Periodic and quasi-periodic orbits for the standard map

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Abstract. *We consider both periodic and quasi-periodic solutions for the standard map, and we study the corresponding conjugating functions, i.e. the functions conjugating the motions to trivial rotations. We compare the invariant curves with rotation numbers* ω *satisfying the Bryuno condition and the sequences of periodic orbits with rotation numbers given by their convergents* $\omega_N = p_N/q_N$*. We prove the following results for* $N \to \infty$*: (1) for rotation numbers* ω_N *we study the radius of convergence of the conjugating functions and we find lower bounds on them, which tend to a limit which is a lower bound on the corresponding quantity for* ω*; (2) the periodic orbits consist of points which are more and more close to the invariant curve with rotation number* ω*; (3) such orbits lie on analytical curves which tend uniformly to the invariant curve.*

1. Introduction

Recently a new approach to KAM theory has been introduced in [1], based directly on the study of the perturbative series (*Lindstedt series*), and without using the standard rapidly convergent iterative procedure.

Here we follow such an approach (in the Renormalization Group interpretation given by [2] and developed in a series of subsequent papers; see [3] for a list of references), and unify the analysis for periodic and quasi-periodic motions of the standard map.

The *standard map* is a rather special system, introduced in [4] and [5], which shows a non-trivial dynamical behaviour and is, at the same time, simple enough to avoid any useless technical intricacies. The method we use should be extended to more general systems: the analysis may become a little more involved, but we think that no extra conceptual difficulties should arise (see [6] for a review of these techniques in a more general context). But, for clarity purposes, we prefer to confine ourselves to a simpler model.

We shall be interested in the relation between the KAM invariant curves (on which the motion is quasi-periodic) and the periodic orbits corresponding to rotation numbers tending to those of the invariant curves. When the perturbation is switched on, it is well known that the invariant curves with rational rotation number disappear, but some trace of them is left: there are curves, which can be interpreted as remnants (or 'ghosts') of the unperturbed invariant curves, on which the points of the periodic orbits have support: we can still define a *conjugating function*, i.e. a function which conjugates the motion to a trivial rotation, with the only difference that now the initial phase has to be fixed to an appropriate value. Even more we can consider functions which parametrize the remnants and which reduce to the conjugating functions in correspondence with the points of the periodic orbits: we shall refer to them as the *interpolating functions*. For rotation numbers ω_N tending to the rotation number ω of a surviving invariant curve along the sequence of best approximants (*convergents*), the remnants are "analytically close" to the invariant curve itself. By "analytically close" we mean that the interpolating functions which define such remnants are analytic and converge uniformly to the (analytic) conjugating function for the corresponding invariant curve: a trivial application of the Cauchy formulae for the derivatives of analytic functions shows then that also the derivatives converge uniformly to the derivatives of the conjugating function of the invariant curve; the convergence therefore happens in quite a strong sense. The precise statements, with the proper setup of the domains of convergence, will be given later as it requires to establish first some definitions.

The relation between periodic and quasi-periodic orbits can be studied also through variational methods, [7]: we think that the interest of our approach relies mostly on the possibility to have an accurate knowledge of the conjugating functions, in particular of their analyticity properties, and to obtain estimates which depend optimally on the involved parameters – in our case on the rotation numbers. We shall be able to provide lower bounds on the radius of convergence of the conjugating functions (in terms of a "truncated" Bryuno function), which are the analogue of what we found in [8] for the conjugating function of the quasi-periodic motions.

Of course it is known that, while the invariant curves "break" at a certain critical threshold, periodic orbits persist for all *real* values of the perturbative parameter (see [9]). However singularities arise in the *complex* plane, and our analysis, together with the numerical results in [10], suggests that, for rotation numbers tending to a Bryuno number along the sequence of the convergents, the singularities of the conjugating functions of the corresponding periodic orbits tend to build up the "natural boundary" (of the analyticity domain) for the conjugating function of the invariant curve with that Bryuno number. We remark that so far the existence of such a natural boundary is only a numerical result and no rigorous proof has been given. This connection between analyticity properties of the conjugating functions is a point which – we think – should deserve further investigation, as it relates to the so-called "Greene's method" to determine numerically the critical threshold and it would help to understand the mechanism of the breakup of the invariant curve itself.

The plan of the paper is as follows. In Section 2 we recall the definition of the standard map and of the Bryuno function, by introducing a natural extension of the latter to rational numbers. In Section 3 we discuss the existence and the analyticity of periodic solutions; in particular we provide lower bounds on the radius of convergence of the conjugating function. In Section 4 we consider a Bryuno number ω and the sequence of its convergents ω_N, we compare the periodic solutions with rotation numbers ω_N with the quasi-periodic solution with rotation number ω, and we state our main result, so making formally more precise the notion of analytical closeness introduced above. The proof of the theorems is achieved in the remaining Sections 5, 6 and 7. We assume that the reader is familiar with the techniques and results of [8], which we rely heavily on.

2. The standard map

2.1. *The standard map.* The standard map is a discrete one-dimensional dynamical system generated by the iteration of the symplectic map of the cylinder to itself, $T_\varepsilon \colon \mathbb{T} \times \mathbb{R} \to \mathbb{T} \times \mathbb{R}$, given by

$$T_\varepsilon \colon \begin{cases} x' = x + y + \varepsilon \sin x, \\ y' = y + \varepsilon \sin x. \end{cases} \tag{2.1}$$

We look for a change of variables of the form

$$\begin{cases} x = \alpha + u(\alpha, \varepsilon, \omega), \\ y = 2\pi\omega + v(\alpha, \varepsilon, \omega), \end{cases} \tag{2.2}$$

such that the dynamics in the α variable is a trivial rotation

$$\alpha' = \alpha + 2\pi\omega, \tag{2.3}$$

where $\omega \in [0,1]$ is called the *rotation number*.

One immediately checks that the function $v(\alpha,\varepsilon,\omega)$ is related to the function $u(\alpha,\varepsilon,\omega)$ by

$$v(\alpha,\varepsilon,\omega) = u(\alpha,\varepsilon,\omega) - u(\alpha - 2\pi\omega,\varepsilon,\omega), \tag{2.4}$$

while $u(\alpha,\varepsilon,\omega)$ is a solution of the functional equation

$$(D_\omega u)(\alpha,\varepsilon,\omega) \equiv u(\alpha + 2\pi\omega,\varepsilon,\omega) + u(\alpha - 2\pi\omega,\varepsilon,\omega) - 2u(\alpha,\varepsilon,\omega) = \varepsilon \sin(\alpha + u(\alpha,\varepsilon,\omega)) . \tag{2.5}$$

We shall call $u = u(\alpha,\varepsilon,\omega)$ the *conjugating function.*

2.2. *Continuous fraction expansion.* For any $\omega \in [0,1]$ let us write $\omega = [0, a_1, a_2, a_3, \ldots]$, where $\{a_n\}$ are the *partial quotients* of ω and call $\{\omega_n\} \equiv \{p_n/q_n\}$ the sequence of *convergents* of ω, [11].

If $\omega \in \mathbb{Q} \cap [0,1]$, i.e. $\omega = p/q$, with $p \le q$ and $\gcd(p,q) = 1$, then there exists $N = N(\omega)$ such that $\omega = [0, a_1, a_2, a_3, \ldots, a_N]$, i.e. such that $a_{N+1} = \infty$: in such a case the sequence of convergents is finite and the last one is given by $p_N/q_N = p/q$. We can eliminate a trivial ambiguity by requesting that $a_N > 1$; in the following, though, we shall be interested essentially in given sequences of convergents, so that the problem does not arise. For such rational ω define

$$B_1(\omega) = \sum_{n=0}^{N-1} \frac{\log q_{n+1}}{q_n}. \tag{2.6}$$

For any $\omega \in [0,1] \cap \mathbb{R} \setminus \mathbb{Q}$ define

$$B_1(\omega) = \sum_{n=0}^{\infty} \frac{\log q_{n+1}}{q_n}, \tag{2.7}$$

and define ω a *Bryuno number* if it is irrational and $B_1(\omega) < \infty$; the latter is called the *Bryuno condition.* With a slight abuse of notation we shall call $B_1(\omega)$ the Bryuno function (see [12]); by analogy we shall define (2.6) the *truncated Bryuno function* of the rational number ω.

If ω is a Bryuno number then there exists a solution of the form (2.2), (2.3), with u, v analytic in α, ε, for ε small enough, and 2π-periodic in α; for the more restrictive Diophantine condition on ω, this follows from the standard KAM theorem. A more formal statement, which will be used later on, is the following.

2.3. THEOREM. *Let $\omega \in (0,1)$ be a Bryuno number. Then there exists $\rho(\omega) > 0$ such that there exists a solution of the form (2.2), (2.3), with $u(\alpha,\varepsilon,\omega)$ periodic in $\alpha \in \mathbb{T}$ and analytic in ε for $|\varepsilon| < \rho(\omega)$. There exists a positive constant C such that*

$$|\log \rho(\omega) + 2B_1(\omega)| < C, \tag{2.8}$$

uniformly in ω.

2.4. *Comments.* The proof of the existence of the invariant curve with rotation numbers satisfying a Diophantine condition is standard, and can be found in any textbook about KAM theory; for instance see [13]. The proof in the case of Bryuno numbers and the explicit derivation of the bound (2.8) are given in [14] and [8].

If $\omega = p/q$ is rational then the functional equation (2.5) admits no solution; however we shall see in Section 3 that it is possible to fix $\alpha = \alpha_0$ in such a way that $x_0 = \alpha_0 + u(\alpha_0, \varepsilon, \omega)$ is the initial datum of a periodic solution with period $2\pi p$, i.e. such that

$$u(\alpha_0 + 2\pi p) = u(\alpha_0). \tag{2.9}$$

This means that, after q iterates of the dynamics, the variable α has been shifted by $2\pi p$, so that the variables (x, y) have come back to their original values (x_0, y_0), up to a shift by $2\pi p$ in the x-direction.

2.5. *Remark.* Note that, if ω is a Bryuno number, and $\{\omega_N\}$ are the convergents of ω, then

$$\lim_{N\to\infty} B_1(\omega_N) = B_1(\omega). \tag{2.10}$$

Note also that $B_1(\omega)$ is still divergent on irrational, non-Bryuno numbers. It would actually be interesting to study the sequence of periodic orbits corresponding to such numbers, to understand the mechanism of divergence of Lindstedt series when the Bryuno condition is violated.

3. Periodic solutions for the standard map

3.1. *Periodic solutions.* When ω is a Bryuno number, it is well known that a quasi-periodic solution with rotation number ω exists: the orbit is a smooth curve, and the trajectory is dense on it (see e.g. [15], and [8] for estimates on the radius of convergence which depend optimally on the rotation number). In the periodic case the trajectory consists in a finite number of points which can be interpolated through a smooth curve in a rather arbitrary way: we shall look for a precise interpolating curve and show that, for rotation numbers of the form $\omega_N = p_N/q_N$, where ω_N are the convergents of the Bryuno number ω, the corresponding curves tend to the invariant KAM curve with rotation number ω.

Fix $\omega = p/q$ and, for any 2π-periodic function,

$$f(\alpha) = \sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha} \hat{f}_\nu, \tag{3.1}$$

write

$$\begin{aligned} &f(\alpha) = \bar{f}(\alpha) + \tilde{f}(\alpha), \\ &\bar{f}(\alpha) = \sum_{\nu\in\mathbb{Z}\setminus q\mathbb{Z}} e^{i\nu\alpha} \hat{f}_\nu, \qquad \tilde{f}(\alpha) = \sum_{\nu\in q\mathbb{Z}} e^{i\nu\alpha} \hat{f}_\nu. \end{aligned} \tag{3.2}$$

Note that, in Fourier space,

$$\left(D_\omega f\right)(\alpha) = \sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha} \gamma(\omega\nu)\, \hat{f}_\nu, \qquad \gamma(\omega\nu) = 2\left[\cos(2\pi\omega\nu) - 1\right] = 2\left[\cos(2\pi p\nu/q) - 1\right], \tag{3.3}$$

so that $(D_\omega \tilde{f})(\alpha) = 0$.

Then we can write

$$\begin{cases} u(\alpha, \varepsilon, \omega) = \bar{u}(\alpha, \varepsilon, \omega) + \tilde{u}(\alpha, \varepsilon, \omega), \\ \varepsilon \sin(\alpha + u(\alpha, \varepsilon, \omega)) \equiv S(\alpha, \varepsilon, \omega) = \bar{S}(\alpha, \varepsilon, \omega) + \tilde{S}(\alpha, \varepsilon, \omega), \end{cases} \tag{3.4}$$

so that (2.5) becomes

$$\left(D_\omega \bar{u}\right)(\alpha, \varepsilon, \omega) \equiv \bar{S}(\alpha, \varepsilon, \omega) + \tilde{S}(\alpha, \varepsilon, \omega). \tag{3.5}$$

We write also, formally, u, S as power series in ε, so that

$$
\begin{aligned}
u(\alpha,\varepsilon,\omega) &= \sum_{k=1}^{\infty} \varepsilon^k u^{(k)}(\alpha,\omega) = \sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha} \hat{u}_\nu(\varepsilon,\omega) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha} \hat{u}_\nu^{(k)}(\omega), \\
S(\alpha,\varepsilon,\omega) &= \sum_{k=1}^{\infty} \varepsilon^k S^{(k)}(\alpha,\omega) = \sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha} \hat{S}_\nu(\varepsilon,\omega) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha} \hat{S}_\nu^{(k)}(\omega),
\end{aligned}
\tag{3.6}
$$

so defining the Taylor-Fourier coefficients $\hat{u}_\nu^{(k)}(\omega)$ and $\hat{S}_\nu^{(k)}(\omega)$.

Then, to all perturbative orders k, (3.5) gives two equations:

$$
\begin{aligned}
(D_\omega \bar{u}^{(k)})(\alpha,\omega) &= \bar{S}^{(k)}(\alpha,\omega), \\
0 &= \tilde{S}^{(k)}(\alpha,\omega).
\end{aligned}
\tag{3.7}
$$

Note that for $k > 1$ we can express $S^{(k)}(\alpha,\omega)$ in terms of all $u^{(k')}(\alpha,\omega)$ with $k' < k$. In fact one has

$$
S^{(k)}(\alpha,\omega) = -\sum_{m=1}^{\infty} \sum_{\substack{k_1,\ldots,k_m\geq 1 \\ k_1+\ldots+k_m=k-1}} \frac{1}{m!} \left(\frac{\partial^{m+1}}{\partial\alpha^{m+1}} \cos\alpha \right) u^{(k_1)}(\alpha,\omega)\ldots u^{(k_m)}(\alpha,\omega). \tag{3.8}
$$

3.2. Lemma. *If the functions u, S are formally well defined, they have to be odd in α.*

3.3. *Proof of the lemma 3.2.* First of all note that the operator D_ω is even. Then the proof is by induction on k. For $k = 1$ one has

$$
S^{(1)}(\alpha,\omega) = \sin\alpha, \qquad u^{(1)}(\alpha,\omega) = (D_\omega^{-1} S^{(1)})(\alpha,\omega), \tag{3.9}
$$

which are obviously odd. If all functions $u^{(k')}(\alpha,\omega)$ are odd for $k' < k$ then, by (3.8), one has for $k > 1$

$$
\begin{aligned}
S^{(k)}(-\alpha,\omega) &= -\sum_{m=1}^{\infty} \sum_{k_1,\ldots,k_m} \frac{1}{m!} \left(\frac{\partial^{m+1}}{\partial\beta^{m+1}} \cos\beta \right)\Big|_{\beta=-\alpha} u^{(k_1)}(-\alpha,\omega)\ldots u^{(k_m)}(-\alpha,\omega) \\
&= (-1)^{m+1+m} S^{(k)}(\alpha,\omega) = -S^{(k)}(\alpha,\omega),
\end{aligned}
\tag{3.10}
$$

so that $S^{(k)}(\alpha,\omega)$ is odd; then also $u^{(k)}(\alpha,\omega)$ is odd. ■

3.4. Corollary. *The functions (3.6), if formally existing, can be written as*

$$
\begin{aligned}
u(\alpha,\varepsilon,\omega) &= \sum_{\nu\in\mathbb{N}} 2i\hat{u}_\nu(\varepsilon,\omega)\sin\nu\alpha = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\nu\in\mathbb{N}} 2i\hat{u}_\nu^{(k)}(\omega)\sin\nu\alpha, \\
S(\alpha,\varepsilon,\omega) &= \sum_{\nu\in\mathbb{N}} 2i\hat{S}_\nu(\varepsilon,\omega)\sin\nu\alpha = \sum_{k=1}^{\infty} \varepsilon^k \sum_{\nu\in\mathbb{N}} 2i\hat{S}_\nu^{(k)}(\omega)\sin\nu\alpha.
\end{aligned}
\tag{3.11}
$$

3.5. Lemma. *In (3.6) one has $|\nu| \leq k$; in other words one has $\hat{u}_\nu^{(k)}(\omega) = 0$ for $|\nu| > k$.*

3.6. *Proof of the lemma 3.5.* From (3.9) one obtains $\nu = \pm 1$ for $k = 1$. Suppose that for all $k' < k$ one has $\hat{u}_\nu^{(k')}(\omega) = 0$ when $|\nu| > k'$: then (3.8) gives

$$
\hat{S}_\nu^{(k)}(\omega) = -\sum_{m=1}^{\infty} \sum_{\substack{k_1,\ldots,k_m \\ k_1+\ldots+k_m=k-1}} \sum_{\substack{\nu_0,\nu_1,\ldots,\nu_m \\ \nu_0+\nu_1+\ldots+\nu_m=\nu}} \left[\left(\frac{(i\nu_0)^{m+1}}{m!2} \right) \hat{u}_{\nu_1}^{(k_1)}(\omega)\ldots\hat{u}_{\nu_m}^{(k_m)}(\omega) \right], \tag{3.12}
$$

where $\nu_0 = \pm 1$, so that $|\nu| \le |\nu_0| + |\nu_1| + \ldots + |\nu_m| \le 1 + (k_1 + \ldots + k_m) \le k$. ∎

3.7. Lemma. *There exists α_0 such that one has formally $\tilde{S}(\alpha_0, \varepsilon, \omega) = \tilde{u}(\alpha_0, \varepsilon, \omega) = 0$, while $\bar{S}(\alpha_0, \varepsilon, \omega)$ and $\bar{u}(\alpha_0, \varepsilon, \omega)$ are formally well defined.*

3.8. *Proof of the lemma 3.7.* From (3.6) and (3.11) one has that, if $\tilde{S}^{(k)}(\alpha_0, \omega)$ exists formally, then

$$\tilde{S}^{(k)}(\alpha_0, \omega) = \sum_{\nu \in q\mathbb{N}} 2i\hat{S}^{(k)}_\nu(\omega) \sin \nu\alpha_0 = 0, \tag{3.13}$$

if we fix $\alpha = \alpha_0$ such that

$$\sin q\alpha_0 = 0. \tag{3.14}$$

Then we can check by induction that the functions $u^{(k)}(\alpha, \omega)$ and $S^{(k)}(\alpha, \omega)$ are well defined for $\alpha = \alpha_0$. By the lemma 3.5 one has $\tilde{S}^{(k)}(\alpha, \omega) = \tilde{u}^{(k)}(\alpha, \omega) = 0$ for all α when $k < q$. Moreover $\bar{S}^{(k)}(\alpha, \omega)$ and $\bar{u}^{(k)}(\alpha, \omega)$ are well defined for all α when $k < q$ as $\gamma(\omega\nu) \neq 0$ for $|\nu| < q$; for $k = q$ one has from (3.12)

$$\hat{S}^{(q)}_\nu(\omega) = -\sum_{m=1}^{\infty} \sum_{\substack{k_1, \ldots, k_m \\ k_1 + \ldots + k_m = q-1}} \sum_{\substack{\nu_0, \nu_1, \ldots, \nu_m \\ \nu_0 + \nu_1 + \ldots + \nu_m = \nu}} \left[\left(\frac{(i\nu_0)^{m+1}}{m!2} \right) \hat{u}^{(k_1)}_{\nu_1}(\omega) \ldots \hat{u}^{(k_m)}_{\nu_m}(\omega) \right], \tag{3.15}$$

where $\nu_0 = \pm 1$. By (3.15) also $\tilde{S}^{(q)}(\alpha, \omega)$ is well defined and, by (3.13) and (3.14), one has $\tilde{S}^{(q)}(\alpha_0, \omega) = 0$. Therefore (3.7) can be solved for $k = q$ and the coefficients $\hat{u}^{(q)}_q(\omega)$ are arbitrary, as (3.3) shows. Moreover by (3.11) and (3.14) one has $\tilde{u}^{(q)}(\alpha_0, \omega) = 0$.

Then suppose that the coefficients $\hat{u}^{(k')}_\nu(\omega)$ are well defined for all $k' < k$ and are arbitrarily chosen for $\nu \in q\mathbb{Z}$: then we can show that also the coefficients $\hat{u}^{(k)}_\nu(\omega)$ are formally well defined. This follows again from (3.12), which shows that $\hat{S}^{(k)}_\nu(\omega)$ is well defined for all ν. Then if $\nu \notin q\mathbb{Z}$ one has

$$\hat{u}^{(k)}_\nu(\omega) = \frac{1}{\gamma(\omega\nu)} \hat{S}^{(k)}_\nu(\omega), \tag{3.16}$$

by the first equation in (3.7), so that also $\hat{u}^{(k)}_\nu(\omega)$ is well defined for $\nu \notin q\mathbb{Z}$.

Moreover if we sum together all Fourier components with $\nu \in q\mathbb{Z}$ and we use (3.11) and (3.15) we see that in the second equation of (3.7) one has $\tilde{S}^{(k)}(\alpha, \omega) = 0$, so that both equations in (3.7) are formally soluble and the coefficients $\hat{u}^{(k)}_\nu(\omega)$, with $\nu \in q\mathbb{Z}$, can be arbitrarily fixed: independently of their values one has $\tilde{u}^{(k)}(\alpha_0, \omega) = 0$ by (3.14). ∎

3.9. *Remark.* The proof of lemma yields, through (3.14), that there are $2q$ values of α_0 in $[0, 2\pi)$ such that there exists a formal 2π-periodic solution of (3.5):

$$\alpha_0 \in \mathcal{A}(\omega) \equiv \left\{ \frac{\pi k}{q} \ : \ k = 0, 1, 2, \ldots, 2q - 1 \right\}. \tag{3.17}$$

As in the variable α the dynamics is a rotation by $2\pi/q$ (see (3.3)), we see that such values of α_0 correspond to two distinct (formal) periodic orbits: one easily checks that, for ε small enough, such orbits are one linearly stable and one unstable.

3.10. Corollary. *In order that the function $u(\alpha_0, \varepsilon, \omega)$ be formally well defined, for all $\nu \in q\mathbb{Z}$ the coefficients $\hat{u}^{(k)}_\nu(\omega)$ can be chosen as arbitrary constants $c^{(k)}_\nu$; in particular they can be chosen as identically vanishing.*

3.11. *Remark.* At a formal level, by choosing the initial datum α_0 in the set $\mathcal{A}(\omega)$ given by (3.17), we see that the corresponding trajectory turns out to be a periodic solution of the equation of motion, of the form (3.2), with

$$u(\alpha_0,\varepsilon,\omega) \equiv \bar{u}(\alpha_0,\varepsilon,\omega) = \sum_{\nu\in\mathbb{Z}\setminus q\mathbb{Z}} e^{i\nu\alpha_0}\hat{u}_\nu(\varepsilon,\omega) = \sum_{k=1}^{\infty}\varepsilon^k \sum_{\nu\in\mathbb{Z}\setminus q\mathbb{Z}} e^{i\nu\alpha_0}\hat{u}_\nu^{(k)}(\omega). \tag{3.18}$$

Of course we are left with the problem of proving the convergence of the series (3.18).

3.12. Theorem. *Let $\omega = p/q$ be a rational number in $[0,1]$, with $\gcd(p,q)=1$. Then there exists $\rho(\omega)>0$ such that the $2\pi p$-periodic solutions of the form (3.18) are analytic in ε for $|\varepsilon|<\rho(\omega)$. One has*

$$\log\rho(\omega) + 2B_1(\omega) > -C, \tag{3.19}$$

for some universal positive constant C, if $B_1(\omega)$ is the truncated Bryuno function (3.6).

3.13. *About the proof of the theorem 3.12.* The actual proof consists in proving that the function

$$u(\alpha,\varepsilon,\omega) \equiv \bar{u}(\alpha,\varepsilon,\omega) = \sum_{\nu\in\mathbb{Z}\setminus q\mathbb{Z}} e^{i\nu\alpha}\hat{u}_\nu(\varepsilon,\omega) = \sum_{k=1}^{\infty}\varepsilon^k \sum_{\nu\in\mathbb{Z}\setminus q\mathbb{Z}} e^{i\nu\alpha}\hat{u}_\nu^{(k)}(\omega), \tag{3.20}$$

is analytic in $(\alpha,\varepsilon)\in\mathcal{D}$, where

$$\mathcal{D} = \left\{(\varepsilon,\alpha)\in\mathbb{C}^2 \;:\; |\varepsilon|<\rho\,,\; |\mathrm{Im}\, a|<\xi \text{ with } e^{\xi}\rho < Ce^{-2B_1(\omega)}\right\}, \tag{3.21}$$

for some universal constant C. For $\alpha\in\mathcal{A}(\omega)$ the function (3.20) interpolates the set of points (3.18), hence the periodic orbits.

The proof of analyticity of (3.20) in the domain $\mathcal{D}$ proceeds exactly as the analogous proof of [8]. Instead of giving the full proof *ex novo* (which would be essentially a repetition of [8]), we assume the reader to be familiar with [8] and we confine ourselves to stress the (few) points in which there is a difference between the case of rational numbers and the case of Bryuno numbers: this will be done in Section 6.

3.14. *Remarks.* (1) By taking into account the Corollary 3.10 we can rewrite (3.20) as

$$u(\alpha,\varepsilon,\omega) \equiv \bar{u}(\alpha,\varepsilon,\omega) = \sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha}\hat{u}_\nu(\varepsilon,\omega) = \sum_{k=1}^{\infty}\varepsilon^k \sum_{\nu\in\mathbb{Z}} e^{i\nu\alpha}\hat{u}_\nu^{(k)}(\omega), \tag{3.22}$$

as $\hat{u}_\nu(\varepsilon,\omega)=0$ for ν such that $\nu\in q\mathbb{Z}$. We shall call (3.22) the *interpolating function* for the periodic solutions with rotation number ω.
(2) The above result gives a lower bound on the radius of convergence of the function (3.18). It would be interesting to see if the radius of convergence admits also an upper bound of the same kind (analogously to what happens in the case of quasi-periodic solutions): the numerical results of [10] suggest that this is the case. [Note that such an upper bound could be easily obtained for the interpolating functions (3.22) by reasoning as in [14].]
(3) For $\alpha\neq\alpha_0$ the function (3.20) does not describe anymore a periodic solution of the equation of motion: it is simply a 2π-periodic analytic function which is equal to the solution only when $\alpha=\alpha_0$, with α_0 satisfying (3.9), i.e. with $\alpha_0\in\mathcal{A}(\omega)$: we call *remnant* the curve described by such a function.

4. Periodic and quasi-periodic solutions

4.1. *Tree formalism.* We refer to [16] and [8] for the a detailed description about the definition and properties of trees. Here we confine ourselves to recall the basic notations, in order to have a self-consistent discussion.

A *tree* θ consists of a family of k *lines* arranged to connect a partially ordered set of points called *nodes*, with the lower nodes to the right. All the lines have two nodes at their extremes, except the highest which has only one node, the *last node* u_0 of the tree (which is the leftmost one); the other extreme r will be called the *root* of the tree and it will not be regarded as a node.

We denote by $\preceq$ the partial ordering relation between nodes: given two nodes u_1 and u_2, we say that $u_2 \preceq u_1$ if u_1 is along the path of lines connecting u_2 to the root r of the tree (they could coincide: we say that $u_2 \prec u_1$ if they do not).

Each line carries an arrow pointing from the node u to the right to the node u' to the left (i.e. directed toward the root): we say that the line exits from u and enters u', and we write $u_0' = r$ even if, strictly speaking, r is not a node. For each node there are only one exiting line and $m_u \geq 0$ entering ones; as there is a one-to-one correspondence between nodes and lines, we can associate to each node u a line ℓ_u exiting from it. The line ℓ_{u_0} connecting the node u_0 to the root r will be called the *root line*. Note that each line ℓ_u can be considered the root line of the subtree consisting of the nodes satisfying $w \preceq u$ and of the lines connecting them: u' will be the root of such subtree. The *order* k of the tree is defined as the number of nodes of the tree.

To each node $u \in \theta$ we associate a *mode label* $\nu_u = \pm 1$, and define the *momentum* flowing through the line ℓ_u as

$$\nu_{\ell_u} = \sum_{w \preceq u} \nu_w, \qquad \nu_w = \pm 1. \tag{4.1}$$

Let us denote by $\mathcal{T}^0_{\nu,k}$ the set of all trees of order k (i.e. with k nodes) and with momentum ν flowing through the root line (*total momentum*), and by $V(\theta)$ and $\Lambda(\theta)$, respectively, the set of nodes and the set of lines of the tree θ.

4.2. Lemma. *Let $\omega \in [0,1]$ and let $u(\alpha,\varepsilon,\omega)$ be a formal solution of the functional equation (3.5); for $\omega \in \mathbb{Q}$ one takes $\alpha = \alpha_0 \in \mathcal{A}(\omega)$, while for ω a Bryuno number α varies in $[0,2\pi]$. Then one has*

$$\hat{u}^{(k)}_\nu(\omega) = \frac{1}{2^k} \sum_{\theta \in \mathcal{T}^0_{\nu,k}} \mathrm{Val}(\theta,\omega), \quad \mathrm{Val}(\theta,\omega) = -i \left(\prod_{u \in V(\theta)} \frac{\nu_u^{m_u+1}}{m_u!} \right) \left(\prod_{\ell \in \Lambda(\theta)} g(\omega\nu_\ell) \right), \tag{4.2}$$

where

$$g(\omega\nu) = \frac{1}{\gamma(\omega\nu_\ell)}, \qquad \gamma(\omega\nu) = 2\left[\cos(2\pi\omega\nu) - 1\right]. \tag{4.3}$$

is the propagator associated to the line ℓ.

4.3. *About the proof of the lemma 4.2.* The proof is iterative and it is left as an (easy) exercise to the reader. [See [16] for details.]

4.4. Lemma. *For any tree θ and for any line $\ell \in \Lambda(\theta)$ one has $\gamma(\omega\nu_\ell) \neq 0$.*

4.5. *Proof of the lemma 4.4.* For ω a Bryuno number the proof reduces to show that $\nu_\ell \neq 0$, and it is given in [17], Section 3 (in a more general situation), while for ω a rational number the proof is a consequence of the discussion in §3. In fact, as noted in §4.1, each line ℓ can be considered the root line of the subtree θ' formed by the nodes and lines preceding ℓ. If k' is the number of nodes of such a subtree and ν' is the momentum flowing through the line ℓ, then the value of such a subtree contributes to $\hat{u}_{\nu'}^{(k')}(\omega)$: the sum of the values $\mathrm{Val}(\theta'',\omega)$ of all subtrees $\theta'' \in \mathcal{T}^0_{\nu',k'}$ gives exactly $\hat{u}_{\nu'}^{(k')}(\omega)$. By the Corollary 3.10, when $\omega = p/q$, no line with $\nu \in q\mathbb{Z}$ can arise, so that, as $\gamma(\omega\nu) = 0$ if and only if $\nu \in q\mathbb{Z}$, the assertion follows. [Note that even if we did not choose the coefficients $\hat{u}_{\nu'}^{(k')}(\omega)$ as vanishing, they would be simply some constants $c_{\nu'}^{(k)}$ not involving any denominator $\gamma(\omega\nu')$.] ■

4.6. LEMMA. *Let $\omega \in [0,1]$ be either a rational number or a Bryuno number. Let $\hat{u}_\nu^{(k)}(\omega)$ be defined as in (4.2). Then there exists a positive constant D such that one has*

$$\left| \hat{u}_\nu^{(k)}(\omega) \right| \le D^k e^{2kB_1(\omega)}, \tag{4.4}$$

where $B_1(\omega)$ is given by (2.7) if ω is a Bryuno number and by (2.6) if ω is rational.

4.7. *About the proof of the lemma.* The proof of such a result can be obtained by reasoning as in [8], and it implies the lower bound in (3.19) of the theorem 3.12; see Section 6 below.

4.8. *Remark.* The result (4.4) is essentially an intermediate step toward the theorem 3.12: we have stated it explicitly as in that form it will be useful in proving the forthcoming theorem (and it will be exploited in Section 5).

4.9. *Remark.* If we take $\omega = 1/q$, with $q \in \mathbb{N}$, then (4.4) reduces to $|\hat{u}_\nu^{(k)}(1/q)| \le cq^{2k}$: such a (trivial) bound could be obtained without any effort simply by noting that any tree to order k has k propagators which can be bounded by a constant times q^2, so that we obtain for the radius of convergence the bound $\rho(1/q) > cq^{-2}$ (for some positive constant c), which is a particular case of the theorem 3.12.

4.10. THEOREM. *Let ω be a Bryuno number; if $\{\omega_N\}$ are the convergents of ω, denote by $u_N \equiv u(\alpha,\varepsilon,\omega_N)$ the functions interpolating the periodic solutions with rotation number ω_N as given by (3.22), and by $u \equiv u(\alpha,\varepsilon,\omega)$ the quasi-periodic solution with rotation number ω. Then there exist two positive constants ρ_0 and β, such that the sequence $\{u_N\}$ converges to the function u, uniformly for $|\varepsilon| < \rho_0 e^{-\beta B_1(\omega)}$; one can choose $\beta = 2$.*

4.11. *Proof of the theorem 4.10.* See the next Section 5.

4.12. *Conclusions.* The theorem 4.10 is our main analytical result. It implies that there is a neighborhood of the origin (in the complex ε-plane) in which the limit u_∞ of the functions u_N exists and coincides with the quasi-periodic solution u. Note that we are heavily using that the sizes of the domains of analyticity of the interpolating functions for ω_N and for the conjugating function for ω admit the same estimates, as it follows from the discussion in §3.13 and from the trivial remark 3.4.

For the uniqueness of the analytical continuation, [18], we can deduce that the functions u_∞ and u have the same analyticity domains in ε: in particular this implies that u_∞ can be extended to the overall analyticity domain of the quasi-periodic solution u, and it coincides with u over there.

By trivial complex-variables arguments, the uniform convergence of the sequence of functions u_N to u implies also that all their derivatives converge uniformly (in α and ε): in this sense we say that the sequence of remnants is "analytical close" to the invariant curve.

Note that the periodic orbits consist of a finite number of points, whose number grows as $\omega_N \to \omega$. Then the content of the theorem 4.10 is the following: for any fixed N such points can be interpolated through a smooth curve which tends (in the analytical way explained above) to the invariant curve corresponding to the quasi-periodic solution with rotation number ω.

Concerning what happens when ω is a Liouville number not satisfying the Bryuno condition, there are two issues that could be addressed. First one can ask if there is something analogous to the remnants seen in the case of rational rotation numbers. Secondly it would be interesting to understand what happens to the sequence of periodic orbits corresponding to the convergents of such a Liouville number.

5. Proof of the theorem 4.10

5.1. *Scales.* As in [8] we can introduce a C^∞ partition of unity by defining a set of functions $\chi_n(x)$ for $x \in \mathbb{R}_+$ and $n \in \mathbb{N}$ in the following way.

Let $\chi(x)$ be a C^∞ non-decreasing compact-support function defined on $\mathbb{R}_+$ such that

$$\chi(x) = \begin{cases} 1, & \text{for } x \le 1 \ , \\ 0, & \text{for } x \ge 2 \ , \end{cases} \tag{5.1}$$

and define

$$\begin{cases} \chi_0(x) = 1 - \chi(96 q_1 x), \\ \chi_n(x) = \chi(96 q_n x) - \chi(96 q_{n+1} x), \quad \text{for } n \ge 1 \ ; \end{cases} \tag{5.2}$$

we shall come back in Section 6 to the meaning of the numerical values of the constants appearing in (5.1) and (5.2). Then for each line ℓ set

$$g(\omega\nu_\ell) \equiv \frac{1}{\gamma(\omega\nu_\ell)} = \sum_{n=0}^{\infty} \frac{\chi_n(\|\omega\nu_\ell\|)}{\gamma(\omega\nu_\ell)} \equiv \sum_{n=0}^{\infty} g^{(n)}(\omega\nu_\ell), \qquad \|x\| = \inf_{p\in\mathbb{Z}} |x-p|, \tag{5.3}$$

and call $g^{(n)}(\omega\nu_\ell)$ the *propagator on scale* n.

Given a tree θ, we can associate to each line $\ell \in \Lambda(\theta)$ a scale label n_ℓ, using the multiscale decomposition (5.3) and singling out the summands with $n = n_\ell$. We shall call n_ℓ the *scale label* of the line ℓ, and we shall say also that the line ℓ is *on scale* n_ℓ.

This leads in a natural way to the definition of *clusters*, see [8], Section 2, pp. 628-629: given a tree θ, a cluster T of θ on scale n is a maximal connected set of lines of lines on scale $\le n$ with at least one line on scale n. Let us denote by $T(\theta)$ the set of clusters in a tree θ.

For any cluster $T \in T(\theta)$ set

$$\nu_T = \sum_{u\in V(T)} \nu_u, \qquad k_T = |V(T)|, \tag{5.4}$$

where $V(T)$ is the set of nodes contained in T and, given a set A, we are denoting by $|A|$ the number of elements of A.

Recall that for $\omega \in \mathbb{Q}$ there is an integer $N = N(\omega)$ such that $p_N/q_N = p/q = \omega$, and for all $\nu \in \mathbb{Z} \setminus q\mathbb{Z}$ one has $\|\omega\nu\| \ge 1/q_N$: therefore for $\omega \in \mathbb{Q}$ there is only a finite number of scales $n = 0, 1, 2, \ldots, N-1$.

Given a line ℓ carrying a momentum ν_ℓ, there can be only two (consecutive) scale labels n_ℓ such that $\chi_{n_\ell}(\omega\nu_\ell) \ne 0$: in such a case one has

$$\frac{1}{96 q_{n_\ell+1}} \le \|\omega\nu_\ell\| \le \frac{1}{48 q_{n_\ell}}. \tag{5.5}$$

Thus we arrive to a slight different definition of tree values, taking into account also the scale labels, so that (4.2) has to be replaced with

$$\hat{u}_\nu^{(k)}(\omega) = \frac{1}{2^k} \sum_{\theta\in\mathcal{T}_{\nu,k}} \mathrm{Val}(\theta,\omega), \quad \mathrm{Val}(\theta,\omega) = -i \left(\prod_{u\in V(\theta)} \frac{\nu_u^{m_u+1}}{m_u!} \right) \left(\prod_{\ell\in\Lambda(\theta)} g^{(n_\ell)}(\omega\nu_\ell) \right), \tag{5.6}$$

where here and henceforth $\mathcal{T}_{\nu,k}$ denotes the set of trees whose lines carry also a scale label.

5.2. *Resonances.* We recall briefly the definition of resonance from [8], Section 2, p. 629.

Given a cluster T, let m_T be the number of entering lines (so that $m_T \geq 0$) and let k_T be the number of nodes in T; we shall denote with n_T the scale of the cluster T, with n_T^i the minimum of the scales of the lines entering T, and with n_T^o the scale of the line exiting T.

Given a tree θ, a cluster V of θ will be called a *resonance* with *resonance-scale* $n = n_V^R \equiv \min\{n_V^i, n_V^o\}$, if

(1) the sum of the mode labels of its nodes is 0,

(2) all the lines entering V are on the same scale except at most one, which can be on a higher scale;

(3) $n_V^i \leq n_V^o$ if $m_V \geq 2$, and $|n_V^i - n_V^o| \leq 1$ for $m_V = 1$;

(4) $k_V < q_n$;

(5) $m_V = 1$ if $q_{n+1} \leq 4q_n$;

(6) if $q_{n+1} > 4q_n$ and $m_V \geq 2$, denoting by k_0 the sum of the orders of the subtrees of order $< q_{n+1}/4$ entering V, either (a) there is only one subtree of order $k_1 \geq q_{n+1}/4$ entering V and $k_0 < q_{n+1}/8$, or (b) there is no such subtree and $k_0 + k_0 < q_{n+1}/4$.

We refer to [8] for further details.

Let us denote by $N_n(\theta)$ the number of lines $\ell \in \Lambda(\theta)$ on scale n and by $P_n(\theta)$ the number of resonances $T \in T(\theta)$ on scale n. Set also $M_n(\theta) = N_n(\theta) + P_n(\theta)$. Finally let us denote by $N_n^R(\theta)$ the number of resonances $T \in T(\theta)$ with resonance-scale n.

5.3. Lemma. *For any tree $\theta \in \mathcal{T}_{\nu,k}$ one has*

$$M_n(\theta) \leq \frac{2k}{q_n} + \frac{8k}{q_{n+1}} + N_n^R(\theta), \tag{5.7}$$

and $M_n(\theta) = 0$ if $k < q_n$.

5.4. *About the proof of the lemma 5.3.* The proof is as in [8], Section 5, for ω an irrational number; it is not difficult to realize that the same proof works also for ω a rational number (see also the comments in Section 6 below).

5.5. *Tree formalism for the function $u_N - u$.* Consider both $u_N \equiv u(\alpha, \varepsilon, \omega_N)$ and $u \equiv u(\alpha, \varepsilon, \omega)$. We can apply the renormalization scheme as in [8]: everything proceeds in the same way. In particular the set $\mathcal{T}_{\nu,k}$ has to be enlarged to a set $\mathcal{T}^*_{\nu,k}$ in which the bound (5.5) can be violated (see [8] and §6.1 below); nevertheless, for any tree $\theta \in \mathcal{T}^*_{\nu,k}$, if a line ℓ carries a momentum ν_ℓ and a scale n_ℓ, then

$$\frac{1}{768 q_{n_\ell+1}} \leq \|\omega \nu_\ell\| \leq \frac{1}{8 q_{n_\ell}}, \tag{5.8}$$

whenever $\chi_{n_\ell}(\omega\nu_\ell) \neq 0$. Then the lemmata 5.3 and 4.6 still apply, as their proofs are based on the bound (5.8); see [8] and Section 6 below for details.

In order to prove the theorem 4.10 we have to consider the function

$$\begin{aligned} u_N - u \equiv u(\alpha, \varepsilon, \omega_N) - u(\alpha, \varepsilon, \omega) &= \sum_{k=1}^{\infty} \varepsilon^k \left(u^{(k)}(\alpha, \omega_N) - u^{(k)}(\alpha, \omega) \right) \\ &= \sum_{k=1}^{\infty} \varepsilon^k \sum_{\nu \in \mathbb{Z}} e^{i\nu\alpha} \left(\hat{u}_\nu^{(k)}(\omega_N) - \hat{u}_\nu^{(k)}(\omega) \right), \end{aligned} \tag{5.9}$$

where both coefficients $\hat{u}_\nu^{(k)}(\omega_N)$ and $\hat{u}_\nu^{(k)}(\omega)$ can be expressed in terms of trees; moreover the sum over the Fourier labels has to satisfy the constraint $|\nu| \leq k$ (see the lemma 3.5), and one has $\hat{u}_\nu^{(k)}(\omega_N) = 0$ for all $\nu \in q\mathbb{Z}$ (see the corollary 3.10).

We want to prove that it is possible to choose a neighborhood of the origin in the ε-plane – call it $\mathcal{B}(0)$ –, such that for all $\eta > 0$ there exists $N_0 \in \mathbb{N}$ such that for all $N > N_0$ and for all $\varepsilon \in \mathcal{B}(0)$ one has

$$|u(\alpha, \varepsilon, \omega_N) - u(\alpha, \varepsilon, \omega)| < \eta. \tag{5.10}$$

We can split the sum over k in (5.9) into two sums, the first one from $k = 1$ to $k = q_N/4$ and the second one over $k > q_N/4$, i.e.

$$\begin{aligned} u(\alpha, \varepsilon, \omega_N) - u(\alpha, \varepsilon, \omega) = & \left[\sum_{k=1}^{q_N/4} \varepsilon^k \left(u^{(k)}(\alpha, \omega_N) - u^{(k)}(\alpha, \omega) \right) \right] \\ & + \left[\sum_{k=(q_N/4)+1}^{\infty} \varepsilon^k \left(u^{(k)}(\alpha, \omega_N) - u^{(k)}(\alpha, \omega) \right) \right]. \end{aligned} \tag{5.11}$$

By the lemma 4.6 (and by the lemma 3.5) one has

$$\begin{aligned} \left| \sum_{k=(q_N/4)+1}^{\infty} \varepsilon^k \left(u^{(k)}(\alpha, \omega_N) - u^{(k)}(\alpha, \omega) \right) \right| \le & \sum_{k=(q_N/4)+1}^{\infty} |\varepsilon|^k \left(\left| u^{(k)}(\alpha, \omega_N) \right| + \left| u^{(k)}(\alpha, \omega) \right| \right) \\ \le \sum_{k=(q_N/4)+1}^{\infty} |\varepsilon|^k \, (2k+1) \, D^k \left(e^{2kB_1(\omega)} + e^{2kB_1(\omega_N)} \right) \le & \left(\frac{1}{2} \right)^{q_N}, \end{aligned} \tag{5.12}$$

provided that one chooses the radius $\rho(\mathcal{B}(0)) \equiv \rho_0 e^{-\beta B_1(\omega)}$, with $\beta \ge 2$ and ρ_0 small enough. Of course one can suppose that N is so large that

$$\left(\frac{1}{2} \right)^{q_N} \le \frac{1}{2} \eta. \tag{5.13}$$

So we are left with the first sum in (5.11), i.e.

$$\sum_{k=1}^{q_N/4} \varepsilon^k \sum_{\nu \in \mathbb{Z}} e^{i\nu\alpha} \left(\hat{u}_\nu^{(k)}(\omega_N) - \hat{u}_\nu^{(k)}(\omega) \right). \tag{5.14}$$

By taking into account (5.6) we can write

$$\hat{u}_\nu^{(k)}(\omega_N) - \hat{u}_\nu^{(k)}(\omega) = \frac{1}{2^k} \sum_{\theta \in \mathcal{T}_{\nu,k}} \overline{\text{Val}}(\theta, \omega_N, \omega), \qquad \overline{\text{Val}}(\theta, \omega_N, \omega) = \text{Val}(\theta, \omega_N) - \text{Val}(\theta, \omega), \tag{5.15}$$

so that

$$\overline{\text{Val}}(\theta, \omega_N, \omega) = -i \left[\prod_{u \in V(\theta)} \frac{\nu_u^{m_u+1}}{m_u!} \right] \left[\left(\prod_{\ell \in \Lambda(\theta)} g^{(n_\ell)}(\omega_N \nu_\ell) \right) - \left(\prod_{\ell \in \Lambda(\theta)} g^{(n_\ell)}(\omega \nu_\ell) \right) \right]. \tag{5.16}$$

Then we can write $\overline{\text{Val}}(\theta, \omega_N, \omega)$ as sum of k terms corresponding to trees whose lines have all the propagators of the form either $g^{(n_\ell)}(\omega_N \nu_\ell)$ or $g^{(n_\ell)}(\omega \nu_\ell)$, up to one which has a new propagator given by the difference $g^{(n_\ell)}(\omega_N \nu_\ell) - g^{(n_\ell)}(\omega \nu_\ell)$; see [19] and [20] for an analogous discussion.

Given a tree θ we can order the lines and construct a set of k subsets $\Lambda_1(\theta), \ldots, \Lambda_k(\theta)$ of $\Lambda(\theta)$, with $|\Lambda_j(\theta)| = j$, in the following way. Set $\Lambda_1(\theta) = \emptyset$, $\Lambda_2(\theta) = \ell_1$, if ℓ_1 is the root line of θ and, inductively for $2 \le j \le k$, $\Lambda_{j+1}(\theta) = \Lambda_j(\theta) \cup \ell_j$, where the line $\ell_j \in \Lambda(\theta) \setminus \Lambda_j(\theta)$ is connected to $\Lambda_j(\theta)$; of course $\Lambda_{k+1}(\theta) = \Lambda(\theta)$. Then

$$\begin{aligned}\overline{\mathrm{Val}}(\theta, \omega_N, \omega) = -i \left[\prod_{u \in V(\theta)} \frac{\nu_u^{m_u+1}}{m_u!} \right] \sum_{j=1}^{k} \Bigg[\Bigg(\prod_{\ell \in \Lambda_j(\theta)} g^{(n_\ell)}(\omega_N \nu_\ell) \Bigg) \\ \left(g^{(n_{\ell_j})}(\omega_N \nu_{\ell_j}) - g^{(n_{\ell_j})}(\omega \nu_{\ell_j}) \right) \Bigg(\prod_{\ell \in \Lambda(\theta) \setminus \Lambda_{j+1}(\theta)} g^{(n_\ell)}(\omega \nu_\ell) \Bigg) \Bigg],\end{aligned} \tag{5.17}$$

where, by construction, the sets $\Lambda_j(\theta)$ are connected (while of course $\Lambda(\theta) \setminus \Lambda_{j+1}(\theta)$ in general are not).

5.6. LEMMA. *With the notations of the lemma 4.6, there exist two positive constants D_0 and D_1 such that one has*

$$\left| \hat{u}_\nu^{(k)}(\omega_N) - \hat{u}_\nu^{(k)}(\omega) \right| \le \frac{1}{q_{N+1}} D_0 D^k D_1^k e^{2kB_1(\omega)}, \tag{5.18}$$

for any $k \le q_N/4$.

5.7. *Proof of the lemma 5.6.* The proof is given in Section 7.

5.8. *Conclusions.* By the lemma 5.6 we can bound (5.14) as

$$\sum_{k=1}^{q_N/4} \varepsilon^k \sum_{\nu \in \mathbb{Z}} e^{i\nu\alpha} \left(\hat{u}_\nu^{(k)}(\omega_N) - \hat{u}_\nu^{(k)}(\omega) \right) \le \sum_{k=1}^{q_N/4} \varepsilon^k \, (2k+1) \, \frac{D_0}{q_{N+1}} \left(D D_1 e^{2B_1(\omega)} \right)^k \le \frac{2D_0}{q_{N+1}}, \tag{5.19}$$

provided that $\beta \ge 2$ and ρ_0 is small enough. Then one can suppose N so large that

$$\frac{2D_0}{q_{N+1}} \le \frac{1}{2}\eta, \tag{5.20}$$

so that, by collecting together (5.13) and (5.20), we obtain (5.10).

6. Comments about the proof of the theorem 3.12

6.1. *Preliminaries.* We recall now the main heuristic ideas behind the proof of [8] for the reader not completely familiar with it. This will also clarify the apparently mysterious choice of the constants in the definitions of the scales.

If we disregard resonances, the proof of [8] becomes very simple: basically one needs to prove lemma 5 in [8], p. 631, relatively easy if $P_n(\theta) = N_n^R(\theta) = 0$ (see the end of §5.2 for notations), as it happens in the case of the semistandard map (see [21]). In fact we would have

$$\begin{aligned}|\mathrm{Val}(\theta)| &\le D_1^k \prod_{n=0}^{\infty} (D_2 q_{n+1})^{2N_n(\theta)} \qquad \text{(because of (5.9), with } D_2 = 96) \\ &\le D_1^k \prod_{n=0}^{\infty} (D_2 q_{n+1})^{2k/q_n + 8k/q_{n+1}}.\end{aligned} \tag{6.1}$$

Now, as it is trivial to see that $\sum_{n=1}^{\infty} (\log q_n)/q_n$ is convergent, it is easy to prove the claim. We recall that the main arithmetic tool behind the proof of the lemma 5 in [8] is Davie's lemma [14], which we quote here in a slightly extended version.

LEMMA. *Let $a > 2$, $b \geq 2a/(a-2)$. Given $\nu \in \mathbb{Z}$ such that $\|\omega\nu\| \leq 1/aq_n$, then*
1. either $\nu = 0$ or $|\nu| \geq q_n$,
2. either $|\nu| \geq q_{n+1}/b$ or $\nu = sq_n$, for some integer s.

For simplicity we choose $a = b = 4$ as in [14], but other choices would be equally good. The choice of the constants a and b sets some rather sharp constraints on all other strange-looking constants, for instance those used in the definition of the scales in §5.1, as we are going to discuss below.

To deal with resonances we need to exploit cancellations arising when summing over trees of given order and total momentum. Suitable resummation must be performed, whose effect is that, for each resonance V, it is as if one of the external lines on scale n_V^R contributed $(D_2 q_{n_V+1})^2$ instead of $(D_2 q_{n_V^R+1})^2$. In the course of exhibiting such cancellations, one needs to perform transformations on trees which extend the set of trees being considered to a larger set $\mathcal{T}^*_{\nu,k}$ (see [8], p. 633).

Now, suppose that the scales had been defined in such a way that for a line ℓ on scale n one obtains

$$\frac{1}{c' q_{n+1}} \leq \|\omega\nu_\ell\| \leq \frac{1}{cq_n}, \tag{6.2}$$

(we had $c' = 96$ and $c = 48$). The effect of the above mentioned transformations is such that, given a resonance V, one has to consider all the resonances which are obtained by shifting its entering lines. This implies that for any line ℓ in V the corresponding momentum ν_ℓ can be changed into a new value ν'_ℓ, as it follows from (4.2); call $\boldsymbol{\nu}(\ell)$ the set of all momenta ν'_ℓ which can be associated to the line ℓ in this way. As a consequence, for each $\nu'_\ell \in \boldsymbol{\nu}(\ell)$, there will be a value n'_ℓ different from the original scale n_ℓ such that $\chi_{n'_\ell}(\|\omega\nu'_\ell\|) \neq 0$. If a line ℓ is contained inside several resonances, the above argument has to be applied for all such resonances.

Now, in order to exploit the cancellations assuring the convergence of the series (3.21), for each line ℓ one has to consider together all the scales n' which are obtained by the above described procedure (see [8], Section 3). The latter scales are defined by the condition

$$\frac{1}{c' q_{n'+1}} \leq \|\omega\nu'_\ell\| \leq \frac{1}{cq_{n'}} \tag{6.3}$$

if $\nu'_\ell \in \boldsymbol{\nu}(\ell)$. The essential fact is that such scales n' are not arbitrary, on the contrary they are are related to the original scale n, as one obtains (see [8], lemma 4 and Sections 3 and 4)

$$\frac{1}{d' q_{n'+1}} \leq \|\omega\nu_\ell\| \leq \frac{1}{dq_{n'}}, \tag{6.4}$$

with $d' > c' > c > d$ (and consequently D_2 grows to d' in (6.1)), provided the constants c and c' in (6.2) are suitably chosen.

More precisely requiring that (6.4) be satisfied imposes two constraints on c and c': in fact they must be chosen in such a way that (i) constants d and d' such that (6.3) holds actually exist, and (ii) one must have $d < 1/2b$ for Davie's lemma to be of some use. We found that $c = 48$ and $c' = 96$ is a choice compatible with those constraints, once one has chosen $a = b = 4$ in Davie's lemma, as follows from the bulk of Sections 3 and 4 of [8]: such a choice, if denoting by $n_\ell \in \boldsymbol{\nu}(\ell)$ the scale n' associated to the line ℓ in (6.4), gives (5.12).

As anticipated in Section §3.13, instead of providing a complete proof of the theorem 3.12, which would require repeating, essentially word by word, the discussion in [8], we prefer to show the points in which the analysis has to be slightly changed, trying to convince the reader why almost the same proof, up to a very few minor adaptations, in fact still works.

6.2. *Technical differences.* The first item of Davie's lemma (see above and [8], p. 630, lemma 1) has to be replaced with: either $\nu \in q\mathbb{Z}$ or $|\nu| \geq q_n$, as $\|\omega n q\| = 0$ for all $n \in \mathbb{Z}$.

As remarked in §5.1 for $\omega = p/q$ there is a finite number of scales, as $n \leq N-1$, if $q_N = q$: this follows from the fact that $\|\omega\nu\| \geq 1/q_N$ for all $n \in \mathbb{Z} \setminus q\mathbb{Z}$ and the very definition of scale.

As a consequence, in the proof of the lemma 5.3 for rational rotation numbers, one can proceeds as for the proof of the lemma 5 of [8], and, when discussing the case [2.2.3.2] in [8], Section 5, one can have either $|\nu - \nu_1| \geq q_{n+1}/4$, or $\nu - \nu_1 = \tilde{s} q_n$ or $|\nu - \nu_1| = q\mathbb{N}$ (as the case $\nu = \nu_1$ can be included in the previous one when $\tilde{s} = 0$). But the last case gives $|\nu - \nu_1| \geq q \geq q_N/4 \geq q_{n+1}/4$ which gives the case [2.2.3.2.1]. These are the only real technical differences in the proof of the lemma: we are left with the problem of verifying that the proof can then be performed by following the analysis of [8]. We shall briefly discuss such issues, by using the notations and concepts introduced in [8], with no further reference to it.

6.3. *Bound (5.7).* We already noticed that the proof of the lemma 5.3 can be carried out as in [8]: here we would want to give an intuitive argument to see why it is so. Basically the bound on the number of lines and resonances on scale n implicit in (5.7), is worked out by finding a bound *which is the worst possible one when all lines which are not on scale n are on a lower scale* (simply go along the proof of lemma 5 of [8] to realize that this is the case). Then even if in the case of irrational rotation numbers ω there are much more scales than in the case of the rational rotation numbers ω_N (for ω irrational there are infinitely many scales in principle, and they can be arbitrarily large for arbitrarily large orders), the quantity $M_n(\theta)$ assumes its largest possible value when there are no lines on scale greater than n, so that the bound (5.7), when $n < N$, holds simultaneously for the Bryuno number ω and for the rational number ω_N.

6.4. *Renormalization.* Moreover the renormalization procedure can be applied exactly in the same way, and no further difference arises between the case of rational numbers and the case of Bryuno numbers. The cancellation between the localized parts of the resonance values is a purely algebraic property which does not depend on the arithmetics of the rotation number, while the control of the renormalized parts is based on dimensional arguments which can be repeated unchanged once the scale labels have been fixed.

6.5. *Remark.* In [8] in fact the bound $|\varepsilon| < \rho(\omega) < Ce^{-2B(\omega)}$ for real values of α was given, but, by simply noting that, for $|\varepsilon| < \rho$ and $|\mathrm{Im}\,\alpha| < \xi$, one has

$$\begin{aligned} |u(\alpha,\varepsilon,\omega)| &\leq \sum_{k=1}^{\infty} \sum_{|\nu|\leq k} \rho^k e^{|\nu|\xi} \left| \hat{u}_\nu^{(k)} \right| \\ &\leq \sum_{k=1}^{\infty} (2k+1)\, D^k e^{2kB_1(\omega)} e^{k\xi} \rho^k \leq \sum_{k=1}^{\infty} \left(C^{-1} e^{2B_1(\omega)} \right)^k \left(e^{\xi} \rho \right)^k , \end{aligned} \tag{6.5}$$

the analyticity in the domain (3.21) easily follows.

7. Proof of the lemma 5.6

7.1. *Set-up.* We can write (5.17) as

$$\begin{aligned}
\overline{\mathrm{Val}}(\theta,\omega_N,\omega) &= \sum_{j=1}^{k} \overline{\mathrm{Val}}_j(\theta,\omega_n,\omega), \\
\overline{\mathrm{Val}}_j(\theta,\omega_N,\omega) &= -i \left[\prod_{u\in V(\theta)} \frac{\nu_u^{m_u+1}}{m_u!} \right] \left[\left(\prod_{\ell\in\Lambda_j(\theta)} g^{(n_\ell)}(\omega_N\nu_\ell) \right) \right. \\
&\quad \left. \left(g^{(n_{\ell_j})}(\omega_N\nu_{\ell_j}) - g^{(n_{\ell_j})}(\omega\nu_{\ell_j}) \right) \left(\prod_{\ell\in\Lambda(\theta)\setminus\Lambda_{j+1}(\theta)} g^{(n_\ell)}(\omega\nu_\ell) \right) \right],
\end{aligned} \tag{7.1}$$

and study separately each term $\overline{\mathrm{Val}}_j(\theta,\omega_N,\omega)$. To any line ℓ we associate a rotation number ω_N if the corresponding propagator is $g^{(n_\ell)}(\omega_N\nu_\ell)$ (i.e. if $\ell\in\Lambda_j(\theta)$), and a rotation number ω if the corresponding propagator is $g^{(n_\ell)}(\omega\nu_\ell)$ (i.e. if $\ell\in\Lambda(\theta)\setminus\Lambda_{j+1}(\theta)$).

The difference propagator $g^{(n_{\ell_j})}(\omega_N\nu_{\ell_j}) - g^{(n_{\ell_j})}(\omega\nu_{\ell_j})$ in (7.1) can be written as follows. Set for simplicity $\nu_{\ell_j}\equiv\nu$ and $n_{\ell_j}=n$. Then

$$\begin{aligned}
g^{(n)}(\omega_N\nu) - g^{(n)}(\omega\nu) &= \frac{1}{2}\left[\chi_n(\omega_N\nu)\left(\frac{1}{\gamma(\omega_N\nu)} - \frac{1}{\gamma(\omega\nu)} \right) \right. \\
&\quad \left. + \chi_n(\omega\nu)\left(\frac{1}{\gamma(\omega_N\nu)} - \frac{1}{\gamma(\omega\nu)} \right) + \left(\chi_n(\omega_N\nu)-\chi_n(\omega\nu)\right)\left(\frac{1}{\gamma(\omega_N\nu)} + \frac{1}{\gamma(\omega\nu)} \right) \right],
\end{aligned} \tag{7.2}$$

where

$$\begin{aligned}
\frac{1}{2}\chi_n(\omega_N\nu)\left(\frac{1}{\gamma(\omega_N\nu)} - \frac{1}{\gamma(\omega\nu)} \right) &= g^{(n)}(\omega_N\nu)\left(\frac{\gamma(\omega\nu)-\gamma(\omega_N\nu)}{\gamma(\omega\nu)} \right) \\
&\equiv g^{(n)}(\omega_N\nu)\, C_1(\omega_N\nu,\omega\nu), \\
\frac{1}{2}\chi_n(\omega\nu)\left(\frac{1}{\gamma(\omega_N\nu)} - \frac{1}{\gamma(\omega\nu)} \right) &= g^{(n)}(\omega\nu)\left(\frac{\gamma(\omega\nu)-\gamma(\omega_N\nu)}{\gamma(\omega_N\nu)} \right) \\
&\equiv g^{(n)}(\omega\nu)\, C_2(\omega_N\nu,\omega\nu),
\end{aligned} \tag{7.3}$$

so that, by defining also

$$C_3(\omega_N\nu,\omega\nu) = \frac{1}{2}\left(\chi_n(\omega_N\nu)-\chi_n(\omega\nu)\right), \tag{7.4}$$

we see that (7.2) becomes

$$\begin{aligned}
g^{(n)}(\omega_N\nu) - g^{(n)}(\omega\nu) &= g^{(n)}(\omega_N\nu)\, C_1(\omega_N\nu,\omega\nu) + g(\omega_N\nu)\, C_3(\omega_N\nu,\omega\nu) \\
&\quad + g^{(n)}(\omega\nu)\, C_2(\omega_N\nu,\omega\nu) + g(\omega\nu)\, C_3(\omega_N\nu,\omega\nu).
\end{aligned} \tag{7.5}$$

In conclusion to the line ℓ_j there corresponds the sum of the four "propagators" in (7.5): if we select the first one or the third one we can associate to the line ℓ_j a rotation number ω_N and ω, respectively. The other two cases can be singled out by assigning a label $*$ to the line ℓ_j; note that in the latter case at least one of the two conditions $\chi_n(\omega_N\nu)\neq 0$ and $\chi_n(\omega\nu)\neq 0$ has to be satisfied, otherwise the quantity $C_3(\omega_N\nu,\omega\nu)$, hence the corresponding propagator, is vanishing.

7.2. Lemma. *There is a constant C_0 such that one has*

$$|C_i(\omega_N\nu,\omega\nu)| \le \frac{kC_0}{q_{N+1}}, \tag{7.6}$$

for $i = 1, 2, 3$.

7.3. *Proof of the lemma 7.2.* Let us denote by C any constant. One has

$$|\omega - \omega_N| = \left|\omega - \frac{p_N}{q_N}\right| = \frac{1}{q_N}\,|\omega q_N - p_N| = \frac{1}{q_N}\|\omega q_N\| < \frac{1}{q_N q_{N+1}}, \tag{7.7}$$

so that

$$|(\omega_N - \omega)\,\nu| \le \frac{k}{q_N q_{N+1}} \tag{7.8}$$

provided that $|\nu| \le k$, as it is the case in a tree $\theta \in \mathcal{T}^*_{\nu,k}$.

Then

$$\begin{aligned} |\chi_n(\omega_N\nu) - \chi_n(\omega\nu)| &\le |(\omega_N - \omega)\,\nu| \int_0^1 \mathrm{d}t \left|\frac{\partial \chi_n}{\partial x}\,(\omega\nu + t(\omega_N - \omega)\nu)\right| \\ &\le \frac{k}{q_N q_{N+1}} C q_{n+1} \le \frac{Ck}{q_{N+1}}, \end{aligned} \tag{7.9}$$

which proves (7.6) for $i = 3$.

Furthermore

$$\begin{aligned} \frac{\gamma(\omega\nu) - \gamma(\omega_N\nu)}{\gamma(\omega_N\nu)} &= \frac{\cos 2\pi\omega\nu - \cos 2\pi\omega_N\nu}{\cos 2\pi\omega_N\nu - 1} \\ &\le \frac{|2\pi\,(\omega_N - \omega)\,\nu|}{|\cos 2\pi\omega_N\nu - 1|} \int_0^1 \mathrm{d}t\,|\sin 2\pi\,(\omega\nu + t(\omega_N - \omega)\nu)| \\ &\le \frac{k}{q_N q_{N+1}} C\|\omega_N\nu\|^{-2} \max\{\|\omega_N\nu\|, \|\omega\nu\|\} \le \frac{k}{q_N q_{N+1}} C q_N, \end{aligned} \tag{7.10}$$

which proves (7.6) for $i = 2$; the proof for $i = 1$ is analogous. ∎

7.4. *Bounds.* We see that each value $\overline{\mathrm{Val}}_j(\theta, \omega_N, \omega)$ splits into the sum of four contributions through (7.5). Each contribution not containing $C_3(\omega_N\nu, \omega\nu)$, up to a factor $C_i(\omega_N\nu, \omega\nu)$, $i = 1, 2$, is of the same form as either $\mathrm{Val}(\theta, \omega_N)$ or $\mathrm{Val}(\theta, \omega)$, with the only difference that for a connected subset $\mathcal{S}_j(\theta)$ of θ the rotation number is ω_N, while for the set $\theta \setminus \mathcal{S}_j(\theta)$ the rotation number is ω; with the notations introduced in §5.5 one has either $\mathcal{S}_j(\theta) = \Lambda_j(\theta)$ or $\mathcal{S}_j(\theta) = \Lambda_j(\theta) \cup \ell_j$.

For the two contributions containing $C_3(\omega_N\nu, \omega\nu)$, a further difference is that the propagator corresponding to the line ℓ_j does not contain neither a factor $\chi_{n_{\ell_j}}(\omega_N\nu_{\ell_j})$ nor a factor $\chi_{n_{\ell_j}}(\omega\nu_{\ell_j})$ (see (3.5)), but this is not important as both functions $1/\gamma(\omega_N\nu_{\ell_j})$ and $1/\gamma(\omega\nu_{\ell_j})$ admit the same bound in terms of the denominators of the convergents (see below), when the factor $C_3(\omega_N\nu, \omega\nu)$ is taken into account.

Then we can try to reason as for the proof of the lemma 5.3, i.e. along the lines of [8], Section 5. The only difference is that, when considering a cluster T, now it can happen that a rotation number ω_N is associated to the line exiting from T, while a rotation number ω is associated to the lines entering T. This does not yield any difference but for the case [2.2.3.2] of [8], Section 5, in which there is only one line entering T: to deal with such a case we shall use the following result.

7.5. Lemma. *Given any tree* $\theta \in \mathcal{T}_{\nu,k}$, *with* $k < q_N/4$, *if* $\omega_N < \omega$ *(respectively* $\omega_N > \omega$*), then one has* $\|\omega_N\nu_\ell\| < \|\omega\nu_\ell\|$ *(respectively* $\|\omega\nu_\ell\| < \|\omega_N\nu_\ell\|$*) for all* $\ell \in \Lambda(\theta)$ *with scales* $n_\ell \ge 1$.

7.6. *Proof of the lemma 7.5.* Suppose that one has $0 < \omega_N < \omega < 1$. If a line ℓ is on scale n, by setting $\nu = \nu_\ell$ one has either $\|\omega_N\nu\| \le 1/8q_n$ or $\|\omega\nu\| \le 1/8q_n$ (see (5.8)), according to which type of rotation number is associated to ℓ, so that, for $k < q_N/4$, one obtains

$$\max\{\|\omega_N\nu\|, \|\omega\nu\|\} \le \frac{1}{8q_n} + \frac{1}{4q_{N+1}}, \tag{7.11}$$

by (7.8).

We can write $\omega_N \nu = \|\omega_N \nu\| + r_N$ and $\omega\nu = \|\omega\nu\| + r$, for suitable $r_N, r \in \mathbb{Z}$. Therefore one has

$$\|\omega_N \nu\| = \omega_N \nu - r_N = \omega\nu + (\omega_N - \omega)\,\nu - r_N = \|\omega\nu\| + (\omega_N - \omega)\,\nu + (r - r_N)\,, \tag{7.12}$$

so that, by using (7.8) and (7.11),

$$\begin{aligned} |r_N - r| &= |\|\omega\nu\| - \|\omega_N\nu\| + (\omega_N - \omega)\,\nu| \\ &\le \frac{1}{8q_n} + \frac{1}{8q_n} + \frac{1}{4q_{N+1}} + \frac{1}{4q_{N+1}} \le \frac{1}{8} + \frac{1}{8} + \frac{1}{4} + \frac{1}{4} < 1, \end{aligned} \tag{7.13}$$

which yields $r_N = r$. Therefore, as we are assuming $\omega_N < \omega$, one has $|\omega_N\nu| < |\omega\nu|$ for all $\nu \in \mathbb{Z} \setminus \{0\}$, so that $\|\omega_N \nu_\ell\| < \|\omega\nu_\ell\|$ for all for all the above considered lines.

In the same way one proves that, if assuming $\omega_N > \omega$, then one obtain the inequality $\|\omega_N\nu_\ell\| > \|\omega\nu_\ell\|$ for all lines $\ell \in \Lambda(\theta)$, $\theta \in \mathcal{T}_{\nu,k}$, on scales $n_\ell \ge 1$. ■

7.7. *Conclusions.* When discussing the case [2.2.3.2] of [8], one can have the line ℓ with associated a rotation number ω_N and the line ℓ_1 with associated a rotation number ω, such that

$$\|\omega_N\nu\| \le \frac{1}{8q_n}, \qquad \|\omega\nu_1\| \le \frac{1}{8q_n}, \tag{7.14}$$

where $\nu = \nu_\ell$ and $\nu_1 = \nu_{\ell_1}$ (see [8], p. 648). By the lemma 7.5 one has either $\|\omega\nu\| \le \|\omega_N\nu\| \le 1/8q_n$ or $\|\omega_N\nu_1\| \le \|\omega\nu_1\| \le 1/8q_n$, so that equation (7.11) in [8] has to be replaced with

$$\min\{\|\omega(\nu - \nu_1)\|, \|\omega_N(\nu - \nu_1)\|\} \le \min\{\|\omega\nu\| + \|\omega\nu_1)\|, \|\omega_N\nu\| + \|\omega_N\nu_1)\|\} \le \frac{1}{4q_n}. \tag{7.15}$$

If the label $*$ is associated to the line ℓ_j, by observing that one has either $\chi_n(\omega_N\nu) \ne 0$ or $\chi_n(\omega\nu) \ne 0$, one can proceed in the same way.

Finally, in order to bound the small divisors, we can use that if a line is on scale n then the corresponding propagator is bounded by a constant times q_{n+1}^2: this is trivial for all lines except, if there is any, for the one carrying the label $*$, about which can reason as follows.

Suppose for concreteness that one has $\omega_N < \omega$ (the case $\omega_N > \omega$ can be dealt with in the same way). It is easy to realize that, besides trivial cases, the only case which really deserves a careful analysis corresponds to the propagator $g(\omega_N\nu_1)C_3(\omega_N\nu_1, \omega\nu_1)$ when $\chi_n(\omega\nu_1) \ne 0$ and $\chi_n(\omega_N\nu_1) = 0$ (so that $\|\omega_N\nu_1\| < \|\omega\nu_1\|$ by the lemma 7.5). We can use that for $k < q_N/4$ one has $\|\omega\nu_1\| - \|\omega_N\nu_1\| > 1/4q_{N+1}$ by (7.8), $\|\omega\nu_1\| > 1/768q_{n+1}$ by (5.8), and $\|\omega_N\nu_1\| > 1/2q_{N+1}$ by [8], (3.15). Then $\|\omega_N\nu_1\| > \|\omega\nu_1\| - 1/4q_{N+1}$, so that, if $1/4q_{N+1} < 1/1536q_{n+1}$ one has $\|\omega_N\nu_1\| > 1/1536q_{n+1}$, while if $1/4q_{N+1} > 1/1536q_{n+1}$ one has $\|\omega_N\nu_1\| > 1/2q_{N+1} > 1/768q_{n+1}$: hence in both cases one has $\|\omega_N\nu_1\| > 1/1536q_{n+1}$.

From here on the discussion proceeds as in [8], with no further difference.

Moreover, by taking into account the sum over j in (7.1) and, for all j, the sum over the four terms in (7.5), we have an extra factor $k^5 \le e^{5k}$.

In conclusion the same bound as (4.4) follows, with the only difference that there is a factor $C_0 k/q_{N+1} \le C_0 e^k/q_{N+1}$, arising from (7.6). Then (5.18) follows with $D_0 = 4C_0$ (where the factor 4 takes into account the fact that for a line carrying a label $*$ one can substitute 768^2 with 1536^2 in bounding the corresponding propagator) and $D_1 = e^6$. This completes the proof of the lemma.

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References

[1] L.H. Eliasson: *Absolutely convergent series expansions for quasi-periodic motions*, Math. Phys. Electron. J. **2** (1996), paper 4, 1–33 (Preprint 1988).

[2] G. Gallavotti: *Twistless KAM tori*, Comm. Math. Phys. **164** (1994), 145–156.

[3] G. Gentile: *Diagrammatic techniques in perturbation theory, and applications*, in *Proceedings of "Symmetry and Perturbation Theory II"* (Rome, 16–22 December 1998), 59–78, Ed. A. Degasperis and G. Gaeta, World Scientific, 1999.

[4] J.M. Greene: A method for determining a stochastic transition, *J. Math. Phys.* **20** (1979), 1183–1201.

[5] B.V. Chirikov: A universal instability of many dimensional oscillator systems, *Phys. Rep.* **52** (1979), 264–379.

[6] G. Gentile, V. Mastropietro: *Methods for the analysis of the Lindstedt series for KAM tori and renormalizability in classical mechanics. A review with some applications*, Rev. Math. Phys. **8** (1996), 393–444.

[7] D. Bernstein, A. Katok: *Birkhoff periodic orbits for small perturbations of completely integrable Hamiltonian systems with convex Hamiltonians*, Invent. Math. **88** (1987), 225–241.

[8] A. Berretti, G. Gentile: *Bryuno function and the standard map*, Comm. Math. Phys. **220** (2001), 623–656 (Preprint 1998).

[9] G.D. Birkhoff: *Dynamical systems*, American Mathematical Society, Providence, 1966; reprinting of the original edition [New York, 1927].

[10] A. Celletti, C. Falcolini: *Singularities of periodic orbits near invariant curves*, Preprint (2001).

[11] W.M. Schmidt: *Diophantine approximation*, Lecture Notes in Mathematics **785**, Springer, Berlin, 1980.

[12] J.-C. Yoccoz: *Théorème de Siegel, Nombres de Bruno et Polinômes Quadratiques*, Astérisque **231** (1995), 3–88.

[13] J.K. Moser: *On invariant curves of area preserving mappings of the annulus*, Nachr. Akad. Wiss. Göttingen, Math. Phys. Kl. II **1962** (1962), 1–20.

[14] A.M. Davie: *The critical function for the semistandard map*, Nonlinearity **7** (1994), 219–229.

[15] H. Rüssmann: *On the frequencies of quasi-periodic solutions of analytic nearly integrable Hamiltonian systems*, Seminar on Dynamical Systems (St. Petersburg, 1991), 160–183, Progr. Nonlinear Differential Equations Appl. **12**, Birkhäuser, Basel, 1994.

[16] A. Berretti, G. Gentile: *Scaling properties for the radius of convergence of Lindstedt series: the standard map*, J. Math. Pures Appl. (9) **78** (1999), 159–176.

[17] A. Berretti, G. Gentile: *Scaling properties for the radius of convergence of Lindstedt series: generalized standard maps*, J. Math. Pures Appl. (9) **79** (2000), 691–713.

[18] E.C. Titchmarsch: *The theory of functions*, Oxford University Press, London, 1933; snd edition 1939.

[19] M. Bartuccelli, G. Gentile: *Lindstedt series for perturbations of isochronous systems. A review of the general theory*, Rev. Math. Phys. **14** (2002), no. 2, 121–171.

[20] G. Gallavotti, G. Gentile: *Hyperbolic low-dimensional invariant tori and summations of divergent series*, to appear in Comm. Math. Phys.

[21] A. Berretti, G. Gentile: *Renormalization Group and field theoretic techniques for the analysis of the Lindstedt series*, Regul. Chaotic Dyn. **6** (2001), no. 4, 389–420.