SCALING PROPERTIES FOR THE RADIUS OF CONVERGENCE OF LINDSTEDT SERIES: GENERALIZED STANDARD MAPS

ALBERTO BERRETTI AND GUIDO GENTILE

Abstract. For a class of symplectic two-dimensional maps which generalize the standard map by allowing more general nonlinear terms, the radius of convergence of the Lindstedt series describing the homotopically non-trivial invariant curves is proved to satisfy a scaling law as the complexified rotation number tends to a rational value non-tangentially to the real axis, thus generalizing previous results of the authors. The function conjugating the dynamics to rotations by ω possesses a limit which is explicitly computed and related to hyperelliptic functions in the case of nonlinear terms which are trigonometric polynomials. The case of the standard map is shown to be non-generic.

Résumé. Pour une classe d'applications symplectiques bidimensionneles qui généralisent l'application standard avec des termes non linéaires plus generaux, on prouve que le rayon de convergence de la série de Lindstedt qui décrit les courbes invariantes homotopiqueement non triviales satisfait une loi d'échelle lorsque le nombre de rotation complexifié tend à une valeur rationelle non tangentiellement à l'axe réel, generalisant ainsi des résultats précédents des mêmes auteurs. La fonction qui conjugue la dynamique à des rotations de ω admet une limite qui est calculée explicitement et reliée à les fonctions hyperelliptiques dans le cas de termes non linéaires qui sont des polinômes trigonometriques. On montre que le cas de l'application standard est non generique.

Keywords: Generalized standard maps, genericity, Lindstedt series, complexified rotation number, hyperelliptic functions, Bryuno function.

AMS Classification: 37J10 37J40 33E05.

1. Introduction

In this paper we generalize the results of [1] considering maps of the kind

$$T_{\varepsilon,f} : \begin{cases} x' = x + y + \varepsilon f(x), \\ y' = y + \varepsilon f(x), \end{cases} \tag{1.1}$$

where $f(x)$ is a 2π-periodic function of x, analytic in a strip $\mathcal{S} = \{|\operatorname{Im}(x)| < \xi\}$ of width 2ξ around the real x axis. The nonlinear term $f(x)$ can be expanded in a

Fourier series:

$$f(x) = \sum_{\nu \in \mathbb{Z}} f_\nu e^{i\nu x}; \tag{1.2}$$

the Fourier coefficients f_ν then decay exponentially:

$$\forall \xi' < \xi \; \exists C_1 : \quad |f_\nu| < C_1 e^{-\xi' |\nu|}. \tag{1.3}$$

If we denote by $\mathcal{B}_\rho$ the Banach space $\mathcal{B}_\rho = \{g | \, \|g\|_\rho < \infty\}$, where

$$\|g\|_\rho = \sum_{\nu \in \mathbb{Z}} |g_\nu| \, e^{\rho |\nu|}, \tag{1.4}$$

then $f \in \mathcal{B}_{\xi'} \; \forall \xi' < \xi$.

As we assume f real, the Fourier coefficients f_ν satisfy $f_\nu^* = f_{-\nu}$. We also assume that f *has zero average*, *i.e.* that $f_0 = 0$. The case where $f(x)$ is a trigonometric polynomial has been especially studied numerically and in this case we call N_f the largest absolute value of the index ν for which the coefficient f_ν is nonzero (that is, the maximal frequency appearing in the Fourier expansion (1.2)).

Maps which fall into the class we are considering (actually with $f(x)$ an *odd* trigonometric polynomial) have been considered in [3] and [4], where the analytic structure of their KAM invariant curves was investigated numerically. In [5] the map with $f(x) = \sin x + (1/20) \sin x$ was studied and the same problem solved in this paper was studied numerically.

The variable y can be easily eliminated from (1.1) to obtain the second order recurrence

$$x_{n+1} - 2x_n + x_{n-1} = \varepsilon f(x_n). \tag{1.5}$$

As it is well known from KAM theory, the homotopically non-trivial invariant curves $\mathcal{C}_{\varepsilon,f}(\omega)$ with rotation number ω of the map $T_{\varepsilon,f}$ may be determined by finding a coordinate transformation

$$x = \alpha + u(\alpha, \varepsilon, \omega), \quad 1 + \frac{\partial u}{\partial \alpha} > 0, \tag{1.6}$$

such that in the coordinate α the dynamics is a rotation by ω:

$$\alpha' = \alpha + 2\pi\omega. \tag{1.7}$$

It is easy to see that the *conjugating function* u satisfies

$$D_\omega u(\alpha, \varepsilon, \omega) = \varepsilon f(\alpha + u(\alpha, \varepsilon, \omega)), \tag{1.8}$$

where the operator D_ω, acting on 2π-periodic functions of α, is given by

$$D_\omega \phi(\alpha) = \phi(\alpha + 2\pi\omega) - 2\phi(\alpha) + \phi(\alpha - 2\pi\omega). \tag{1.9}$$

The fact that the perturbation $f(x)$ is given by a general Fourier series and is not necessarily odd in x raises the issue of the existence of the formal solution of (1.8). While this problem, in a different context, was basically solved by Poincaré, we believe that a simple proof of this fact using the same combinatorial tools we use for the rest of the paper (*i.e.* trees) is interesting, especially as it provides additional clues to the renormalization method used to deal with resonances. The proof of the existence of the formal solution of (1.8) will be achieved in sect. 3.

The *Lindstedt series* for the problem is the formal expansion of u as a Taylor series in ε and a Fourier series in α:

$$u(\alpha,\varepsilon,\omega)=\sum_{k\geq 1}\varepsilon^k u^{(k)}(\alpha,\omega)=\sum_{k\geq 1}\sum_{\nu\in\mathbb{Z}}\varepsilon^k e^{i\nu\alpha}\,u_\nu^{(k)}(\omega). \tag{1.10}$$

More generally, given any function F of ε and α, 2π-periodic in α, we write $[F(\varepsilon,\alpha)]_\nu^{(k)}$ the k-th Taylor coefficient of the ν-th Fourier coefficient of its (formal) Lindstedt-type expansion:

$$F(\varepsilon,\alpha)=\sum_{k=0}^{\infty}\sum_{\nu\in\mathbb{Z}}\varepsilon^k e^{i\nu\alpha}[F(\varepsilon,\alpha)]_\nu^{(k)}.$$

The *radius of convergence* of the Lindstedt series is defined as

$$\rho_f(\omega)\equiv\rho(\omega)=\inf_{\alpha\in\mathbb{T}}\left(\limsup_{k\to\infty}\left|u^{(k)}(\alpha,\varepsilon,\omega)\right|^{1/k}\right)^{-1}. \tag{1.11}$$

See *e.g.* [1], [5], [6] for further details and references on these rather standard matters.

As in [1], to which we refer for more introductory details and motivations, we are interested in the behaviour of the radius of convergence $\rho(\omega)$ of the Lindstedt series (1.10) as the complexified rotation number ω tends to a *resonant* value p/q, with $\gcd(p,q)=1$.

In the class of analytic perturbations $f\in\mathcal{B}_\rho$ the property that $f_\nu\neq 0\ \forall\nu\in\mathbb{Z}$ is generic (*i.e.* it holds on a set of second category, [7], [8]). We first state our results for generic perturbations, then we shall consider more general situations which require a deeper discussion, also only in stating the results, as the latter will be shown to depend on the arithmetic properties of the Fourier labels with respect to the value p/q.

Theorem 1. *Let $f\in\mathcal{B}_\rho$, with $\rho>0$, such that $f_\nu\neq 0\ \forall\nu\in\mathbb{Z}$. Consider the cone $\mathcal{C}_{p/q,\beta}=\{z\in\mathbb{C}:\ |\operatorname{Im}z|>0,\,|\operatorname{Re}z-p/q|\leq\beta|\operatorname{Im}z|,\ \beta\geq 0\}$; let $\omega\in\mathcal{C}_{p/q,\beta}$. Then the* rescaled conjugating function

$$\bar u_{p/q}(\alpha,\varepsilon,\omega)=u\left(\alpha,\varepsilon\left(\omega-\frac{p}{q}\right)^2,\omega\right) \tag{1.12}$$

extends to a function continuous in ω in the closure of the cone $\mathcal{C}_{p/q,\beta}$ and analytic in ω in the interior of $\mathcal{C}_{p/q,\beta}$, for any $\beta\geq 0$, analytic in ε for $|\varepsilon|<a$ and analytic in α for $|\operatorname{Im}\alpha|<b$, with a, b two positive constants. In particular, the following

limit exists:

$$\bar{\bar{u}}_{p/q}(\alpha,\varepsilon) = \lim_{\omega\to p/q} \bar{u}(\alpha,\varepsilon,\omega), \tag{1.13}$$

and it is independent from the non-tangential path chosen either in the complex upper half plane $\operatorname{Im}\omega > 0$ *or in the complex lower half plane* $\operatorname{Im}\omega < 0$.

To state our results for more general perturbations, we need first some arithmetic definitions. Let $\omega \to p/q$, where $\gcd(p,q) = 1$, in the complex plane. We then consider the q sequences $\tilde{I}_c(f) = \{f_{lq+c}\}_{l\in\mathbb{Z}_+}$, $c = 1,\dots,q$ (recall that f_0 is assumed to vanish). For each sequence $\tilde{I}_c(f)$, $c = 1,\dots,q$, let

$$I_c(f) = \{f_\nu \in \tilde{I}_c(f) | f_\nu \neq 0\}$$

be the set of nonzero values of the sequence $\tilde{I}_c(f)$.

We define the following sets of integers:

$$A_{p/q}(f) = \{c \in \{1,\dots,q\} | I_c(f) \neq \emptyset\}. \tag{1.14}$$

and

$$B_c(f) = \{l \in \mathbb{Z}_+ | f_{lq+c} \neq 0\}. \tag{1.15}$$

Of course $A_{p/q}(f) = \{c_1,\dots,c_M\}$, where $1 \le c_1 < \cdots < c_M \le q$, and $M \le q$. Note that $A_{p/q}(f)$ is the set of equivalence classes modulo q of frequencies actually appearing in the Fourier expansion of the perturbation,

Lemma 1. *If* $q \notin A_{p/q}(f)$ *and* $|I_c(f)| = 1$ $\forall c \in A_{p/q}(f)$, *define*

$$\mathcal{A}(f) = \{\nu \in \mathbb{N} | f_\nu \neq 0\}; \tag{1.16}$$

then $\mathcal{A}(f) = \{\nu_1,\dots,\nu_M\}$, *with* $1 \le M \le q-1$ *and* $\nu_i - \nu_j \notin q\mathbb{Z}$ $\forall i \neq j$.

Proof. If $q \notin A_{p/q}(f)$ then $f_\nu = 0$ for all $\nu \in q\mathbb{Z}$, so that $I_q(f) = \emptyset$: then $M \equiv |A_{p/q}(f)| \le q-1$. As $|I_c(f)| = 1$ $\forall c \in A_{p/q}(f)$, for any $c \in A_{p/q}(f)$ there is only one $\nu \in \mathbb{N}$ of the form $\nu = c + lq$, with $l \in \mathbb{Z}_+$, such that $f_\nu \neq 0$. So the number of Fourier labels ν's for which $f_\nu \neq 0$ is given by $M \le q-1$; of course the bound $M \ge 1$ is obvious. □

If $q \notin A_{p/q}(f)$ and $|I_c(f)| = 1$ $\forall c \in A_{p/q}(f)$, we define $2M$ integers $r_1,\dots,r_M$, $r'_1,\dots,r'_M$, with $r_i, r'_i \ge 0$, and an integer $R > 0$ as those integers which satisfy the following conditions:

$$(r_1 - r'_1)\,\nu_1 + \cdots + (r_M - r'_M)\,\nu_M = Rq, \tag{1.17a}$$

$$r_1 + r'_1 + \cdots + r_M + r'_M = r_0 \ge 2, \tag{1.17b}$$

$$r_0 \text{ is minimal,} \tag{1.17c}$$

where $\{\nu_1,\dots,\nu_M\} = \mathcal{A}(f)$.

We observe that this problem is reminiscent of the so called "knapsack problem", which is believed to be computationally "hard" (see [10] for details on knapsacks and on what "hard" really means).

The meaning of the above definition will be clear after proposition 1 in sect. 2, while its relevance for the problem we are studying will appear in theorem 2 below.

As we now show, integers satisfying the conditions (1.17) always exist. Some properties of the solutions will be explicitly discussed in sect. 6.

Lemma 2. *The Diophantine problem* (1.17) *has always a finite, nonzero number of solutions and* $r_0 \le q$.

Proof. Solutions to (1.17a) and (1.17b) do exist: one is given for example by $R = \nu_i$ for some i and $r_i = q$, while $r_j = 0 \ \forall j \ne i$ and $r'_j = 0 \ \forall j$. Note that in this case (1.17b) yields $r_0 = q$. As the set of solutions to (1.17a) and (1.17b) is not empty and as each solution has associated a *positive* value of r_0, there must be at least one such that r_0 is minimal. On the other hand, a minimal solution has $r_0 \le q$ because of the above note, so there can be only a finite number of them. □

Remark 1. Of course one can exhibit functions in which $r_0 = q$: an example is the standard map itself (for which $M = 1$ and $\nu_1 = 1$).

Define $r^*(f) = r^*$ as

$$r^* = \begin{cases} 1 & \text{if } q \in A_{p/q}(f), \\ 2 & \text{if } q \notin A_{p/q}(f) \text{ and } \exists \bar{c} \in A_{p/q}(f) \text{ such that } |I_{\bar{c}}(f)| \ge 2, \\ r_0 & \text{otherwise,} \end{cases} \tag{1.18}$$

where r_0 is defined as in (1.17). Note that $1 \le r^* \le q$.

We can now state our main result.

Theorem 2. *Let f be any function in $\mathcal{B}_\rho$, with $\rho > 0$. Consider the cone* $\mathcal{C}_{p/q,\beta} = \{z \in \mathbb{C} : \ |\operatorname{Im} z| > 0, |\operatorname{Re} z - p/q| \le \beta |\operatorname{Im} z|, \ \beta \ge 0\}$*; let $\omega \in \mathcal{C}_{p/q,\beta}$. Then the* rescaled conjugating function

$$\bar{u}_{p/q}(\alpha, \varepsilon, \omega) = u\left(\alpha, \varepsilon\left(\omega - \frac{p}{q}\right)^{2/r^*}, \omega\right) \tag{1.19}$$

extends to a function continuous in ω in the closure of the cone $\mathcal{C}_{p/q,\beta}$ and analytic in ω in the interior of $\mathcal{C}_{p/q,\beta}$, for any $\beta \ge 0$, analytic in ε for $|\varepsilon| < a$ and analytic in α for $|\operatorname{Im} \alpha| < b$, with a, b two positive constants. In particular, the following limit exists:

$$\bar{\bar{u}}_{p/q}(\alpha, \varepsilon) = \lim_{\omega \to p/q} \bar{u}(\alpha, \varepsilon, \omega), \tag{1.20}$$

and it is independent from the non-tangential path chosen either in the complex upper half plane $\operatorname{Im} \omega > 0$ *or in the complex lower half plane* $\operatorname{Im} \omega < 0$.

We can therefore let $\bar{u}(\alpha, \varepsilon, p/q) = \bar{\bar{u}}_{p/q}(\alpha, \varepsilon)$. We shall see in sect. 6 that the function (1.20) will be defined as the solution of a differential equation; in particular if $f(x)$ is a trigonometric polynomial, then $\bar{\bar{u}}_{p/q}(\alpha, \varepsilon)$ is related to hyperelliptic functions.

Remark 2. Note that the limit in (1.20) is taken along any path *inside the cone* $\mathcal{C}_{p/q,\beta}$, therefore *non-tangential* to the real axis; consideration of *tangential* limits would be much more difficult.

Remark 3. If the limit function $\bar{\bar{u}}_{p/q}(\alpha, \varepsilon)$ has a finite radius of convergence, the above theorem implies a scaling law for the radius of convergence $\rho(\omega)$ of the Lindstedt series; in particular

$$\rho(\omega) = O\bigg(\Big(\omega - \frac{p}{q}\Big)^{2/r^*}\bigg). \tag{1.21}$$

However we do not know in general if the radius of convergence is finite. For the case of trigonometric polynomials this should follow from the theory of hyperelliptic functions.

The proof of theorem 2 is achieved through a series of lemmata in the following sections. Theorem 1 then follows as a particular case: we stated it first for clarity and to emphasise the generic case.

2. Trees, Clusters and other combinatorial tools

By inserting (1.10) into the functional equation (1.8) we can find the recursion relation satisfied by the coefficients $u_\nu^{(k)} \equiv u_\nu^{(k)}(\omega)$

$$u_\nu^{(k)} = g(\nu) \sum_{m \ge 1} \frac{1}{m!} \sum_{\substack{k_1 + \dots + k_m = k-1, \\ k_j \in \mathbb{N}}} \sum_{\substack{\nu_0 + \nu_1 + \dots + \nu_k = \nu, \\ \nu_j \in \mathbb{Z}}} f_{\nu_0} (i\nu_0)^m \prod_{j=1}^m u_{\nu_j}^{(k_j)}, \tag{2.1}$$

where

$$g(\nu) \equiv \frac{1}{\gamma(\nu)} = \frac{1}{2(\cos 2\pi\omega\nu - 1)} \tag{2.2}$$

is called *propagator*.

The coefficients $u_\nu^{(k)}$ can be expressed graphically in terms of *semitopological labelled trees* by iterating (2.1). We refer the reader to the now large literature on the topic, and in particular to [11] for a review, to [12] and [14] where the formalism we use was first introduced and to [1] where a result analogous to the one of this paper was proved, in the specific case of the standard map (that is, $f(x) = \sin x$); the first proof of existence of invariant tori for Hamiltonian systems by tree expansions of the Lindstedt series is due to Eliasson, [15].

The trees are defined as in [1] (see also [2] and the references quoted therein). Note that, as our perturbation $f(x)$ is more general than the one considered in [1],

[2], the *mode label* ν_u associated to each node $u \in \vartheta$ can now assume all values in $\mathbb{Z}$, except of course 0 since f has zero average. In case the perturbation is a trigonometric polynomial, *i.e.* only a finite number of Fourier coefficients f_ν are different from 0, the mode label can assume any value such that the corresponding Fourier coefficient is nonzero; in particular, $|\nu_u| \le N_f$.

For our models we have

$$u_\nu^{(k)} = \sum_{\vartheta \in \mathcal{T}_{\nu,k}} \mathrm{Val}(\vartheta), \tag{2.3}$$

where

$$\mathrm{Val}(\vartheta) = i^{k-1} \Big[\prod_{u \in \vartheta} \frac{f_{\nu_u} \nu_u^{m_u}}{m_u!} \Big] \Big[\prod_{\ell \in \vartheta} g(\nu_\ell) \Big] \tag{2.4}$$

and $\mathcal{T}_{\nu,k}$ is the set of semitopological trees with k nodes and *momentum* ν flowing through the root line; compare with eq. (2.2) of [1], which is obtained from (2.4) by setting $f_{\nu_u} = -i\nu_u$, with $\nu_u = \pm 1$.

As in [1], m_u denotes the number of lines entering the node u; one has

$$\sum_{u \in \vartheta} m_u = k - 1. \tag{2.5}$$

The estimates on the propagators are fairly trivial and can be borrowed by [1].

Lemma 3. *In the cone $\mathcal{C}_{p/q,\beta}$, the propagators $g(\nu)$ satisfy the estimates:*

$$|g(\nu)| \le \begin{cases} \dfrac{C}{|\nu\zeta|^2}, & \text{for } \nu \text{ a non-zero multiple of } q, \\[2ex] \dfrac{C}{q^2}, & \text{otherwise}, \end{cases} \tag{2.6}$$

where $\zeta = \omega - p/q$ and C is a constant (depending on β).

Proof. The proof can be done as in [1], sects. 3 and 7; or one could use dimensional (Cauchy) estimates in a fairly obvious way. □

To each line ℓ of a tree ϑ, we associate a *scale label* n_ℓ which takes values 0 or 1: $n_\ell = 0$ if its momentum ν_ℓ is a multiple of q (so that the propagator associated to that line diverges as $\omega - p/q = \zeta \to 0$ as $O(\zeta^{-2})$, see lemma 3), while $n_\ell = 1$ otherwise. If we could neglect the problem of resonances (see below), we could just count the number of lines on scale 0 to obtain the scaling behaviour of the radius of convergence. We now define our main combinatorial tools.

Definition. A *cluster* T of a tree ϑ is a maximal connected set of nodes connected by lines on scale $n = 1$; such lines are called *internal* lines, and we shall write $\ell \in T$. The line exiting and the lines entering a cluster are all on scale $n = 0$ and they are called the *external lines* of T. Let k_T be the number of nodes in T: the case $k_T = 1$ is possible and corresponds to the case of a single node such that the exiting line and the entering ones are all on scale $n = 0$. We define also m_T as the number of lines entering the cluster T (of course there is always a single line exiting T). A node u is said to be *internal* to T if $u \in T$.

Definition. A *resonance* (or *resonant cluster*) is a cluster V such that:

$$\sum_{u \in V} \nu_u = 0; \tag{2.7}$$

in this case, we shall say that the line exiting the resonance is a *resonant line*.

Given a resonance V in the tree ϑ, with resonant line ℓ_V, we define its *resonance factor* as

$$\mathcal{V}_V(\vartheta) = \Big[\prod_{u \in V} \frac{f_{\nu_u} \nu_u^{m_u}}{m_u!} \Big] \Big[\prod_{\ell \in V} \frac{1}{\gamma(\nu_{\ell_u})} \Big]. \tag{2.8}$$

We now prove a simple but important fact about the minimal size of clusters.

Proposition 1. *A nonresonant cluster has at least r^* nodes. If $q > N_f$, then $r^* = r_0$.*

Proof. A cluster T has m_T entering lines, carrying momenta $\nu_i = s_i q$, $i = 1, \ldots, m_T$, as they are on scale 0, and one exiting line carrying a momentum $\nu_0 = s_0 q$, as it is also on scale 0; of course s_i, s_0 are all different from 0. Each node $u \in T$ has a mode label $\nu_u = s_u q + \operatorname{sign}(\nu_u)\, c_u$, where $s_u \in \mathbb{Z}$ and $c_u \in A_{p/q}(f)$; note that ν_u uniquely identifies the integers s_u, c_u. Then

$$\nu_0 = \sum_{u \in T} \nu_u + \sum_{i=1}^{m_T} \nu_i = \left(\sum_{u \in T} s_u + \sum_{i=1}^{m_T} s_i \right) q + \sum_{u \in T} \operatorname{sign}(\nu_u)\, c_u = s_0 q.$$

This implies that

$$\sum_{u \in T} \operatorname{sign}(\nu_u)\, c_u = S q, \tag{2.9}$$

where

$$S = s_0 - \left(\sum_{u \in T} s_u + \sum_{i=1}^{m_T} s_i \right) \in \mathbb{Z}. \tag{2.10}$$

If $q \in A_{p/q}(f)$, then we can make a cluster containing only one node u with mode label $\nu_u \in q\mathbb{Z}$, which yields $S = \pm 1$ in (2.9).

If $q \notin A_{p/q}(f)$ and $|I_{\bar{c}}(f)| \geq 2$ for some $\bar{c} \in A_{p/q}(f)$, then we can make a cluster with only two nodes which is not a resonance; in fact, there are at least two frequencies $\nu_1 = s_1 q + \bar{c}$, $\nu_2 = s_2 q + \bar{c}$, $s_1, s_2 \geq 0$, $s_1 \neq s_2$ and a cluster can be built with only two nodes u_1, u_2 with mode labels $\nu_{u_1} = \nu_1$ and $\nu_{u_2} = -\nu_2$: it is a cluster since the sum $\nu_1 - \nu_2$ is a multiple of q, while neither ν_1 nor ν_2 are (so the only internal line of the cluster is on scale 1), but it is not a resonance since $\nu_1 - \nu_2 \neq 0$. Note that $S = 0$ in (2.10), in such a case.

If $q \notin A_{p/q}(f)$ and $|I_c(f)| = 1 \ \forall c \in A_{p/q}(f)$, set $\mathcal{A}(f) = \{\nu_1, \ldots, \nu_M\}$; then

$$\sum_{u \in T} \nu_u = \sum_{i=1}^{M} (r_i - r_i')\, \nu_i = Rq,$$

where r_i is the number of ν_u's equal to $+\nu_i$ and r_i' is the number of ν_u's equal to $-\nu_i$. Of course $R \neq 0$ as T is not a resonance. Then (1.17a) must hold. Moreover, a cluster must have at least two nodes in it if $q \notin A_{p/q}(f)$, so that $r_0 \geq 2$ and also (1.17b) holds; and of course, a cluster has the least possible number of nodes if (1.17c) is satified.

Finally, if $q > N_f$, then $q \notin A_{p/q}(f)$ and $|I_c(f)| = 1 \ \forall c \in A_{p/q}(f)$, so that $r^* = r_0$. In fact the integers c_i are the actual frequencies appearing in the Fourier expansion of the perturbation $f(x)$, that is $s_u = 0$ and $\mathrm{sign}(\nu_u)\, c_u = \nu_u$. □

3. Existence of the formal solution

The first step is, of course, to show that (1.8) actually has a *formal* solution, *i.e.* that a Taylor series in ε, whose coefficients admit a Fourier expansion in α, satisfies order by order eq. (1.8), disregarding any issue of actual convergence.

This fact was essentially proved by Poincaré, [16], in the case of Hamiltonian flows and it was explicitly worked out by [18] in the case of area-preserving maps. We believe that a direct, purely combinatorial proof of the existence of the formal solution is of some interest, especially as it helps to understand the mechanism of the renormalization of resonances in the following sections.

Proposition 2. *If* $f_0 = \langle f \rangle = 0$*, then* (1.8) *has a formal solution* u*, such that* $\langle u \rangle = (1/2\pi) \int_0^{2\pi} \mathrm{d}\alpha\, u(\alpha, \varepsilon, \omega) = 0$.

Proof. First, note that it is always possible to choose $u_0^{(0)} = 0$, since (1.8) is invariant with respect to the transformation

$$\begin{cases} \alpha \mapsto \alpha + c, \\ u \mapsto u - c. \end{cases}$$

Next, for $k = 1$, we let

$$u_0^{(1)} = 0, \tag{3.1}$$

$$u_\nu^{(1)} = g(\nu) f_\nu, \quad \nu \neq 0, \tag{3.2}$$

since $Du^{(1)}(\alpha, \omega) = f(\alpha)$ and $f_0 = 0$.

To order k in ε, equation (1.8) becomes

$$\gamma(\nu) u_\nu^{(k)} = [f(\alpha + u(\alpha))]_\nu^{(k-1)}.$$

For this to be soluble, one must have

$$[f(\alpha + u(\alpha))]_0^{(k)} = 0 \quad \forall k, \tag{3.3}$$

since $\gamma(0)=0$. We prove this by induction on k, considering that the case $k=1$ is trivial since $[f(\alpha+u(\alpha))]_0^{(0)}=f_0=0$ (see above).

Of course, once (3.3) has been proved, the formal solubility of (1.8) immediately follows, as $|\gamma(\nu)|\ge C^{-1}|\zeta|^2$ $\forall\nu\in\mathbb{Z}$, for $\zeta=\omega-p/q$ fixed with $|\operatorname{Im}\zeta|\neq 0$; see lemma 3 in sect. 2.

We therefore assume that (3.3) holds $\forall k'<k$. Note that $[f(\alpha+u(\alpha))]_\nu^{(k)}$ has the same representation, in term of trees, of $u_\nu^{(k+1)}$, with the only difference that the propagator associated to the root line is neglected – *i.e.* taken to be identically 1. We denote with $\widetilde{\operatorname{Val}}(\vartheta)$ the value of a tree ϑ according to this new rule; then

$$[f(\alpha+u(\alpha))]_0^{(k)}=\sum_{\vartheta\in\mathcal{T}_{k,0}}\widetilde{\operatorname{Val}}(\vartheta),$$

and, by the induction hypothesis, all lines of ϑ have nonzero momentum.

We now define a group $\mathcal{G}$ of transformations acting on $\mathcal{T}_{k,0}$. A generator of $\mathcal{G}$ is a transformation of the following type: detach the root line of ϑ and reattach it to a node of ϑ. The action of $\mathcal{G}$ on a tree ϑ defines the *$\mathcal{G}$-family* $\mathcal{F}_{\mathcal{G}}(\vartheta)$ of the tree ϑ.

Then we have the following.

(1) $\mathcal{G}$ leaves $\mathcal{T}_{k,0}$ invariant: that is, all trees obtained from trees in $\mathcal{T}_{k,0}$ by transformations of $\mathcal{G}$ still belong to $\mathcal{T}_{k,0}$.

(2) If $G\in\mathcal{G}$, $\vartheta'=G\vartheta$, then on some lines the arrows indicating the flow of the momentum change direction so that the partial ordering relation defined in ϑ' is different than the one in ϑ; if $\ell\in\vartheta$, denote by ℓ' the same line seen as an element of ϑ'. Consequently, on lines ℓ where the direction did not change $\nu_{\ell'}=\nu_\ell$, while on lines ℓ where the direction did change $\nu_{\ell'}=-\nu_\ell$; in fact, if ℓ is the exiting line of u, $\nu_\ell=\sum_{w\preccurlyeq u}\nu_w$ and $\nu_{\ell'}=\sum_{w\succ u}\nu_w$ (using the ordering of ϑ in both cases), so that

$$\nu_\ell+\nu_{\ell'}=\sum_{w\preccurlyeq u}\nu_w+\sum_{w\succ u}\nu_w=\sum_{w\in\vartheta}\nu_w=0.$$

(3) Suppose that the root line is detached from the node u_0 and reattached to the node u_1: set $P(u_0,u_1)=\{w\in\vartheta|u_1\preccurlyeq w\preccurlyeq u_0\}$. Then, forgetting for a moment the combinatorial factor, the value of the tree changes because a factor ν_{u_0} is replaced with a factor ν_{u_1}. This follows from the fact that for each node $u\in\vartheta$ there is a factor $(i\nu_u)^{m_u}$ and, by shifting the root line from u_0 to u_1, for each node $w\in P(u_0,u_1)$ an entering line becomes an exiting one and *vice versa*, except for u_0 and u_1: the node u_0 simply loses an entering line, while the node u_1 turns out to have an extra entering line. So m_u keeps the same value for all $u\in\vartheta$, except for u_0 and u_1: m_{u_0} becomes $m_{u_0}-1$ and m_{u_1} becomes $m_{u_1}+1$. Therefore when we sum over all the trees in the same $\mathcal{G}$-family, *assuming that all combinatorial factors*

are the same, we get a factor $\sum_{u\in\vartheta}\nu_u = 0$ and therefore we obtain 0:

$$\sum_{\vartheta\in\mathcal{T}_{k,0}} \widetilde{\mathrm{Val}}(\vartheta) = \sum_{\vartheta\in\mathcal{T}_{k,0}} \frac{1}{|\mathcal{F}_{\mathcal{G}}(\vartheta)|} \underbrace{\sum_{\vartheta'\in\mathcal{F}_{\mathcal{G}}(\vartheta)} \widetilde{\mathrm{Val}}(\vartheta')}_{=0} = 0, \tag{3.4}$$

and this concludes the proof of theorem 2, *provided everything goes well with the combinatorial factors.*

We are therefore left with the task of proving that all combinatorial factors inside a $\mathcal{G}$-family do match. First, we note that it is convenient to use *topological* trees instead of the usual *semitopological* ones used throughout the paper (in [1] only semitopological trees were used).

We briefly outline the differences between the two kinds of trees, deferring to [13] and [11] for a more detailed discussion of the differences between what finally amounts to a different way to count trees. Define a group of transformations acting on trees generated by the following operations: fix any node $u \in \vartheta$ and permute the subtrees entering such a node. We shall call semitopological trees the trees which are superposable up to a continuous deformation of the lines, and topological trees the trees for which the same happens modulo the action of the just defined group of transformations (see Fig. 3.2 in [11] for a concrete example of trees which are different as semitopological trees and not as topological trees).

We define *equivalent* two trees which are equal as topological trees.

Then we can still write $u_\nu^{(k)}$ in the form (2.3), restricting the sum over the set of all nonequivalent topological trees with k nodes and momentum ν flowing through the root line – we can denote it by $\mathcal{T}^{\mathrm{top}}_{\nu,k}$ –, *provided that to each node* $u \in \vartheta$ *we associate a combinatorial factor which is not the* $1/m_u!$ *appearing in* (2.4).

In fact for topological trees the combinatorial factor associated to each node is different, because we have to look now to how the subtrees emerging from each node differ. For semitopological trees we have a factor $1/m_u!$ for each node u, where m_u is the number of lines entering u *disregarding the kinds of the subtrees entering* u, so counting as different many trees otherwise identical, while in the case of topological trees we consider one and the same tree those trees that are different as semitopological trees, but have the same value because they just differ in the order in which *identical* subtrees enter each node u: therefore, if $s_{u,1}, \ldots, s_{u,j_u}$ are the number of entering lines to which are attached subtrees of a given shape and with the same labels (so that $s_{u,1} + \cdots + s_{u,j_u} = m_u$, $1 \le j_u \le m_u$), the combinatorial factor, for each node, becomes

$$\frac{1}{m_u!}\cdot\frac{m_u!}{s_{u,1}!\cdot s_{u,j_u}!} = \frac{1}{s_{u,1}!\cdot s_{u,j_u}!};$$

note in the second factor in the above formula the multinomial coefficient corresponding to the number of different *semitopological* trees corresponding to the same *topological* tree, for each node.

So in terms of topological trees $u_\nu^{(k)}$ can be expressed as

$$u_\nu^{(k)} = \sum_{\vartheta \in \mathcal{T}_{\nu,k}^{\rm top}} \overline{\rm Val}(\vartheta), \tag{3.5}$$

where

$$\overline{\rm Val}(\vartheta) = i^{k-1} \Big[\prod_{u\in\vartheta} \frac{f_{\nu_u} \nu_u^{m_u}}{s_{u,1}! \ldots s_{u,j_u}!} \Big] \Big[\prod_{\ell\in\vartheta} g(\nu_\ell) \Big]. \tag{3.6}$$

Still, when computing the combinatorial factors inside each $\mathcal{G}$-family, they do differ. But this is actually an apparent, not a real discrepancy. In fact, due to symmetries in the tree (that is, to the fact that the subtrees emerging from some node are sometimes equal, *i.e.* that some $s_{u,i}$ are greater than 1), the actual number of topological trees in a given $\mathcal{G}$-family is less than the total number of trees obtained by the action of the group $\mathcal{G}$: in other words some trees obtained by the action of $\mathcal{G}$ are equivalent as topological trees. When moving the root line from a node u_0 to another node u_1, so transforming a tree ϑ into a tree $\vartheta_1 \in \mathcal{F}_{\mathcal{G}}(\vartheta)$, for some nodes $w \in P(u_0, u_1)$ the factor $1/s_{w,i}!$ can turn into $1/(s_{w,i}-1)!$, but then this means that the same topological tree could be formed by the action of $s_{w,i}$ different transformations of the group $\mathcal{G}$: each of the $s_{w,i}$ equivalent subtrees entering w contains a node such that, by attaching to it the root line, the same topological tree is obtained. Therefore, *by counting all trees obtained by the action of the group* $\mathcal{G}$ (as done to implement the cancellation (3.4)), the corresponding topological tree value is in fact counted $s_{w,i}$ times, so to avoid overcounting one needs a factor $1/s_{w,i}$: this gives back the same combinatorial factor $1/s_{w,i}!$. Analogously one discusses the case of a factor $1/s_{w,i}!$ turning into $1/(s_{w,i}+1)!$, simply by noting that the same argument as above can be followed also in this case by changing the rôles of the two nodes u_0 and u_1.

Considering the argument in pts. 1, 2, 3, the proof is complete. □

Remark 4. If one wants to prove the formal solubility of the equation (1.8) for $\omega \in \mathbb{R}$, one has to impose some condition on ω. In the case of trigonometric polynomials ($N_f < \infty$), it is enough to require the rotation number ω to be irrational. For really analytic perturbations ($N_f = \infty$) the weakest condition is

$$\lim_{n\to\infty} \frac{\log q_{n+1}}{q_n} = 0, \tag{3.7}$$

where $\{q_n\}$ are the denominators of the convergents arising from the continuous fraction expansion for ω: in this way the Fourier labels can be summed over. Note that such a condition is guaranteed by the Bryuno condition,

$$\sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n} < \infty, \tag{3.8}$$

under which the formal series defining the conjugating function can be proved to converge (for ε small enough).

4. Estimates on the radius of convergence

Let $N_n(\vartheta)$, $n = 0, 1$, be the number of lines of ϑ on scale n and let $N_0^*(\vartheta)$ be the number of lines of ϑ on scale 0 which are not resonant lines, *i.e.* exiting lines of any resonance. We then have the following fundamental estimate (the argument is adapted from [17]).

Lemma 4. *The number of nonresonant lines satisfies of any tree ϑ:*

$$N_0^*(\vartheta) \le \left\lfloor \frac{k(\vartheta)}{r^*} \right\rfloor .$$

if $k(\vartheta)$ is the order of ϑ.

Proof. We prove by induction on the order of the tree that $N_0^*(\vartheta) \le k(\vartheta)/r^*$.

For $k(\vartheta) = 1$ the bound is trivially satisfied.

If the root line of ϑ is on scale 1 or is on scale 0 but is resonant, let $\vartheta_1, \ldots, \vartheta_m$ be the subtrees of ϑ entering the last node u_0 of ϑ. Then

$$N_0^*(\vartheta) = \sum_{j=1}^{m} N_0^*(\vartheta_j) \le \sum_{j=1}^{m} \frac{k(\vartheta_j)}{r^*} < \frac{k(\vartheta)}{r^*}. \tag{4.1}$$

If the root line of ϑ is on scale 0 and is not resonant, then let $\vartheta_1, \ldots, \vartheta_m$ be the subtrees entering the cluster T that the root line of ϑ exits. Then

$$N_0^*(\vartheta) = 1 + \sum_{j=1}^{m} N_0^*(\vartheta_j) \le 1 + \sum_{j=1}^{m} \frac{k(\vartheta_j)}{r^*} = 1 + \frac{k(\vartheta) - k_T}{r^*} \le \frac{k(\vartheta)}{r^*},$$

where in the last step k_T is the size of the cluster T and use was made of proposition 1 in sect. 2. Then, by taking into account that $N_0^*(\vartheta)$ has to be an integer, the lemma follows. □

We now have to deal with the problem of resonances, *i.e.* we have to produce a partial resummation of the trees expansion in such a way that extra ζ^2 factors are produced, one for each resonant line (as in [1]). The proof goes much along the way of the analogous proof in [1], except that now we have to take into account that the perturbation is no longer necessarily odd in the angle variable α, so one cannot use symmetry arguments but rather has to enlarge the group of transformations (with respect to [1]) to include transformations of the type used in sect. 3 to show the existence of the formal series (where the problem was just the absence of parity in the perturbation).

Like in sect. 3, to better implement the cancellation mechanisms, let us use topological trees, so that a factor

$$\frac{1}{s_{u,1}!\dots s_{u,j_u}!} \tag{4.2}$$

will be associated to each node $u \in \vartheta$ (we use the same notation as in sect. 3). In particular, if V is a resonance in ϑ and $u \in V$, then $j_u = j'_u + j''_u$, where $s_{u,1} + \dots + s_{u,j'_u}$ is the number of non-equivalent subtrees entering u and internal to V, while $s_{u,j'_u} + \dots + s_{u,j'_u+j''_u}$ is the number of non-equivalent subtrees entering u and entering V; in fact a subtree entering V cannot be equivalent to a subtree internal to V.

This allows us to rewrite the resonance factor $\mathcal{V}_V(\vartheta)$ in (2.8) as

$$\mathcal{V}_V(\vartheta) = \Big[\prod_{u\in V} \frac{f_{\nu_u}\nu_u^{m_u}}{s_{u,1}!\dots s_{u,j_u}}\Big]\Big[\prod_{\ell\in V}\frac{1}{\gamma(\nu_{\ell_u})}\Big]. \tag{4.3}$$

Note also that, for any tree ϑ and for any resonance V in ϑ, one has

$$\mathrm{Val}(\vartheta) = \mathcal{V}_V(\vartheta)\mathcal{S}_V(\vartheta), \qquad \mathcal{S}_V(\vartheta) = \Big[\prod_{u\in\vartheta\setminus V} \frac{f_{\nu_u}\nu_u^{m_u}}{s_{u,1}!\dots s_{u,j_u}}\Big]\Big[\prod_{\ell\in\vartheta\setminus V}\frac{1}{\gamma(\nu_{\ell_u})}\Big]. \tag{4.4}$$

Given a tree ϑ and a resonance V with m_V incoming lines $\ell_1, \dots, \ell_{m_V}$ and with k_V nodes, we define the family $\mathcal{F}_V(\vartheta)$ of V in ϑ as the set of trees obtained from ϑ by the action of a group of transformations $\mathcal{P}_V$ on ϑ, generated by the following operations.

(1) For each entering line, detach it and reattach it to all nodes of the resonance.
(2) We include in the group $\mathcal{P}_V$ transformations like those of the group $\mathcal{G}$ defined in sect. 3: detach the exiting line of V and reattach it to all nodes of V.

We shall call transformation of type 1 and 2 the operations described above. We note also that the idea of including the transformations of type 2 in order to implement the cancellation mechanism comes back to [19], in the context of continuous Hamiltonian systems.

We then have the following lemma, where the cancellations are exhibited for a resonance, and its corollary, for trees which contain more than one resonance (as it generally happens).

Lemma 5. *Given a tree ϑ with a resonance V, let $\nu_1, \dots, \nu_{m_V}$ be the momenta flowing through the entering lines $\ell_1, \dots, \ell_{m_V}$ of V; then we have, for any $\xi > \delta > 0$,*

$$\frac{1}{|\mathcal{F}_V(\vartheta)|}\Big|\sum_{\vartheta'\in\mathcal{F}_V(\vartheta)} \mathcal{V}_V(\vartheta')\Big| \le$$

$$\le q^{2(k_V+1)} D_0 C_1^{k_V} D_1^{k_V-1+m_V}(\delta) k_V^2 e^{-(\xi-\delta)|\nu_V|} \sum_{m,m'=1}^{m_V} |\nu_m \nu_{m'} \zeta^2|, \tag{4.5}$$

for some constant D_0 and with $D_1(\delta)$ diverging as δ^{-1} for $\delta \to 0^+$.

The proof of this lemma is fairly standard but rather technical, so we defer it to the next section.

Corollary. *For a tree $\vartheta \in \mathcal{T}_{\nu.k}$ with resonances $V_1, \ldots, V_s$, consider the family $\mathcal{F}(\vartheta)$ obtained by the action on ϑ of all the groups $\mathcal{P}_{V_1}, \ldots, \mathcal{P}_{V_s}$; the number of trees in this family is then given by*

$$k_{\mathcal{F}(\vartheta)} = \prod_{i=1}^{s} |\mathcal{F}_{V_i}(\vartheta)|,$$

and we have the estimate:

$$\frac{1}{k_{\mathcal{F}(\vartheta)}} \Big| \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \mathrm{Val}(\vartheta') \Big| \le q^{2N_1(\vartheta)} |\zeta|^{-2N_0(\vartheta)} q^{2s} D_0^s C_1^k D_1^k(\delta) k^4 |\zeta|^{2s} e^{-(\xi-\delta)|\nu|} \tag{4.6}$$

where D_0 and $D_1(\delta)$ are the same constants as in lemma 5.

Proof. Note that this is possible since the mechanisms generating the cancellations for each resonance V_i, $i = 1, \ldots, s$, do not interfere with each other. Note also that in our case, as well as in [1], no combinatorial problems arise due to resonances contained inside others, as there are only two scales: that is, scales are used just to account for singular (divergent) and regular propagators. So we have just to use lemma 5 for each resonance V_i: see [1] for more details, as the proof is identical in that case. A minor observation is that, if $\mathbf{V}$ denotes the set of resonances, $\mathbf{V} = \{V_1, \ldots, V_s\}$, then

$$\sum_{V \in \mathbf{V}} (k_V - 1 + m_V) + \sum_{u \notin \mathbf{V}} m_u = k, \tag{4.7}$$

where $u \notin \mathbf{V}$ means $u \notin V_j$ $\forall j = 1, \ldots, s$. □

Next, we estimate $|u^{(k)}(\alpha, \omega)|$. Let $N_0^R(\vartheta) = N_0(\vartheta) - N_0^*(\vartheta)$ be the number of resonant lines in a given tree ϑ (as each resonance has *one* exiting line, the number of resonances is the same as $N_0^R(\vartheta)$, of course). By the corollary to lemma 5, with $s = N_0^*(\vartheta)$, we obtain

$$\begin{aligned} \frac{1}{k_{\mathcal{F}(\vartheta)}} \Big| \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \mathrm{Val}(\vartheta') \Big| \le \\ \le q^{2k} |\zeta|^{-2(N_0^R(\vartheta) + N_0^*(\vartheta))} D_0^{N_0^R(\vartheta)} C_1^k D_1^k(\delta) k^4 |\zeta|^{2N_0^R(\vartheta)} e^{-(\xi-\delta)|\nu|} \le \\ \le \left(e^4 C_1 D_2 D_1(\delta) q^2 \right)^k |\zeta|^{-2\lfloor k/r^* \rfloor} e^{-(\xi-\delta)|\nu|}, \end{aligned} \tag{4.8}$$

where lemma 4 was used in the last step and $D_2 = \max\{1, D_0\}$. Then we can write the sum on the trees in the following way:

$$\sum_{\vartheta \in \mathcal{T}_{\nu,k}} \mathrm{Val}(\vartheta) = \sum_{\vartheta \in \mathcal{T}_{\nu,k}} \frac{1}{k_{\mathcal{F}(\vartheta)}} \sum_{\vartheta' \in \mathcal{F}(\vartheta)} \mathrm{Val}(\vartheta'),$$

from which it easily follows that

$$|u_\nu^{(k)}| \le B_0^k C_1^k D_1^k(\delta) |\zeta|^{-2\lfloor k/r^* \rfloor} e^{-\xi|\nu|}, \tag{4.9}$$

for some constant B_0, as the number of trees of order k is bounded by a constant to the power k.

So from (4.9) one deduces that, for any $\delta > 0$, $u^{(k)}(\alpha,\omega)$ is analytic in α in a strip of width $\xi - \delta$ around the real axis and satisfies the bound

$$\|u^{(k)}(\cdot,\omega)\|_{\xi-\delta} \le B_1(\delta) C_1^k D_1^k(\delta) |\zeta|^{-2\lfloor k/r^* \rfloor}, \tag{4.10}$$

for some constant $B_1(\delta)$, diverging as δ^{-1} for $\delta \to 0^+$. Therefore, for any $\delta > 0$, the conjugating function u is analytic in α for α in a strip $|\alpha| < \xi - \delta$ and in ε for $|\varepsilon| < \epsilon_0 \equiv C_1^{-1} D_1^{-1}(\delta) |\zeta|^{2\lfloor k/r^* \rfloor}$.

This implies that the limit (1.20) exists and, under the additional assumption that the radius of convergence is finite, that the scaling property (1.21) holds. The full proof of theorem 2 will be achieved in sect. 6, where it will be shown that the limit (1.20) is independent on the non-tangential path chosen in the complex ω plane and where this limit can be explicitly computed (at least in principle) as solution of a differential equation, proving that it is actually a function of ε^{r^*} and $q\alpha$.

5. Renormalization of resonances

For each resonance V in ϑ and for each node $u \in V$, write

$$\nu_{\ell_u} = \nu_{\ell_u}^0 + \sum_{\ell' \in L_u(V)} \nu_{\ell'}, \tag{5.1}$$

where

$$\nu_{\ell_u}^0 = \sum_{\substack{w \in V, \\ w \preccurlyeq u}} \nu_w \tag{5.2}$$

and, if $L(V) = \{\ell_1, \ldots, \ell_{m_V}\}$ denotes the set of lines entering V and $u \in V$, one sets

$$L(V) = L_u(V) \cup \tilde{L}_u(V), \tag{5.3a}$$

$$L_u(V) = \{\ell \in L(V) | \ell \text{ enters a node } w \in V \text{ such that } w \preccurlyeq u\}, \tag{5.3b}$$

$$\tilde{L}_u(V) = L(V) \setminus L_u(V). \tag{5.3c}$$

Note that each line $\ell_u \in V$ induces a natural splitting of V into two disjoint sets:

$$V = V_1(\ell_u) \cup V_2(\ell_u), \tag{5.4a}$$

$$V_1(\ell_u) = \{w \in V | w \preccurlyeq u\}, \tag{5.4b}$$

$$V_2(\ell_u) = V \setminus V_1(\ell_u). \tag{5.4c}$$

Under the action of the group $\mathcal{P}_V$ the propagators of any line $\ell_u \in V$ can change for the following two reasons.

(1) The set L_u can change as the entering lines of V are shifted.
(2) The arrow superimposed to ℓ_u can change its direction if the root line is shifted from a node in $V_2(\ell_u)$ to a node in $V_1(\ell_u)$.

To avoid confusion, in the following discussion, as in sect. 3, for any tree $\vartheta' \in \mathcal{F}_V(\vartheta)$ we use the partial ordering relations defined in ϑ (which of course can be different from those of ϑ').

It is easy to reason as in sect. 3 to conclude that the action of the transformations of the group $\mathcal{P}_V$ does not modify the combinatorial factors (4.2) if each tree obtained through a transformation is counted once and only once: simply repeat iteratively the same argument as in sect. 3, by noting that each transformation in $\mathcal{P}_V$ can be obtained by combining a finite number of elementary operations consisting just in shifting a single line. So in the following analysis the combinatorial factors play no rôle as they are invariant under the action of the group $\mathcal{P}_V$.

Then for any line $\ell \in L(V)$, write

$$\omega\nu_\ell = \frac{p}{q}\nu_\ell + \mu_\ell, \qquad \mu_\ell \equiv \zeta\nu_\ell, \tag{5.5}$$

and, for any tree $\vartheta' \in \mathcal{F}_V(\vartheta)$, consider the resonance factor (4.3) as a function of the quantities $\mu_{\ell_1}, \ldots, \mu_{\ell_{m_V}}$:

$$\mathcal{V}_V(\vartheta') \equiv \mathcal{V}_V(\vartheta'; \mu_{\ell_1}, \ldots, \mu_{\ell_{m_V}}). \tag{5.6}$$

As in [1] we can write

$$\begin{aligned} \mathcal{V}_V(\vartheta'; \mu_{\ell_1}, \ldots, \mu_{\ell_{m_V}}) = & \\ = \mathcal{V}_V(\vartheta'; 0, \ldots, 0) + \sum_{j=1}^{m_V} \mu_{\ell_j} \frac{\partial}{\partial \mu_{\ell_j}} \mathcal{V}_V(\vartheta'; 0, \ldots, 0) + & \\ + \sum_{i,j=1}^{m_V} \mu_{\ell_i}\mu_{\ell_j} \int_0^1 \mathrm{d}t\,(1-t) \frac{\partial^2}{\partial\mu_{\ell_i}\partial\mu_{\ell_j}} \mathcal{V}_V(\vartheta'; t\mu_{\ell_1}, \ldots, t\mu_{\ell_{m_V}}), & \end{aligned} \tag{5.7}$$

with the same notations as in [1] to denote the derivatives.

Now we pass to the proof of lemma 5.

Proof. We can reason as in sect. 3 to deduce that

$$\sum_{\vartheta' \in \mathcal{F}_V(\vartheta)} \mathcal{S}_V(\vartheta')\,[\mathcal{V}_V(\vartheta'; 0, \ldots, 0)] = 0 \tag{5.8}$$

simply noting that $\mathcal{S}_V(\vartheta')$, defined in (4.4), assumes the same value for any tree $\vartheta' \in \mathcal{F}_V(\vartheta)$ and that the argument given in sect. 3 applies identically because of (2.7).

To prove that also the terms to first order cancel in (5.7), we shall prove now that

$$\sum_{\vartheta'\in\mathcal{F}_V(\vartheta)} \mathcal{S}_V(\vartheta') \left[\frac{\partial}{\partial\mu_{\ell_j}}\mathcal{V}_V(\vartheta';0,\dots,0)\right] = 0 \tag{5.9}$$

for any $j=1,\dots,m_V$. First note that

$$\frac{\partial}{\partial\mu_{\ell_j}}\mathcal{V}_V(\vartheta';0,\dots,0) = \\ = \sum_{\ell\in V}\left\{\left[\frac{\partial}{\partial\mu_{\ell_j}}g(\nu_\ell^0)\right]\left[\prod_{\substack{\ell'\in V,\\ \ell'\neq\ell}} g(\nu_{\ell'})\right]\left[\prod_{u\in V}\frac{f_{\nu_u}\nu_u^{m_u}}{s_{u,1}!\dots s_{u,j_u}}\right]\right\}. \tag{5.10}$$

Then take the term in (5.10) in which the derivative acts on the line $\ell\equiv\ell_u$ exiting the node $u\in V$. Consider together all the trees $\vartheta'\in\mathcal{F}_V(\vartheta)$ obtained through the following operations in $\mathcal{P}_V$.

(1) Either attach the root line to any node $w_0\in V_2(\ell_u)$ and the entering line ℓ_j to any node $w_j\in V_1(\ell_u)$;
(2) or attach the root line to any node $w_0\in V_1(\ell_u)$ and the entering line ℓ_j to any node $w_j\in V_2(\ell_u)$;
(3) in both cases attach any line ℓ_i, $i\neq j$, to any node $w_i\in V$.

If we define, analogously to (2.7),

$$\nu_{V_1} = \sum_{w\in V_1(\ell_u)}\nu_w \equiv \nu_{\ell_u}^0, \qquad \nu_{V_2} = \sum_{w\in V_2(\ell_u)}\nu_w. \tag{5.11}$$

Each tree ϑ' obtained in this way has a value given by a common factor – call it $\Theta(\vartheta')\equiv\Theta(\vartheta)$ – times a factor which, on the contrary, depends on the particular transformations which have been performed – call it $\Omega(\vartheta')$ –.

The factor $\Omega(\vartheta')$ contains the derived propagator in (5.10) times a factor which is given as follows. Either the operation 1 produces a factor ν_{w_0}, $w_0\in V_2(\ell_u)$, the operation 2 produces a factor ν_{w_j}, $w_j\in V_1(\ell_u)$ and the operation 3 produces a factor ν_{w_i}, $w_i\in V$ for each entering line ℓ_i (case 1), or the operation 1 produces a factor ν_{w_0}, $w_0\in V_1(\ell_u)$, the operation 2 produces a factor ν_{w_j}, $w_j\in V_2(\ell_u)$ and the operation 3 produces a factor ν_{w_i}, $w_i\in V$ for each entering line ℓ_i (case 2).

Then by summing over all the considered trees we obtain

$$\Theta(\vartheta)\sum_{w_0\in V_2(\ell_u)}\sum_{w_j\in V_1(\ell_u)}\sum_{\substack{i=1,\\ i\neq j}}^{m_V}\sum_{w_i\in V}\nu_{w_0}\nu_{w_j}\nu_{w_i}\left[\frac{\partial}{\partial\mu_{\ell_j}}g(\nu_{\ell_j}^0)\right] = \\ = \Theta(\vartheta)\nu_{V_1}\nu_{V_2}\nu_V^{m_V-1}\left[\frac{\partial}{\partial\mu_{\ell_j}}g(\nu_{\ell_j}^0)\right] \tag{5.12}$$

for case 1 and

$$- \Theta(\vartheta) \sum_{w_0 \in V_1(\ell_u)} \sum_{w_j \in V_2(\ell_u)} \sum_{w_i \in V} \sum_{\substack{i=1, \\ i \neq j}}^{m_V} \sum_{w_i \in V} \nu_{w_0} \nu_{w_j} \nu_{w_i} \Big[\frac{\partial}{\partial \mu_{\ell_j}} g(\nu^0_{\ell_j}) \Big] =$$

$$= -\Theta(\vartheta) \nu_{V_2} \nu_{V_1} \nu_V^{m_V - 1} \Big[\frac{\partial}{\partial \mu_{\ell_j}} g(\nu^0_{\ell_j}) \Big] \qquad (5.13)$$

for case 2. We used that in the second sum (5.13) the derived propagator has changed its sign: the propagator is indeed even (so that its derivative is odd) and the arrow superimposed to the line ℓ_u has a different direction in the sets of trees corresponding to the two sums above.

Then the sum of (5.12) and (5.13) gives zero, hence implies (5.9).

In order to conclude the proof of lemma 5, it is enough to use the following properties.

By using (1.3), (2.5) and the fact that $x^m \leq m! \rho^{-m} e^{\rho x}$ for all $x, \rho \in \mathbb{R}_+$, one finds

$$\Big(\prod_{u \in V} |f_{\nu_u}| \frac{|\nu_u|^{m_u}}{m_u!} \Big) \leq C_1^k D_1^{k_V - 1 + m_V}(\delta) e^{-(\xi - \delta)|\nu|}, \qquad (5.14)$$

for any $\xi > \delta > 0$ and some constant $D_1(\delta)$ such that

$$0 < \lim_{\delta \to 0^+} D_1(\delta)\, \delta < \infty.$$

Furthermore the second derivative of a propagator on scale 1 is trivially bounded by $CD_0 q^{-4}$ (C is the same bound as in (2.6)), so that the second line in (5.7) gives a contribution which cancels exactly when summed together with the values of all trees in $\mathcal{F}_V(\vartheta)$, while the third line admits the bound (4.5). □

6. Asympotics

Define $\bar{u}(\alpha, \varepsilon, \omega) \equiv \bar{u}_{p/q}(\alpha, \varepsilon, \omega)$ and $\bar{\bar{u}}(\alpha, \varepsilon) \equiv \bar{\bar{u}}_{p/q}(\alpha, \varepsilon)$ as in (1.19) and (1.20) and write

$$\bar{\bar{u}}(\alpha, \varepsilon) = \sum_{k \geq 1} \sum_{\nu \in \mathbb{Z}} \varepsilon^k e^{i \nu \alpha}\, \bar{\bar{u}}_\nu^{(k)}. \qquad (6.1)$$

As we shall show now, only the trees without resonances can contribute to $\bar{\bar{u}}(\alpha, \varepsilon)$.

Lemma 6. *If $\vartheta \in \mathcal{T}_{\nu,k}$ and $N_0^R(\vartheta) \neq 0$, then $N_0^*(\vartheta) < k(\vartheta)/r^*$.*

Proof. The proof is by induction on the order of the tree. Of course for $k = 1$ there can be no resonance, so that the bound is trivially satisfied.

If the root line of ϑ is either on scale 1 or on scale 0 and resonant, then (4.1) shows that $N_0^*(\vartheta) < k(\vartheta)/r^*$.

If the root line of ϑ is on scale 0 and nonresonant, then

$$N_0^*(\vartheta) = 1 + \sum_{j=1}^m N_0^*(\vartheta_j),$$

where $\vartheta_1, \dots, \vartheta_m$ are the subtrees entering the cluster T: by lemma 4 one has $k(\vartheta_1) + \ldots + k(\vartheta_m) \leq k - r^*$. Then we can iterate the procedure by considering the trees $\vartheta_1, \ldots, \vartheta_m$. If at least one of them, say $\vartheta_{\bar{m}}$, with $1 \le \bar{m} \le m$, has the root line either on scale 1 or on scale 0 and resonant, one can reason as above to deduce that $N_0^*(\vartheta_{\bar{m}}) < k(\vartheta_{\bar{m}})/r^*$ and the statement is proven. Otherwise one continues until a line ℓ on scale 0 and resonant is reached (as $N_0^R(\vartheta) \neq 0$ such a line exists). For the subtree having ℓ as root line one can reason as above and the statement follows again. □

Then the following result can be proven.

Lemma 7. *The function* $\bar{\bar{u}}(\alpha, \varepsilon)$ *admits the tree representation*

$$\bar{\bar{u}}_\nu^{(k)} = \begin{cases} \sum'_{\vartheta \in \mathcal{T}_{\nu,k}} \mathrm{Val}'(\vartheta), & \text{if } \nu \in q\mathbb{Z} \setminus \{0\}, k \in r^*\mathbb{N}, \\ 0 & \text{otherwise}, \end{cases} \tag{6.2}$$

where $\sum'$ *means that only trees* ϑ *with* $N_0^R(\vartheta) = 0$ *and* $N_0^*(\vartheta) = k/r^*$ *have to be summed over and* $\mathrm{Val}'(\vartheta)$ *differs from* $\mathrm{Val}(\vartheta)$ *as, for* ν *multiple of* q, $\gamma(\nu)$ *has to be replaced with* $(2\pi i\nu)^2$.

Proof. As $\gamma(\nu) = O(\zeta^2)$ for $\nu \in q\mathbb{Z}$ and $\gamma(\nu) = O(1)$ otherwise (see (2.6)), by lemma 4, lemma 6 the cancellation mechanisms, one has that

$$\lim_{\zeta \to 0} |\mathrm{Val}(\vartheta)| \, \zeta^{2k/r^*} = 0,$$

whenever $N_0^R(\vartheta) \ge 1$, so that only trees without resonances are not vanishing for $\zeta \to 0$; this means that one must have $N_0(\vartheta) = N_0^*(\vartheta)$. Moreover $N_0(\vartheta) \le \lfloor k(\vartheta)/r^* \rfloor$ and

$$\lim_{\zeta \to 0} \left| \bar{\bar{u}}_\nu^{(k)} \right| < \infty$$

for any $k \ge 1$ and for any $\nu \in \mathbb{Z}$. Then if $k \in r^*\mathbb{N}$ and $N_0(\vartheta) = k/r^*$ one has

$$\lim_{\zeta \to 0} \left| \bar{\bar{u}}_\nu^{(k)} \right| \ge 0,$$

while

$$\lim_{\zeta \to 0} \bar{\bar{u}}_\nu^{(k)} = 0$$

otherwise. So rescaling $\varepsilon \to \varepsilon\zeta^{2/r^*}$ and taking the limit $\zeta \to 0$, one finds that only the trees with $k \in r^*\mathbb{Z}$ and $N_0(\vartheta) = k/r^*$ can have a nonvanishing value contributing to $\bar{\bar{u}}_\nu^{(k)}$. Finally, for such trees to have really a nonvanishing contribution, one can easily see, by using that $N_0^*(\vartheta) = k/r$ and $N_0^R(\vartheta) = 0$, the scale of the root line must be 0 (otherwise $N_0^*(\vartheta) < k/r^*$ by (4.1)), so that the momentum ν flowing through the root line has to be a multiple of q. □

Now we come back to (1.17) and prove the following trivial properties.

Lemma 8. *There exists a solution to* (1.17) *with* $r_0 = 2$ *if and only if there exist* $\nu_i, \nu_j \in \mathcal{A}(f)$, *with* $\nu_i < \nu_j$, *such that* $\nu_i = c_i + l_i q$ *and* $\nu_j = c_j + l_j q$, *with* $c_i + c_j = q$.

Proof. If (1.17) has a solution with $r_0 = 2$ then there exist $i \neq j$ such that either $r_i + r_j = 2$ or $r_i + r'_j = 2$. The first relation gives $\nu_i + \nu_j = Rq$ for some $R > 0$, while the second one would give $\nu_i - \nu_j = Rq$, so that it has to be discarded, if $|I_c(f)| = 1$ for any $c \in A_{p/q}(f)$, by lemma 1. Writing $\nu_i = c_i + l_i q$ and $\nu_j = c_j + l_j q$, with $l_i, l_j \in \mathbb{Z}_+$ and $c_i, c_j < q$, one has $\nu_i + \nu_j = (c_i + c_j) + (l_i + l_j)q = Rq$ so that $c_i + c_j = sq$, with $s \in \mathbb{N}$, and $c_i + c_j < 2q$: then $c_i + c_j = q$. □

Lemma 9. *For any solution to* (1.17) *one has* $r_i r'_i = 0 \ \forall i = 1, \ldots, M$.

Proof. Suppose both $r_i \neq 0$ and $r'_i \neq 0$. Set $s_1 = \min\{r_i, r'_i\}$ and $s_2 = \max\{r_i, r'_i\} - s_1$: then (1.17a) does not change by replacing $(r_i, r'_i) = (s_1, s_1 + s_2)$ with $(0, s_2)$, while (1.17b) is decreased by $2s_1$. □

Remark 5. Note that when $q \notin A_{p/q}(f)$ and $|I_{\bar{c}}(f)| \geq 2$ for some $\bar{c}$, then we can also consider the Diophantine problem (1.17) and it is easy to see that in general there can be solutions to (1.17) with $r_0 = 2$. This does not modify the value of r^* in (1.18), but, as we shall see below, it is a case which has to be explicitly taken into account if one wants to construct all possible clusters with two nodes.

Given any tree ϑ, to any node $u \in \vartheta$ we associate the labels

$$\sigma_u = \pm 1, \tag{6.3a}$$

$$c_u \in A_{p/q}(f), \tag{6.3b}$$

$$l_u \in B_{c_u}(f), \tag{6.3c}$$

where $A_{p/q}(f)$ and $B_c(f)$ are defined in (1.14) and (1.15), respectively; then for any node $u \in \vartheta$ one has that its mode label ν_u can be expressed as

$$\nu_u = \sigma_u \left(c_u + l_u q \right); \tag{6.4}$$

note that $\sigma_u = \mathrm{sign}(\nu_u)$ and $\sigma_u l_u = s_u$ with the notations used in the proof of proposition 1.

Given $A_{p/q}(f)$, define the set

$$C(f) = \{ c \in A_{p/q}(f) | \, |I_c(f)| \geq 2 \} \, . \tag{6.5}$$

Then the minimal trees ϑ (*i.e.* the trees with the minimal number of nodes) such that $\mathrm{Val}'(\vartheta) \neq 0$ have order $k(\vartheta) = r^*$ and they are obtained in the following way.

- If $r^* = 1$, fix $\sigma = \pm 1$ and $l \in B_q(f)$: the only line $\ell \in \vartheta$ has momentum $\nu_\ell = \sigma(q + lq)$.
- If $r^* = 2$, fix $\sigma = \pm 1$. For any $c \in C(f)$, fix $l_1, l_2 \in B_c(f)$ with $\sigma(l_1 - l_2) > 0$: then the two nodes u_1 and u_2 of ϑ have mode labels ν_1 and $-\nu_2$ such that $\nu_i = c + l_i q$ for $i = 1, 2$. Otherwise fix $\{c_1, c_2\}$ such that $c_1 + c_2 = q$ and set

$\nu_i = c_i + l_i q$ with $l_i \in B_{c_i}(f)$ for $i = 1, 2$. Of course, if ℓ is the root line of ϑ, $\nu_\ell = \sigma(l_1 - l_2)q$ in the first case, while $\nu_\ell = q + (l_1 + l_2)q$ in the latter.

- If $r^* = r_0 \geq 3$, fix $\sigma = \pm 1$ and $\{\nu_1, \ldots, \nu_{r_0}\}$, with $\nu_j \in \mathcal{A}(f)$ for $j = 1, \ldots, r_0$, such that $\nu_1 + \ldots + \nu_{r_0} = Rq$, with $R > 0$: the root line ℓ of ϑ has momentum $\nu_\ell = Rq$.

Remark 6. Note that, in the case $r^* \geq 3$, $\forall c \in A_{p/q}(f)$ one has $|I_c(f)| = 1$ and $|B_c(f)| = 1$, that is there is only one integer $l \in B_c(f)$ and one integer $\nu \in \mathcal{A}(f)$ such that $\nu = c + lq$.

By using the above results and reasoning as in [1], sect. 6, we can prove the following result, analogous to lemma 5 of [1].

Lemma 10. *The function* $\bar{\bar{u}}(\alpha, \varepsilon)$ *defined in* (1.20) *satisfies the differential equation:*

$$\frac{\mathrm{d}^2 \bar{\bar{u}}(\alpha,\varepsilon)}{\mathrm{d}\alpha^2} = \varepsilon \sum_{\sigma=\pm 1} \sum_{l \in B_q(f)} \mathcal{C}_{p/q}^{\sigma(l+1)q}(f) \, \exp\left[i(\sigma(l+1)q\,(\alpha + \bar{\bar{u}}(\alpha,\varepsilon)) \right], \tag{6.6}$$

for $r^* = 1$,

$$\frac{\mathrm{d}^2 \bar{\bar{u}}(\alpha,\varepsilon)}{\mathrm{d}\alpha^2} = \varepsilon^2 \sum_{\sigma=\pm 1} \Big[\sum_{c \in C(f)} \sum_{\substack{l_1, l_2 \in B_c(f), \\ \mu \equiv \sigma(l_1 - l_2) > 0}} \mathcal{C}_{p/q}^{\mu q}(f) \, \exp\left[i\mu q\,(\alpha + \bar{\bar{u}}(\alpha,\varepsilon)) \right] +$$

$$+ \sum_{\substack{c_1, c_2 \in A_{p/q}(f), \\ c_1 + c_2 = q}} \sum_{\substack{l_1 \in B_{c_1}(f), \\ l_2 \in B_{c_2}(f)}} \mathcal{C}_{p/q}^{\sigma(l_1+l_2+1)q}(f) \, \exp\left[i\sigma(l_1+l_2+1)q\,(\alpha + \bar{\bar{u}}(\alpha,\varepsilon)) \right] \Big], \tag{6.7}$$

for $r^* = 2$ *and*

$$\frac{\mathrm{d}^2 \bar{\bar{u}}(\alpha,\varepsilon)}{\mathrm{d}\alpha^2} = \varepsilon^{r_0} \sum_{\sigma=\pm 1} \sum_{R > 0} \sum_{\substack{\nu_1, \ldots, \nu_{r_0} \in \mathcal{A}(f), \\ \nu_1 + \ldots + \nu_{r_0} = Rq}} \mathcal{C}_{p/q}^{\sigma R q}(f) \, \exp\left[i\sigma R q\,(\alpha + \bar{\bar{u}}(\alpha,\varepsilon)) \right], \tag{6.8}$$

for $r^* \geq 3$, *with boundary conditions* $\bar{\bar{u}}(0,\varepsilon) = \bar{\bar{u}}(2\pi,\varepsilon) = 0$ *and*

$$\mathcal{C}_{p/q}^{\nu}(f) = \sum_{\vartheta \in \mathcal{T}_{r^*,\nu}} \mathrm{Val}'(\vartheta), \tag{6.9}$$

where $r^* = r^*(f)$ *is defined in* (1.18).

We can define

$$\mathcal{N}(f) = \begin{cases} \{\nu \in \mathbb{N} | \nu \in I_q(f)\}, & \text{if } r^* = 1, \\ \{\nu = \nu_1 + \nu_2 | (\nu_1, \nu_2) \in \mathbb{N}^2 & \\ \qquad \text{and either } \nu_1 + \nu_2 = q \text{ or } \nu_1 - \nu_2 \in q\mathbb{Z}\}, & \text{if } r^* = 2, \\ \{\nu = \nu_1 + \ldots + \nu_{r_0} | (\nu_1, \ldots, \nu_{r_0}) \in \mathbb{Z}^{r_0} & \\ \qquad \text{and } \nu_1 + \ldots + \nu_{r_0} = Rq \text{ with } R > 0\}, & \text{if } r^* = r_0 \geq 3, \end{cases} \tag{6.10}$$

so that the three differential equations appearing in the statement of lemma 10 can be all expressed as

$$\frac{\mathrm{d}^2\bar{\bar{u}}(\alpha,\varepsilon)}{\mathrm{d}\alpha^2} = \varepsilon^{r^*} \sum_{\sigma=\pm 1} \sum_{\nu\in\mathcal{N}(f)} \mathcal{C}^{\sigma\nu}_{p/q}(f)\, \exp\left[i\sigma\nu\left(\alpha+\bar{\bar{u}}(\alpha,\varepsilon)\right)\right], \tag{6.11}$$

where $\nu \in q\mathbb{N}$.

Remark 7. For $f(x) = \sin x$ (standard map), (6.11) reduces to Eq. (6.13) of [1] and (6.9) reduces to Eq. (6.14) of [1]. So lemma 10 extends lemma 5 of [1] to more general perturbations.

Remark 8. In the case $r^* = 1$, one immediately sees that $C^{\nu}_{p/q} = (2\pi i\nu)^{-1} f_\nu$.

Note that, by the bounds of sect. 4, one has

$$\left|\mathcal{C}^{\sigma\nu}_{p/q}(f)\right| \le B_0^{r^*} C_1^{r^*} D_1^{r^*}(\delta)\, e^{-(\xi-\delta)|\nu|},$$

for any $\xi > \delta > 0$ (see (1.3)), assuring summability on $\nu \in \mathcal{N}(f)$ in (6.11).

Eq. (6.11) is easily reducible to an ordinary differential equation with separable variables (just take $\xi = \alpha + \bar{\bar{u}}(\alpha,\varepsilon)$ to reduce it to the Hamilton equations of a one-dimensional system); moreover if $f(x)$ is a trigonometric polynomial then

$$\sum_{\nu\in\mathcal{N}(f)} C^{\sigma\nu}_{p/q}(f)\, \exp\left[i\sigma\nu\left(\alpha+\bar{\bar{u}}(\alpha,\varepsilon)\right)\right]$$

is a trigonometric polynomial, so that - by a classical change of variable, see *e.g.* [20], p. 72 - the solution of (6.11) is the inverse of an hyperelliptic integral (which reduces to an elliptic function in the case of the standard map explicitly considered in [1]); see [21], [22] for an introduction to the theory of hyperelliptic functions.

7. Conclusions

Theorem 1 shows that generically the radius of convergence of the KAM invariant curves for the generalized standard maps of the form (1.1) scales as $\rho(\omega) = O(\zeta^2)$, for $\omega = p/q+\zeta$, because for generic analytic perturbations $f(x)$ one has $q \in A_{p/q}(f)$.

The situation is quite different if trigonometric polynomials are explicitly considered. In such a case there is strong dependence of the scaling factor on the rational rotation number p/q (more precisely on q); compare with the numerical results in [5]. So a first conclusion we can draw from our analysis is that the case of the standard map is non-generic.

Note that proposition 1 yields $r^* \le q$, so that the best behaviour one can hope to get for the radius of convergence is of the form $O(\zeta^{2/q})$.

Consider the set of analytic functions f with (arbitrarily) prefixed norm, say $\|f\|_0 = 1$: we shall denote it by $\mathcal{S}_1$, with

$$\mathcal{S}_1 = \{f \in \mathcal{B}_{\xi'}\ \forall \xi' < \xi \,|\, \|f\|_0 = 1\}, \tag{7.1}$$

using the notations introduced in sect. 1. Define

$$R(\omega) = \inf_{f \in \mathcal{S}_1} \rho_f(\omega), \qquad R_r(\omega) = \inf_{\substack{f \in \mathcal{S}_1, \\ r^*(f)=r}} \rho_f(\omega). \tag{7.2}$$

Theorem 2 shows that

$$R_r(\omega) \geq C_r |\zeta|^{2/r}, \qquad R(\omega) \geq C_0 |\zeta|^2, \tag{7.3}$$

for some constants C_r and C_0.

On the other hand if $r^* = q$ one has at least one function $f \in \mathcal{S}_1$ such that $\rho_f(\omega) \leq C|\zeta|^{2/q}$ for some constant C: just take $f(x) = \sin x$, see [1].

More generally, for any $r \in \{1, \dots, q\}$, one can consider a function f which is a trigonometric polynomial of degree $N_f < q$ and with $r^* = r$: the radius of convergence of the solution $\bar{\bar{u}}$ to the equation (1.20) is nonvanishing, as this solution must depend analytically on ε in a neighborhood of the origin, and cannot be infinite, by the properties of the hyperelliptic functions, [22]. As the radius of convergence $\rho_f(\omega)$ is related to the radius of convergence of $\bar{\bar{u}}$ by the scaling law $\varepsilon \to \varepsilon \zeta^{2/r^*}$, this means that there exists a constant C_f such that $\rho_f(\omega) \leq C_f |\zeta|^{2/r^*}$ for such a function f.

So we can conclude that

$$R_r(\omega) = D_r(\omega) |\zeta|^{2/r}, \qquad R(\omega) = D_0(\omega) |\zeta|^2, \tag{7.4}$$

where $D_r(\omega)$ and $D_0(\omega)$ are bounded functions such that

$$0 < D_{r1} < D_r(\omega) < D_{r2} < \infty, \qquad 0 < D_{01} < D_0(\omega) < D_{02} < \infty, \tag{7.5}$$

for some constants $D_{r1}, D_{r2}, D_{01}, D_{02}$, independent on ω.

Note that the behaviour $\rho_f(\omega) = C_f(\omega)|\zeta|^{2/q}$, with $C_f(\omega)$ uniformily bounded, holding for the standard map, $f(x) = \sin x$, is related to the interpolation through the Bryuno function of the radius of convergence $\rho_f(\omega)$ for real ω which has been proven in [2]; the relation can be exactly formulated by using [23], where the extension to the complex plane of the Bryuno function is discussed.

Therefore no universal dependence of the radius of convergence on p/q is expected in general for the invariant curves of maps of the forms (1.1): therefore no interpolation through the Bryuno function (or possible generalizations of it) can be attempted generically for $\rho_f(\omega)$ or $R(\omega)$, in the case of real rotation numbers, as done in [2] in the case of the standard map. Compare also the results holding for the Siegel problem, [24], for which the situation is quite different.

On the contrary if we restrict ourselves to the space $\{f \in \mathcal{S}_1 | r^*(f) = r\}$, at least for $r^* = q$ the interpolation through the Bryuno function holds for $R_q(\omega)$; see above. Then one could conjecture that some possible generalization of the Bryuno function interpolates $R_r(\omega)$, but of course one needs to work it out.

As a final comment, we note that, instead of generalized standard map like (1.1), one can study also generalized semistandard maps, by considering perturbations $f(x)$ of the form (1.2), with the sum *only* over the integers $\nu \geq 1$, *i.e.*,

$$f(x) = \sum_{\nu \geq 1} f_\nu e^{i\nu x}, \tag{7.6}$$

where the coefficients f_ν satisfy (1.3). If $f_1 = 1$ and $f_\nu = 0$ $\forall \nu \geq 2$, (7.6) reduces to the semistandard map.

Note that in such a case the analysis performed in this paper could be easily adapted (and less work would be required, as there would not be the problem of resonances: all clusters are nonresonant for perturbations of the form (7.6)) and results similar to the theorems in sect. 1 would immediately follow.

However the absence of negative Fourier labels induces a condition which can be different to (1.17) to determine the scaling of the radius of convergence, as it can be easily checked. For instance it is straightforward to see that, even if there is a $c \in A_{p/q}(f)$ such that $c \neq q$ and $|I_c(f)| \geq 2$ (the same symbols introduced for the maps (1.1) can be used with the same meaning), it is no longer possible to form a (nonresonant) cluster with two nodes u_1 and u_2 such that $\nu_{u_1} - \nu_{u_2} \in q\mathbb{Z}$ and $\nu_{u_1} + \nu_{u_2} \in q\mathbb{Z}$, simply because all mode labels are strictly positive. An explicit example can be the following. Consider $f(x) = 2\sin x + 2\sin 4x$ as generalized standard map and $f(x) = e^{ix} + e^{4ix}$ as generalized semistandard map, for $q = 3$: one obtains $r^*(f) = 2$ for the generalized standard map, while the analogous exponent for the generalized semistandard map would be 3.

This means that the scaling behaviour of the radii of convergence for generalized standard and semistandard maps having the same Fourier labels (*i.e.*, admitting the same sequences $\tilde{I}_m(f)$) can be completely different (as the above example shows): this is a feature quite new with respect to the case of the standard and semistandard maps, for which the scaling behaviour is the same (and whose radii of convergence, for real rotation numbers, admit the same interpolation in terms of the Bryuno function; see [2] and [9]). Note that such differences may arise only in the non-generic case (of theorem 2).

Ackowlegdments. We thank M. Kontsevich and S. Marmi for useful discussions. One of us (GG) thanks IHES for hospitality while part of this work was done.

References

[1] Berretti, A. and Gentile, G., *Scaling Properties for the Radius of Convergence of a Lindstedt Series: the Standard Map*, J. Math. Pures. Appl. (9) **78** (1999), no. 2, 159–176

[2] Berretti, A. and Gentile, G., *Bryuno Function and the Standard Map*, Preprint (1998).

[3] Berretti, A., Celletti, A., Chierchia, L. and Falcolini, C., *Natural Boundaries for Area-Preserving Twist Maps*, J. Stat. Phys. **66** (1992), no. 5–6, 1613–1630

[4] Falcolini, C. and de la Llave, R., *Numerical Calculation of Domains of Analyticity for Perturbation Theories in presence of Small Divisors*, J. Stat. Phys. **67** (1992), no. 3–4, 645–666.

[5] Berretti, A. and Marmi, S., *Scaling, Perturbative Renormalization and Analyticity for the Standard Map and Some Generalizations*, Chaos Solitons Fractals **5** (1995), no. 2, 257–269

[6] Berretti, A., *Analyticity, Scaling and Renormalization for Some Complex Analytic Dynamical Systems*, Planet. Space Sci. **46** (1998), 1487–????

[7] Anosov, D. V. and Arnol'd, V. I., *Dynamical Systems I. Ordinary Differential Equations and Smooth Dynamical Systems*, Encyclopaedia of Mathematical Sciences **1**, Springer, Berlin, 1988

[8] Katok, A. and Hasselblatt, B., *Introduction to the Modern Theory of Dynamical Systems*, Encyclopedia of Mathematics and its Applications **54**, Cambridge University Press, Cambridge, 1995.

[9] Davie, A. M., *The critical function for the semistandard map*, Nonlinearity **7** (1994), no. 1, 219–229

[10] Garey, M. R., and Johnson, D. S., *Computers and Intractability: A Guide to the Theory of NP-Completeness*, Freeman, San Francisco, 1979

[11] Gentile, G. and Mastropietro, V., *Methods for the Analysis of the Lindstedt Series for KAM tori and Renormalizability in Classical Mechanics. A Review with Some Applications*, Rev. Math. Phys. **8** (1996), no. 3, 393–444

[12] Gallavotti, G., *Twistless KAM tori*, Comm. Math. Phys. **164** (1994), no. 1, 145–156

[13] Gallavotti, G., *Twistless KAM tori, quasiflat homoclinic intersections, and other cancellations in the perturbation series of certain completely integrable Hamiltonian systems. A review*, Rev. Math. Phys. **6** (1994), no. 3, 343–411

[14] Gentile, G. and Mastropietro, V., *KAM theorem revisited*, Phys. D **90** (1996), no. 3, 225–234

[15] Eliasson, L. H., *Absolutely Convergent Series Expansions for Quasi-periodic Motions*, University of Stockholm preprint (1988), and Math. Phys. Electron. J. **2** (1996), paper 4, 1–33, `<http://mpej.unige.ch/mpej/Vol/2/4.ps>`

[16] Poincaré, H., *Les méthodes nouvelles de la mécanique céleste*, Gauthier-Villars, Paris, Vol. I, 1892, Vol. II, 1893, Vol. III, 1899

[17] Pöschel, J., *Invariant manifolds of complex analytic mappings*, Les Houches, XLIII (1984), K. Osterwalder & R. Stora Eds., North-Holland, Amsterdam, Vol. II, 949–964, 1986

[18] Moser, J., *On invariant curves of area preserving mappings of the annulus*, Nachr. Akad. Wiss. Göttingen, Math. Phys. Kl. II **1962** (1962), 1–20

[19] Chierchia, L., and Falcolini, C., *A Direct Proof of a Theorem by Kolmogorov in Hamiltonian Systems*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **21** (1994), no. 4, 541–593

[20] Whittaker, E. T., *A Treatise on the Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press, Cambridge, 1947

[21] Baker, H. F., *Abelian functions*, reprint of the 1897 original, Cambridge University Press, Cambridge (1995)

[22] Siegel, C. L., *Topics in Complex Function Theory*, John Wiley & Sons, New York, Vol. I. Elliptic Functions and Uniformization Theory, 1969 Vol. II. Automorphic functions and Abelian Integrals, 1971 Vol. III. Abelian Functions and Modular Functions of Several Variables, 1973

[23] Marmi, S., Moussa, P. and Yoccoz, J.-C., *Complex Brjuno function*, Preprint (1999)

[24] Yoccoz, J.-C., *Théorème de Siegel, Nombres de Bruno et Polinômes Quadratiques*, Astérisque **231** (1995), 3–88

Alberto Berretti, Dipartimento di Matematica, II Università di Roma (Tor Vergata), Via della Ricerca Scientifica, 00133 Roma, Italy and Istituto Nazionale di Fisica Nucleare, Sez. Tor Vergata

E-mail address: `berretti@roma2.infn.it`

Guido Gentile, Dipartimento di Matematica, Università di Roma 3, Largo S. Leonardo Murialdo 1, 00146 Roma, Italy

E-mail address: `gentile@matrm3.mat.uniroma3.it`