SCALING OF THE CRITICAL FUNCTION FOR THE STANDARD MAP: SOME NUMERICAL RESULTS

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A . The behavior of the critical function for the breakdown of the homotopically non-trivial invariant (KAM) curves for the standard map, as the rotation number tends to a rational number, is investigated using a version of Greene's residue criterion. The results are compared to the analogous ones for the radius of convergence of the Lindstedt series, in which case rigorous theorems have been proved. The conjectured interpolation of the critical function in terms of the Bryuno function is discussed.

1. I

A long-standing problem in the study of quasi-integrable Hamiltonian systems is the characterization of the threshold for the break-down of KAM invariant surfaces in terms of the arithmetic properties of the frequencies vectors. In this context, we consider a simple, yet paradigmatic, discrete-time model, the so called *standard map*, introduced originally in [17, 22]. The standard map is the dynamical system defined by the iteration of the map

$$T_\varepsilon : \begin{cases} x' = x + y + \varepsilon \sin x \,, \\ y' = y + \varepsilon \sin x \,. \end{cases} \tag{1.1}$$

Here $(x, y) \in \mathbb{T} \times \mathbb{R}$; but of course the map T_ε could be lifted to a map

$$T^*_\varepsilon : (\xi, \eta) \mapsto (\xi', \eta')$$

on the plane $\mathbb{R}^2$ given by the same formula as (1.1) with (ξ, η) replacing (x, y). For some background information, we refer the reader to the enormous literature on the topic, and in particular to [27] for a review.

Despite its apparent simplicity, there are only a few properties of the standard map which can be considered really well understood to full extent, especially from an analytical point of view. For instance the existence of KAM invariant curves, for values of the parameter ε small enough and Diophantine rotation numbers, has been proved a long time ago, but only recently the dependence of the radius of convergence on the rotation number has been obtained [18, 5] as an interpolation formula in terms of the Bryuno function (see below). Also for the studying of the separatrix splitting, only recently the original program by Lazutkin [24] has been completely achieved in a rigorous way [21].

In particular no rigorous analysis has been implemented for detecting the critical value of ε at which the KAM invariant curve breaks down, and only numerical results and heuristic theories exist on that subject; see [27, 29, 1].

In (1.1) we can eliminate the y variable by writing the dynamics "in Lagrangian form" as a second order recursion:

$$x_{n+1} - 2x_n + x_{n-1} = \varepsilon \sin x_n \,, \tag{1.2}$$

for all $n \in \mathbb{Z}$.

For $\varepsilon = 0$, the circles $y = (\text{const.})$ are invariant curves on which the dynamics is given by rotation with angular velocity $\omega = y/2\pi$; we call ω the *rotation number*. Without generality loss we can choose $\omega \in (0, 1)$ as the invariant curves of the standard map are invariant under translation of 2π in the y-direction.

As the perturbation is turned on, we face the classical KAM problem of determining which invariant curves survive and up to which size of the perturbative parameter ε. Such invariant curves are given parametrically by the equation

$$C_{\varepsilon,\omega} : \begin{cases} x = \alpha + u(\alpha, \varepsilon, \omega) \,, \\ y = 2\pi\omega + u(\alpha, \varepsilon, \omega) - u(\alpha - 2\pi\omega, \varepsilon, \omega) \,, \end{cases}$$

where in the α variable the dynamics on the curve $C_{\varepsilon,\omega}$ is given by rotations $\alpha_{n+1} = \alpha_n + 2\pi\omega$ (which solve (1.2) for $\varepsilon = 0$). The function $u(\alpha, \varepsilon, \omega)$ is called the *conjugating function* or *linearization,* and satisfies the functional equation

$$\begin{aligned} \left(D^2_\omega u\right)(\alpha, \varepsilon, \omega) &\equiv u(\alpha + 2\pi\omega, \varepsilon, \omega) - 2u(\alpha, \varepsilon, \omega) + u(\alpha - 2\pi\omega, \varepsilon, \omega) \\ &= \varepsilon \sin(\alpha + u(\alpha, \varepsilon, \omega)) \,, \end{aligned} \tag{1.3}$$

whose solutions are formally unique if we impose that $u(\alpha, \varepsilon, \omega)$ has zero average in the α variable. Therefore the study of the invariant curves $C_{\varepsilon,\omega}$ and of their smoothness properties may be reduced to the study of the existence and smoothness of the solutions of the functional equation (1.3).

The solutions of (1.3) can be studied perturbatively by formally expanding $u(\alpha, \varepsilon, \omega)$ in Taylor series in ε and in Fourier series in α; the resulting series is what is traditionally called the *Lindstedt series*:

$$u(\alpha, \varepsilon, \omega) = \sum_{k=1}^{\infty} \varepsilon^k \, u^{(k)}(\alpha, \omega) = \sum_{k=1}^{\infty} \varepsilon^k \sum_{|\nu| \le k} e^{i\nu\alpha} \, u^{(k)}_\nu(\omega) \,. \tag{1.4}$$

To characterize the breakdown of an invariant curve $C_{\varepsilon,\omega}$ we introduce the *radius of convergence* of the Lindstedt series

$$\rho(\omega) = \inf_{\alpha \in \mathbb{T}} \left(\limsup_{k \to \infty} \left| u^{(k)}(\alpha, \omega) \right|^{1/k} \right)^{-1} \,, \tag{1.5}$$

the *lower (analytic) critical function*

$$\varepsilon_{\rm c}(\omega) = \sup\{\varepsilon' \ge 0 : \ \forall\, \varepsilon'' < \varepsilon' \ C_{\varepsilon'',\omega} \text{ exists and is analytic}\}, \tag{1.6}$$

and the *upper (analytic) critical function*

$$\tilde{\varepsilon}_{\rm c}(\omega) = \inf\{\varepsilon' \ge 0 : \ \forall\, \varepsilon'' > \varepsilon' \ C_{\varepsilon'',\omega} \text{ does not exists as an analytic curve}\}. \tag{1.7}$$

In general one could define analogous functions for negative values of ε; for the standard map they would be anyhow identical (by symmetry properties).

Clearly $\rho(\omega) \leq \varepsilon_c(\omega)$ (in the early papers on the subject some confusion was often made between ρ and ε_c). It is instead *believed* that, *for the standard map*, $\varepsilon_c(\omega) = \tilde{\varepsilon}_c(\omega)$, so we can speak generically of *one* critical function $\varepsilon_c(\omega)$ without further qualification. Note that for similar maps with more general perturbations numerical results [35] suggest that the two critical functions may be indeed different. Note also that one could define breakdown thresholds with the analyticity condition in (1.6), (1.7) replaced by a weaker one (such as C^∞ or C^k); again those thresholds could, in principle, be different from the analytical one, though for the standard map it is *believed* that no such difference exists, so that the analytic category is the right one to investigate the breakdown phenomenon.

The radius of convergence of the series (1.4) is zero – and no KAM invariant curve exists – when ω is rational. When ω satisfies an irrationality condition known as the *Bryuno condition* (see below), instead, it can be proved that $\rho(\omega) > 0$ – so that analytic invariant curves exist for ε small – and even precise upper and lower bounds on the dependence of $\rho(\omega)$ on ω can be given, up to a bounded function of ω [18, 5]. More precisely for any rotation number ω one can define the *Bryuno function* $B(\omega)$, as the solution of the functional equation [36, 31]

$$\begin{cases} B(\omega) = -\log\omega + \omega B(\omega^{-1}) & \text{for } \omega \in (0,1) \text{ and irrational}, \\ B(\omega+1) = B(\omega)\,. \end{cases} \tag{1.8}$$

By a fixed-point argument it can be proved that a solution to (1.8) exists and is unique in $L^p(\mathbb{T})$ for each $p \geq 1$.

We shall call *Bryuno number* a number ω satisfying the *Bryuno condition* $B(\omega) < \infty$. Then for any Bryuno number ω one has

$$\left|\log\rho(\omega) + 2B(\omega)\right| < C_0\,, \tag{1.9}$$

for a universal constant C_0, that is for a constant C_0 independent of ω (see [18, 5] for a proof); in particular this implies that an invariant curve with rotation number ω exists if and only if ω satisfies the Bryuno condition. Equation (1.9) and similar formulas are referred to as “Bryuno’s interpolation formulas”.

The claim is often made that a formula analogous to (1.9) should hold for $\varepsilon_c(\omega)$: for any ω satisfying the Bryuno condition one should have

$$\left|\log\varepsilon_c(\omega) + \beta B(\omega)\right| < C_1\,, \tag{1.10}$$

for a universal constant C_1, with an exponent $\beta \leq 2$; it is conjectured that $\beta = 1$ (see *e.g.* [32, 14]).

Equation (1.10) implies a scaling law for the critical function $\varepsilon_c(\omega)$ as $\omega \to p/q$ *on suitable sequences of Bryuno numbers*. In fact, given (1.10), there are sequences of Bryuno (even Diophantine) rotation numbers chosen in such a way that along them the critical function tends to zero *in any arbitrarily fast way*. For example, we can consider the two sequences of Diophantine (even noble) rotation numbers

$$\omega_k = \frac{1}{k+\gamma}, \qquad \tilde{\omega}_k = \cfrac{1}{k + \cfrac{1}{2^{k^2}+\gamma}}, \tag{1.11}$$

where $\gamma = (\sqrt{5}-1)/2 = [1^\infty]$ is the golden mean; then (1.10), with $\beta = 1$, would imply that $\varepsilon_c(\omega_k) = O(1/k)$ while $\varepsilon_c(\tilde{\omega}_k) = O(e^{-k}/k)$, that is *much faster* (see [5], p. 625-626). It is therefore essential to have a good control over the arithmetic properties of the rotation numbers one considers when speaking of scaling properties of the critical function $\varepsilon_c(\omega)$.

The conjecture of Bryuno's interpolation was actually made for the critical function $\varepsilon_c(\omega)$ more than 10 years ago in [32] (motivated also by the numerical results in [30] for related complex area-preserving maps and in [12] for modular smoothing); in that paper, Bryuno's interpolation is stated formally for the radius of convergence, but the numerical calculations, with Greene's method, compute the critical function. The main motivation behind [32] was the comparison with the work of Yoccoz in [36], together with the claims of universality coming from the renormalization description of the critical invariant circle. In [18, 5] (see also [3]) Bryuno's interpolation for the radius of convergence was indeed proved; the mechanism of proof in [5], based on the multiscale decomposition of the propagators in the tree expansion, naturally generates an estimate of $\rho(\omega)$ in terms of the Bryuno function for the semi-standard and standard maps. On the other hand, there is no compelling *a priori* heuristic reason for the critical function $\varepsilon_c(\omega)$ for the standard map to satisfy an interpolation formula in terms of the same arithmetical function as the radius of convergence $\rho(\omega)$; put it in another way, it is by no means obvious that $|\log\rho(\omega) - (2/\beta)\log\varepsilon_c(\omega)|$ should be bounded.

From this point of view, it would be interesting to consider generalized standard maps, *i.e.* maps where the nonlinear term in (1.1) is an arbitrary analytic, periodic function of x (see [4, 6]). In these cases a Bryuno's interpolation formula for the radius of convergence of their Lindstedt series is not known.

The method used in [2] cannot be pushed so far to get reasonable numerical data on the critical function, for some rather obvious reasons; in fact, the method we used there (Padé approximants) attempts at modeling the *whole* natural boundary, giving particular weight at those regions of the boundary where the singularity is "strongest": that is, to those regions closer to the origin (the *first order* or *dominant singularities* as defined in [2]), which determine $\rho(\omega)$; so that part of the natural boundary near the real ε axis, which determines $\varepsilon_c(\omega)$, is represented, as ω is closer and closer to a rational value, as a few scattered points from which no reliable information can be extracted: this happens already for rotation numbers as little close to a rational value as, for instance, $1/(50+\gamma)$ is to 0, that is still quite far from the rational value. One clearly needs a method in which all the computing power is dedicated to the calculation of the quantity one is interested in, that is $\varepsilon_c(\omega)$.

To this aim, two methods have been used previously in the literature: Greene's method (also known as residue criterion; more about it in the next section), used in [32], and the frequency map analysis [23], used in [14]. As we also use Greene's method, we shall postpone a more thorough analysis to the next section, and go on to a discussion of the results of [14].

In [14] the following functions are defined:

$$\omega^+_{p/q}(\varepsilon) = \inf\left\{\omega > \frac{p}{q} : C_{\varepsilon,\omega} \text{ exists and is } C^1\right\}, \tag{1.12}$$

and

$$\omega^-_{p/q}(\varepsilon) = \sup\left\{\omega < \frac{p}{q} : \mathcal{C}_{\varepsilon,\omega} \text{ exists and is } C^1\right\}. \tag{1.13}$$

The meaning of those functions is that, for the given value of ε, no (C^1) invariant curves exist with rotation numbers between $\omega^-_{p/q}(\varepsilon)$ and $\omega^+_{p/q}(\varepsilon)$. The frequency map analysis method computes $\Delta_{p/q}(\varepsilon) = \omega^+_{p/q}(\varepsilon) - \omega^-_{p/q}(\varepsilon)$ for selected values of ε; $\Delta_{p/q}(\varepsilon)$ should tend to 0 with ε and in this way a lower bound on β should be obtained (see below). Note that ε is fixed, and correspondingly some rotation numbers are *computed numerically*, therefore losing any strict control over their arithmetical properties.

We remark that the regularity properties of the functions $\omega^\pm_{p/q}(\varepsilon)$ are quite hard to understand, and in particular their relation with the critical function is far from obvious. In fact, while it is certainly true that

$$\varepsilon_c(\tilde\omega) < \varepsilon \quad \forall \tilde\omega \in (\omega^-_{p/q}(\varepsilon), \omega^+_{p/q}(\varepsilon)), \tag{1.14}$$

the formulas at p. 2037 and p. 2052 of [14], that is $\varepsilon_c(\omega^\pm_{p/q}(\varepsilon)) = \varepsilon$, cannot be claimed in full rigor since an invariant curve with rotation number very close to p/q can be broken by the effect of *another* resonance $p'/q' \approx p/q$, but distinct, so that we can at most claim that

$$\varepsilon_c(\omega^\pm_{p/q}(\varepsilon)) \le \varepsilon. \tag{1.15}$$

This implies that the law $|\omega^\pm_{p/q}(\varepsilon) - p/q| \approx \varepsilon^q$, numerically determined in [14], provides for the critical exponent an estimate from below of the actual value β, which in principle could be higher (if it does exist at all). Equality in (1.15) can be safely assumed at best for ε such that the corresponding value $\omega^\pm_{p/q}(\varepsilon)$ belongs to a special class of rotation numbers tending to p/q (in some sense the "best ones", that is the ones whose partial quotients grow as slow as possible), which are indeed the ones considered in [14] and in the present paper (and which are the only really accessible to a numerical investigation). Note also that to saturate (1.15) one should assume other qualitative features (like monotonicity) on the functions $\omega^\pm_{p/q}(\varepsilon)$, which are far from being proved. However *for the case* $\omega \to 0$ *only* this is enough, since estimates in [34] imply *an upper bound* on the critical exponent, which closes the gap (the analytical estimates in [14] are indeed obtained for such a case by combining the results of [34] for the upper bound with those of [21] for the lower bound).

The numerical lower bounds found in [14] for β are consistent with $\beta = 1$ with errors of orders 4% for ω close to $0/1$, 10% for ω close to $1/2$, 5% for ω close to $1/3$, 10% for ω close to $1/4$, 8% for ω close to $1/5$ and 10% for ω close to $2/5$.

Establishing a condition like (1.10) is out of reach from the numerical point of view if one wants to take into account *arbitrary* sequences of Bryuno numbers. In fact for the frequency map analysis this is a limitation intrinsic to the method itself, since it automatically sort of chooses the best sequence of Diophantine numbers tending to any given rational value. For any other method, like Greene's residue criterion, to investigate Bryuno non-Diophantine numbers would require computer resources far beyond current availability, while computing the critical function even for Diophantine numbers with large partial quotients becomes substantially hard. So the question of establishing a Bryuno interpolation formula for $\varepsilon_{\rm c}(\omega)$, and obtaining the correct critical exponent β if such a formula is indeed established, is still quite open.

In this paper we use Greene's method to compute the critical function when the distance of the rotation numbers from the resonances is of order 10^{-5}. As the computations close to resonances become very time-consuming we look at only three resonances (0, 1/2 and 1/3). We then use the conjectured Hölder-continuity property of the function $\log \varepsilon_{\rm c}(\omega) + \beta B(\omega)$ to derive the corrections to the asymptotic behavior of $\log \varepsilon_{\rm c}(\omega)$, so improving significantly the agreement of the data with the conjectured value of $\beta = 1$.

Of course the problem is not completely solved, even from the numerical point of view, for two reasons. The first is that we consider only three resonances, so that a more exhaustive investigation would be needed. The second is the aforementioned very special choice of the sequences of rotation numbers tending to the resonances that we have to use. Nevertheless we improve the results existing in literature by one order of magnitude both in the distance from the resonance and in the value of β, finding further support for the conjectured Bryuno's interpolation formula for the critical function.

2. G ’

The main tool we use to determine numerically the break-down thresholds for analytic invariant curves for the standard map is Greene's method, known also as *residue criterion*. We now recall the main properties of the periodic solutions of the standard map used to formulate Greene's method, and sketch briefly its foundations, referring to the original paper [22] for more details.

We also recall that in [19] and [28] some theorems are proved that go some way in the direction of proving the validity of Greene's method, at least in special cases. While a full rigorous justification of its use has not yet been achieved, Greene's method is considered one of the most accurate way to compute the critical function $\varepsilon_{\rm c}$ for the standard map.

If ω is a rational number, given as the irreducible fraction p/q, then Birkhoff theory [10, 11] applies; its consequences for maps like the standard map T_ε are the following. If $\varepsilon = 0$ (unperturbed, linear case) then there are trivially invariant curves with rational rotation number p/q, such that every point on them is a fixed point of the iterated map $T_\varepsilon^{\circ q}$. As the perturbation is turned on, only $2kq$, $k \in \mathbb{N}$, points survive as fixed points of the q-th iterate of the map T_ε. These correspond to an even number ($2k$) of periodic orbits of period q. Such orbits – that we call *perturbative* – are the ones which will be studied within a perturbative framework; a simple perturbative calculation (see *e.g.* [7]) shows that for the standard map the even number of such periodic orbits is indeed just 2.

Of course this does not mean that such orbits are the only periodic ones for the standard map, but they are those which are obtained by continuation (in ε) from unperturbed ones. In other words such a scenario does not consider the new periodic orbits arising when the perturbation is switched on. If we pass to the plane $\mathbb{R}^2$ and consider the map T_ε^*, then the situation can be clarified in the following way. When ω is irrational and satisfies the Bryuno condition, then the invariant curve with rotation number ω of the unperturbed map survives for small values of ε, while an invariant curve with rational rotation number p/q is suddenly destroyed; instead, only two discrete invariant sets of points $\{(\xi_j^{(\ell)}, \eta_j^{(\ell)})\}_{j\in\mathbb{Z}}$,

$\ell = 1, 2$ survive, such that

$$\begin{cases} \xi_{j+1}^{(\ell)} > \xi_j^{(\ell)}, \\ \xi_{j+q}^{(\ell)} = \xi_j^{(\ell)} + 2\pi p, \quad \ell = 1, 2. \end{cases} \tag{2.1}$$

By taking the quotient in the first variable by the group of discrete translations by multiples of 2π, we obviously get two periodic orbits of period q, on which the motion has rotation number p/q.

In [7] it is also proved that, for small values of ε, each such periodic orbit lies on an analytic curve – called a *remnant* of the rational invariant curve of the unperturbed map –, and for rational numbers which approximate a Bryuno number ω such remnants approximate the invariant curve with rotation number ω.

The basic idea of Greene's method consists in relating the break-down of an invariant curve with the loss of stability of nearby perturbative periodic orbits. In practice, the hypothesis behind Greene's method is that, if $\varepsilon < \varepsilon_c(\omega)$, then there is a sequence of stable perturbative periodic orbits with rotation numbers p_k/q_k; as ε grows beyond $\varepsilon_c(\omega)$, these periodic orbits lose stability in the large k limit.

The criterion can be formulated more precisely in the following way. Let $\{(x_i^{(k)}, y_i^{(k)})\}_{i=1}^q$ be a perturbative periodic orbit with rotation number p_k/q_k, approximating the irrational rotation number ω. Let $\mathcal{T}_k(\varepsilon)$ be the trace of the tangent dynamics along the periodic orbit:

$$\mathcal{T}_k(\varepsilon) = \operatorname{tr} \prod_{i=1}^{q_k} \begin{bmatrix} 1 + \varepsilon \cos x_i^{(k)} & 1 \\ \varepsilon \cos x_i^{(k)} & 1 \end{bmatrix}. \tag{2.2}$$

Then the periodic orbit is stable if $-2 < \mathcal{T}_k(\varepsilon) < 2$, unstable otherwise. For historical reasons, the criterion is usually formulated in term of the *residue* $\mathcal{R}_k(\varepsilon)$ of the orbit, related to the above trace by

$$\mathcal{R}_k(\varepsilon) = \frac{2 - \mathcal{T}_k(\varepsilon)}{4}.$$

Therefore in terms of the residue the orbit is stable if $0 < \mathcal{R}_k(\varepsilon) < 1$, unstable otherwise. We then track, for a fixed value of ε, the residue of those perturbative periodic orbits with rotation numbers p_k/q_k which are stable for $\varepsilon = 0$; if the residue diverges as $k \to \infty$, then $\varepsilon > \varepsilon_c(\omega)$, while if the residue tends to 0 then $\varepsilon < \varepsilon_c(\omega)$.

It is actually conjectured (see [28]) that if $\varepsilon < \varepsilon_c(\omega)$, then the residue $\mathcal{R}_k(\varepsilon)$ tends exponentially to zero as $k \to \infty$, with a rate of decay proportional to the width of the analyticity strip of the conjugating function $u(\alpha, \varepsilon, \omega)$ on the complex α plane for the values of ε and ω considered. So Greene's method can also be used also to provide numerical information on the analytic properties of u in α, assuming this conjecture.

An interesting question is what actually happens to the residue *at the critical function* $\varepsilon_c(\omega)$. It was originally conjectured that for *noble* rotation numbers (that is, rotation numbers which are obtained by applying a modular transformation to the golden mean, so that their continued fraction expansion has a "tail" of 1's) it tends to a limit value, which should be about 0.25. We present below some numerical results which show that generally the situation is more complicate, and that such *limit residue* $\mathcal{R}_\infty(\varepsilon)$ could be not only different for different classes of rotation numbers, but could also be non-existent, and relate the behavior of the sequence of residues $\mathcal{R}_k(\varepsilon)$ for a fixed value of ε along the sequence of perturbative periodic orbits of rotation numbers p_k/q_k to the arithmetic properties of the rotation number.

From the practical, computational standpoint, the implementation of Greene's method faces some challenges if we wish to use it near resonances. The first is that, if ω is near a resonance, then the q_k become soon quite large, that is we have to find many *long* periodic orbits, which takes a lot of computer time.

The second, hardest, challenge is more subtle. In fact, if it happens that the rotation number of a periodic orbit is $p/q \approx p'/q'$, with $q \gg q'$ (the typical situation arising when approximating irrational rotation numbers close to small-denominator rationals) then it appears numerically that the periodic orbit of rotation number p/q tends to consist in lots (q is supposed to be large) of points accumulating near the points making the periodic orbit of rotation number p'/q'. The consequences for the computation of the residue are dire, as in this case the matrix in (2.2) has two very large, opposite, nearly equal in absolute value diagonal elements, so that when computing the trace the real data cancels and one is left with just the numerical error. Note that using a low precision with Greene's method so close to resonances gives essentially noise instead of the residue, so we get no values at all for β. We choose a brute-force solution to this precision problem, which consists in increasing the number of digits in the calculations until some data is left when computing the trace. Empirically, this could mean that one has to use *hundreds* of digits of precision in computing (2.2) numerically, *therefore also the periodic orbits must be known with such a precision*: considering that one easily needs periodic orbits of period in the range of several tens of thousands – we actually reach orbits of length of the order of 150000 –, the calculation of a single value of $\varepsilon_{\rm c}$ can require a great amount of computer time.

3. N

3.1. **Rotation numbers close to 0.** Consider rotation numbers $\omega_n = 1/(n+\gamma) = [n, 1^\infty]$, with $n \in \mathbb{N}$: in table 1 we give the values of the Bryuno function and of the critical function for rotation numbers ω_{n_k}, with $\{n_k\}$ a finite increasing sequence. Note that we reach values of rotation number close more than 2×10^{-5} to the resonance value (0 in this case), which corresponds to values of n up to 60000.

By fitting $y = -\log \varepsilon_{\rm c}(\omega_{n_k})$ as a linear function of $x = B(\omega_{n_k})$, we obtain

$$y = ax + b \qquad a = 0.9705, \qquad b = -1.9553. \tag{3.1}$$

As we see the slope is close to (but different from) 1: the relative difference is about 3.0%.

One also realizes that the slope of the line increases if we neglect the rotation numbers ω_n corresponding to smaller values of n: this suggests that, if we consider just pairs of successive rotation numbers and evaluate the slope of the line passing through them, then we obtain an increasing function. This can be formulated more precisely as follows. For $n, m \in \mathbb{N}$ define

$$A(\omega_n, \omega_m) = -\frac{\log \varepsilon_{\rm c}(\omega_n) - \log \varepsilon_{\rm c}(\omega_m)}{B(\omega_n) - B(\omega_m)}, \tag{3.2}$$

which measures the slope a of the line

$$-\log \varepsilon_{\rm c}(\omega) = aB(\omega) + b, \tag{3.3}$$

passing through the points $(B(\omega_n), -\log \varepsilon_c(\omega_n))$ and $(B(\omega_m), -\log \varepsilon_c(\omega_m))$. We set $A_k = A(\omega_{n_{k+1}}, \omega_{n_k})$. In table 1 we give also the values of the slopes A_k: as we noted A_k steadily increases.

The value $\beta = 1$ is anyhow still far from being reached: at best, just considering the last value of A_k in table 1 we obtain a value whose relative difference from 1 is greater than 1%. Moreover, though the values of the slopes increase as $n \to \infty$, the convergence to 1 is very slow. In the next section we shall provide a heuristic argument which allows to guess the correction to the asymptotic behavior and so try to extrapolate a better value of β; this applies also to the cases considered in the next subsections.

In table 2 we give the values of the Bryuno function and of the critical function for a finite sequence of rotation numbers ω_{n_k}, with $\omega_{n_k} = 1/(n_k + 1/(20 + \gamma)) = [n_k, 20, 1^\infty]$: such numbers tend to 0 as the previously considered ones, and share with them, essentially, the same Diophantine properties, as they have the same "tail" of 1's in their continued fraction expansion, with the only difference that there is a partial quotient 20 before such a "tail". The distance of the rotation numbers considered is up to 2×10^{-4} from 0, i.e. an order less than in the previous case: this is due to the fact the partial quotients go faster, and it becomes longer for the residue to reach the asymptotic value (so that periodic orbits with larger periods should be considered in order to obtain for the rotation numbers the same distance from the resonance value).

As one can see, the values of the Bryuno function and of the critical function are comparable with those listed in table 1: the introduction of a larger partial quotient does not introduce any relevant change. As a consequence, also the slopes A_k, defined as before with the new definition of ω_{n_k}, are very similar (as a look at the last column of table 2 immediately confirms).

Note however that to compute numerically the critical function for rotation numbers of the form $[n_k, 20, 1^\infty]$ for given k is much more time consuming, since, in general, to obtain a reliable precision we are forced to reach periodic orbits with very high periods (say more than a hundred thousand), which requires a precision of about 600 digits.

3.2. **Rotation numbers close to 1/2.** In table 3 we consider a sequence of rotation numbers tending to 1/2 of the form $\omega_n = 1/(2 + 1/(n + \gamma)) = [2, n, 1^\infty]$. The rotation numbers considered are up to 10^{-5} close to the resonance value 1/2 (which correspond to values of n up to 20000).

The fit for $y = -\log \varepsilon_c(\omega_{n_k})$ as a linear function of $x = B(\omega_{n_k})$ gives

$$y = ax + b \qquad a = 0.9641, \qquad b = -1.6203. \tag{3.4}$$

Again we see the slope is not 1, and the relative error is now about 3.6%. It is greater than in the previous case because we stopped to smaller values of n; in fact the values of the slopes listed in table 3 show that again the function A_k, defined exactly as before with the new definition for the rotation numbers ω_{n_k}, is increasing in k. The relative difference from 1 of the last value of A_k is about 1.7%.

3.3. **Rotation numbers close to 1/3.** In table 4 we consider a sequence of rotation numbers tending to 1/3 of the form $\omega_n = 1/(3 + 1/(n + \gamma)) = [3, n, 1^\infty]$. The rotation numbers considered are up to 5×10^{-6} close to the resonance value 1/3 (which correspond to values of n up to 20000).

The fit for $y = -\log \varepsilon_c(\omega_{n_k})$ as a linear function of $x = B(\omega_{n_k})$ gives

$$y = ax + b \qquad a = 0.9637, \qquad b = -1.6526. \tag{3.5}$$

while in the last column of table 4 we list the slopes A_k, again defined as before with the new definition for the rotation numbers ω_{n_k}; the relative difference of the slope with respect to 1 is more than 3.7%, while the relative difference from 1 of the last value of A_k is about 1.7%.

3.4. **Behavior of the critical residues and other rotation numbers.** The behavior of the residue for ε *exactly* equal to the critical function $\varepsilon_c(\omega)$ when the rotation number of the approximating periodic orbits tends to ω has been considered since the very first papers on the subjects (for example, in [22] itself). In particular, one considers the sequence of residues $\mathcal{R}_k(\varepsilon_c(\omega))$ when $k \to \infty$; it appears that this sequence has a limit only when ω is a number of so called "constant type", *i.e.* when ω can be written as $[a_1, \dots, a_N, d^\infty]$. This limit moreover seems to depend only on the integer d, and not at all from the "head" of the continued fraction expansion $[a_1, \dots, a_N]$. Unfortunately, a sound numerical evidence can be obtained only for $d = 1$ and for short "heads" in the continued fraction expansion, otherwise the partial quotients gets soon large and it becomes difficult to compute the critical residues with the accuracy required: therefore we state this more as a somewhat numerically founded and reasonable conjecture than else. In table 5 we give some values of the critical residue for a few values of d.

If a rational number is not of constant type, then a limit does not seem to be achieved for the sequence of critical residues. In fact, it seems to happen that if ω is a quadratic irrational, so that the sequence of the partial quotients a_k is eventually periodic, the sequence of critical residues is itself eventually periodic with the same period. In tables 6, 7, 8 we can see the sequence of critical residues for some quadratic irrationals with short periods (resp. 2, 2 and 3). If the rotation number is not a quadratic irrational, so the partial quotients are aperiodic, the sequence of critical residues does not seem to have any regularity (but see below for a numerical difficulty).

So far only quadratic irrational ω have been considered. This is of course a limitation, due mainly to practical reasons; in fact, quadratic irrationals are the only irrationals with an eventually periodic continued fraction expansion, so they are particularly suited to Greene's residue criterion for two reasons: (1) the partial quotients a_k are bounded, since they are periodic, so the approximants p_k/q_k have denominators which do not grow too much and (2) if the period is reasonably small, one can tell whether the critical function has been reasonably approximated by looking at the sequence of residues over a span of periods and easily see whether it decreases or increases instead of being periodic. Instead, if the sequence of the partial quotients is aperiodic (and worst yet, unbounded) one can never be sure that the critical function has been obtained since the next periodic orbit to be considered (corresponding to the next approximant p_k/q_k) could come from an abnormally high (or low) partial quotient a_k. Note that in [15], where general irrationals are also considered in Subsection 3.3, in the numerical calculations of the critical function, only the first ten partial quotients of the rotation numbers are retained, and all the others are set to 1, so one practically comes back to the case of noble numbers like ours.

This raises the question whether the *algebraic* (rather than just number-theoretic) properties of the rotation number have any role in the properties of the corresponding invariant curve. Lindstedt series expansion methods for instance do not care about the algebraic properties of ω, as the only relevant property is whether ω is a Bryuno number or not. "Phase space" renormalization group methods instead seem to work (or at least they have been applied) only in the case of quadratic irrational rotation numbers, so their results could depend on the algebraic layer. We expect that the algebraic properties of ω could show up, maybe, in discussing the smoothness of the natural boundary in the complex ε plane, but of course this is just a speculation.

4. D

Despite intensive numerical calculations, the problem of confirming the conjecture expressed in (1.10) and estimating the exponent β cannot be considered completely settled even from the numerical point of view. In fact, as we noted earlier, only the three resonances $0/1$, $1/2$ and $1/3$ have been considered, and only very special sequences of rotation numbers tending to such rational numbers have been used: considering other sequences of rotation numbers, in fact, means using numbers which have quite soon very large partial quotients, so that they are very bad from the numerical point of view.

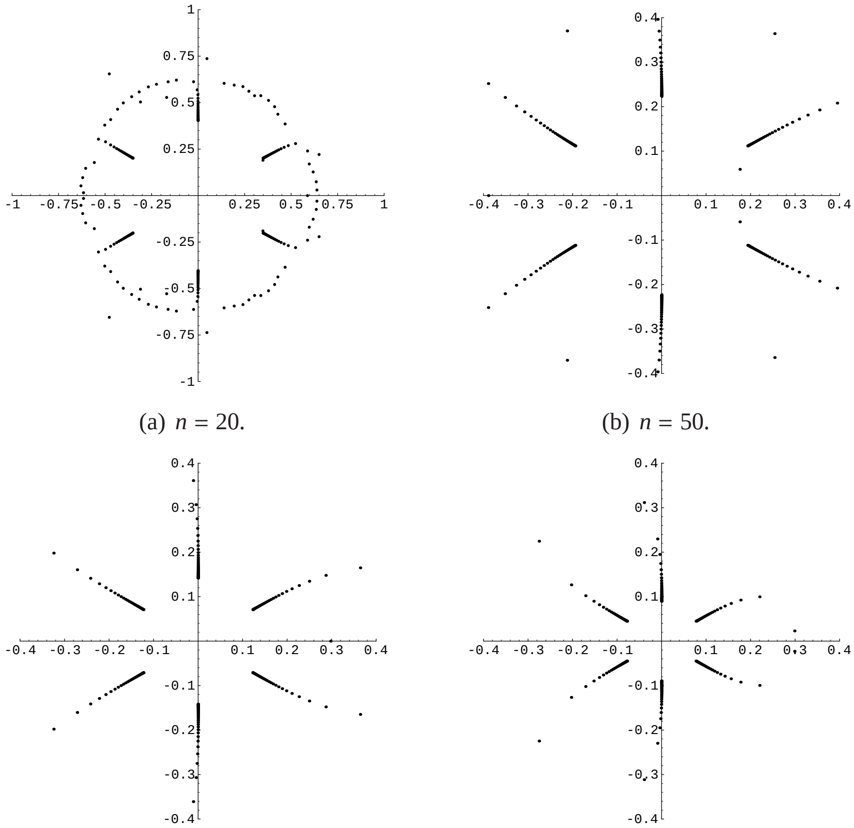
Moreover, a simple linear fit of $\log \varepsilon_c(\omega)$ against $B(\omega)$, that is a fit which takes into account only the leading conjectured asymptotic behavior without any corrections, still gives results which are quite unsatisfying, as the difference between the estimated value of β and the conjectured value $\beta = 1$ is still of the order of a few percent. What is worst, the "running slopes" A_k defined in the preceding section continue to grow monotonically from below, slowly but steadily, so that one cannot even conclude that the conjecture is false or that the value of β is actually smaller than 1. Clearly corrections must be taken into account, or otherwise rotation numbers even closer to the resonances (and significantly such) must be considered, which is numerically unfeasible with current resources.

Note also the apparently quite singular fact that for $\rho(\omega)$ the value 2 of the corresponding critical exponent seems to be obtained within a few percent *much earlier*. For instance for the rotation numbers ω close to $1/2$ listed in table 9, by using for the corresponding radii of convergence the values $\rho_P(\omega)$ computed by Padé approximants, we obtain for the slopes $A'_k = A'(\omega_{n_{k+1}}, \omega_{n_k})$, with

$$A'(\omega_n, \omega_m) = -\frac{\log \rho(\omega_n) - \log \rho(\omega_m)}{B(\omega_n) - B(\omega_m)}, \tag{4.1}$$

the values in the last column of table 9. Analogously for the rotation numbers ω close to $1/3$ listed in table 10, again by using the values $\rho_P(\omega)$ computed by Padé approximants for the corresponding radii of convergence, we obtain the slopes A'_k in the last column of the same table.

In figure 1 we represent the analyticity domains for $\omega = [3, 20, 1^\infty]$, $[3, 50, 1^\infty]$, $[3, 100, 1^\infty]$ and $[3, 200, 1^\infty]$ as given by the poles of the Padé approximants $[240/240]$. As noted in Section 1 for ω getting closer to $1/2$ the poles tend to accumulate near the strongest singularity: therefore Padé approximants are not suitable for determining the critical function, but they can be fruitfully used in order to detect the radius of convergence.



(a) $n = 20$. (b) $n = 50$.

(c) $n = 100$. (d) $n = 200$.

Figure 1: Poles of the Padé approximant [240/240] for $\omega = [3, n, 1^{\infty}]$ and $\alpha = 1$.

The relative errors with respect to 2 for the values corresponding to $n = 40$, taken from tables 9 and 10, are about 1.4% and 3.0%, respectively, therefore they are comparable with the errors for the last entries of the corresponding tables 3 and 4 for the critical function for rotation numbers *much closer* to the resonance values: in the latter case indeed such errors are about 1.7%. And for larger values of n the relative errors become much smaller: for instance, for $n = 100$ and $n = 200$, we find from table 10 errors about 1.0% and 0.5%, respectively.

Of course it would be also interesting to have the slopes for the rotations numbers appearing in tables 3 and 4. To obtain the values of $\rho(\omega)$ numerically can be as hard as to determine the critical function $\varepsilon_{\rm c}(\omega)$. Also using the method of Padé approximants can be delicate, as in order to obtain reliable results a very high precision could be necessary. One could think of using the complex extension of Greene's method envisaged in [20], and the analysis, at best, could be as delicate as in the present paper, where real values of ε have been studied. We have also two more difficulties with respect to the case of $\varepsilon_{\rm c}(\omega)$. First one has to guess the direction in the complex plane where the singularities of the boundary of the analyticity domain are the closest to the origin; in this respect the results of [2] suggest, as a natural Ansatz, that, for rotation numbers close to p/q, they such singularities lie along the directions of the $2q$th roots of -1. Next, for fixed ω close to a resonance value, the value of $\rho(\omega)$ should be much smaller than the value of $\varepsilon_c(\omega)$, again as a byproduct of the numerical analysis of [2] (and also that of [14]), so that the value of $\rho(\omega)$ is expected to be harder to detect than $\varepsilon_c(\omega)$, as it should require more precision and hence more computing time. However we prefer to avoid any technical difficulties and to circumvent the problem by using the heuristic formula introduced in [2], say $\rho(\omega) \approx \rho_1(\omega)$, with

$$\rho_1(\omega) = \eta^{2/q} \left(q|C_{p/q}|^{-1} \lambda_c \right)^{1/q}, \tag{4.2}$$

where $\eta = |\omega - p/q|$ if ω is close to the resonance p/q, $C_{p/q}$ is the numerical constant introduced in [3] (one has $C_{0/1} = 1$, $C_{1/2} = -1/8$ and $C_{1/3} = -1/24$), and $\lambda_c = 4\pi^2 \times 0.827524 \approx 32.669338$.

In [2] we have already seen that there is a good agreement between the value $\rho_P(\omega)$ of the radius of convergence found by Padé approximants and the value $\rho_1(\omega)$ predicted by the formula (4.2). Furthermore the formula (4.2) becomes more and more reliable as ω approaches an rational value. See for instance tables 9 and 10, which show how the difference between the two values $\rho_P(\omega)$ and $\rho_1(\omega)$ tend to shrink to zero when making the rotation number ω closer to the rational values $1/2$ and $1/3$, respectively. So we can expect that the approximation we make by evaluating the radius of convergence $\rho(\omega)$ with $\rho_1(\omega)$ is very good for values much closer to the resonance values, as the ones we have considered are.

Then we obtain the values listed in tables 11, 12 and 13 for values of ω close, respectively, to 0, 1/2 and 1/3 (the same for which we determined numerically the critical function); the slopes A'_k are listed in the last columns of these tables. Of course, if we use the formula (4.2), a slope approximately equal to 2 is expected, by the definition itself of $\rho_1(\omega)$. The important fact is, in any case, that the discrepancy with respect to the value $\beta = 2$ (which in such a case *is known* to be the right one) is much smaller. In other words the asymptotic formula (1.9) is reached much earlier than the one which is believed to hold for the critical function.

This different speed in reaching the asymptotic behavior of $\rho(\omega)$ and $\varepsilon_c(\omega)$ can be explained in terms of different corrections to the leading order when $\omega \to p/q$ (and therefore $B(\omega) \to \infty$). We shall now try to compute such correction, at least heuristically, both for $\rho(\omega)$ and for $\varepsilon_c(\omega)$, and try to use them to extrapolate a better value of β.

For what concerns $\rho(\omega)$, we shall assume the validity of the heuristic formula (4.2); this of course can introduce further corrections not accounted for, which we neglect assuming that they are smaller.

Consider for example $\omega_n = 1/(n + \gamma)$, $n \to \infty$. Then $\eta = \omega_n$ in (4.2) and

$$B(\omega_n) = -\log \omega_n + \omega_n B(1/\omega_n),$$

which implies that $\log(n + \gamma) = B(\omega_n) - B(\gamma)/(n + \gamma)$. Therefore in first approximation we have that

$$\log(n + \gamma) \approx B(\omega_n) - B(\gamma) e^{-B(\omega_n)},$$

as the leading behavior of $B(\omega_n)$ is just $\log(n+\gamma)$. This gives

$$\log \rho_1(\omega_n) \approx \log(|C_{0/1}|^{-1}\lambda_c) - 2B(\omega_n) + 2B(\gamma)e^{-B(\omega_n)}, \tag{4.3}$$

that is the correction to the linearly growing asymptotic behavior is *exponentially small*. An analogous, slightly more complicated, computation for $\omega_n = 1/(q+1/(n+\gamma))$ gives a correction of the form $B(\omega_n)\exp(-qB(\omega_n))$, that is still essentially exponentially small. This explains the exceptional rapidity of the approach to the scaling behavior for $\rho(\omega)$.

Quantitatively, a fit of the numerical data of table 11 using (4.3) to model the data gives for β the value 1.9999989, whose difference from the correct value of 2 is of the order of 10^{-7}, while a straight linear fit gives 2.00091, whose error is three orders of magnitude larger.

Analogously, a fit of the numerical data of table 13 using

$$\log \rho_1(\omega_n) \approx \log(|C_{0/1}|^{-1}\lambda_c) - \beta B(\omega_n) + (b + cB(\omega))^{-3B(\omega_n)}, \tag{4.4}$$

gives for β the value 2.000000287, whose difference from the correct value of 2 is of the order of 10^{-7}, while a straight linear fit gives 1.99984, whose error is again three orders of magnitude larger. Also a comparison between the mean-square distances of the data from the corresponding fits is remarkable: we obtain 2.389×10^{-8} for the fit by (4.4), and 0.0000858 for the linear fit. Note that also fits with either $b = 0$ or $c = 0$ in (4.4) are worse: for $b = 0$ we obtain a value $\beta = 1.99966$ (with mean-square distance 0.0000413), while for $c = 0$ we obtain a value $\beta = 1.99967$ with mean-square distance 0.0000393).

To compute the correction to the leading behavior of $\varepsilon_{\rm c}(\omega)$ is of course quite another matter, since we don't even have a proof or at least a very strong theoretical argument for the leading order. So the following argument is more a qualitative explanation rather than a quantitative attempt to extrapolate seriously the value of β (but we shall try nevertheless).

As before we shall consider only the case $\omega_n = 1/(q+1/(n+\gamma))$, $n \to \infty$, and we shall set $\eta_n = |\omega_n - 1/q|$. Let

$$\log \varepsilon_{\rm c}(\omega_n) + B(\omega_n) = C(\omega_n), \tag{4.5}$$

where the function $C(\omega)$ is believed to be continuous (see [30, 32]). Let then $\overline{c} = \lim_{\omega\to 0} C(\omega)$. It is also conjectured (see [31]) that $C(\omega)$ is Hölder-continuous with some exponent α (in the quoted paper it is suggested that α could be $1/2$): so, by recalling that $\eta_n \approx e^{-qB(\omega_n)} \to 0$, a reasonable guess in (4.5) could be

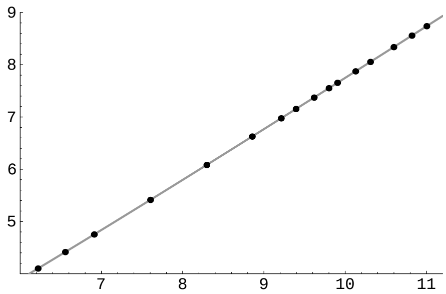
$$\log \varepsilon_{\rm c}(\omega_n) = (\text{const.}) - \beta B(\omega_n) + O(e^{-\alpha q B(\omega_n)}). \tag{4.6}$$

Despite the rough, qualitative nature of the argument above, we can try to fit the data with the formula (4.6) and see whether the growth of the slopes is such that the value of 1 can actually be reached. The "best" value of α is obtained by choosing it in such a way that the mean square distance of the experimental data from the values obtained from the fit is minimal. As an alternative, we performed also nonlinear fits using Levenberg-Marquardt method (see [25, 33]), obtaining consistent results.

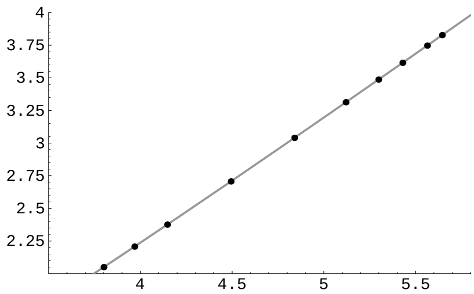
Fitting the data relative to $\varepsilon_c(\omega_n)$ for the sequence considered in table 1 with the formula (4.6), we obtain

$$\log \varepsilon_c(\omega_n) \approx -2.34630 + 1.00359\, B(\omega_n) + 1.59684\, e^{-0.3302\, B(\omega_n)}, \tag{4.7}$$

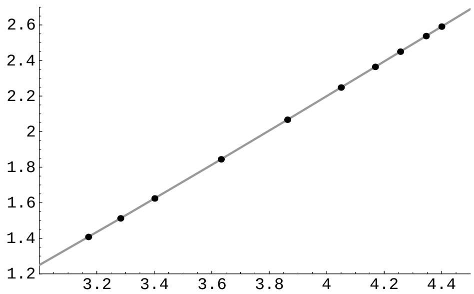
that is finally a value much closer to 1 than the straight linear fit, which gave 0.97052. Moreover, the mean-square distance of the data from the fit is 0.000210 in the case of the fit with corrections, while is much larger, that is 0.0396, in the case of the linear fit (see figure 2a).



(a) Sequence ω_{n_k} listed in table 1.



(b) Sequence ω_{n_k} listed in table 3.



(c) Sequence ω_{n_k} listed in table 4.

Figure 2: Numerical values of $-\log \varepsilon_c(\omega_{n_k})$, obtained with Greene's method, versus $B(\omega_{n_k})$ for the sequence ω_{n_k} listed in tables 1, 3; and 4; the error bars are less than the size of the points. The solid curve corresponds to the fits (4.7), (4.8) and (4.9).

If we consider the case $\omega \to 1/2$ as in subsection 3.2, the fit with formula (4.6) gives

$$\log \varepsilon_c(\omega_n) \approx -1.86364 + 1.00308\, B(\omega_n) + 1.43766\, e^{-0.69671\, B(\omega_n)} \tag{4.8}$$

with mean-square distance $d = 0.0000512$ (the linear fit would give $\beta = 0.96413$ and $d = 0.0124$); in figure 2b we plot the data together with the fit.

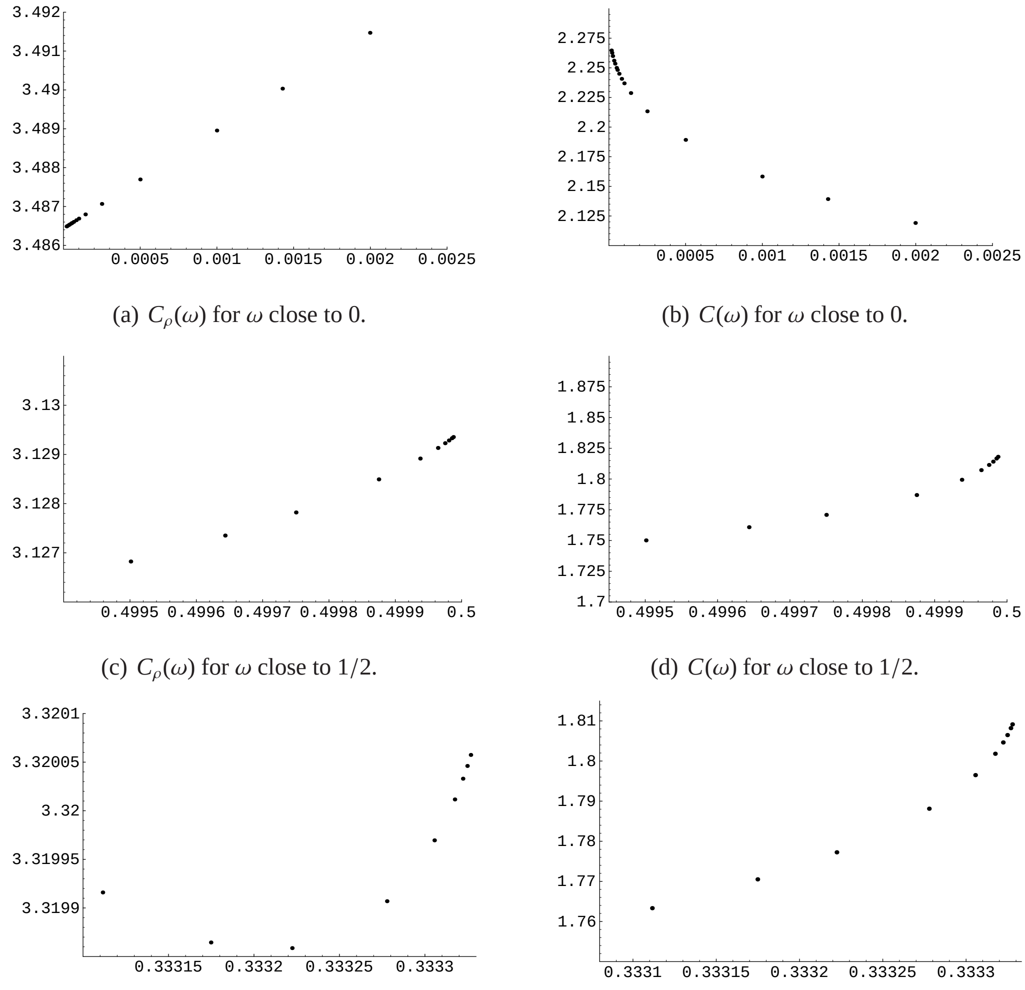
Finally, in the case $\omega \to 1/3$ as in subsection 3.3, the fit with formula (4.6) gives

$$\log \varepsilon_c(\omega_n) \approx -1.84393 + 1.00344\, B(\omega_n) + 1.82643\, e^{-1.0300\, B(\omega_n)}, \tag{4.9}$$

with mean-square distance 0.0000403 (the linear fit would give $\beta = 0.96369$ and $d = 0.00832$); in figure 2c we plot the data together with the fit.

The results, together with the interpolation formula (4.6), hint at a value of α close to $1/3$, while $C_\rho(\omega)$, according to the formula (4.4), seems to be Hölder-continuous with any exponent $\alpha_\rho < 1$ (and $\alpha_\rho = 1$ in 0). So while in both cases the corrections are exponentially small in $B(\omega)$, the coefficient in the exponential is about three times larger for $\log\rho(\omega)$, leading to smaller corrections and faster approach to the asymptotic regime.

If we try to plot $C_\rho(\omega)$ for the values of ω close to 0 listed in table 1, and use the values of $\rho(\omega)$ in table 11 we find the behavior represented in figure 3a, which also support the smoothness conjecture. Analogously, if we plot $C(\omega)$ for the same set of values of ω, by using the values of $\varepsilon(\omega)$ listed in table 1, we find the behavior in figure 3b.



(a) $C_\rho(\omega)$ for ω close to 0.

(b) $C(\omega)$ for ω close to 0.

(c) $C_\rho(\omega)$ for ω close to $1/2$.

(d) $C(\omega)$ for ω close to $1/2$.

(e) $C_\rho(\omega)$ for ω close to $1/3$.

(f) $C(\omega)$ for ω close to $1/3$.

Figure 3: Plot of $C_\rho(\omega) = \log\rho(\omega) + 2B(\omega)$ and $C(\omega) = \log\varepsilon_c(\omega) + B(\omega)$ versus ω for the sequence ω_{n_k} listed in tables 1 (plots a and b), 3 (plots c and d) and 4 (plots e and f).

In figures 3c and 3d we represent the functions $C_\rho(\omega)$ and $C(\omega)$ for the values of ω close to $1/2$ listed in table 3, and in figures 3e and 3f we represent the functions $C_\rho(\omega)$ and $C(\omega)$ for the values of ω close to $1/3$ listed in table 4.

While the variation in the case of $\varepsilon_c(\omega)$ is larger, all plots support the conjecture of a function which is not only bounded but also Hölder continuous close to the resonances, but of course a deeper numerical investigation is needed in order to draw more quantitative deductions.

To conclude, we note also that the value $\beta = 2$, which holds for the radius of convergence, is found both in [14] and in [15] for the critical function, if the computations are made without requiring a high precision (this is attributed to the low precision in [14] and to the truncations and approximations due to the numerical implementation of the renormalization group method in [15]). We find this phenomenon at least very curious: it would be in fact quite interesting to understand why truncation and approximation errors in the numerical computations give a different value of β (and exactly the one holding for the radius of convergence), while in our case numerical errors due to lack of precision give just gibberish.

5. C

We conclude by some general remarks about the advances which have been made and the conclusions which can be drawn from our analysis.

(1) The numerical results of [14] have been improved by an order of magnitude, both in the size of $\Delta\omega$ and in the order of the numerical errors. Moreover, an heuristic argument providing corrections to the leading order has been given: the analysis of the numerical data, taking into account the conjectured form of the corrections, supports both (1.10) with $\beta = 1$ and the continuity of the function $C(\omega)$. A stronger support would require getting closer to the resonances, and considering more resonances rather than just $0/1$, $1/2$ and $1/3$: all these actions are clearly feasible but would require significantly more computer time, which has already reached the order of several CPU years on Compaq Alpha computers for the calculations of the present paper.

(2) While a reasonably complete analysis of what happens for sequences of rotation numbers which are *not* the best ones cannot be practically done, the study of a sequence like $[n, 20, 1^\infty]$ shows that the behavior of the critical function along this sequence is the same that along the "best" sequence $[n, 1^\infty]$. It is likely that this holds, in the limit of long periodic orbits, at least for all sequences of noble numbers tending to a rational value. We could investigate very few non-noble sequences and no sequence at all made by something different than quadratic irrationals (which have measure 0). For a sequence like $[n, 2^\infty]$ the values of $\varepsilon_c(\omega)$ seem to be comparable to the ones of the "best" sequence quoted above.

(3) The study of the functions $C(\omega) = \log \varepsilon_c(\omega) + B(\omega)$ and $C_\rho(\omega) = \log \rho(\omega) + 2B(\omega)$ seem to suggest that they depend smoothly on ω. In general such functions look as continuous in their arguments, as also the comparison between the two sequences $[n, 1^\infty]$ and $[20, n, 1^\infty]$ seem

to support. This is in contrast with the conclusions made in [26], where doubts were raised about continuity of the function $\log \varepsilon_c(\omega) + \beta B(\omega)$ for any choice of β.

(4) Some interesting conclusions can be drawn for the behavior of the critical residues. It appears that a limit value is obtained only when ω is of constant type, and this limit seems to depend only on the "tail" of the expansion. If ω is not of that form but still a quadratic irrational, then the sequence of the partial quotients is eventually periodic. In this case it appears that $\mathcal{R}_k(\varepsilon_c(\omega))$, for large k, approaches a periodic sequence of values with the same period of the partial quotients. If ω is not a quadratic irrational, then it is difficult to draw any conclusion at all. If the partial quotients are bounded, then the critical residues seem to be bounded away from zero and oscillating in an apparent random way, but bounded; for unbounded sequences of partial quotients no numerical data at all could be obtained. We note that such a scenario is consistent with that arising within the renormalization group approach as described, for instance, in [15].

After submission of our paper we become aware of [13], where the problem of interpolation of the radius $r(\omega)$ of the Siegel disk in terms of the $1/2$-Bryuno function $B_{1/2}(\omega)$ is numerically studied (The $1/2$-Bryuno function solves a function equation of the kind of (1.8) and differs from $B(\omega)$ by an essentially bounded function [31]). There numerical evidence is found that the function $\log r(\omega) + B_{1/2}(\omega)$ is $1/2$-Hölder continuous. The main differences between our paper and [13] are the following ones. First of all in the case of Siegel's problem the radius $r(\omega)$ is equal to the modulus of the so called Yoccoz function [36], which is more accessible from a numerical point of view, so that [13] can not be used to compute the critical function of the standard map. Second: complex values of ω are considered, with small imaginary part, so that the the validity of the interpolation formula on the real axis has to be considered as an extrapolation from the really available numerical results. Third: Hölder continuity is checked by using the Littlewood-Paley theory, which requires to consider all together all the values the function assumes (as one needs to know its Fourier transform), while we check Hölder continuity where the function is expected to be less smooth, that is in correspondence of the real resonant values; of course this method could be pursued in principle also in our case, and it would be interesting to compare the results found with the two different approaches.

Another interesting issue could be to study what happens for more general maps, like the generalized standard maps considered in [4, 6]. In our case no interpolation formula in terms of the Bryuno function is expected to hold for the radius of convergence, and one can think that the same occurs for the critical function. One can imagine that, for fixed rotation number ω, either an interpolation formula in terms of the Bryuno function with a rotation number different from ω (but related to it!) can be obtained (as it occurs in a somehow similar situation in [16], where the Escande-Doveil's pendulum was considered), or an interpolation formula in terms of another function holds. But of course the problem is open, and one can not even exclude that no interpolation formula holds at all, that is there, in the case of more general maps, there is no key arithmetical function playing the rôle of the Bryuno function for special models as the Siegel problem and the standard map are.

A

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R

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Table 1: Values of the Bryuno function, of the critical function and of the running slopes $A_k = A(\omega_{n_k}, \omega_{n_{k-1}})$ corresponding to a finite sequence of rotation numbers $\omega_{n_k} = 1/(n_k + \gamma) = [n_k, 1^\infty]$. The error on $\varepsilon_c(\omega_{n_k})$ is of 1 unit on the last digit, and the corresponding slopes are computed with consistent accuracy.

k	ω_{n_k}	$B(\omega_{n_k})$	$\varepsilon_c(\omega_{n_k})$	A_k
1	$[500, 1^\infty]$	6.21836	0.016585	
2	$[700, 1^\infty]$	6.55376	0.0121005	0.9399±0.0002
3	$[1000, 1^\infty]$	6.90963	0.0086401	0.9465±0.0001
4	$[2000, 1^\infty]$	7.60184	0.0044599	0.9553±0.0001
5	$[4000, 1^\infty]$	8.29452	0.0022854	0.9652±0.0001
6	$[7000, 1^\infty]$	8.85393	0.0013265	0.9724±0.0002
7	$[10000, 1^\infty]$	9.21053	0.00093627	0.9770±0.0002
8	$[12000, 1^\infty]$	9.39284	0.00078320	0.9793±0.0001
9	$[15000, 1^\infty]$	9.61593	0.00062927	0.9808±0.0001
10	$[18000, 1^\infty]$	9.79823	0.00052610	0.9823±0.0002
11	$[20000, 1^\infty]$	9.90358	0.00047433	0.9833±0.0004
12	$[25000, 1^\infty]$	10.12671	0.00038081	0.9842±0.0002
13	$[30000, 1^\infty]$	10.30902	0.00031816	0.9859±0.0003
14	$[40000, 1^\infty]$	10.59668	0.00023955	0.9865±0.0003
15	$[50000, 1^\infty]$	10.81982	0.000192161	0.9879±0.0003
16	$[60000, 1^\infty]$	11.00213	0.000160443	0.9895±0.0002

Table 2: Values of the Bryuno function, of the critical function and of the running slopes $A_k = A(\omega_{n_k}, \omega_{n_{k-1}})$ corresponding to a finite sequence of rotation numbers $\omega_{n_k} = 1/(n_k + 1/(20 + \gamma)) = [n_k, 20, 1^\infty]$. The error on $\varepsilon_c(\omega_{n_k})$ is of 1 unit on the last digit, and the corresponding slopes are computed with consistent accuracy.

k	ω_{n_k}	$B(\omega_{n_k})$	$\varepsilon_c(\omega_{n_k})$	A_k
1	$[500, 20, 1^\infty]$	6.22088	0.016303	
2	$[700, 20, 1^\infty]$	6.55556	0.011926	0.9341±0.0004
3	$[1000, 20, 1^\infty]$	6.91089	0.008535	0.9415±0.0006
4	$[2000, 20, 1^\infty]$	7.60247	0.004421	0.9512±0.0005
5	$[4000, 20, 1^\infty]$	8.29483	0.002271	0.962 ±0.001

Table 3: Values of the Bryuno function, of the critical function and of the running slopes $A_k = A(\omega_{n_k}, \omega_{n_{k-1}})$ corresponding to a finite sequence of rotation numbers $\omega_{n_k} = 1/(2 + 1/(n_k + \gamma)) = [2, n_k, 1^\infty]$. The error on $\varepsilon_c(\omega_{n_k})$ is of 1 unit on the last digit, and the corresponding quantities are computed with consistent accuracy.

k	ω_{n_k}	$B(\omega_{n_k})$	$\varepsilon_c(\omega_{n_k})$	A_k
1	$[2, 500, 1^\infty]$	3.80022	0.12872	
2	$[2, 700, 1^\infty]$	3.96840	0.109967	0.9362±0.0005
3	$[2, 1000, 1^\infty]$	4.14674	0.092932	0.9438±0.0001
4	$[2, 2000, 1^\infty]$	4.49337	0.066777	0.9535±0.0001
5	$[2, 4000, 1^\infty]$	4.84001	0.047805	0.9642±0.0001
6	$[2, 7000, 1^\infty]$	5.11987	0.036420	0.9720±0.0002
7	$[2, 10000, 1^\infty]$	5.29823	0.030598	0.9766±0.0003
8	$[2, 13000, 1^\infty]$	5.42943	0.026909	0.9792±0.0006
9	$[2, 17000, 1^\infty]$	5.56357	0.023591	0.9810±0.0006
10	$[2, 20000, 1^\infty]$	5.64484	0.021780	0.983 ±0.001

Table 4: Values of the Bryuno function, of the critical function and of the running slopes $A_k = A(\omega_{n_k}, \omega_{n_{k-1}})$ corresponding to a finite sequence of rotation numbers $\omega_{n_k} = 1/(3 + 1/(n_k + \gamma)) = [3, n_k, 1^\infty]$. The error on $\varepsilon_c(\omega_{n_k})$ is of 1 unit on the last digit, and the corresponding quantities are computed with consistent accuracy.

k	ω_{n_k}	$B(\omega_{n_k})$	$\varepsilon_c(\omega_{n_k})$	A_k
1	$[3, 500, 1^\infty]$	3.17069	0.244787	
2	$[3, 700, 1^\infty]$	3.28264	0.22044	0.9358±0.0001
3	$[3, 1000, 1^\infty]$	3.40139	0.197080	0.9433±0.0001
4	$[3, 2000, 1^\infty]$	3.63230	0.158153	0.9529±0.0001
5	$[3, 4000, 1^\infty]$	3.86330	0.126588	0.9637±0.0001
6	$[3, 7000, 1^\infty]$	4.04983	0.105608	0.9715±0.0001
7	$[3, 10000, 1^\infty]$	4.16872	0.094035	0.9763±0.0002
8	$[3, 13000, 1^\infty]$	4.25617	0.086319	0.9787±0.0003
9	$[3, 17000, 1^\infty]$	4.34559	0.079072	0.9807±0.0003
10	$[3, 20000, 1^\infty]$	4.39977	0.074973	0.9826±0.0005

Table 5: Critical residues $\mathcal{R}_\infty(\omega)$ for some rotation numbers ω. The error on $\varepsilon_c(\omega_{n_k})$ is of 1 unit on the last digit.

ω	$\varepsilon_c(\omega)$	$\mathcal{R}_\infty(\omega)$
$[1^\infty]$	0.971635406	0.250088
$[2^\infty]$	0.957445408	0.2275138
$[3^\infty]$	0.890863502	0.202230
$[4^\infty]$	0.80472544	0.17923
$[10, 2^\infty]$	0.481985986	0.22751
$[1, 3, 2^\infty]$	0.829500533	0.22751
$[7, 3^\infty]$	0.615071885	0.2022
$[1, 2, 4^\infty]$	0.86423037	0.1792

Table 6: Residues of critical periodic orbits for $\omega = \sqrt{3} - 1 = [1, 2, 1, 2, 1, 2, \dots]$.

$\varepsilon_c(\omega)$	0.876067426
approximant	residue
3/4	0.24871
8/11	0.18612
11/15	0.25216
30/41	0.18516
41/56	0.25275
112/153	0.18493
153/209	0.25288
418/571	0.18487

$\varepsilon_c(\omega)$	0.876067426
approximant	residue
571/780	0.25291
1560/2131	0.18486
2131/2911	0.25292
5822/7953	0.18485
7953/10864	0.25292
21728/29681	0.18485
29681/40545	0.25292
81090/110771	0.18486

Table 7: Residues of critical periodic orbits for $\omega = (\sqrt{3} - 1)/2 = [2, 1, 2, 1, 2, 1, \ldots]$.

$\varepsilon_c(\omega)$	0.9402827
approximant	residue
3/8	0.19574
4/11	0.24746
11/30	0.18763
15/41	0.25145
41/112	0.18556
56/153	0.25254
153/418	0.18503
209/571	0.25282

$\varepsilon_c(\omega)$	0.9402827
approximant	residue
571/1560	0.18490
780/2131	0.25290
2131/5822	0.18486
2911/7953	0.25292
7953/21728	0.18486
10864/29681	0.25292
29681/81090	0.18486
40545/110771	0.25293

Table 8: Residues of critical periodic orbits for $\omega = \sqrt{5/2} - 1 = [1, 1, 2, 1, 1, 2, 1, 1, 2 \ldots]$.

$\varepsilon_c(\omega)$	0.9402827
approximant	residue
3/5	0.2242
4/7	0.2639
7/12	0.2278
18/31	0.2222
25/43	0.2660
43/74	0.2270
111/191	0.2227
154/265	0.2656
265/456	0.2272

$\varepsilon_c(\omega)$	0.9402827
approximant	residue
684/1177	0.2226
949/1633	0.2656
1633/2810	0.2271
4215/7253	0.2227
5848/10063	0.2656
10063/17316	0.2271
25974/44695	0.2227
36037/62011	0.2656
62011/106706	0.2272

Table 9: Radius of convergence for some values of the rotation number ω close to $1/2$ and slopes $A'_k = A'(\omega_{n_k}, \omega_{n_{k-1}})$. The value $\rho_1(\omega)$ is given by the formula (4.2), while $\rho_P(\omega)$ is the value obtained numerically by using Padé approximants. the two values for the slopes correspond to the values $\rho_1(\omega)$ and $\rho_P(\omega)$, respectively. One has $\eta = |\omega - 1/2|$.

k	ω_{n_k}	η	$\rho_1(\omega_{n_k})$	$\rho_P(\omega_{n_k})$	A'_k
1	$[2, 10, 1^\infty]$	0.0224860	0.51409	0.51052	
2	$[2, 12, 1^\infty]$	0.0190577	0.43571	0.43355	2.19667/2.17013
3	$[2, 15, 1^\infty]$	0.0155106	0.35462	0.35352	2.14426/2.12464
4	$[2, 20, 1^\infty]$	0.0118382	0.27066	0.27024	2.09658/2.08449
5	$[2, 30, 1^\infty]$	0.0080339	0.18368	0.18361	2.05439/2.04822
6	$[2, 40, 1^\infty]$	0.0060801	0.13901	0.13902	2.02821/2.02484
6	$[2, 50, 1^\infty]$	0.0048906	0.11181	0.11184	2.01612/2.01480

Table 10: Radius of convergence for some values of the rotation number ω close to $1/3$ and slopes $A'_k = A'(\omega_{n_k}, \omega_{n_{k-1}})$. The value $\rho_1(\omega)$ is given by the formula (4.2), while $\rho_P(\omega)$ is the value obtained numerically by using Padé approximants; the two values for the slopes correspond to the values $\rho_1(\omega)$ and $\rho_P(\omega)$, respectively. One has $\eta = |\omega - 1/3|$.

k	ω_{n_k}	η	$\rho_1(\omega_{n_k})$	$\rho_P(\omega_{n_k})$	A'_k
1	$[3, 10, 1^\infty]$	0.0101459	0.62329	0.61993	
2	$[3, 12, 1^\infty]$	0.0085791	0.55734	0.55524	2.28295/2.24934
3	$[3, 13, 1^\infty]$	0.0079642	0.53038	0.52858	2.23762/2.22067
4	$[3, 20, 1^\infty]$	0.0053033	0.40444	0.40400	2.17212/2.15360
5	$[3, 30, 1^\infty]$	0.0035899	0.31180	0.31182	2.09982/2.09051
6	$[3, 40, 1^\infty]$	0.0027132	0.25871	0.25872	2.06311/2.06339
7	$[3, 50, 1^\infty]$	0.0021807	0.22364	0.22360	2.04490/2.04795
8	$[3, 100, 1^\infty]$	0.0011006	0.14177	0.14179	2.02455/2.02313
9	$[3, 200, 1^\infty]$	0.0005529	0.08959	0.08961	2.00902/2.00866

Table 11: Values of the radius of convergence and of the slopes $A'_k = A'(\omega_{n_k}, \omega_{n_{k-1}})$ corresponding to a finite sequence of rotation numbers $\omega_{n_k} = 1/(n_k + \gamma) = [n_k, 1^\infty]$. The radius of convergence is computed with the formula (4.2).

k	ω_{n_k}	$\rho(\omega_{n_k})$	A'_k
1	$[500, 1^\infty]$	0.000130355	
2	$[700, 1^\infty]$	0.0000665545	2.0042837
3	$[1000, 1^\infty]$	0.000032629	2.0030298
4	$[2000, 1^\infty]$	0.00000816229	2.0018183
5	$[4000, 1^\infty]$	0.0000020412	2.0009090
6	$[7000, 1^\infty]$	0.000000666603	2.0004825
7	$[10000, 1^\infty]$	0.000000326653	2.0003028
8	$[12000, 1^\infty]$	0.000000226847	2.0002303
9	$[15000, 1^\infty]$	0.000000145185	2.0001882
10	$[18000, 1^\infty]$	0.000000100824	2.0001536
11	$[20000, 1^\infty]$	0.0000000816683	2.0001329
12	$[25000, 1^\infty]$	0.0000000522684	2.0001129
13	$[30000, 1^\infty]$	0.0000000362978	2.0000921
14	$[40000, 1^\infty]$	0.0000000204177	2.0000730
15	$[50000, 1^\infty]$	0.0000000130674	2.0000565

Table 12: Values of the radius of convergence and of the slopes $A'_k = A'(\omega_{n_k}, \omega_{n_{k-1}})$ corresponding to a finite sequence of rotation numbers $\omega_{n_k} = 1/(2 + 1/(n_k + \gamma)) = [2, n_k, 1^\infty]$. The radius of convergence is computed with the formula (4.2).

k	ω_{n_k}	$\rho(\omega_{n_k})$	A'_k
1	$[2, 500, 1^\infty]$	0.011405915	
2	$[2, 700, 1^\infty]$	0.008152279	1.9968638
3	$[2, 1000, 1^\infty]$	0.005709327	1.9973651
4	$[2, 2000, 1^\infty]$	0.002856258	1.9980597
5	$[2, 4000, 1^\infty]$	0.001428528	1.9987793
6	$[2, 7000, 1^\infty]$	0.000816400	1.9992292
7	$[2, 10000, 1^\infty]$	0.000571507	1.9994593
8	$[2, 13000, 1^\infty]$	0.000439632	1.9995765
9	$[2, 17000, 1^\infty]$	0.000336196	1.9996572
10	$[2, 20000, 1^\infty]$	0.000285770	1.9997123

Table 13: Values of the radius of convergence and of the slopes $A'_k = A'(\omega_{n_k}, \omega_{n_{k-1}})$ corresponding to a finite sequence of rotation numbers $\omega_{n_k} = 1/(3 + 1/(n_k + \gamma)) = [3, n_k, 1^\infty]$. The error on $\varepsilon_c(\omega_{n_k})$ is of 1 unit on the last digit, and the corresponding logarithm is computed with consistent accuracy.

k	ω_{n_k}	$\rho(\omega_{n_k})$	A'_k
1	$[3, 500, 1^\infty]$	0.04873028	
2	$[3, 700, 1^\infty]$	0.03895268	2.0004598
3	$[3, 1000, 1^\infty]$	0.03071760	2.0000489
4	$[3, 2000, 1^\infty]$	0.01935701	1.9997910
5	$[3, 4000, 1^\infty]$	0.01219611	1.9997289
6	$[3, 7000, 1^\infty]$	0.00839894	1.9997744
7	$[3, 10000, 1^\infty]$	0.00662168	1.9998204
8	$[3, 13000, 1^\infty]$	0.00555920	1.9998501
9	$[3, 17000, 1^\infty]$	0.00464887	1.9998731
10	$[3, 20000, 1^\infty]$	0.00417153	1.9998900

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