solution itself.

More precisely the situation is as follows. We can consider systems described by

$$\ddot{x} + \gamma\dot{x} + g(x) = f(\boldsymbol{\omega}t), \tag{1.4}$$

where $f$ is given by (1.2) and $g$ is an analytic function. The case $g(x) = x^2$ corresponds to the varactor equation studied in [4]. Studying the behaviour of the system (1.4) for $\gamma$ large enough suggests to introduce a new parameter $\varepsilon = 1/\gamma$, in terms of which the differential equation (1.4) becomes

$$\varepsilon\ddot{x} + \dot{x} + \varepsilon g(x) = \varepsilon f(\boldsymbol{\omega}t), \tag{1.5}$$

and study what happens for $\varepsilon$ small enough.

If we assume that there exists $c_0 \in \mathbb{R}$ such that $g(c_0) = f_{\mathbf{0}}$ and $g'(c_0) := \partial_x g(c_0) \neq 0$, then the system (1.5) admits a quasi-periodic solution $x_0(t)$, analytic in $t$, with the same frequency vector $\boldsymbol{\omega}$ as the forcing $f$, and furthermore $x_0(t) = c_0 + O(\varepsilon)$. This was proved in [12], where the solution $x_0(t)$ was explicitly constructed through a suitable summation of the perturbation series

$$x_P(t) = \sum_{k=0}^{\infty} \varepsilon^k x^{(k)}(\boldsymbol{\omega}t), \qquad x^{(k)}(\boldsymbol{\psi}) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \mathrm{e}^{i\boldsymbol{\nu}\cdot\boldsymbol{\psi}} x^{(k)}_{\boldsymbol{\nu}}, \tag{1.6}$$

for a function formally solving the equations of motion.

As a drawback of the construction we were not able to prove any uniqueness result about $x_0(t)$. In fact, in principle, there could be other quasi-periodic solutions near $x_0(t)$, possibly with the same frequency vector $\boldsymbol{\omega}$. Neither could we exclude the existence of other solutions reducing to $c_0$ as $\varepsilon \to 0$ or even admitting the same formal expansion (1.6) in powers of $\varepsilon$. In this paper, under the further positivity condition $g'(c_0) > 0$, we eliminate these possibilities, and we prove that there exists, in the plane $(x, \dot{x})$, a neighborhood $\mathcal{B}$ of the point $(c_0, 0)$ where $(x_0(t), \dot{x}_0(t))$ is the only stable solution of (1.5). Moreover it turns out to be asymptotically stable, that is it attracts any trajectory starting in $\mathcal{B}$. Therefore, this allows us to formulate a strengthened version of the theorem proved in [12]; cf. Theorem 4 in Section 4.

In general the neighbourhood $\mathcal{B}$ can be very small. In [2] we show that in specific cases, in particular in the case $g(x) = x^2$, one obtains improved estimates of $\mathcal{B}$.

More formal statements of the results will be formulated in the forthcoming sections. Some open problems will be discussed at the end. Here we confine ourselves to noting that, while in the case of periodic forcing standard techniques, like those based on Poincaré sections [15, 18], could be applied, this is not the case for quasi-periodic forcing, where no Poincaré maps can be introduced.

The rest of the paper is organised as follows. Sections 2 and 3 are devoted to the global study of the system (1.1) for $\gamma$ large enough, whereas in Section 4 we draw the conclusions. In Section 5 we pass to the study of the system (1.4), and we prove existence of a local attractor under the non-degeneracy conditions mentioned after (1.5). Finally, in Section 6 we mention some open problems, and possible directions of future research.

## 2 Existence of the quasi-periodic solution

First we show that for $\gamma$ large enough there exists a quasi-periodic solution $x_0(t)$. The proof of existence of a periodic solution in the case of periodic forcing is relatively easy, whereas in the case of quasi-periodic forcing to prove existence of a quasi-periodic solution becomes a little more subtle. By using a formal power series expansion, as we do, we need to introduce a suitable summation of the series: this requires a careful multiscale analysis and employs techniques of renormalisation group theory. The analysis is explicitly performed in [12], and uses ideas and notions introduced first in [8] and exploited in [9, 11, 10] in similar contexts.

The periodic solutions can be proved to be Borel summable at the origin [12]. We briefly recall here the notion of Borel summability [19]. Let $f(\varepsilon) = \sum_{n=1}^{\infty} a_n \varepsilon^n$ a formal power series. We say that $f(\varepsilon)$ is Borel summable if

1. $B(p) := \sum_{n=1}^{\infty} a_n p^n / n!$ converges in some circle $|p| < \delta$,
2. $B(p)$ has an analytic continuation to a neighbourhood of the positive real axis, and
3. $g(\varepsilon) = \int_0^{\infty} \mathrm{e}^{-p/\varepsilon} B(p) \, \mathrm{d}p$ converges for some $\varepsilon > 0$.

A function which admits the formal power series expansion $f(\varepsilon)$ is called Borel summable if $f(\varepsilon)$ is Borel summable; in that case the function equals $g(\varepsilon)$.

Borel summability also holds in the quasi-periodic case for two-dimensional frequency vectors of constant type [13]. In [13] it is also shown that the results on the existence of periodic or quasi-periodic solutions can be extended to any frequency vector which satisfies the Bryuno condition (weaker than the standard Diophantine condition (1.3) usually assumed).

**Theorem 1** *Consider the equation (1.1), with $f$ a non-zero average quasi-periodic function analytic in its argument and with $\boldsymbol{\omega}$ satisfying the Diophantine condition (1.3). There exists $\gamma_0 > 0$ such that for all $\gamma > \gamma_0$ there is a quasi-periodic solution $x_0(t)$ with the same frequency vector as the forcing term. Such a solution extends to a function analytic in $1/\gamma$ in a disc $\mathcal{D}$ of the complex plane tangent to the imaginary axis at the origin and centered on the real axis. Furthermore, $x_0(t) = \alpha + O(1/\gamma)$, where $\alpha = f_{\mathbf{0}}^{1/(2p+1)} \neq 0$.*

*Proof.* We can apply the results of Section 7 in [12]. If we set $g(x) = x^{2p+1}$, then $g(c_0) = f_{\mathbf{0}}$ yields $c_0 = f_{\mathbf{0}}^{1/(2p+1)}$, so that $g'(c_0) \neq 0$ as by assumption one has $f_{\mathbf{0}} \neq 0$. Both the existence of the analyticity domain $\mathcal{D}$ and the form of the solution itself follow from the analysis in [12]. ■

If $\gamma$ is large enough, say $\gamma > \overline{\gamma}_0 \geq \gamma_0$, then the solution $x_0(t)$ is of definite sign. In the following we shall assume that this is the case: hence $x_0(t) \neq 0$ for all $t \in \mathbb{R}$.

Note that if we write $f(\boldsymbol{\omega} t) = f_{\mathbf{0}} + \eta \tilde{f}(\boldsymbol{\omega} t)$, where $\tilde{f}$ is a function having zero-average, we can interpret our solution as arising by bifurcation from the constant solution $x(t) = \alpha$, with $\alpha = f_{\mathbf{0}}^{1/(2p+1)}$, as the bifurcation parameter $\eta$ (not necessarily small) moves away from zero.

Earlier studies, such as in the classical book by Stoker [20], have treated cases of self-sustained oscillations, like for example in the van der Pol equation. In theses cases a limit cycle is already present in the dynamics of the system when the time-periodic forcing term is absent, and the problem consists in studying the persistence of the self-sustained limit cycle. In our case, the problem is different: in the absence of the forcing all the solutions are attracted by the unique fixed point, whereas the non-trivial attractor is generated by the presence of the forcing. Note that our approach includes also quasi-periodic forcing, which is generally not analysed: from a technical point of view a quasi-periodic forcing, instead of a purely periodic one, introduces additional subtle difficulties, related to the appearence of small divisors [12, 13].

## 3 Convergence to the quasi-periodic solution

Given the quasi-periodic solution $x_0(t)$ one can write $x(t) = x_0(t)+\xi(t)$, with $\xi(t)$ satisfying the differential equation

$$\ddot{\xi} + \gamma\dot{\xi} + \xi F(\xi, x_0(t)) = 0, \tag{3.1}$$

where we have defined

$$F(\xi, x) := \frac{1}{\xi}\left((x+\xi)^{2p+1} - x^{2p+1}\right) = \sum_{j=0}^{2p} \binom{2p+1}{j} \xi^{2p+1-j} x^j. \tag{3.2}$$

We can write (3.1) as

$$\begin{cases} \dot{\xi} = y, \\ \dot{y} = -\gamma y - \xi F(\xi, x_0(t)), \end{cases} \tag{3.3}$$

that is $\dot{z} = \Phi(z)$, if we define $z = (\xi, y)$ and $\Phi(z) = (y, -\gamma y - \xi F(\xi, x_0(t))$. We denote by $\varphi(t, z_0)$ the solution of (3.3) with initial datum $z_0$. Define also $P(\xi, t) := F(\xi, x_0(t))$ and $Q(\xi) := F(\xi, \alpha)$ and set $R(\xi, t) := P(\xi, t)/Q(\xi)$.

Here we prove the following result.

**Theorem 2** *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given in Theorem 1. There exists $\gamma_1 > 0$ such that for all $\gamma > \gamma_1$ all trajectories in phase space converge toward the origin as time goes to infinity.*

The proof will pass through several lemmata.

**Lemma 1** *Assume $\gamma > \overline{\gamma}_0$ so that $x_0(t)$ exists and $x_0(t) \neq 0$ for all $t \in \mathbb{R}$. There exist two positive constants $R_1$ and $R_2$ such that $R_1 < R(\xi, t) < R_2$ for all $\xi \in \mathbb{R}$ and for all $t \in \mathbb{R}$.*

*Proof.* By (3.2) we can write

$$F(\xi, x) = (2p+1) \int_0^1 \mathrm{d}s\, (x + s\,\xi)^{2p}, \tag{3.4}$$

so that $F(\xi, x) \geq 0$ for all $(\xi, x) \in \mathbb{R}^2$. Moreover $F(0, 0) = 0$ and $F(\xi, x) > 0$ for all $\xi \in \mathbb{R}$ if $x \neq 0$, and $\lim_{|\xi|\to\infty} F(\xi, x) = \infty$ for all $x \in \mathbb{R}$. Hence for $\alpha \neq 0$ and $\gamma > \overline{\gamma}_0$, one has both $P(\xi, t) > 0$ and $Q(\xi) > 0$, hence also $R(\xi, t) > 0$ for all $(\xi, t) \in \mathbb{R}^2$. Moreover $\lim_{|\xi|\to\infty} R(\xi, t) = 1$ for all $t \in \mathbb{R}$, so that the assertion follows. ■

**Lemma 2** *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given in Theorem 1. There exists $\gamma_2 > 0$ such that for all $\gamma > \gamma_2$ there is a convex set $\mathcal{S}$ containing the origin such that any trajectory starting inside $\mathcal{S}$ tends to the origin as time goes to infinity. One can take $\mathcal{S}$ such that $\partial\mathcal{S}$ crosses both the positive and negative $y$-axis at a distance $O(\gamma^2)$ from the origin and both the positive and negative $\xi$-axis at a distance $O(\gamma^{2/(p+1)})$ from the origin.*

*Proof.* Rescale time through the Liouville transformation

$$\tau = \int_0^t \mathrm{d}t' \sqrt{R(\xi(t'), t')}, \tag{3.5}$$

which is well-defined by Lemma 1. Then, if we introduce the coordinate transformation $\psi : (\xi, y) \to (v, y)$ by setting $\xi(t) = v(\tau(t))$ and $y(t) = \sqrt{R(\xi(t), t)}\, w(\tau(t))$, equation (3.3) is transformed into

$$\begin{cases} v' = w, \\ w' = -\dfrac{w}{\sqrt{R}}\left(\gamma + \dfrac{R'}{2\sqrt{R}}\right) - v\, Q(v), \end{cases} \tag{3.6}$$

where primes denote differentiation with respect to $\tau$, $Q(v(\tau)) = Q(\xi(t(\tau))$ and $R = R(v(\tau), t(\tau)) = R(\xi(t(\tau)), t(\tau))$.

The autonomous system

$$\begin{cases} v' = w, \\ w' = -v\,Q(v), \end{cases} \tag{3.7}$$

can be explicitly solved: all trajectories move on the level curves of the function

$$H(v,w) = \frac{1}{2}w^2 + \int_0^v \mathrm{d}v'\, v'Q(v'). \tag{3.8}$$

In (3.6) one has $R'/\sqrt{R} = \dot{R}/R$, with $\dot{R}/R = \dot{P}/P - \dot{Q}/Q$, and it is easy to see (Appendix A) that there are two $\gamma$-independent positive constants $B_1$ and $B_2$ such that

$$\left| \frac{\dot{R}}{2R} \right| < \frac{1}{\gamma}\left(B_1 + B_2|w|\right). \tag{3.9}$$

If $\gamma$ satisfies $\gamma^2 > 2B_1$ we can define $\widetilde{w}$ as

$$\widetilde{w} = \frac{\gamma^2 - B_1}{B_2} > \frac{\gamma^2}{2B_2}, \tag{3.10}$$

so that $\overline{\gamma} := (\gamma + R'/2\sqrt{R})/\sqrt{R} > 0$ for $|w| \le \widetilde{w}$.

Consider the compact set $\widetilde{\mathcal{P}}$ whose boundary $\partial\widetilde{\mathcal{P}}$ is given by the level curve $H(v,w) = \widetilde{w}^2/2$ of the system (3.7). Such a curve crosses the $w$-axis at $w = \pm\widetilde{w} = O(\gamma^2)$ and the $v$-axis at $v = O(\gamma^{2/(p+1)})$. If we take an initial datum $(v(0), w(0)) \in \widetilde{\mathcal{P}}$ then the dissipation coefficient $\overline{\gamma}$ in (3.6), even if it changes with time, always remains strictly positive. Moreover $H' = -\overline{\gamma}w^2 \le 0$ and $H' = 0$ only for $w = 0$, and for $w = 0$ the vector field in (3.6) vanishes only at $v = 0$, because $Q(v) > 0$ for all $v$ (cf. the proof of Lemma 1). Then we can apply Barbashin-Krasovsky theorem [1, 17], and conclude that the origin is asymptotically stable and that $\widetilde{\mathcal{P}}$ belongs to its basin of attraction.

Let $\mathcal{P}(t)$ be the time-dependent preimage of $\widetilde{\mathcal{P}}$ under the transformation $\psi$. By Lemma 1 if $\gamma$ is large enough there is a compact set $\mathcal{S} \subset \mathcal{P}(t)$ for all $t \in \mathbb{R}$, such that the boundary $\partial\mathcal{S}$ crosses the positive and negative $y$- and $\xi$-axes at a distances of order $\gamma^2$ and $\gamma^{2/(p+1)}$ from the origin, respectively. All trajectories starting from points inside $\mathcal{S}$ are attracted by the origin. ■

**Lemma 3** *Consider the curve $g(\xi,t) = -\gamma^{-1}\xi F(\xi, x_0(t))$ in the plane $(\xi, y)$. There exists $\gamma_3 > 0$ such that for $\gamma > \gamma_3$, outside the set $\mathcal{S}$ defined in Lemma 2, one has*

$$-\frac{1}{2\gamma}\xi^{2p+1} \ge g(\xi,t) \ge -\frac{2}{\gamma}\xi^{2p+1} \tag{3.11}$$

*for all $t \in \mathbb{R}$.*

*Proof.* Consider $\xi \ge 0$ (the case $\xi < 0$ can be discussed in the same way). By (3.2) one has

$$\xi F(\xi, x_0(t)) = \xi^{2p+1} + \sum_{j=1}^{2p} \binom{2p+1}{j} \xi^{2p+1-j} x_0^j(t),$$

and, if $\gamma$ is sufficiently large so that $|x_0(t)| < 2|\alpha|$, then for $\xi \ge 2|\alpha|$ the sum can be bounded by $2^{2p+1}(2|\alpha|)\,\xi^{2p} \equiv C_p\xi^{2p}$. Hence one has

$$\frac{1}{2}\xi^{2p+1} \le \xi F(\xi, x_0(t)) \le 2\xi^{2p+1}, \tag{3.12}$$

as soon as $\xi \geq 2C_p$ (note that if $\xi \geq 2C_p$ then one has automatically $\xi \geq 2|\alpha|$). Next, we want to show that the latter inequality is satisfied outside $\mathcal{S}$.

Consider the intersection of the graph of $g(\xi, t)$ with $\partial\mathcal{S}$. Let II be the quadrant $\{(\xi, y) \in \mathbb{R}^2 : \xi \geq 0, y < 0\}$; cf. figure 1. The curve $\partial\mathcal{S}\cap$II is below the line

$$y_1(\xi) = y_0 \left(1 - \frac{\xi}{\xi_0}\right), \tag{3.13}$$

where $y_0 := -b\gamma^2$, with $b > 0$, is the $y$-coordinate of the point at which $\partial\mathcal{S}$ crosses the $y$-axis, and $\xi_0 := a\gamma^{2/(p+1)}$, with $a > 0$, is the $\xi$-coordinate of the point at which $\partial\mathcal{S}$ crosses the $\xi$-axis. On the other hand the graph of $g(\xi, t)$ in II is above the curve

$$y_2(\xi) = -\frac{2}{\gamma} (\xi + 2|\alpha|)^{2p+1}, \tag{3.14}$$

because in (3.2) one has $\xi F(\xi, x) \leq |x + \xi|^{2p+1} + |x|^{2p+1}$.

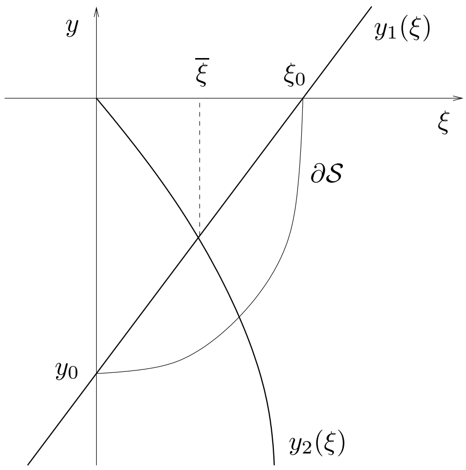


Figure 1: Construction used in the proof of Lemma 3.

As a consequence in II the two curves $\partial\mathcal{S}$ and $g(\xi, t)$ cannot cross each other for $\xi \in (0, a\gamma^\beta/2]$, with $\beta \leq 2/(2p+1)$. The latter assertion can be proved by *reductio ad absurdum.* First note that $a\gamma^\beta \leq \xi_0$ for $\gamma$ large enough. Suppose that there exists $\overline{\xi} \leq a\gamma^\beta/2$ such that $y_1(\overline{\xi}) = y_2(\overline{\xi})$. Then one has

$$b\frac{\gamma^2}{2} \leq b\gamma^2 \left(1 - \frac{\overline{\xi}}{\xi_0}\right) = \frac{2}{\gamma} \left(\overline{\xi} + 2|\alpha|\right)^{2p+1} \leq \frac{4}{\gamma} \max\{\overline{\xi}, 2|\alpha|\}^{2p+1} \tag{3.15}$$

that is $b\gamma^3 \leq 8 \max\{\overline{\xi}, 2|\alpha|\}^{2p+1}$, which is not possible if $\beta \leq 2/(2p+1)$ and $\gamma$ is large enough.

Therefore in II the graph of $g(\xi, t)$ can be outside $\mathcal{S}$ only for $\xi > a\gamma^\beta/2$, which is greater than $2C_p$ for $\gamma$ large enough. Hence (3.12) is satisfied outside $\mathcal{S}$, so that (3.11) follows. ∎

**Lemma 4** *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given in Theorem 1. There exists $\gamma_4 > 0$ such that for all $\gamma > \gamma_4$, if $z \notin \mathcal{S}$,then either $\varphi(t, z)$ enters $\mathcal{S}$ or crosses the y-axis outside $\mathcal{S}$ in a finite positive time.*

*Proof.* Consider the four quadrants

$$\begin{aligned} \text{I} &= \{(\xi, y) \in \mathbb{R}^2 : \xi > 0, y \geq 0\}, \qquad \text{II} = \{(\xi, y) \in \mathbb{R}^2 : \xi \geq 0, y < 0\}, \\ \text{III} &= \{(\xi, y) \in \mathbb{R}^2 : \xi < 0, y \leq 0\}, \qquad \text{IV} = \{(\xi, y) \in \mathbb{R}^2 : \xi \leq 0, y > 0\}. \end{aligned} \tag{3.16}$$

In I one has $\dot{\xi} \geq 0, \dot{y} < 0$, in II one has $\dot{\xi} < 0$, in III one has $\dot{\xi} \leq 0, \dot{y} > 0$, and in IV one has $\dot{\xi} > 0$. It is easy to see that each trajectory starting in I enters II and each trajectory starting from III enters IV in a finite time (see Appendix B).

Consider now an initial datum $z$ in II but not in $\mathcal{S}$. Let $\mathcal{C}_1$ be a continuous curve $\xi \to y(\xi)$ in II such that $\dot{y} < 0$ for $z$ in II above $\mathcal{C}_1$; cf. figure 2. Existence of such a curve follows from Lemma 3, which also implies that $\mathcal{C}_1$ is decreasing outside $\mathcal{S}$ (see Appendix C). The curve $\mathcal{C}_1$ divides II into two sets IIa and IIb, with IIa above IIb. Denote by $\mathcal{T}_1$ and $\mathcal{T}_2$ the parts of IIa and IIb, respectively, outside $\mathcal{S}$. Hence for $z \in \mathcal{T}_1$ the trajectory $\varphi(t, z)$ either enters $\mathcal{S}$ or enters $\mathcal{T}_2$. In the latter case it cannot come back to $\mathcal{T}_1$, hence $y(t) \leq \overline{y}$, if $(\overline{\xi}, \overline{y}) = \mathcal{C}_1 \cap \partial\mathcal{S}$. This means that if the solution does not enter $\mathcal{S}$ then it has to cross the vertical axis and enter III.

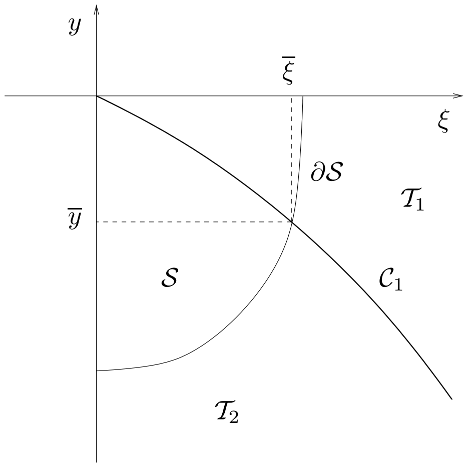


Figure 2: Construction used in the proof of Lemma 4.

Analogously one discusses the case of initial data $z$ in IV, outside $\mathcal{S}$: their evolution leads either to $\mathcal{S}$ or to I. Hence the lemma is proved. ■

**Lemma 5** *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given in Theorem 1. There exists $\gamma_5 > 0$ such that for all $\gamma > \gamma_5$ and for all $z \notin \mathcal{S}$ on the vertical axis, either $\varphi(t, z)$ enters $\mathcal{S}$ or the trajectory $\varphi(t, z)$ re-crosses the vertical axis at a point $z_1$ which is such that $|z| - |z_1| > \delta$ for some positive $z$-independent constant $\delta$.*

*Proof.* Fix an initial datum $z \notin \mathcal{S}$ on the vertical axis. This means that at $t = 0$ one has $z = z(0) = (0, y(0))$ outside $\mathcal{S}$. Assume for concreteness $y(0) > 0$ and set $y(0) = 1/\varepsilon^{p+1}$, with $\varepsilon > 0$. As $z(0) \notin \mathcal{S}$ there exists a constant $C_2$ such that $\gamma^2 \varepsilon^{p+1} \leq C_2$. Consider the change of coordinates

$$X = \varepsilon\xi, \qquad Y = \varepsilon^{p+1} y, \qquad T = \varepsilon^{-p} t. \tag{3.17}$$

Then the system (3.3) becomes

$$\begin{cases} X' = Y, \\ Y' = -\gamma\varepsilon^p Y - XF(X, \varepsilon x_0(\varepsilon^p T), \end{cases} \tag{3.18}$$

where primes denote differentiation with respect to $T$. Note that $Y(0) = 1$ and $XF(X, \varepsilon x_0(\varepsilon^p T) = X^{2p+1} + O(\varepsilon\, X^{2p})$. Call $S$ the image of $\mathcal{S}$ under the transformation (3.17); cf. figure 3.

We can rewrite the system (3.18) as

$$\begin{cases} X' = Y, \\ Y' = \Psi(X, Y) \equiv \Psi_1(X, Y) + \Psi_2(X, Y) + \Psi_3(X, Y), \end{cases} \tag{3.19}$$

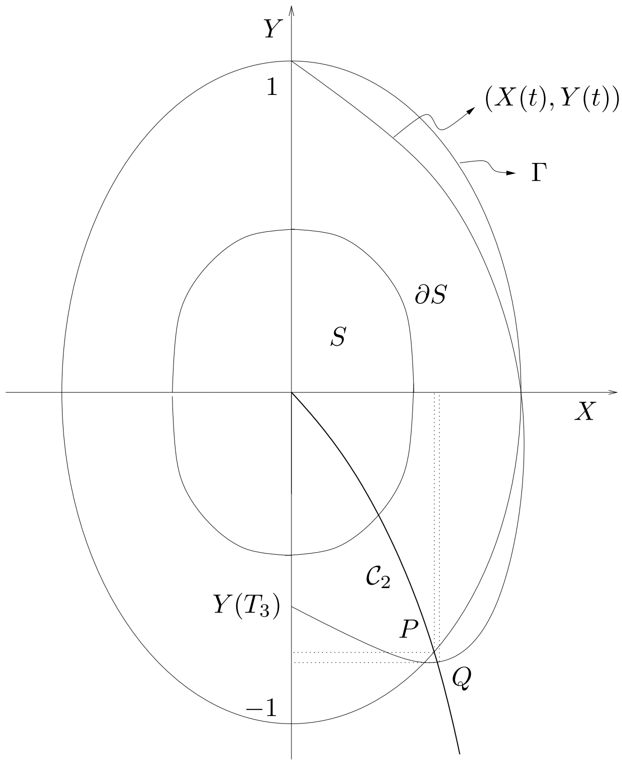


Figure 3: Construction used in the proof of Lemma 5.

with $\Psi_1(X,Y) = -XF(X,\varepsilon x_0(0))$, $\Psi_2(X,Y) = -X\left(F(X,\varepsilon x_0(\varepsilon^p T)) - F(X,\varepsilon x_0(0))\right)$, and $\Psi_3(X,Y) = -\gamma\varepsilon^p Y$.

If we replace $\Psi(X,Y)$ with $\Psi_1(X,Y)$ in (3.19) the trajectory moves on the level curve $\Gamma = \{(X,Y) \in \mathbb{R}^2 : H(X,Y) = 1/2\}$ for the function

$$H(X,Y) = \frac{1}{2}Y^2 + \int_0^X \mathrm{d}X\, XF(X,\varepsilon x_0(0)) \tag{3.20}$$

and crosses the vertical axis at the point $(0,-1)$, hence at the same distance from the origin as at $t=0$. By Lemma 3 there exists in II a curve $\mathcal{C}_2$, decreasing outside $S$, such that $-\gamma\varepsilon^p Y \geq \Psi(X,Y) \geq -\gamma\varepsilon^p Y/2$, for $Y$ in II below $\mathcal{C}_2$ (see Appendix C). Such a curve can be chosen in such a way that it crosses the level curve $\Gamma$ in a point $P = (X_P, Y_P)$, with $X_P = 2D_1(\gamma\varepsilon^p)^{1/(2p+1)}$, for some constant $D_1$ (see Appendix C). Note that the time $T_1$ necessary to reach such a point is of order 1.

If we take into account the component $\Psi_2(X,Y)$ of the vector field in (3.19), we can move from $P$ at most by a quantity of order $\varepsilon^{p+1}$. Indeed, as long as the motion remains close to that generated by the vector field $\Psi_1(X,Y)$, one has

$$|F(X,\varepsilon x_0(\varepsilon^p T_1)) - F(X,\varepsilon x_0(0))| \leq D_2'\varepsilon\,|x_0(\varepsilon^p T_1) - x_0(0)| \leq D_2''\varepsilon^{p+1}, \tag{3.21}$$

for suitable positive constants $D_2'$ and $D_2''$, so that the points reached at time $T_1$ by moving according to the vector fields $\Psi_1$ and $\Psi_1 + \Psi_2$ cannot be more distant than $D_2\varepsilon^{p+1}$ for some constant $D_2$. This follows from the fact that the system is quasi-integrable, so that in a time of order 1 the action variable can change at most by a quantity of order of the perturbation as bounded in (3.21); see Appendix D.

Finally the component $\Psi_3(X,Y)$ points inward along the full length of the curve $\Gamma$. Define $T_2$ as the time at which the the trajectory of the full system (3.19) crosses the curve $\mathcal{C}_2$ in a point $Q$ near $P$. Of course $T_2$ is near $T_1$, and so is of order 1, and $X(T_2) \geq X_P/2$ by construction, while $Y(T_2) \geq Y_P - D_2\varepsilon^{p+1}$, with $Y_P > -1$.

From time $T_2$ onwards, we have

$$\begin{cases} X' = Y, \\ Y' \geq -\gamma\varepsilon^p Y/2, \end{cases} \tag{3.22}$$

as long as the motion remains below $\mathcal{C}_2$. The latter property is easily checked to hold (see Appendix C). Then the trajectory crosses the vertical axis and meanwhile, at least, moves upward in the vertical direction by a quantity $\gamma\varepsilon^p X_P/2 = D_1\gamma\varepsilon^p(\gamma\varepsilon^p)^{1/(2p+1)}$.

Therefore when the trajectory again crosses the vertical axis, this happens at a time $T_3$ such that $Y(T_3) \geq Y_P - D_2\varepsilon^{p+1} + D_1\gamma\varepsilon^p(\gamma\varepsilon^p)^{1/(2p+1)} > -1 + \Delta Y$, with $\Delta Y = D_1\gamma\varepsilon^p(\gamma\varepsilon^p)^{1/(2p+1)} - D_2\varepsilon^{p+1} \geq D_2\varepsilon^{p+1}$, where the latter inequality holds provided $(\gamma\varepsilon^p)^{1+1/(2p+1)} \geq 2D_2\varepsilon^{p+1}$, that is provided

$$\gamma^{2(p+1)} \geq D_0\varepsilon^{p+1}, \qquad D_0 = (2D_2)^{2p+1}. \tag{3.23}$$

Since $\varepsilon^{p+1}\gamma^2 \leq C_2$, inequality (3.23) is satisfied if $\gamma^{2(p+1)} \geq D_0C_2\gamma^{-2}$, which requires $\gamma^{2(p+2)} \geq D_0C_2$, that is $\gamma \geq (D_0C_2)^{1/2(p+2)}$, with $D_0 = (2D_2)^{2p+1}$ Under this condition one has $|Y(0)| - |Y(T_3)| = 1 - |Y(T_3)| \geq \Delta Y \geq D_2\varepsilon^{p+1}$, so that, in terms of the original coordinate $y$, one has $|y(0)| - |y(t_3)| \geq D_2$.

Then, if the trajectory crosses the vertical axis once more in the positive direction (and this necessarily happens if it does not enter $\mathcal{S}$, by Lemma 4), this occurs at a time $t_4$ such that $|y(0)| - |y(t_4)| \geq 2D_2$, where we recall that the constant $D_2$ is independent of the initial datum $y(0)$. Simply one can repeat the argument above by taking $(0, y(t_3))$ as initial datum and calling $t_4$ the time of crossing of the positive $\xi$-axis. This means that the trajectory either enters $\mathcal{S}$ or, after a complete cycle, moves closer to the origin by a finite positive distance $\delta = 2D_0$. ■

**Lemma 6** *Consider the equation (3.1), with $x_0(t)$ the quasi-periodic solution of (1.1) given as in Theorem 1. There exists $\gamma_6 > 0$ such that for all $\gamma > \gamma_6$ for all $z \notin \mathcal{S}$ there is a finite time $t(z)$ such that $\varphi(t(z), z) \in \mathcal{S}$.*

*Proof.* Consider $z \notin \mathcal{S}$. By Lemma 4 either $\varphi(t, z)$ enters $\mathcal{S}$ or there exists a time $t_1$ such that $\varphi(t_1, z)$ is on the vertical axis outside $\mathcal{S}$. Hence, without loss of generality, we can consider only initial data $z = (\xi, y)$ outside $\mathcal{S}$ such that $\xi = 0$. Assume $y > 0$ (if $y < 0$ the discussion proceeds in the same way): we can apply Lemma 4 and we find that, as far as the trajectory does not enter $\mathcal{S}$, at each turn it gets closer to $\mathcal{S}$ by a finite quantity. Hence sooner or later it enters $\mathcal{S}$. ■

Theorem 2 follows from the lemmata above: it is enough to take $\gamma_1 = \max\{\overline{\gamma}_0, \gamma_2, \gamma_3, \gamma_4, \gamma_5, \gamma_6\}$, so that all lemmata apply.

# 4 Uniqueness of the quasi-periodic solution

Let $\gamma$ be $\gamma > \max\{\gamma_0, \gamma_1\}$. Then there exists a quasi-periodic solution $x_0(t)$ for the system (1.1), and such a solution is a global attractor.

In [12] we explicitly constructed a solution $x_0(t)$ with the properties stated in Theorem 1. Such a solution turns out to be Borel summable for $d = 1$. In general the solution is obtained from the formal series through a suitable summation procedure. Since Theorem 2 implies that if there exists a quasi-periodic solution $x_0(t)$ this has to be unique, we can conclude that for real $\gamma$ large enough there exists a unique quasi-periodic solution $x_0(t)$ with the same frequency vector as the forcing $f$. By setting $\varepsilon = 1/\gamma$, in the complex $\varepsilon$-plane, there is a solution $x_1(t)$ which is analytic in a domain containing a disk $\mathcal{D}$ with centre on the real axis and tangent at the origin to the imaginary axis. For real $\gamma$ such a solution coincides with $x_0(t)$ (as the latter is the only one), hence, by uniqueness of the analytic continuation, the function

$x_1(t)$ is the only solution of (1.1) in all the domain $\mathcal{D}$. In particular it is the only one which admits the formal expansion given by perturbation theory.

We can summarise the discussion above through the following statement.

**Theorem 3** *Consider the equation (1.1), with $f$ a non-zero average quasi-periodic function analytic in its argument and with $\boldsymbol{\omega}$ satisfying the Diophantine condition (1.3). There exists $\gamma_0 > 0$ such that for all real $\gamma > \gamma_0$ there is a unique quasi-periodic solution $x_0(t)$ with the same frequency vector as the forcing term. Such a solution describes a limit cycle in the plane $(x, \dot{x})$ which is a global attractor.*

Note that the hypotheses made in Theorem 1 are more restrictive than those considered in [12]. In particular we have excluded both polynomial nonlinearities and monomial nonlinearities with even degree. We come back to this in the next section.

# 5 Local attractors for more general nonlinearities

In the previous analysis, the restriction to monomials only resulted in a strictly positive function $R(\xi, t)$, which was used to construct the positively invariant set $\mathcal{S}$. We leave as an open problem the study of what happens if the nonlinearity $x^{2p+1}$ in (1.1) is replaced with

$$g(x) = \sum_{j=1}^{2p+1} a_j\, x^j, \qquad a_j \in \mathbb{R}, \qquad a_{2p+1} > 0. \tag{5.1}$$

For $d = 1$ and $\gamma = 0$ it is known that all motions are bounded, also replacing the constants $a_j$ with periodic functions [6, 14, 5]. The same holds for $d > 1$ [16]. One could expect that the presence of friction tends to contract phase space toward some periodic solution (which certainly exists for $\gamma$ large enough, as proved in [12]), but our results do not allow us to treat, in general, such a case.

If the nonlinearity $x^{2p+1}$ is replaced with an even monomial $x^{2p}$, with $p \in \mathbb{N}$, then, under the further condition that $f_{\mathbf{0}} > 0$, there is a quasi-periodic solution $x_0(t)$: again this follows from [12]. In such a case $x_0(t)$ is not a global attractor, as there are unbounded solutions; cf. for example [4] for $p = 1$. Still one can prove that the solution found in [12] is unique, in the sense that it is the only attractor in a neighbourhood of the solution itself, and it is a local attractor. The same result holds, more generally, for any analytic $g(x)$ in (1.4) such that $g(c_0) = f_{\mathbf{0}}$ and $g'(c_0) > 0$ for some $c_0 \in \mathbb{R}$. A more formal statement is as follows.

**Theorem 4** *Consider the equation (1.4), with $f$, given by (1.2), and $g$ both analytic in their arguments, and with $\boldsymbol{\omega}$ satisfying the Diophantine condition (1.3). Assume that there exists $c_0 \in \mathbb{R}$ such that $g(c_0) = f_{\mathbf{0}}$ and $g'(c_0) := \partial_x g(c_0) > 0$. There exists $\gamma_0 > 0$ such that for $\gamma > \gamma_0$ there is a unique quasi-periodic solution $x_0(t)$ which has the same frequency vector as $f$, reduces to $c_0$ in the limit $\gamma \to \infty$, and extends to a function analytic in a disk with center on the positive real axis and boundary tangent to the vertical axis at the origin. Furthermore, there exists $\gamma_1 \geq \gamma_0$ such that for $\gamma > \gamma_1$ there is a neighbourhood $\mathcal{B}$ of the point $(c_0, 0)$, containing the orbit described by $x_0(t)$, with the property that all trajectories starting in $\mathcal{B}$ are attracted to the solution described by $x_0(t)$ in the plane.*

*Proof.* The existence of a quasi-periodic solution $x_0(t)$ with the same frequency vector $\boldsymbol{\omega}$ as the forcing was proved in [12]. As a byproduct of the proof, one can write $x_0(t) = c_0 + x_1(\boldsymbol{\omega} t)$, with $x_1(\boldsymbol{\psi})$ analytic in $\boldsymbol{\psi}$ and of order $\varepsilon$, if $\varepsilon = 1/\gamma$ (that is for $\varepsilon$ small enough one has $|x_1(\boldsymbol{\psi})| \leq C|\varepsilon|$ for all $\boldsymbol{\psi}$ and for a suitable $C$). Therefore we can write $x(t) = x_0(t) + \xi(t)$, where $\xi(t)$ satisfies the differential equation

$$\ddot{\xi} + \gamma\dot{\xi} + \xi F(\xi, x_0(t)) = 0, \tag{5.2}$$

with

$$F(\xi, x) = \frac{1}{\xi}\Big(g(x + \xi) - g(x)\Big) = \partial_x g(x) + O(\xi). \tag{5.3}$$

Then we can write (5.2) as a system of first order differential equations,

$$\begin{cases} \dot{\xi} = y, \\ \dot{y} = -\gamma y - \xi F(\xi, x_0(t)), \end{cases} \tag{5.4}$$

and define $R(\xi,t) = F(\xi, x_0(t))/F(\xi, c_0)$. It is easy to see that one has $\lim_{\xi\to 0} R(\xi,t) = 1 + O(\varepsilon)$, so that for $\varepsilon$ small enough one has $R_1 < R(\xi,t) < R_2$, for two suitable positive constants $R_1$ and $R_2$.

Then we can rescale time and variables by setting

$$\tau = \int_0^t \mathrm{d}t \, \sqrt{R(\xi(t'),t')}, \qquad \xi(t) = v(\tau(t)), \qquad y(t) = \sqrt{R(\xi(t),t)}\, w(\tau(t)), \tag{5.5}$$

which transforms the system (5.4) into

$$\begin{cases} v' = w, \\ w' = -\dfrac{w}{\sqrt{R}} \left( \gamma + \dfrac{R'}{2\sqrt{R}} \right) - v\, F(v, c_0), \end{cases} \tag{5.6}$$

where primes denote differentiation with respect to $\tau$.

If we neglect the friction term in (5.6) we obtain the autonomous system

$$\begin{cases} v' = w, \\ w' = -vF(v,c_0)) = -\partial_x g(c_0)\, v + O(v^2), \end{cases} \tag{5.7}$$

which admits the constant of motion

$$H(v,w) = \frac{1}{2} w^2 + \frac{1}{2} \partial_x g(c_0)\, v^2 + O(v^3), \qquad \partial_x g(c_0) > 0. \tag{5.8}$$

Hence the origin is a stable equilibrium point for (5.7), and the level curves for $H$ are close to ellipses in a neighbourhood $\widetilde{\mathcal{P}}$ of the origin. It is easy to check that in $\widetilde{\mathcal{P}}$ the coefficient of the friction term is strictly positive, for $\gamma$ large enough, because $R'/2\sqrt{R}$ is (in $\widetilde{\mathcal{P}}$) less than a constant. Hence we can apply Barbashin-Krasovsky's theorem and conclude that the origin is asymptotically stable and $\widetilde{\mathcal{P}}$ belongs to its basin of attraction. If we go back to the original variables $(\xi, y)$ we find that $\widetilde{\mathcal{P}}$ is transformed back to a time-dependent set $\mathcal{P}(t)$. But the dependence on $t$ of $\mathcal{P}(t)$ is very weak (as $R$ is close to 1 for $\gamma$ large enough), so that there exists a convex set $\mathcal{S} \subset \mathcal{P}(t)$ for all $t \in \mathbb{R}$. Hence any trajectory starting from $\mathcal{S}$ is attracted toward the origin. In terms of the variables $(x, y)$, using once more that the solution $x_0(t)$ is close to $c_0$ within $O(\varepsilon)$, we can say that, for $\varepsilon$ small enough (that is for $\gamma$ large enough) there exists a neighbourhood $\mathcal{B}$ of the point $(c_0, 0)$ such that it contains the projection of the quasi-periodic solution $x_0(t)$ into the plane $(x, \dot{x})$, and any trajectory starting from $\mathcal{B}$ is attracted by such a solution.

In particular the solution $x_0(t)$ is the only quasi-periodic solution which tends to $c_0$ as $\varepsilon \to 0$, and for $\varepsilon > 0$ small enough, say $\varepsilon < \varepsilon_0$, it is the only one which admits the formal power expansion (1.6). Such a solution was proved in [12] to be analytic in a domain $\mathcal{D}$ containing the interval $(0, \varepsilon_0)$, hence by the uniqueness of the analytic continuation, we can conclude that $x_0(t)$ is unique in all $\mathcal{D}$. ∎

By looking at the proof of Theorem 4, we see that it proceeds along the same lines of the proof of Lemma 2 in Section 3. In the extended phase space $(x, \dot{x}, t)$ the orbit described by the solution $x_0(t)$ is not a bounded set. However, this is easily remedied by replacing $t \in \mathbb{R}$ with $\boldsymbol{\tau} \in \mathbb{T}^d$ and using $\dot{\boldsymbol{\tau}} = \boldsymbol{\omega}$ instead of $\dot{t} = 1$ when augmenting (3.3), so that the corresponding orbit becomes bounded in $\mathbb{R}^2 \times \mathbb{T}^d$. By using the definitions of [7], Theorem 4 says that the orbit described by $x_0(t)$ is an attractor, and its closure is an attracting set with fundamental neighbourhood $\mathcal{B}$.

## 6 Conclusions, extensions and open problems

We conclude with a list of open problems (some of which have already been mentioned in the previous sections).

The first one concerns possible extensions of the proof of Theorem 3 to the case of more general polynomials of the form (5.1). A natural question is: under what conditions is there still a global attractor, in these cases, when the dissipation coefficient is large enough?

A characterisation of the set $\mathcal{B}$ can be given in some concrete cases, such as that of the varactor equation mentioned in Section 1. In [2] we show that we can improve the estimate by obtaining a set whose size increases linearly in $\gamma$ in the vertical direction, but in such a way that it is still expected to be strictly included inside the actual basin of attraction. It would be worthwhile to attempt constructions of sets contained inside the basins of attraction that are as large as possible.

We also leave as an open problem for the varactor equation the proof that any bounded solution is attracted by $x_0(t)$. On the basis of numerical simulations, we conjecture that this is the case.

Another interesting problem is whether one can weaken the hypotheses on the function $g$, both for determining the existence of a quasi-periodic solution with the same frequency vector as the forcing and, in that case, for proving its uniqueness and attractivity.

Finally, extensions to higher dimensional cases would be desirable.

## A Proof of (3.9)

One has $\dot{P} = \partial_\xi P\,\dot{\xi} + \partial_t P$ and $\dot{Q} = \partial_\xi Q\,\dot{\xi}$, so that

$$\frac{\dot{P}}{P} - \frac{\dot{Q}}{Q} = \left(\frac{\partial_\xi P}{P} - \frac{\partial_\xi Q}{Q}\right)\dot{\xi} + \frac{\partial_t P}{P} = \frac{\partial_\xi P - \partial_\xi Q}{P} + \frac{\partial_\xi Q}{PQ}\,(Q-P) + \frac{\partial_t P}{P}. \tag{A.1}$$

One can write

$$\begin{aligned} \partial_t P &= \sum_{j=1}^{2p} \binom{2p+1}{j} j\xi^{2p+1-j} x_0^{j-1}(t)\,\dot{x}_0(t), \\ Q-P &= \sum_{j=1}^{2p} \binom{2p+1}{j} \xi^{2p+1-j}\,(\alpha - x_0(t))^j\,, \\ \partial_\xi P - \partial_\xi Q &= \sum_{j=1}^{2p} \binom{2p+1}{j} (2p+1-j)\,\xi^{2p-j}\,(x_0(t)-\alpha)^j\,, \end{aligned} \tag{A.2}$$

where $\dot{x}_0(t) = O(1/\gamma)$ and $x_0(t) - \alpha = O(1/\gamma)$.

Finally $|x_0(t)| \le 2|\alpha|$ for all $t \in \mathbb{R}$ if $\gamma$ is large enough, and both $\xi^{2p+1-j}/P$ and $\xi^{2p+1-j}/Q$ tend to zero as $\xi \to \infty$ for $j \ge 1$. Hence (3.9) follows, with the constants $B_1$ and $B_2$ depending on $p$ but not on $\gamma$.

## B Initial data in I and III

Take an initial datum $z = (\xi, y)$ in I. If $y = 0$ then $\dot{\xi} = 0$ and $\dot{y} = -\xi F(\xi, x_0(t)) < 0$, so that the trajectory enters II. If $y > 0$ then $\dot{\xi} > 0$ and $\dot{y} < 0$.

Moreover $\partial_\xi(\xi F(\xi,x)) = \partial_\xi(x+\xi)^{2p+1} = (2p+1)\,(x+\xi)^{2p} \ge 0$ for all $x \in \mathbb{R}$, so that, by using the fact that $\xi(t) \ge \xi(0)$ as long as $(\xi(t), y(t))$ remains in I, one has in I

$$\xi F(\xi, x_0(t)) \ge \inf_{t\in\mathbb{R}} \xi F(\xi, x_0(t)) \ge \xi(0) \inf_{t\in\mathbb{R}} F(\xi(0), x_0(t)) \ge c > 0, \tag{B.1}$$

where we used that $F(\xi, x)$ is strictly greater than a positive constant for $x \neq 0$ (see the proof of Lemma 1). Therefore we obtain

$$\dot{y} \leq -\gamma y - c, \tag{B.2}$$

which implies that $y(t)$ reaches the $\xi$-axis in a finite time.

Analogously one discusses the case of initial data $z$ in III.

# C On the curves $\mathcal{C}_1$ and $\mathcal{C}_2$

Call $\mathcal{T}$ the subset of II outside $\mathcal{S}$.

Define $\mathcal{C}_1$ as a continuous curve in II such that in $\mathcal{T}$ it is given by the graph of the function $\xi \to -\xi^{2p+1}/4\gamma$. In (3.3) one can write $\dot{y} = \gamma(-y + g(\xi, t))$, with $g(\xi, t)$ defined in Lemma 3. By Lemma 3, in $\mathcal{T}$ one has $g(\xi, t) \leq -\xi^{2p+1}/2\gamma$, so that at all points in $\mathcal{T}$ above $\mathcal{C}_1$ one has

$$-y + g(\xi, t)) = |y| + g(\xi, t) \leq \frac{1}{4\gamma}\,\xi^{2p+1} - \frac{1}{2\gamma}\,\xi^{2p+1} \leq -\frac{1}{4\gamma}\,\xi^{2p+1},$$

hence $\dot{y} < 0$.

Define $\mathcal{C}_2$ as a continuous curve in II such that in $\mathcal{T}$ it is given by the graph of the function $\xi \to -4\xi^{2p+1}/\gamma$. By Lemma 3, one has $g(\xi, t) \geq -2\xi^{2p+1}/\gamma$, so that in all points of $\mathcal{T}$ below $\mathcal{C}_2$ one has $y \leq -4\xi^{2p+1}/\gamma \leq 2g(\xi, t)$, hence $-\gamma y \geq \gamma(-y + g(\xi, t)) \geq -\gamma y/2$, so that $-\gamma y \geq \dot{y} \geq -\gamma y/2$. In terms of the rescaled variables $(X, Y)$ this yields $Y' \equiv \Psi(X, Y)$, with

$$-\gamma\varepsilon^p Y \geq \Psi(X, Y) \geq -\gamma\varepsilon^p Y/2, \tag{C.1}$$

as asserted after (3.20).

The point $P$ is given by the intersection of the curve $Y_1(X) = -4X^{2p+1}/\gamma\varepsilon^p$ withe the level curve $\Gamma$. Hence

$$\frac{1}{2} = \frac{1}{2}\left(\frac{4X_P^{2p+1}}{\gamma\varepsilon^p}\right)^2 + X\,F(X, \varepsilon x_0(0)) = \frac{2^{4p+1}}{\gamma^2\varepsilon^{2p}}X_P^{4p+2} + X_P^{2p+1} + O(\varepsilon\,X^{2p}),$$

hence $X_P = O((\gamma\varepsilon^p)^{1/(2p+1)})$.

Now consider the solution of (3.19) with initial datum $Z(T_2) = (X(T_2), Y(T_2))$. We want to check that the solution remains below $\mathcal{C}_2$ until it crosses the $Y$-axis. The solution of

$$\begin{cases} X' = Y, \\ Y' = \Psi(X, Y), \end{cases} \tag{C.2}$$

with $\Psi(X, Y)$ satisfying the bounds (C.1), moves below the line with slope $-\gamma\varepsilon^p$ passing through $Z(T_2)$, that is below the line of equation $Y = Y_1(X) := Y_0 - \gamma\varepsilon^p X$, with $Y_0$ determined by the request that for $X = \overline{X} \equiv X(T_2)$ one has $Y_0 - \gamma\varepsilon^p\overline{X} = -4X^{2p+1}/\gamma\varepsilon^p$, where the graph of $-4X^{2p+1}/\gamma\varepsilon^p$ describes the curve $\mathcal{C}_2$ in the coordinates $(X, Y)$. By using that $\overline{X}$ is close to $X_P$ one realises that $Y_0$ has to be negative. In turn this implies that the line of equation $Y = Y_1(X)$ is below the curve $\mathcal{C}_2$, so that also the assertion after (3.22) is proved.

# D Variations in finite times for quasi-integrable systems

The system obtained from (3.19) by replacing $\Psi(X, Y)$ with $\Psi_1(X, Y)$ is an integrable Hamiltonian system, with Hamiltonian (3.20). For $\varepsilon = 0$ the Hamiltonian reduces to

$$H_0(X, Y) = \frac{1}{2}Y^2 + \frac{1}{2p+2}X^{2p+2}, \tag{D.1}$$

which can be written in terms of the action-angle variables $(I, \varphi)$ as $H_0(X,Y) = \mathcal{H}_0(I) = c_p I^{(2n+2)/(n+2)}$, where $c_p$ is a suitable $p$-dependent positive constant. By taking into account the other terms of the vector field, we obtain

$$H(X,Y) = \mathcal{H}(I) = c_p I^{(2n+2)/(n+2)} + O(I^{(2n+1)/(n+2)}). \tag{D.2}$$

The equations obtained by adding to $\Psi_1(X,Y)$ the vector field $\Psi_2(X,Y)$ are still Hamiltonian, and are described by the non-autonomous Hamiltonian $\mathcal{H}(I) + \mathcal{H}_1(I,\varphi,t)$, with $\mathcal{H}_0$ given as in (D.2) and $\mathcal{H}_1$ of order $\varepsilon^{p+1}$ as long as the action variables remain of order 1.

The corresponding equations of motion are

$$\begin{cases} \dot{I} = -\partial_\varphi \mathcal{H}_1(I,\varphi), \\ \dot{\varphi} = \omega_0(I) + \partial_I \mathcal{H}_1(I,\varphi), \end{cases}$$

with $\omega_0(I) = \partial_I \mathcal{H}_0(I)$. Then one immediately realises that in a time of order 1 the action variables remain close to their initial values. In turn this implies that also the angle variables are changed by order $\varepsilon^{p+1}$ with respect their unperturbed values. In terms of the original coordinates $(X,Y)$ this means that the solution remains within a distance $O(\varepsilon^{p+1})$ with respect the unperturbed value.

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