SCALING PROPERTIES FOR THE RADIUS OF CONVERGENCE OF A LINDSTEDT SERIES: THE STANDARD MAP

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Abstract. By using a version of the tree expansion for the standard map, we prove that the radius of convergence of the corresponding Lindstedt series satisfies a scaling property as the (complex) rotation number tends to any rational (resonant) value, non-tangentially to the real axis. By suitably rescaling the perturbative parameter ε, the function conjugating the dynamic on the (KAM) invariant curve with given rotation number to a linear rotation has a well defined limit, which can be explicitly computed.

Résumé. *En utilisant une version de l'expansion en arbres pour la série de Lindstedt de l'application standard, nous montrons que son rayon de convergence satisfait une proprieté d'invariance d'échelle lorsque le nombre de rotation (complexe) tend à n'importe quelle valeur rationelle (résonante), non tangentiellement à l'axe réel. Si on fait un changement d'échelle convenable sur le paramètre perturbatif, la fonction qui conjugue à une rotation linéaire la dynamique sur la courbe (KAM) invariante, avec un nombre de rotation fixé, a une limite bien definie, qui peut être explicitement calculée.*

1. Introduction

The standard map is a discrete-time, one-dimensional dynamical system generated by the iteration of the area-preserving (symplectic) map of the cylinder into itself, $T_\varepsilon : \mathbb{T}\times\mathbb{R} \mapsto \mathbb{T}\times\mathbb{R}$, given by:

$$T_\varepsilon : \begin{cases} x' &= x + y + \varepsilon \sin x, \\ y' &= y + \varepsilon \sin x. \end{cases} \tag{1.1}$$

For some background information, we refer the reader to the enormous literature on the topic, and to [1] for a review.

The homotopically non-trivial invariant curves $\mathcal{C}_{\varepsilon,\omega}$ with rotation number ω of the map T_ε may be determined by changing coordinates on $\mathbb{T}\times\mathbb{R}$:

$$\begin{cases} x = \alpha + u(\alpha,\varepsilon,\omega), \\ y = 2\pi\omega + v(\alpha,\varepsilon,\omega), \end{cases} \tag{1.2}$$

and imposing that the dynamics induced in the variables (α,ω) is given by the unperturbed map:

$$\begin{cases} \alpha' = \alpha + 2\pi\omega, \\ \omega' = \omega. \end{cases} \tag{1.3}$$

By (1.1), $y' = x' - x$ so that $v(\alpha,\varepsilon,\omega) = u(\alpha,\varepsilon,\omega) - u(\alpha - 2\pi\omega,\varepsilon,\omega)$; we can therefore consider only the function u.

The coordinate transformation (1.2) *conjugates* the dynamics on the invariant curve to a rotation, and the function u is called the *conjugating function.* It satisfies the functional equation:

$$D^2_\omega u(\alpha,\varepsilon,\omega) = \varepsilon \sin(\alpha + u(\alpha)), \tag{1.4}$$

where the operator D^2_ω acts on functions of α as follows:

$$D^2_\omega \phi(\alpha) = \phi(\alpha + 2\pi\omega) - 2\phi(\alpha) + \phi(\alpha - 2\pi\omega).$$

By imposing that the average of u over α be 0, the (formal) solutions of (1.4) are unique and odd as functions of α.

To each smooth solution to (1.4) corresponds an invariant curve $\mathcal{C}_{\varepsilon,\omega}$ whose parametric equations are:

$$\mathcal{C}_{\varepsilon,\omega} : \begin{cases} x = \alpha + u(\alpha,\varepsilon,\omega), \\ y = 2\pi\omega + u(\alpha,\varepsilon,\omega) - u(\alpha - 2\pi\omega,\varepsilon,\omega); \end{cases}$$

it is trivial to prove that is has the same smoothness properties as those of $u(\cdot,\varepsilon,\omega)$. To simplify the notations, we shall drop the dependence on ω and write just $u(\alpha,\varepsilon)$.

The conjugating function u has a formal expansion – the *Lindstedt Series* – of the form:

$$u(\alpha,\varepsilon) = \sum_{\nu\in\mathbb{Z}} u_\nu(\varepsilon)\, e^{i\nu\alpha} = \sum_{k\geq 1} u^{(k)}(\alpha)\varepsilon^k = \sum_{k\geq 1}\sum_{\nu\in\mathbb{Z}} u^{(k)}_\nu e^{i\nu\alpha}\, \varepsilon^k. \tag{1.5}$$

As it can be easily verified from (1.4), by inserting in it the formal series (1.5) and equating the Fourier and Taylor coefficients of both sides, one finds that the coefficients $u^{(k)}_\nu$ are defined by the recursion relations:

$$u^{(k)}_\nu = \frac{1}{\gamma(\nu)} \sum_{m\geq 0} \frac{1}{m!} \sum_{\substack{\nu_0+\ldots+\nu_m=\nu \\ k_1+\ldots+k_m=k-1}} \frac{1}{2}(-i\nu_0)(i\nu_0)^m \prod_{j=1}^m u^{(k_j)}_{\nu_j}, \tag{1.6}$$

with $\nu_0 = \pm 1$ and:

$$\gamma(\nu) = 2\left(\cos 2\pi\omega\nu - 1\right) \tag{1.7}$$

for $\nu \neq 0$, while $u^{(k)}_0 = 0$ for all $k \geq 1$. The case $m = 0$ in (1.6) has to be interpreted as $u^{(k)}_\nu = (-i\nu_0)/\gamma(\nu)$, which imposes $k = 1$ and $\nu = \nu_0$.

The radius of convergence of the Lindstedt series is defined as follows:

$$\rho = \inf_{\alpha\in\mathbb{T}} \left(\limsup_{k\to\infty} \left| u^{(k)}(\alpha) \right|^{\frac{1}{k}} \right)^{-1} . \tag{1.8}$$

The Lindstedt series is plagued by the *small divisors* problem, due to the fact that $\gamma(\nu)$ can be arbitrarily close to 0 for $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and can be 0 for $\omega \in \mathbb{Q}$; note though that, for $\mathrm{Im}(\omega) \neq 0$, $\gamma(\nu)$ is bounded away from 0 and therefore no small divisors appear.

As it is well known from KAM theory, if $\omega \in \mathbb{R}$ satisfies a Diophantine condition and ε is sufficiently small, ρ is strictly greater than 0 and therefore we have an analytic invariant curve and analytic conjugation to smooth rotations; this could be proved also using the tree expansion, *e.g.* by the methods of [2], [3] (and [4]), which inspired the authors of this paper (note that the first proof of existence of invariant tori for Hamiltonian systems by tree expansions of the Lindstedt series is due to Eliasson [5]).

We are interested to the behaviour of ρ as the rotation number ω tends to a *resonant* value, *i.e.* $\omega \to p/q$, with $p, q \in \mathbb{Z}$ and $p \wedge q = 1$. As shown in [6], the behaviour of the radius of convergence near a resonant value of the rotation number is related to the problem of *Bryuno's interpolation* and to the problem of determining the optimal arithmetic condition on ω to have an analytic invariant curve. The reader is referred to [6] and to the review [7] for a more complete discussion on the subject. We also refer to [8] where analogous results are proved for Siegel's problem and for the semi-standard map, by using different techniques. We refer the interested reader also to [9], where the conjecture of Bryuno's interpolation for the standard map was first introduced, and to [10] and its references for more details on Bryuno's function.

We consider:

$$\omega = \frac{p}{q} + i\eta, \tag{1.9}$$

with $p, q \in \mathbb{Z}$, $p \wedge q = 1$ and $\eta \in \mathbb{R}$, in the limit $\eta \to 0$. In sect. 7 we show how to extend our results to the case in which ω tends to p/q *along any path on the complex* ω *plane non-tangential to the real axis.*

We are interested to the exact (asymptotic) dependence of ρ on η, in the limit $\eta \to 0$. In particular, we shall prove the following theorem.

Theorem. *Consider the standard map* (1.1) *with* $\omega = p/q + i\eta$, $p, q \in \mathbb{Z}$, $p \wedge q = 1$ *and* $\eta \in \mathbb{R}$. *Then the following results hold.*

(1) *For fixed* $\eta \neq 0$ *the function* $u(\alpha, \varepsilon)$, *defined by* (1.2), *is divisible by* ε *and jointly analytic in* (α, ε) *in the product of a strip around the real axis in the complex* α *plane and a neighborhood* $|\varepsilon| < \varepsilon_0$ *of the origin in the complex* ε *plane, with* $\varepsilon_0 = O(\eta^{2/q})$.

(2) *The function* $v(\alpha,\varepsilon) = u(\alpha,(2\pi\eta)^{2/q}\varepsilon)$ *is well defined for* $\eta \to 0$ *and converges to a function* $\bar{u}(\alpha,\varepsilon)$*, divisible by* ε^q *and analytic in* ε^q *in a neighborhood of the origin, which is* $2\pi/q$ *periodic and solves the differential equation:*

$$\frac{d^2\bar{u}(\alpha)}{d\alpha^2} = C_{p/q}\varepsilon^q \sin\left(q(\alpha+\bar{u}(\alpha))\right) \tag{1.10}$$

with boundary conditions $\bar{u}(0) = \bar{u}(2\pi) = 0$*, for some constant* $C_{p/q}$.

Note that (1.10) was obtained in [6] in the special cases $p/q = 0/1$ and $p/q = 1/2$ and conjectured to hold in all cases.

The fact that, modulo rescalings, the limit function is essentially the same is of course not *a priori* obvious, not even heuristically, and comes basically from the numerical analysis of [6]. Preliminary numerical analysis seems to suggest that for more general maps a different, more complicated picture arises in the limit.

The theorem will be proved through a series of lemmata, using the formalism introduced in the following section.

2. Tree expansion for the Lindstedt series

It is possible to express graphically the coefficients $u_\nu^{(k)}$ in terms of *labeled trees* (or simply trees), defined as in [2] (see also [4], where the formalism was originally introduced).

A tree ϑ consists of a family of k lines arranged to connect a partially ordered set of points – *nodes* –, with the lower nodes to the right. All the lines have two nodes at their extremes, except the highest which has only one node, the *last node* u_0 of the tree; the other extreme r will be called the *root* of the tree and it will not be regarded as a node.

We denote by $\leq$ the partial ordering relation between nodes: given two nodes u, w, we say that $w \leq u$ if u is along the path of lines connecting w to the root r of the tree (they could coincide: we say that $u < w$ if they do not).

Each line carries an arrow pointing from the node u to the right to the node u' to the left (*i.e.* directed toward the root): we say that the line exits from u and enters u', and we write $u_0' = r$ even if, strictly speaking, r is not a node. For each node there is only one exiting line, and $m_u \geq 0$ entering ones; as there is a one-to-one correspondence between nodes and lines, we can associate to each node u a line ℓ_u exiting from it. The line ℓ_{u_0} exiting the node u_0 to the root r will be called the *root line*. Note that each line ℓ_u can be considered the root line of the subtree consisting of the nodes satisfying $w \leq u$: u' will be the

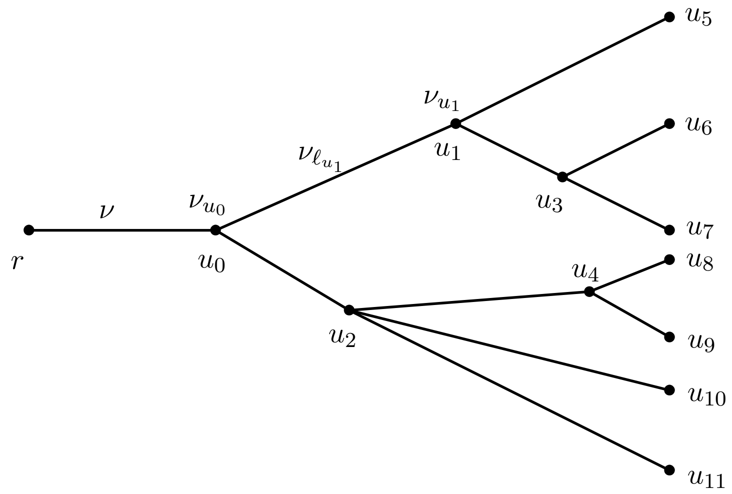

FIGURE 1. A tree ϑ with $m_{u_0} = 2$, $m_{u_1} = 2$, $m_{u_2} = 3$, $m_{u_3} = 2$, $m_{u_4} = 2$ and order $k = 12$; note that only a few labels are explicitly shown. The partial ordering relation $\leq$ implies $u_1 \leq u_0$, $u_5 \leq u_1$, and so on.

root of such subtree. The *order* k of the tree is defined as the number of nodes of the tree. Fig. 1 provides an example of tree of order 12.

To each node $u \in \vartheta$ we associate a *mode label* $\nu_u = \pm 1$, and define the *momentum* flowing through the line ℓ_u as:

$$\nu_{\ell_u} = \sum_{w \leq u} \nu_w, \quad \nu_w = \pm 1; \tag{2.1}$$

the condition $u_0^{(k)} = 0$ implies that no line ℓ can have momentum $\nu_\ell = 0$.

A group $\mathcal{G}$ of transformations acts on the trees, generated by the permutations of all the subtrees emerging from each node with at least one entering line: $\mathcal{G}$ is therefore a cartesian product of copies of the symmetric groups of various orders. Two trees that can be transformed into each other by the action of the group $\mathcal{G}$ are considered identical. Note that the definition of tree given here corresponds to the labeled *semitopological trees* in [3], sect. 3.1 ÷ 3.3. The number of trees of order k is bounded by $2^k \cdot 2^{2k}$.

The recursion relation (1.6) can be represented graphically as in fig. 2. By iterating this graphic, we can represent $u_\nu^{(k)}$ in terms of trees of order k with

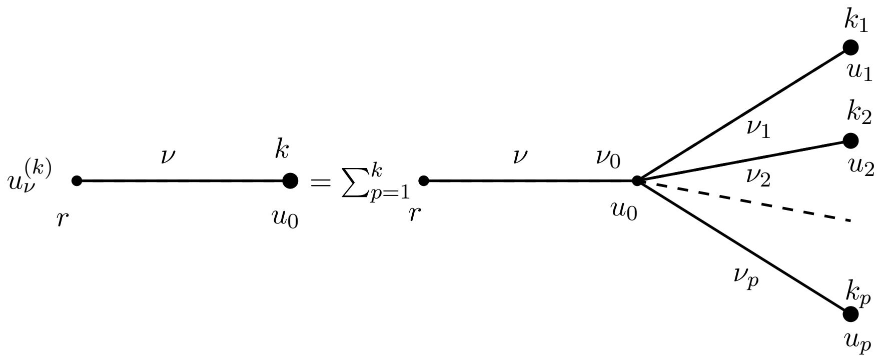

FIGURE 2. A graphic representation of (1.6)

momentum ν flowing through the root line (*total momentum*). So we can write:

$$\begin{aligned} u_\nu^{(k)} &= \frac{1}{2^k} \sum_{\vartheta \in \mathcal{T}_{\nu,k}} \mathrm{Val}(\vartheta), \\ \mathrm{Val}(\vartheta) &= -i \prod_{u \in \vartheta} \frac{1}{m_u!} \frac{\nu_u^{m_u+1}}{\gamma(\nu_{\ell_u})}, \end{aligned} \tag{2.2}$$

where $\mathcal{T}_{\nu,k}$ is the set of trees with k nodes and total momentum $\nu_{\ell_{u_0}} = \nu$, if u_0 is the last node of the tree. By ignoring the constraint on the sum of the mode labels, the cardinality of $\mathcal{T}_{\nu,k}$ is bounded again by 2^{3k}. The factors $1/\gamma(\nu_{\ell_u})$ in (2.2) are called *propagators* or *small divisors*, and the quantity $\mathrm{Val}(\vartheta)$ will be called the *value* of the tree ϑ.

For $\eta \neq 0$ the power series (1.5) is well defined and no small divisors appear: the denominators in $\mathrm{Val}(\vartheta)$ are all bounded from below by $|\eta|^2$. Nevertheless the convergence radius ρ is not uniform in η, and it shrinks to 0 when $\eta \to 0$. To understand the extent of this statement, consider that the most simple series with small divisors:

$$\sum_{k=1}^{\infty} \frac{z^k}{1-\lambda^k}, \quad \lambda = e^{2\pi i \omega} \tag{2.3}$$

has radius of convergence equal to 1 $\forall |\lambda| < 1$, *i.e.* whenever $\mathrm{Im}(\omega) > 0$, while by trivial Diophantine estimates its radius of convergence can be any value between 0 and 1 inclusive if $\omega \in \mathbb{R} \setminus \mathbb{Q}$; but (2.3) is the solution of a *linear* functional equation, while $u(\alpha, \varepsilon)$ is the solution of the nonlinear functional equation (1.4), so Bryuno's interpolation is related both to small divisors and to their accumulation.

3. Estimates on the Radius of Convergence

The small divisor $\gamma(\nu)$ defined by (1.7) satisfies the bound:

$$|\gamma(\nu)| \geq \begin{cases} c|\nu\eta|^2, & \text{for } \nu \in q\mathbb{Z} \setminus \{0\}, \\ cq^{-2}, & \text{otherwise}, \end{cases} \tag{3.1}$$

for some positive constant c, as it can be easily checked from the inequalities:

$$\begin{aligned} |\cos z - 1| &\geq \frac{|z|^2}{4}, \quad \text{for } |\operatorname{Re} z| \leq \frac{\pi}{4}, \\ |\cos z - 1| &\geq \frac{|\operatorname{Im} z|^2}{2}, \\ |\cos z - 1| &\geq \frac{1}{2}, \quad \text{for } |\operatorname{Re} z \bmod 2\pi| \geq \frac{\pi}{4}, \end{aligned} \tag{3.2}$$

holding for any complex z; for example, one can take $c = \min\{q^2/2, 2\pi^2\} \geq 1/2$.

Then, given a tree ϑ, we can associate to each line ℓ of ϑ a scale label n_ℓ, setting $n_\ell = 0$ if its momentum ν_ℓ is a multiple of q, and $n_\ell = 1$ otherwise. Given a tree ϑ, a *cluster* T of ϑ is a maximal connected set of lines on scale $n = 1$; we shall say that such lines are *internal* to T, and write sometimes $\ell \in T$. The lines outside the clusters are all on scale $n = 0$, and each cluster has an arbitrary number $m_T \geq 0$ of entering lines but only one exiting line. A node u will be considered internal to T, and we shall write $u \in T$, if at least the exiting line or one of its entering lines is in T.

A cluster V will be called a *resonance* if:

$$\sum_{u \in V} \nu_u = 0, \tag{3.3}$$

and, in such a case, the exiting line of the cluster V will be called a *resonant line*; we also denote with k_V the number of nodes internal to V.

Given a resonance V, with resonant line ℓ_V, we can define its *resonance factor* $\mathcal{V}_V(\vartheta)$ as:

$$\mathcal{V}_V(\vartheta) = \gamma(\nu_{\ell_V}) \prod_{u \in V} \frac{1}{m_u!} \frac{\nu_u^{m_u+1}}{\gamma(\nu_{\ell_u})} = \left[\prod_{u \in V} \frac{\nu_u^{m_u+1}}{m_u!} \right] \left[\prod_{\ell \in V} \frac{1}{\gamma(\nu_{\ell_u})} \right]; \tag{3.4}$$

of course the resonance factor will depend on ϑ only through the momenta of the incoming lines of V and on the mode labels ν_u's of the nodes inside the resonance V. The factor $\gamma(\nu_{\ell_V})$ in (3.4) simply cancels the propagator in the product corresponding to the exiting line (which is the resonant line ℓ_V).

Note that, by (3.1), we have the bound:

$$|\mathcal{V}_V(\vartheta)| \leq c^{-k_V} q^{2k_V}, \tag{3.5}$$

as all lines inside V are on scale $n = 1$.

If $N_n(\vartheta)$, $n=0,1$, denotes the number of lines in ϑ on scale n, we have trivially using again (3.1) that for a fixed tree ϑ:

$$|\mathrm{Val}(\vartheta)| \le c^{-k} q^{2N_1} |\eta|^{-2N_0}. \tag{3.6}$$

However, even if the bound (3.6) cannot be improved *for a single tree*, we shall see that by performing a partial resummation a cancellation mechanism appears, allowing us to obtain, for the coefficients (2.2) of the Lindstedt series, a better estimate, and thus leading to the proof of the theorem.

Lemma 1. *Let $N_0^*(\vartheta)$ be the number of lines on scale $n=0$ which are not resonant, in a given tree ϑ. We then have the bound:*

$$N_0^*(\vartheta) \le \left\lfloor \frac{k}{q} \right\rfloor, \tag{3.7}$$

where k is the order of the tree and $\lfloor x \rfloor$ is the highest integer smaller or equal to x.

Proof. We can confine ourselves to the cases $q \ge 2$, as for $q=1$ one has $\lfloor k/q \rfloor = k$ and the bound (3.6) gives immediately (3.7).

For $k<q$, $N_0^*(\vartheta)$ is clearly 0, and for $k=q$, $N_0^*(\vartheta) \le 1$, so in these cases the bound (3.7) is trivially satisfied. So consider the case $k>q$.

If the tree ϑ has the root line on scale $n=1$, or on scale $n=0$ and resonant, then the bound $N_0^*(\vartheta) \le k/q$ follows inductively. In fact, calling $\vartheta_1, \ldots, \vartheta_{m_{u_0}}$ the subtrees of ϑ which have as root the last node u_0 of ϑ, and $k_1, \ldots, k_{m_{u_0}}$ their orders, one has:

$$N_0^*(\vartheta) = \sum_{j=1}^{m_{u_0}} N_0^*(\vartheta_j) \le \frac{k-1}{q} < \frac{k}{q}.$$

On the other hand, if the root line is on scale $n=0$ and non resonant, then the lines entering u_0 cannot be all on scale $n=0$, otherwise $\nu_{u_0}=0$ (as $q>1$), which is not allowed. Then at least one line will have scale $n=1$: let T be the cluster containing it. The cluster T will have m_T entering lines, with $m_T \ge 0$, and – as we just assumed – it is not a resonance; then $\sum_{u\in T} \nu_u \ne 0$, so there must be at least q nodes, hence $q-1$ lines, inside T. The subtrees entering into T will have, respectively, $k_1, \ldots, k_{m_T}$ nodes, with $q + \sum_{j=1}^{m_T} k_j \le k$ so that, again inductively:

$$N_0^*(\vartheta) = 1 + \sum_{j=1}^{m_T} N_0^*(\vartheta_j) \le 1 + \frac{k-q}{q} \le \frac{k}{q}.$$

Then N_0^* will be bounded by k/q. As N_0^* has to be an integer number, the assertion follows. □

We now show how to construct a suitable partial resummation of the Lindstedt series, that is we shall define families of trees to be grouped and bounded together to obtain extra η factors.

Given a tree ϑ and a resonance V with m_V incoming lines $\ell_1, \ldots, \ell_{m_V}$ and k_V nodes, we define the family $\mathcal{F}_V(\vartheta)$ of V in ϑ as the set of trees obtained from ϑ by the action of a group of transformation $\mathcal{P}_V$ on ϑ, generated by the following operations.

(1) Detach the line ℓ_1, and reattach it to all of the nodes of the resonance; for each of the so obtained trees, detach the line ℓ_2 and reattach it to all of the nodes of the resonance; and so on for each entering line of the resonance.

(2) In a given tree, each node $u \in V$ will have m_u entering lines, of which s_u are inside V and $r_u = m_u - s_u$ are outside V (*i.e.* are entering lines in V). Then we can apply to the set of lines entering u a transformation in the group obtained as the quotient of the group of permutations of the m_u lines by the groups of permutations of the s_u internal entering lines and of permutations of the r_u entering lines outside V: in this way for each node $u \in V$, a number of trees equal to:

$$\binom{m_u}{s_u} = \frac{m_u!}{s_u! r_u!} \tag{3.8}$$

is obtained.

(3) Flip all the mode labels inside V simultaneously.

We shall call transformations of type 1, 2, 3 the operations described in the tree items above.

The following lemma is the crucial one, where cancellations between trees in the same family are exploited. We state it here and use it to prove the main estimate on the radius of convergence of the Lindstedt series, and postpone its proof to the next section.

Lemma 2. *Given a tree ϑ, with a resonance V, if $\nu_1, \ldots, \nu_{m_V}$ are the momenta flowing through the entering lines $\ell_1, \ldots, \ell_{m_V}$ of V, we have the bound:*

$$\frac{1}{\mathcal{F}_V(\vartheta)} \left| \sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \mathcal{V}_V(\vartheta') \right| \le c^{-k_V} q^{2k_V} D_0 \sum_{m,m'=1}^{m_V} \left| \nu_m \nu_{m'} \eta^2 \right| , \tag{3.9}$$

for some constant D_0.

As the typical tree may contain more than one resonance, we need the following easy corollary of the above lemma.

Corollary. *Given a tree* ϑ *with resonances* $V_1, \ldots, V_s$*, consider the family* $\mathcal{F}(\vartheta)$ *obtained by the action on* ϑ *of all the groups* $\mathcal{P}_{V_1}, \ldots, \mathcal{P}_{V_s}$*; the number of trees in this family is given by* $k_{\mathcal{F}(\vartheta)} = \prod_{i=1}^{s} |\mathcal{F}_{V_i}(\vartheta)|$*, and:*

$$\frac{1}{k_{\mathcal{F}(\vartheta)}} \left| \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \mathrm{Val}(\vartheta') \right| \le c^{-k} q^{2N_1} |\eta|^{-2N_0} D_0^s k^2 |\eta|^{2s}, \tag{3.10}$$

with the same constant D_0 *of lemma 2.*

Proof. Simply use that for each resonance V a factor $\left|\nu_m \nu_{m'} \eta^2\right|$ arises from lemma 2, while the propagators corresponding to the two lines ℓ_m and $\ell_{m'}$ can be bounded, respectively, by $c^{-1} |\nu_m \eta|^{-2}$ and $c^{-1} |\nu_{m'} \eta|^{-2}$, by (3.1). Then just note that the cancellation mechanisms operating for each resonance do not interfere with each other (*i.e.* there is no *cancellation overlap*): this follows from the fact that, for any tree $\vartheta' \in \mathcal{F}(\vartheta)$, we can write:

$$\mathrm{Val}(\vartheta') = \mathcal{B}(\vartheta) \prod_{i=1}^{s} \mathcal{V}_{V_i}(\vartheta'), \tag{3.11}$$

where the factor $\mathcal{B}(\vartheta)$ assumes the same value independently from ϑ'. Therefore we can use the trivial inequality $\left|\nu_m \nu_{m'} \eta^2\right| |\nu_{m'} \eta|^{-2} |\nu_m \eta|^{-2} \le |\eta|^2$ for any resonance V_i. Moreover the number of addends appearing in the sum in (3.9) may be bounded by $k_{V_i}^2$ for each resonance, and $\prod_i k_{V_i}^2 \le k^2$. □

Finally we have the main lemma on the convergence radius of the Lindstedt series.

Lemma 3. *Let* $\eta \neq 0$*, and* $\rho(\eta)$ *the radius of convergence of the Lindstedt series* (1.8) *as a function of* η*. Then there exists a positive constant* r *such that* $\rho(\eta) \ge r|\eta|^{2/q}$*.*

Proof. Let N_0^R be the number of resonant lines on scale $n = 0$, so that $N_0 = N_0^* + N_0^R$. For the resonances an overall gain $D_0^s k^2 |\eta|^{2s}$ is obtained, by the corollary to lemma 2. So we have:

$$\begin{aligned} \frac{1}{k_{\mathcal{F}(\vartheta)}} \left| \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \mathrm{Val}(\vartheta) \right| &\le c^{-k} q^{2N_1} |\eta|^{-2N_0^*} |\eta|^{-2N_0^R} D_0^{N_0^R} k^2 |\eta|^{2N_0^R} \\ &\le \left(c^{-1} e^2 D_0 q^2\right)^k |\eta|^{-2k/q}, \end{aligned} \tag{3.12}$$

as $k^2 \le e^{2k}$ and the number of resonances is equal to the number of resonant lines, *i.e.* $s = N_0^R$. Therefore, writing the sum over all trees as:

$$\sum_{\vartheta \in \mathcal{T}_{\nu,k}} \mathrm{Val}(\vartheta) = \sum_{\vartheta \in \mathcal{T}_{\nu,k}} \frac{1}{k_{\mathcal{F}(\vartheta)}} \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \mathrm{Val}(\vartheta'), \tag{3.13}$$

we can conclude that:

$$\left|u_\nu^{(k)}\right| \le D_1^k |\eta|^{-2k/q}, \qquad \left|\sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha} u_\nu^{(k)}\right| \le D_1^k |\eta|^{-2k/q} \tag{3.14}$$

for some constants D_1 (see comments after (2.2) on the number of trees in $\mathcal{T}_{\nu,k}$). A comparison between (2.2), (3.12) and (3.13) gives $D_1 = 4cD_0q^2$; this implies that $\rho(\eta) \ge r|\eta|^{2/q}$, for some – explicitly computable – constant r. □

4. Cancellations

In this section we prove lemma 2, by exhibiting the cancellation mechanisms between trees belonging to the same family.

Proof of lemma 2. By neglecting the trees obtained flipping all the mode labels ν_u, $u \in V$, the resonance factors (3.4) associated to the trees in $\mathcal{F}_V(\vartheta)$ can differ from each other because of the following three reasons.

(1) Some of the propagators are different, as the momentum flowing through a line ℓ internal to V depends on the momentum flowing through the entering line ℓ_m only if ℓ is on the path leading from ℓ_m to the exiting line of V.
(2) The numerator is given by a common factor (*i.e.* a factor which is the same for all trees) times $\nu_{w_1} \dots \nu_{w_{m_V}}$, if $w_{m_1}, \dots, w_{m_V}$ are the nodes inside V which the entering lines $\ell_1, \dots, \ell_{m_V}$ are attached to.
(3) Some of the combinatorial factors $m_u!^{-1}$ are different, as the number of lines entering the nodes can change, and the permutations of the exiting lines in the definition of the group $\mathcal{P}_V$ produces, for each node u, a binomial factor as in (3.8).

If we perform also the mode flipping of the labels ν_u, $u \in V$, other changes are introduced: the numerator will change by a multiplicative factor $(-1)^{m_V+1}$, and the arguments of the propagators will be modified in the obvious way. Anyway we shall not consider cancellations between the trees obtained by mode labels flipping, except for the case in which there is only one incoming line (*i.e.* $m_V = 1$), and we shall see that in such a case the resonance factor enjoys a strong parity property.

The momentum flowing through a line ℓ_u inside a resonance will depend on the modes of the nodes $w \in V$ such that $w \le u$ and on the momenta of the entering lines of the resonance *only if the latter end into nodes which precede* u; we call L_u the set of such lines.

For any ℓ_u internal to a resonance V we have:

$$\nu_{\ell_u} = \sum_{\substack{w\in V \\ w\le u}} \nu_w + \sum_{\ell'\in L_u} \nu_{\ell'}, \tag{4.1}$$

and in the corresponding propagator, we can write the argument of the cosine as:

$$\frac{2\pi p}{q} \sum_{\substack{w\in V \\ w\le u}} \nu_w + 2\pi i \left[\sum_{\substack{w\in V \\ w\le u}} \eta\nu_w + \sum_{\ell'\in L_u} \eta\nu_{\ell'} \right]. \tag{4.2}$$

We shall consider the resonance factor $\mathcal{V}_V(\vartheta')$ as a function of the quantities $\mu_1 \equiv \eta\nu_{\ell_1}, \ldots, \mu_{m_V} \equiv \eta\nu_{\ell_{m_V}}$ of the incoming lines $\ell_1, \ldots, \ell_{m_V}$, *i.e.* $\mathcal{V}_V(\vartheta') \equiv \mathcal{V}_V(\vartheta'; \eta\nu_{\ell_1}, \ldots, \eta\nu_{\ell_{m_V}})$. For any $u \in V$, we have $L_u \subseteq \{\ell_1, \ldots, \ell_{m_V}\}$.

We can write:

$$\begin{aligned} \mathcal{V}_V(\vartheta'; \eta\nu_{\ell_1}, \ldots, \eta\nu_{\ell_{m_V}}) &= \mathcal{V}_V(\vartheta'; 0, \ldots, 0) + \sum_{m=1}^{m_V} \eta\nu_{\ell_m} \frac{\partial}{\partial\mu_m} \mathcal{V}_V(\vartheta'; 0, \ldots, 0) \\ &+ \sum_{m,m'=1}^{m_V} \eta^2 \nu_{\ell_m}\nu_{\ell_{m'}} \int_0^1 \mathrm{d}t\,(1-t) \frac{\partial^2}{\partial\mu_m\partial\mu_{m'}} \mathcal{V}_V(\vartheta'; t\eta\nu_{\ell_1}, \ldots, t\eta\nu_{\ell_{m_V}}), \end{aligned} \tag{4.3}$$

where:

$$\frac{\partial}{\partial\mu_m} \mathcal{V}_V(\vartheta'; 0, \ldots, 0)$$

denotes the first derivative of $\mathcal{V}_V(\vartheta'; \mu_1, \ldots, \mu_{m_V})$ with respect to the argument μ_m, computed in $\mu_1 = \ldots = \mu_{m_V} = 0$, while the term in the second line is the integral interpolation formula for the second order remainder.

Of course we use here the fact that the resonance factor is a function of class C^2 – actually analytic – in the parameters $\mu_1, \ldots, \mu_{m_V}$. As it can be seen from (4.2), the resonance value depends on $\mu_1, \ldots, \mu_{m_V}$ only through the last sum appearing in the argument of the propagators.

Note that, as the incoming lines are on scale $n = 0$, ν_{ℓ_m} is a multiple of q for each $m = 1, \ldots, m_V$. Note also that the tree values $\mathrm{Val}(\vartheta')$ differ only as far as the resonance factors are concerned, because all the other factors contributing to the tree value are equal, *i.e.* $\mathrm{Val}(\vartheta') = \mathcal{A}(\vartheta)\mathcal{V}_V(\vartheta')$, where $\mathcal{A}(\vartheta)$ is a factor which has the same value for all $\vartheta' \in \mathcal{F}_V(\vartheta)$.

The first term in (4.3) is the term which arises from the resonance factor by neglecting the change in the momenta.

Define $\tilde{\mathcal{F}}_V(\vartheta)$ as the equivalence class of the trees in $\mathcal{F}_V(\vartheta)$, such that two trees in $\mathcal{F}_V(\vartheta)$ are considered equivalent if each one can be obtained from the other by the action of a transformation of the group $\mathcal{P}_V$ of type 2.

Then, summing the values of the trees belonging to $\mathcal{F}_V(\vartheta)$, we can group them into subfamilies of inequivalent trees whose contributions are different as for each node there is a factor:

$$\frac{1}{m_u!}\binom{m_u}{s_u} = \frac{1}{s_u! r_u!}, \tag{4.4}$$

as all terms which are obtained by permutations are summed together (this gives the binomial coefficient in the left hand side of the above equation), times a factor:

$$\nu_u^{m_u+1} = \nu_u^{(s_u+1)+r_u}, \tag{4.5}$$

times a propagator $1/\gamma(\nu_{\ell_u})$.

Then for $\mu_1 = \ldots = \mu_{m_V} = 0$ we can write:

$$\begin{aligned}\sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \mathrm{Val}(\vartheta') &= A(\vartheta) \sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \mathcal{V}_V(\vartheta') \\ &= A(\vartheta) \sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \left[\left(\prod_{u\in V} \frac{\nu_u^{s_u+1}}{s_u!}\right)\left(\prod_{\ell \in V}\frac{1}{\gamma(\nu_\ell)}\right)\right]\left(\prod_{u\in V}\frac{\nu_u^{r_u}}{r_u!}\right) \\ &= A(\vartheta)\left[\left(\prod_{u\in V} \frac{\nu_u^{s_u+1}}{s_u!}\right)\left(\prod_{\ell \in V}\frac{1}{\gamma(\nu_\ell)}\right)\right] \sum_{\vartheta' \in \tilde{\mathcal{F}}_V(\vartheta)} \left(\prod_{u\in V}\frac{\nu_u^{r_u}}{r_u!}\right),\end{aligned} \tag{4.6}$$

where we have used that for $\mu = 0$ the factors in square brackets have the same value for all $\vartheta' \in \tilde{\mathcal{F}}_V(\vartheta)$. The last sum in (4.6) can be rewritten as:

$$\sum_{\vartheta' \in \tilde{\mathcal{F}}_V(\vartheta)} \left[\prod_{u\in V}\frac{\nu_u^{r_u}}{r_u!}\right] = \sum_{\substack{\{r_u \geq 0\} \\ \sum_{u\in V} r_u = m_V}} \prod_{u\in V}\frac{\nu_u^{r_u}}{r_u!} = \frac{1}{m_V!}\left(\sum_{u\in V}\nu_u\right)^{m_V}, \tag{4.7}$$

so that a quantity proportional to the m_V-th power of $\sum_{u\in V}\nu_u$ is obtained. But such a sum is zero by definition of resonance – see (3.3).

Also the second term in (4.3) vanishes, after summing over the trees $\vartheta' \in \mathcal{F}_V(\vartheta)$. To prove this we shall consider separately the cases $m_V \geq 2$ and $m_V = 1$.

In the first case, when the derivative $(\partial/\partial\mu_m)\,\mathcal{V}_V(\vartheta; 0, \ldots, 0)$ is considered, let us compare all the trees ϑ' in the subfamily of $\tilde{\mathcal{F}}_V(\vartheta)$ in which the line ℓ_m is kept fixed (call $\bar{u}$ the node which such a line enters), while all other lines are shifted (*i.e.* detached and reattached to all nodes inside the resonance). The difference with respect to the previous case, discussed above, is that the line with momentum ν_{ℓ_m} can be chosen in $r_{\bar{u}}$ ways among the $r_{\bar{u}}$ lines entering the node $\bar{u} \in V$ and outside V. This means that we can write:

$$\frac{\nu_u^{m_u+1}}{m_u!}\binom{m_u}{s_u} = \frac{\nu_u^{(s_u+1)+r_u}}{s_u! r_u!} \tag{4.8}$$

for all nodes $u \neq \bar{u}$, and:

$$\frac{\nu_{\bar{u}}^{m_{\bar{u}}+1}}{m_{\bar{u}}!}\binom{m_{\bar{u}}}{s_{\bar{u}}} r_{\bar{u}} = \frac{\nu_{\bar{u}}^{(s_{\bar{u}}+1)+(r_{\bar{u}}-1)}}{s_{\bar{u}}!(r_{\bar{u}}-1)!}\nu_{\bar{u}}, \tag{4.9}$$

for $\bar{u}$. Then we have an expression analogous to (4.6), with the only difference that the labels $\{r_u\}$ have to be replaced with labels $\{r'_u\}$, defined as $r'_u = r_u - \delta_{u\bar{u}}$ $\forall u \in V$, such that $\sum_{u\in V} r'_u = m_V - 1$: so the last sum in the third line of (4.6) has to be replaced by:

$$\begin{aligned} \nu_{\bar{u}} \sum_{\vartheta' \in \tilde{\mathcal{F}}_V(\vartheta)} \prod_{u\in V} \frac{\nu_u^{r'_u}}{r'_u!} &= \nu_{\bar{u}} \sum_{\substack{\{r'_u\}\geq 0 \\ \sum_{u\in V} r'_u = m_V - 1}} \prod_{u\in V} \frac{\nu_u^{r'_u}}{r'_u!} \\ &= \frac{\nu_{\bar{u}}}{(m_V-1)!}\left(\sum_{u\in V}\nu_u\right)^{m_V-1}, \end{aligned} \tag{4.10}$$

so that we have again vanishing contributions.

On the contrary, if $m_V = 1$, the above reasoning does not apply (as there is only one incoming line). Anyway the function $(\partial/\partial\mu_1)\,\mathcal{V}_V(\vartheta; 0)$ is an odd function, as all the propagators are even in their arguments (so that the derived one becomes odd) and the numerator contains an even number of ν_u's. Then by reversing the signs of the labels ν_u, $u \in V$, the numerator will not change (as $m_V + 1 = 2$), while the overall sign of the denominator changes, so that the sum over the two considered tree values gives zero.

The integral appearing in the third term in (4.3) gives, for each $\vartheta' \in \mathcal{F}_V(\vartheta)$, a contribution bounded by $d_0 q^2$ times the original bound on $|\mathcal{V}_V(\vartheta)|$. In fact the propagators of all lines inside V are bounded by $c^{-1}q^2$, and their first and second derivatives, respectively, by $d_1 q^3$ and $d_2 q^4$ for some constants d_1 and d_2 (such lines remain on scale $n = 1$: the incoming lines contribute a quantity that modifies only the imaginary part of the momenta of lines inside V).

Then the lemma follows, with $D_0 = \max\{c d_1^2, d_2\} q^2$. □

5. Computation of the critical exponent

Let us consider the coefficient $u_\nu(\varepsilon)$ in (1.5). By definition of momentum, in (2.2) only trees of order $k \geq |\nu|$ can contribute to $u_\nu(\varepsilon)$, so that $u_\nu^{(k)} = 0$ for

$k < \nu$. Therefore we can write:

$$u_\nu(\varepsilon) = \sum_{k \geq |\nu|} \varepsilon^k u_\nu^{(k)} = \varepsilon^{|\nu|} u_\nu^{(|\nu|)} + \sum_{k=|\nu|+1}^{\infty} \varepsilon^k u_\nu^{(k)}, \tag{5.1}$$

and use the first bound in (3.14) for $u_\nu^{(k)}$, $k > |\nu|$, in order to bound the last sum in (5.1) by:

$$\left| \sum_{k=|\nu|+1}^{\infty} \varepsilon^k u_\nu^{(k)} \right| \leq 2 \left(D_1 |\eta|^{-2/q} \varepsilon \right)^{|\nu|+1}, \tag{5.2}$$

provided that $|\varepsilon| < D_1^{-1} |\eta|^{2/q}/2$. The coefficient $u_\nu^{(|\nu|)}$ in (5.1) can be expressed in terms of trees having all the modes $\nu_u = \sigma$, where $\sigma = \operatorname{sign} \nu$. From the definition of tree value in (2.2) it is easy to see that such trees have the same product $\prod_{u \in \vartheta} \nu_u^{m_u+1}$ appearing in (2.2), which is given by $\sigma^{2|\nu|+1} = \sigma$. Furthermore among such trees there will also be trees having $\lfloor |\nu|/q \rfloor$ propagators with momentum q, $2q$, $\dots$, $\lfloor |\nu|/q \rfloor q$: for instance the linear tree (*i.e.* the tree whose nodes have all only one entering line). Therefore there will be trees whose value will be bounded *from below* by $O\left(|\eta|^{-2\lfloor |\nu|/q \rfloor}\right)$. When the limit $\eta \to 0$ is taken, as the quantities $\gamma(\nu)$ in (1.6) for $\eta = 0$ are always non positive, then each of such trees ϑ will have a value which is formally:

$$\begin{aligned} \operatorname{Val}(\vartheta) &= -i\sigma B(\vartheta) \eta^{-2\lfloor |\nu|/q \rfloor}, \\ B(\vartheta) &= (-1)^k \left[\prod_{u \in \vartheta} \frac{1}{m_u!} \right] \left[\prod_{\substack{\ell \in \vartheta \\ n_\ell = 0}} \frac{1}{(2\pi \nu_\ell)^2} \right] \left[\prod_{\substack{\ell \in \vartheta \\ n_\ell = 1}} \frac{1}{|\gamma(\nu_\ell)|}, \right] \end{aligned} \tag{5.3}$$

as the propagators (1.7) diverge as $(2\pi i \nu \eta)^{-2}$ for $\eta \to 0$.

This means that no cancellation is possible between such trees, so that:

$$u_\nu^{(|\nu|)} = A_\nu |\eta|^{-2\lfloor |\nu|/q \rfloor}, \quad |A_\nu| > 0. \tag{5.4}$$

If we want that, in the limit $\eta \to 0$, the coefficient $u_\nu(\varepsilon)$ non only does not diverge but also does not vanish, we have to impose that:

$$0 < \lim_{\eta \to 0} \varepsilon^{|\nu|} u_\nu^{(|\nu|)} < \infty, \tag{5.5}$$

so that (5.4) implies that ε has to be taken of order $O(|\eta|^{2/q})$. In such a way all the limits (5.5) exist, for any ν, and they are vanishing except for $|\nu|$ multiple of q.

Moreover, when we compute $u_\nu(\varepsilon)$, with ν multiple of q, only the coefficients $u_\nu^{(k)}$ with k multiple of $|\nu|$ will contribute to the limit $\eta \to 0$, as all other contributions arise from trees containing resonances, and the analysis of the sect. 3 shows that the propagators corresponding to the resonant lines do not introduce new denominators small in ε – as a consequence of the cancellation mechanism discussed in sect. 4 – while each new node contributes a factor $|\eta|^{2/q}$, by (5.4) and (5.5).

6. Asymptotics

Lemma 3 implies that the function $v(\alpha,\varepsilon) = u(\alpha,(2\pi\eta)^{2/q}\varepsilon)$ admits a Taylor series in ε convergent for $|\varepsilon| < \varepsilon_0 = O(1)$ (in η). The k-th order Fourier coefficients of v are defined by:

$$v_\nu^{(k)} = u_\nu^{(k)}(2\pi\eta)^{2k/q}, \tag{6.1}$$

so that, if we introduce the function:

$$\bar u(\alpha,\varepsilon) = \lim_{\eta\to 0} v(\alpha,\varepsilon), \tag{6.2}$$

we have that:

$$\bar u(\alpha,\varepsilon) = \sum_{k=1}^{\infty}\sum_{\nu\in\mathbb{Z}} \varepsilon^k e^{i\nu\alpha}\bar u_\nu^{(k)}, \tag{6.3a}$$

$$\bar u_\nu^{(k)} = \lim_{\eta\to 0} u_\nu^{(k)}(2\pi\eta)^{2k/q}, \tag{6.3b}$$

and the series in (6.3a) converges absolutely by (3.14): this means that the coefficients $\bar u_\nu^{(k)}$ are well defined, and, from the analysis of the last section, we know that only the Fourier coefficients with modes multiples of q survive when the limit $\eta\to 0$ is taken; furthermore, the function $\bar u(\alpha,\varepsilon)$ is analytic in ε for ε small enough and η independent, and periodic in α with period $2\pi/q$.

Summarizing, we have:

$$\bar u_\nu^{(k)} = \begin{cases} -\frac{(2\pi)^{k/q}}{2^k}\sum'_{\vartheta\in\mathcal{T}_{\nu,k}} \mathrm{Val}'(\vartheta), & \text{if } \nu\in q\mathbb{Z}\setminus\{0\},\ k\in q\mathbb{Z}\setminus\{0\}, \\ 0, & \text{otherwise}, \end{cases} \tag{6.4}$$

where $\sum'$ means that only trees without resonances have to be summed over and $\mathrm{Val}'(\vartheta)$ differs from $\mathrm{Val}(\vartheta)$ as it is defined in (2.2) inasmuch as, for ν multiple of q, the denominator $\gamma(\nu)$ has to be replaced with $(2\pi i\nu)^2$.

Let us now consider all trees of order q contributing to $\nu = \sigma q$, with $\sigma = \operatorname{sign}\nu$: the values of such trees can be read from (2.2), and one sees that the numerator is identically $-i\sigma$, so that:

$$\sum_{\vartheta\in\mathcal{T}_{\sigma q,q}} \mathrm{Val}'(\vartheta) = (2\pi)^2 \sum_{\vartheta\in\mathcal{T}_{\sigma q,q}} (-i\sigma)\prod_{u\in\vartheta}\frac{1}{m_u!}\frac{1}{\gamma(\nu_{\ell_u})} = \frac{1}{(i\nu)^2}(-i\sigma)\mathcal{S}_{p/q}, \tag{6.5}$$

so defining the expression $\mathcal{S}_{p/q}$ (which does not depends in σ): the factor $(i\nu)^{-2}$ arises from $(2\pi)^2$ times the propagator $\gamma(\nu)$, which appears in any tree in $\mathcal{T}_{\sigma q,q}$ so it can be put in evidence. For instance, for $p/q = 0/1$, $p/q = 1/2$ and $p/q = 1/3$, by explicit computation, we find:

$$\begin{aligned}
&\mathcal{S}_{0/1} = 1,\\
&\mathcal{S}_{1/2} = \frac{1}{2(\cos\pi - 1)} = -\frac{1}{4},\\
&\mathcal{S}_{1/3} = \frac{1}{2(\cos(4\pi/3)-1)}\frac{1}{2(\cos(2\pi/3)-1)} + \frac{1}{2}\frac{1}{[2(\cos(4\pi/3)-1)]^2} = \frac{1}{6},
\end{aligned} \tag{6.6}$$

as it can be easily computed from the definition (6.5).

We have the following lemma.

Lemma 4. *The coefficients* (6.3b) *satisfy the recursion relation:*

$$\bar{u}_\nu^{(k)} = \frac{1}{2^{q-1}}\frac{1}{(i\nu)^2}\sum_{m=1}^{\infty}\frac{1}{m!}\frac{\mathcal{S}_{p/q}}{2}(-i\sigma)(i\sigma q)^m \sum_{\substack{\sigma q+\nu_1+\ldots+\nu_m \\ k_1+\ldots+k_m=k-q}} \prod_{i=1}^{m} \bar{u}_{\nu_i}^{(k_i)}, \tag{6.7}$$

where the constant $\mathcal{S}_{p/q}$ *is defined in* (6.5).

Proof. A generic tree of order $k = \kappa q$, $\kappa \geq 1$, and momentum $\nu = nq$, $n \geq 1$, can be obtained starting from a tree ϑ_0 of order q by attaching to its nodes $m \geq 0$ trees $\vartheta_1, \ldots, \vartheta_m$ of orders $k_1 = \kappa_1 q, \ldots, k_m = \kappa_m q$, with $\kappa_1 + \ldots + \kappa_m = \kappa - 1$ and total momenta $\nu_1 = n_1 q,\ldots,\nu_m = n_m q$ with $n_1+\ldots+n_m = n-1$. Each tree can be attached to any node of ϑ_0 so that the combinatorial factor associated to any node $u \in \vartheta_0$will be $m_u!^{-1}$, with $m_u = s_u + r_u$, if s_u is the number of lines connecting u to other nodes of ϑ_0 and r_u is the number of trees attached to u. Then by construction $\sum_{u\in\vartheta_0} r_u = m$. Note that two trees in which one of the first s_u subtrees (with root line belonging to ϑ_0) is permuted with one of the remaining r_u can not be identical, as they have a different number of nodes: only the latter will have a number of nodes which is multiple of q. If we sum together all trees which can be obtained from each other by choosing in a different way the s_u subtrees with root line belonging to ϑ_0 and the remaining r_u subtrees, we have $\prod_{u\in\vartheta_0} m_u!s_u!^{-1}r_u!^{-1}$ terms, so that, by taking into account that (see (6.5)):

$$\mathrm{Val}'(\vartheta_0) = (2\pi)^2(-i\sigma)\prod_{u\in\vartheta_0}\frac{1}{s_u!}\frac{1}{\gamma(\nu_{\ell_u})}, \tag{6.8}$$

and that, by shifting the subtrees attached to the nodes of ϑ_0, the momentum flowing through any line of ϑ_0 can vary by an amount proportional to a multiple of q, so that the corresponding propagator does not change, we can write:

$$\mathrm{Val}'(\vartheta) = \sum_{\vartheta_0 \in \mathcal{T}_{\sigma q,q}} \mathrm{Val}(\vartheta_0) \sum_{m=0}^{\infty} \sum_{\substack{\{r_u \geq 0\} \\ \sum_{u\in\vartheta_0} r_u = m}} \prod_{u\in\vartheta_0} \frac{1}{r_u!} (i\sigma)^m \sum_{\vartheta_1,\ldots,\vartheta_m}{}' \prod_{i=1}^{m} \mathrm{Val}'(\vartheta_i), \tag{6.9}$$

where $'$ recalls the constraint on the trees $\vartheta_1, \ldots, \vartheta_p$ described above. Then just note that:

$$\prod_{\substack{\{r_u \geq 0\} \\ \sum_{u\in\vartheta_0} r_u = m}} \frac{1}{r_u!} = \frac{q^m}{m!}, \tag{6.10}$$

to deduce from (6.9) that:

$$\mathrm{Val}'(\vartheta) = \sum_{\vartheta_0 \in \mathcal{T}_{\sigma q,q}} \mathrm{Val}'(\vartheta_0) \sum_{m=0}^{\infty} \frac{1}{m!} (i\sigma q)^m \sum_{\vartheta_1,\ldots,\vartheta_m}{}' \prod_{i=1}^{m} (i\sigma q)\, \mathrm{Val}'(\vartheta_i). \tag{6.11}$$

Then from (6.4) and (6.11) we read that:

$$\bar{u}_{\nu}^{(k)} = \frac{1}{2^q} \sum_{\vartheta_0 \in \mathcal{T}_{q,q}} \mathrm{Val}'(\vartheta_0) \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\substack{k_1+\ldots+k_m = k-q \\ \sigma+\nu_1+\ldots+\nu_m=\nu}} \prod_{i=1}^{m} (i\sigma q) \bar{u}_{\nu_i}^{(k_i)}. \tag{6.12}$$

and using (6.5) we obtain (6.7). □

Lemma 5. *The function $\bar{u}(\alpha) \equiv \bar{u}(\alpha,\varepsilon)$ in (6.2) satisfies the differential equation:*

$$\frac{d^2\bar{u}(\alpha)}{d\alpha^2} = C_{p/q}\varepsilon^q \sin(q(\alpha+\bar{u}(\alpha))), \tag{6.13}$$

with boundary conditions $\bar{u}(0) = \bar{u}(2\pi) = 0$, and:

$$C_{p/q} = 2^{-(q-1)} \mathcal{S}_{p/q} = 2^{-(q-1)} \sum_{\vartheta\in\mathcal{T}_{q,q}} (2\pi i\nu)^2 \prod_{u\in\vartheta} \frac{1}{m_u!} \frac{1}{\gamma(\nu_{\ell_u})}, \tag{6.14}$$

where the factor $(2\pi i \nu)^2$ simply cancels the propagator of the root line of ϑ.

Proof. Simply write (6.13) in Fourier space, and write the recursion relations defining the coefficients, by taking into account that only orders and momenta multiples of q can occur, as it can be immediately verified by induction. Then we obtain:

$$\bar{u}_{\nu}^{(k)} = \frac{1}{(i\nu)^2} \sum_{m=0}^{\infty} \frac{1}{m!} \frac{C_{p/q}}{2} (-i\sigma)(i\sigma q)^m \sum_{\substack{\sigma q+\nu_1+\ldots+\nu_m=\nu \\ k_1+\ldots+k_m=k-q}} \prod_{i=1}^{m} \bar{u}_{\nu_i}^{(k_i)}, \tag{6.15}$$

with $\sigma = \pm 1$. Therefore we have the same expression as (6.7) provided the constant $\mathcal{C}_{p/q}$ is chosen as in (6.14). □

We note that lemma 5 gives the values $C_{0/1} = 1$, $C_{1/2} = -1/8$ and $C_{1/3} = 1/24$, which are consistent with the results of [6]. We therefore prove the conjecture proposed in [6], and give an explicit formula for the constant $C_{p/q}$ appearing in the differential equation.

7. Conclusions

The above lemmata prove all the claims in the main theorem in sect. 1.

Proof of the theorem. As for $\eta \neq 0$ there are no small divisors and the convergence of the Lindstedt series can be proved by elementary means, the only non obvious part of statement (1) is the behaviour of the radius of convergence as $\eta \to 0$. This result is a consequence of lemma 3 and statement (2): in fact the solution of (1.10), with the given boundary conditions, has a finite positive radius of convergence, as a simple calculation based on the theory of ϑ functions and on Rouché's theorem shows (see [6]).

Statement (2) is proved by lemma 3, by the combinatorics in sect. 5 and by lemma 5. $\square$

We also note that the considerations of [6] about the analyticity of the limit function $\bar{u}$ in both α and ε apply, thus giving a rigorous explanation to the numerical findings of [11], [6].

We observe that the restriction on the way we take the limit $\omega \to p/q$ in the complex plane is taken only for the sake of simplicity: in fact, it is easy to modify the proofs in such a way that any path in the complex plane, *provided it is not tangent to the real axis*, can be taken.

More precisely, let:

$$\omega = \frac{p}{q} + \zeta + i\eta, \tag{7.1}$$

with $p, q \in \mathbb{Z}$, $p \wedge q = 1$, and $\zeta, \eta \in \mathbb{R}$, with:

$$|\eta| \geq a|\zeta|, \qquad a > 0, \tag{7.2}$$

in the limit $\eta \to 0$. Condition (7.2) defines a cone in the complex ω plane, with its vertex in p/q and its slope equal to a; any path inside this cone tends to p/q non-tangentially.

First we show that inequalities like (3.1) can be derived under the condition (7.2). The first inequality holds in fact trivially also in this more general case, as, for ν multiple of q, we can write:

$$\cos(2\pi[p/q + \zeta + i\eta]\nu) = \cos(2\pi[\zeta + i\eta]\nu), \tag{7.3}$$

so that:

$$2|\cos(2\pi[p/q+\zeta+i\eta]\nu)-1| \geq 4\pi^2|\nu\eta|^2, \tag{7.4}$$

by the second inequality in (3.2).

If $\nu \neq 0 \pmod q$, we can use the third inequality in (3.2) to deduce that, by denoting $x=[p/q+\zeta]\nu$, $|\gamma(\nu)| \geq 1/2$ for $2\pi|x| \geq \pi/4$.

If $2\pi|x| \leq \pi/4$ we can use the first inequality in (3.2) and write:

$$|\gamma(\nu)| \geq 2\pi^2(|x|^2+|\eta\nu|^2). \tag{7.5}$$

Then, if $|x| \leq (2q)^{-1}$, one has $|\zeta\nu| \geq (2q)^{-1}$ as $|[p/q]\nu| \geq 1$, so that $|\nu| \geq (2q|\zeta|)^{-1}$, *i.e.* $|\eta\nu| \geq (2q)^{-1}|\eta/\zeta|$: this means that in such a case $|\gamma(\nu)| \geq 4\pi^2(2q)^{-1}a$. If $|x| \geq (2q)^{-1}$, then $|\gamma(\nu)| \geq \pi^2 q^{-2}/2$.

Then we can write again the same inequalities as in (3.1), with the only difference that now $c=\min\{a\pi^2/2, q^2/2\} \geq a/2$.

As our analysis is based on the inequalities (3.1) and on the definition of resonance (3.3), it can be repeated essentially unchanged in the case (7.1), and the same results hold. In fact, the proof of lemma 2 can be carried out in a similar manner, by expressing the resonance value as a function of the quantities $\xi\nu_{\ell_1}, \ldots, \xi\nu_{\ell_{m_V}}$, with $\xi=\zeta+i\eta$; then by taking into account that for any ν such that $\nu=0 \pmod q$:

$$|\gamma(\nu)| \geq 4\pi|\nu\eta|^2, \tag{7.6}$$

and $|\nu\zeta| \leq \sqrt{\zeta^2+\eta^2} \leq |\eta|(1+a^{-1})$, one sees that the cancellation mechanisms operate exactly in the same way as before, and the second order terms can be dealt with as in sect. 4, with the only difference that now $D_0 = \max\{cd_1^2, d_2\}q^2(1+a^{-1})$.

Once the perturbation parameter ε has been scaled to $(2\pi\xi)^{2/q}\varepsilon$, the surviving terms are exactly the same as before, so that all of the above discussions apply *verbatim*; in fact:

$$\lim_{\eta\to 0}\gamma(\nu)[2\pi(\zeta+i\eta)]^{-2}=1 \tag{7.7}$$

for ν multiple of q and ζ, η satisfying (7.2).

Our main results therefore still apply provided the path taken by ω while tending to p/q is not tangential to the real axis, so that (7.2) applies for some a. When ω approaches p/q tangentially, the numerical evidence of [6] suggests that the scaling properties of the radius of convergence are the same; however an analytical proof is still lacking; we believe that in such a case the full strenght of the multiscale analysis of [4], [3] is needed, as in the case where the limit is taken along a sequence of real, Diophantine numbers tending to p/q.

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