# Lower-dimensional invariant tori for perturbations of a class of non-convex Hamiltonian functions

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**Abstract**

We consider a class of quasi-integrable Hamiltonian systems obtained by adding to a non-convex Hamiltonian function of an integrable system a perturbation depending only on the angle variables. We focus on a resonant maximal torus of the unperturbed system, foliated into a family of lower-dimensional tori of codimension 1, invariant under a quasi-periodic flow with rotation vector satisfying some mild Diophantine condition. We show that at least one lower-dimensional torus with that rotation vector always exists also for the perturbed system. The proof is based on multiscale analysis and resummation procedures of divergent series. A crucial role is played by suitable symmetries and cancellations, ultimately due to the Hamiltonian structure of the system.

## 1 Introduction

Consider the Hamiltonian dynamical system described, in action-angle variables, by the Hamiltonian function

$$H(\boldsymbol{\alpha},\beta,\boldsymbol{A},B) = -\frac{1}{2}\boldsymbol{A}\cdot\boldsymbol{A} + \frac{1}{2}B^2 + \varepsilon f(\boldsymbol{\alpha},\beta), \tag{1.1}$$

where $(\boldsymbol{\alpha},\beta)\in\mathbb{T}^d\times\mathbb{T}$, $(\boldsymbol{A},B)\in\mathbb{R}^d\times\mathbb{R}$, $f:\mathbb{T}^{d+1}\to\mathbb{R}$ is real-analytic, $\varepsilon\in\mathbb{R}$ is a small parameter (the *perturbation parameter*) and $\cdot$ is the standard scalar product in $\mathbb{R}^d$. The corresponding Hamilton equations can be written as closed equations for the angle variables $(\boldsymbol{\alpha},\beta)$,

$$\begin{cases} \ddot{\boldsymbol{\alpha}} = \varepsilon\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha},\beta), \\ \ddot{\beta} = -\varepsilon\partial_\beta f(\boldsymbol{\alpha},\beta). \end{cases} \tag{1.2}$$

For $\varepsilon=0$ all the solutions of (1.2) are trivially of the form $(\boldsymbol{\alpha}(t),\beta(t))=(\boldsymbol{\alpha}_0-\boldsymbol{A}_0t,\beta_0+B_0t)$ where $(\boldsymbol{\alpha}_0,\beta_0)$ and $(\boldsymbol{A}_0,B_0)$ are the initial phases and actions, respectively. Fix the initial actions as $(\boldsymbol{A}_0,B_0)=(-\boldsymbol{\omega},0)$ with $\boldsymbol{\omega}\in\mathbb{R}^d$ such that $\boldsymbol{\omega}\cdot\boldsymbol{\nu}\neq 0$ for all $\boldsymbol{\nu}\in\mathbb{Z}^d_*:=\mathbb{Z}^d\setminus\{\boldsymbol{0}\}$. Then the solutions of the unperturbed system lie on a $(d+1)$-dimensional invariant torus foliated into $d$-dimensional invariant tori parametrized by $\beta_0$.

For $\varepsilon \neq 0$ we say that the system has an invariant $d$-dimensional torus with frequency $\boldsymbol{\omega}$ if there is an invariant manifold for (1.2) where the motion is conjugated to a rotation with frequency vector $\boldsymbol{\omega}$ on $\mathbb{T}^d$, more precisely if there exists $\beta_0 \in \mathbb{T}$ and two analytic functions $\boldsymbol{\alpha}_\varepsilon : \mathbb{T}^d \to \mathbb{T}^d$ and $\beta_\varepsilon : \mathbb{T}^d \to \mathbb{T}$ such that $\boldsymbol{\alpha}_0(\boldsymbol{\psi}) = \mathbf{0}$ and $\beta_0(\boldsymbol{\psi}) = 0$, the submanifold $\mathfrak{M}$ of the form $\boldsymbol{\alpha} = \boldsymbol{\psi} + \boldsymbol{\alpha}_\varepsilon(\boldsymbol{\psi})$ and $\beta = \beta_0 + \beta_\varepsilon(\boldsymbol{\psi})$ is invariant for (1.2) and the flow on $\mathfrak{M}$ is given by $\boldsymbol{\psi} \to \boldsymbol{\psi} + \boldsymbol{\omega} t$.

For $\boldsymbol{\omega} \in \mathbb{R}^d$ define the *Bryuno function* as [1]

$$\mathcal{B}(\boldsymbol{\omega}) := \sum_{m \geq 0} \frac{1}{2^m} \log \frac{1}{\alpha_m(\boldsymbol{\omega})}, \qquad \alpha_m(\boldsymbol{\omega}) := \inf_{\substack{\boldsymbol{\nu} \in \mathbb{Z}^d \\ 0 < |\boldsymbol{\nu}| \leq 2^m}} |\boldsymbol{\omega} \cdot \boldsymbol{\nu}|.$$

We shall prove the following result.

**Theorem 1.1.** *For any $\boldsymbol{\omega} \in \mathbb{R}^d$ such that $\mathcal{B}(\boldsymbol{\omega}) < \infty$, there exists $\varepsilon_0 > 0$ such that for any $|\varepsilon| < \varepsilon_0$ the system (1.2) admits at least one invariant $d$-dimensional torus with frequency $\boldsymbol{\omega}$.*

**Remark 1.2.** It will turn out from the proof that it may happen that the system admits a whole $(d+1)$-dimensional torus foliated into $d$-dimensional invariant tori. In such a highly non-generic case, the solution is analytic in both the initial data and in the perturbation parameter.

Theorem 1.1 can be seen as a particular case of the result announced in [5], where the problem of existence of $d$-dimensional tori is considered for Hamiltonian systems with $(d+1)$-degrees of freedom described by Hamiltonian functions $H(\boldsymbol{\alpha}, \beta, I) = H_0(I) + \varepsilon H_1(\boldsymbol{\alpha}, \beta, I)$, where $I = (\boldsymbol{A}, B) \in \mathfrak{D}_1$, with $\mathfrak{D}_1$ a neighbourhood of zero in $\mathbb{R}^{d+1}$, and the functions $H_0, H_1$ are real-analytic in all their arguments and $2\pi$-periodic in $\boldsymbol{\alpha}, \beta$ and (modulo a canonical transformation) satisfy the following conditions:

1. $\partial_{\boldsymbol{A}} H_0(\mathbf{0}, 0) = \boldsymbol{\omega}$, with $\boldsymbol{\omega}$ such that $|\boldsymbol{\omega} \cdot \boldsymbol{\nu}| \geq c\, |\boldsymbol{\nu}|^{-\tau}$ for $c > 0$, $\tau > d - 1$ and all $\boldsymbol{\nu} \in \mathbb{Z}^d_*$;
2. $\det \partial_I^2 H_0(0) \neq 0$ and $\partial_B^2 H_0(0) = 1$;
3. the function $H_0(\boldsymbol{A}, B) - \boldsymbol{\omega} \cdot \boldsymbol{A}$ has a saddle point of signature 1 in zero, that is

$$\boldsymbol{\eta} \cdot S_0 \boldsymbol{\eta} < |\boldsymbol{t}_0 \cdot \boldsymbol{\eta}|^2 \qquad \forall \boldsymbol{\eta} \in \mathbb{R}^d \setminus \{\mathbf{0}\},$$

   where $S_0 := \partial_{\boldsymbol{A}}^2 H_0(0)$ and $\boldsymbol{t}_0 := \partial_{\boldsymbol{A}} \partial_B H_0(0)$.

Indeed the Hamiltonian function (1.1) satisfies the conditions above, with $S_0 = -\mathbb{1}$ and $\boldsymbol{t}_0 = \mathbf{0}$ (in fact the Bryuno condition $\mathcal{B}(\boldsymbol{\omega}) < \infty$ is weaker than the standard Diophantine condition in item 1 above). Our method should apply to the more general case considered in [5]: we prefer to focus on a particular class of systems, to avoid technical complications and put emphasis on the method, rather than the result itself — already stated in [5]. We shall show that the result can be credited to the existence of remarkable symmetries of suitable quantities, the so-called self-energies, that will be introduced in the proof. In turn such symmetries are related to the Hamiltonian form of the equations of motion.

The construction envisaged below, as well as the method of [5], does not allow us to obtain the existence of invariant $d$-dimensional tori in the case of convex $H_0(I)$ treated by Cheng [2]. At the end we shall try to briefly illustrate where problems arise when dealing with convex Hamiltonians. In particular we shall see that the aforementioned symmetries are not sufficient in that case, and other cancellation mechanisms should be looked for.

## 2 The formal expansion

Fix $\boldsymbol{\omega} \in \mathbb{R}^d$ such that $\mathcal{B}(\boldsymbol{\omega}) < \infty$. We look for a quasi-periodic solution to (1.2) of the form $(\boldsymbol{\alpha}(t), \beta(t)) = (\boldsymbol{\alpha}_0 + \boldsymbol{\omega} t + \boldsymbol{a}(t), \beta_0 + b(t))$, with

$$\boldsymbol{a}(t) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega} t}\boldsymbol{a}_{\boldsymbol{\nu}}, \qquad b(t) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega} t} b_{\boldsymbol{\nu}}, \tag{2.1}$$

and $(\boldsymbol{a}(t), b(t)) \to (\boldsymbol{0}, 0)$ as $\varepsilon \to 0$, so that in the Fourier space (1.2) becomes

$$(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 \boldsymbol{a}_{\boldsymbol{\nu}} = -[\varepsilon\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}}, \qquad \boldsymbol{\nu} \neq \boldsymbol{0}, \tag{2.2a}$$

$$(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2 b_{\boldsymbol{\nu}} = [\varepsilon\partial_{\beta} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}}, \qquad \boldsymbol{\nu} \neq \boldsymbol{0}, \tag{2.2b}$$

$$[\varepsilon\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{0}} = \boldsymbol{0}, \tag{2.2c}$$

$$[\varepsilon\partial_{\beta} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{0}} = 0, \tag{2.2d}$$

where

$$\begin{aligned}
[\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}} &= \sum_{\substack{p\ge 0\\ q\ge 0}} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_{p+q}=\boldsymbol{\nu}\\ \boldsymbol{\nu}_0\in\mathbb{Z}^d\\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*, i=1,\ldots,p+q}} \frac{1}{p!q!}(\mathrm{i}\boldsymbol{\nu}_0)^{p+1}\partial_\beta^q f_{\boldsymbol{\nu}_0}(\boldsymbol{\alpha}_0,\beta_0) \prod_{i=1}^{p} \boldsymbol{a}_{\boldsymbol{\nu}_i} \prod_{j=p+1}^{p+q} b_{\boldsymbol{\nu}_j},\\
[\partial_{\beta} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}} &= \sum_{\substack{p\ge 0\\ q\ge 0}} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_{p+q}=\boldsymbol{\nu}\\ \boldsymbol{\nu}_0\in\mathbb{Z}^d\\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*, i=1,\ldots,p+q}} \frac{1}{p!q!}(\mathrm{i}\boldsymbol{\nu}_0)^{p}\partial_\beta^{q+1} f_{\boldsymbol{\nu}_0}(\boldsymbol{\alpha}_0,\beta_0) \prod_{i=1}^{p} \boldsymbol{a}_{\boldsymbol{\nu}_i} \prod_{j=p+1}^{p+q} b_{\boldsymbol{\nu}_j},
\end{aligned} \tag{2.3}$$

and $f_{\boldsymbol{\nu}}(\boldsymbol{\alpha}_0,\beta_0) = \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\alpha}_0}\hat{f}_{\boldsymbol{\nu}}(\beta_0)$, where we denoted

$$f(\boldsymbol{\alpha},\beta) = \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d} \hat{f}_{\boldsymbol{\nu}}(\beta)\mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\alpha}}.$$

Throughout the paper, the sums and the products over the empty set have to be considered as 0 and 1, respectively. Equations (2.2a) and (2.2b) are called the *range equations*, while (2.2c) and (2.2d) are called the *bifurcation equations*.

We start by writing formally

$$\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_0) = \boldsymbol{\alpha}_0 + \boldsymbol{\omega} t + \sum_{k\ge 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega} t}\boldsymbol{a}^{(k)}_{\boldsymbol{\nu}}(\boldsymbol{\alpha}_0,\beta_0), \tag{2.4a}$$

$$\beta(t) = \beta(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_0) = \beta_0 + \sum_{k\ge 1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega} t} b^{(k)}_{\boldsymbol{\nu}}(\boldsymbol{\alpha}_0,\beta_0). \tag{2.4b}$$

If we define recursively for $k \geq 1$ and $\boldsymbol{\nu} \neq \mathbf{0}$

$$\boldsymbol{a}_{\boldsymbol{\nu}}^{(k)} = -\frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}[\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}}^{(k-1)}, \qquad b_{\boldsymbol{\nu}}^{(k)} = \frac{1}{(\boldsymbol{\omega}\cdot\boldsymbol{\nu})^2}[\partial_{\beta} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}}^{(k-1)},$$

with $[\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}}^{(0)} = \mathrm{i}\boldsymbol{\nu}\, f_{\boldsymbol{\nu}}(\boldsymbol{\alpha}_0,\beta_0)$, $[\partial_{\beta} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}}^{(0)} = \partial_\beta f_{\boldsymbol{\nu}}(\boldsymbol{\alpha}_0,\beta_0)$ and, for $k \geq 1$,

$$[\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}}^{(k)} = \sum_{\substack{p\geq 0\\ q\geq 0}} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_{p+q}=\boldsymbol{\nu}\\ \boldsymbol{\nu}_0\in\mathbb{Z}^d\\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*, i=1,\ldots,p+q}} \frac{1}{p!q!}(\mathrm{i}\boldsymbol{\nu}_0)^{p+1}\partial_\beta^q f_{\boldsymbol{\nu}_0}(\boldsymbol{\alpha}_0,\beta_0) \sum_{\substack{k_1+\ldots+k_{p+q}=k\\ k_i\geq 1}} \prod_{i=1}^{p} \boldsymbol{a}_{\boldsymbol{\nu}_i}^{(k_i)} \prod_{j=p+1}^{p+q} b_{\boldsymbol{\nu}_j}^{(k_j)},$$

$$[\partial_{\beta} f(\boldsymbol{\alpha},\beta)]_{\boldsymbol{\nu}}^{(k)} = \sum_{\substack{p\geq 0\\ q\geq 0}} \sum_{\substack{\boldsymbol{\nu}_0+\ldots+\boldsymbol{\nu}_{s+r}=\boldsymbol{\nu}\\ \boldsymbol{\nu}_0\in\mathbb{Z}^d\\ \boldsymbol{\nu}_i\in\mathbb{Z}^d_*, i=1,\ldots,p+q}} \frac{1}{p!q!}(\mathrm{i}\boldsymbol{\nu}_0)^{p}\partial_\beta^{q+1} f_{\boldsymbol{\nu}_0}(\boldsymbol{\alpha}_0,\beta_0) \sum_{\substack{k_1+\ldots+k_{p+q}=k\\ k_i\geq 1}} \prod_{i=1}^{p} \boldsymbol{a}_{\boldsymbol{\nu}_i}^{(k_i)} \prod_{j=p+1}^{p+q} b_{\boldsymbol{\nu}_j}^{(k_j)},$$

then the series (2.4) turn out to be a formal solution of the range equations for any values of the parameters $\boldsymbol{\alpha}_0$ and $\beta_0$.

Unfortunately in general we are not able to prove the convergence of the series (2.4) and moreover we also have to solve the bifurcation equations. As we shall see the two problems are somehow related.

# 3 Conditions of convergence for the formal expansion

In this section we shall see how to represent graphically the formal solutions (2.4). We shall see that under suitable (quite non-generic) hypotheses the two series converge. However, in general, a resummation is needed to give the series a meaning: this will be discussed in Section 4.

## 3.1 Diagrammatic rules

Our aim is to represent the formal series as a "sum over trees", so first of all we need some definitions. (We closely follow [3, 4], with obvious adaptations).

A graph is a set of points and lines connecting them. A *rooted tree* $\theta$ is a graph with no cycle, such that all the lines are oriented toward a single point (*root*) which has only one incident line $\ell_\theta$ (*root line*); we will omit the adjective "rooted" in the following. All the points in a tree except the root are called *nodes*. The orientation of the lines in a tree induces a partial ordering relation ($\preceq$) between the nodes and the lines: we can imagine that each line carries an arrow pointing toward the root. Given two nodes $v$ and $w$, we shall write $w \prec v$ every time $v$ is along the path (of lines) which connects $w$ to the root.

We denote by $N(\theta)$ and $L(\theta)$ the sets of nodes and lines in $\theta$, respectively. Since a line $\ell \in L(\theta)$ is uniquely identified by the node $v$ which it leaves, we may write $\ell = \ell_v$. We write $\ell_w \prec \ell_v$ if $w \prec v$, and $w \prec \ell = \ell_v$ if $w \preceq v$; if $\ell$ and $\ell'$ are two comparable lines, i.e. $\ell' \prec \ell$, we denote by $\mathcal{P}(\ell,\ell')$ the (unique) path of lines connecting $\ell'$ to $\ell$, with $\ell$ and $\ell'$ not included (in particular $\mathcal{P}(\ell,\ell')=\emptyset$ if $\ell'$ enters the node $\ell$ exits).

With each node $v\in N(\theta)$ we associate a *mode* label $\boldsymbol{\nu}_v\in\mathbb{Z}^d$ and a *component* label $h_v\in\{\alpha_1,\ldots,\alpha_d,\beta\}$, and we denote by $s_v$ the number of lines entering $v$. With each line $\ell=\ell_v$ we associate a component label $h_{\ell_v}=h_v$ and a *momentum* $\boldsymbol{\nu}_\ell\in\mathbb{Z}^d_*$, except for the root line which can have either zero momentum or not, i.e. $\boldsymbol{\nu}_{\ell_\theta}\in\mathbb{Z}^d$. For any node $v\in N(\theta)$ we denote by $p_{j,v},q_v$ the number of lines entering $v$ with component $\alpha_j$ and $\beta$, respectively, and set $p_v=p_{1,v}+\ldots+p_{d,v}$; of course $s_v=p_v+q_v$. Finally, we associate with each line $\ell$ also a *scale label* such that $n_\ell=-1$ if $\boldsymbol{\nu}_\ell=\mathbf{0}$, while $n_\ell\in\mathbb{Z}_+$ if $\boldsymbol{\nu}_\ell\neq\mathbf{0}$ (so far there is no relation between non-zero momenta and scale labels: a constraint will appear shortly). Note that one can have $n_\ell=-1$ only if $\ell$ is the root line of $\theta$. We force the following *conservation law*

$$\boldsymbol{\nu}_\ell=\sum_{\substack{w\in N(\theta)\\ w\prec\ell}}\boldsymbol{\nu}_w. \tag{3.1}$$

We shall call trees tout court the trees with labels, and we shall use the term *unlabelled tree* for the trees without labels. We shall say that two trees are *equivalent* if they can be transformed into each other by continuously deforming the lines in such a way that these do not cross each other and also labels match. This provides an equivalence relation on the set of the trees. From now on we shall call trees such equivalence classes.

Given a tree $\theta$ we call *order* of $\theta$ the number $k(\theta)=|N(\theta)|=|L(\theta)|$ (for any finite set $S$ we denote by $|S|$ its cardinality), *total momentum* of $\theta$ the momentum associated with $\ell_\theta$ and *total component* of $\theta$ the component associated with $\ell_\theta$. We shall denote by $\Theta_{k,\boldsymbol{\nu},h}$ the set of trees with order $k$, total momentum $\boldsymbol{\nu}$ and total component $h$. A subset $T\subset\theta$ is a *subgraph* of $\theta$ if it is formed by set of nodes $N(T)\subseteq N(\theta)$ and lines $L(T)\subseteq L(\theta)$ connecting them (possibly including the root line: in such a case we say that the root is included in $T$) in such a way that $N(T)\cup L(T)$ is connected. If $T$ is a subgraph of $\theta$ we call *order* of $T$ the number $k(T)=|N(T)|$. We say that a line enters $T$ if it connects a node $v\notin N(T)$ to a node $w\in N(T)$, and we say that a line exits $T$ if it connects a node $v\in N(T)$ to a node $w\notin N(T)$ or to the root (which is not included in $T$ in this case). Of course, if a line $\ell$ enters or exits $T$, then $\ell\notin L(T)$. If $T$ is a subgraph of $\theta$ with only one entering line $\ell'$ and one exiting line $\ell$, we set $\mathcal{P}_T:=\mathcal{P}(\ell,\ell')$.

A *cluster* $T$ on scale $n$ is a maximal subgraph of a tree $\theta$ such that all the lines have scales $n'\leq n$ and there is at least a line with scale $n$. The lines entering the cluster $T$ and the line coming out from it (unique if existing at all) are called the *external* lines of $T$.

A *self-energy cluster* is a cluster $T$ such that (i) $T$ has only one entering line $\ell_T'$ and one exiting line $\ell_T$, (ii) $\boldsymbol{\nu}_\ell\neq\boldsymbol{\nu}_{\ell_T'}$ for all $\ell\in\mathcal{P}_T$, (iii) one has $\boldsymbol{\nu}_{\ell_T}=\boldsymbol{\nu}_{\ell_T'}$ and hence $\sum_{v\in N(T)}\boldsymbol{\nu}_v=\mathbf{0}$.

We shall say that a self-energy cluster is on scale $-1$, if $N(T)=\{v\}$, with of course $\boldsymbol{\nu}_v=\mathbf{0}$ (so that $\mathcal{P}_T=\emptyset$).

**Remark 3.1.** Given a self-energy cluster $T$, the momenta of the lines in $\mathcal{P}_T$ depend on $\boldsymbol{\nu}_{\ell_T'}$ because of the conservation law (3.1). More precisely, for all $\ell\in\mathcal{P}_T$ one has $\boldsymbol{\nu}_\ell=\boldsymbol{\nu}_\ell^0+\boldsymbol{\nu}_{\ell_T'}$ with $\boldsymbol{\nu}_\ell^0=\sum_{w\in N(T),w\prec\ell}\boldsymbol{\nu}_w$, while all the other labels in $T$ do not depend on $\boldsymbol{\nu}_{\ell_T'}$.

We say that two self-energy clusters $T_1,T_2$ have the same *structure* if setting $\boldsymbol{\nu}_{\ell_{T_1}'}=\boldsymbol{\nu}_{\ell_{T_2}'}=\mathbf{0}$

one has $T_1 = T_2$. Of course this provides an equivalence relation on the set of all self-energy clusters. From now on we shall call self-energy clusters tout court such equivalence classes and we shall denote by $\mathfrak{S}^k_{n,u,e}$ the set of self-energy clusters with order $k$, scale $n$ and such that $h_{\ell'_T} = e$ and $h_{\ell_T} = u$, with $e, u \in \{\alpha_1, \dots, \alpha_d, \beta\}$.

Given any tree $\theta \in \Theta_{k,\boldsymbol{\nu},h}$ we associate with each node $v \in N(\theta)$ a *node factor*

$$\mathcal{F}_v := \begin{cases} -\dfrac{1}{p_v! q_v!} (\mathrm{i}\boldsymbol{\nu}_v)^{p_v+1} \partial_\beta^{q_v} f_{\boldsymbol{\nu}_v}(\boldsymbol{\alpha}_0, \beta_0), & h_v = \alpha_j, j = 1, \dots, d, \\ \\ \dfrac{1}{p_v! q_v!} (\mathrm{i}\boldsymbol{\nu}_v)^{p_v} \partial_\beta^{q_v+1} f_{\boldsymbol{\nu}_v}(\boldsymbol{\alpha}_0, \beta_0), & h_v = \beta \end{cases} \tag{3.2}$$

which is a tensor of rank $s_v + 1$. We associate with each line $\ell \in L(\theta)$ a *propagator* defined as follows. Let us introduce the sequences $\{m_n, p_n\}_{n \ge 0}$, with $m_0 = 0$ and, for all $n \ge 0$, $m_{n+1} = m_n + p_n + 1$, where $p_n := \max\{q \in \mathbb{Z}_+ : \alpha_{m_n}(\boldsymbol{\omega}) < 2\alpha_{m_n+q}(\boldsymbol{\omega})\}$. Then the subsequence $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n \ge 0}$ of $\{\alpha_m(\boldsymbol{\omega})\}_{m \ge 0}$ is decreasing. Let $\chi : \mathbb{R} \to \mathbb{R}$ be a $C^\infty$ even function, non-increasing for $x \ge 0$, such that

$$\chi(x) = \begin{cases} 1, & |x| \le 1/2, \\ 0, & |x| \ge 1. \end{cases} \tag{3.3}$$

Set $\chi_{-1}(x) = 1$ and $\chi_n(x) = \chi(8x/\alpha_{m_n}(\boldsymbol{\omega}))$ for $n \ge 0$. Set also $\psi(x) = 1 - \chi(x)$, $\psi_n(x) = \psi(8x/\alpha_{m_n}(\boldsymbol{\omega}))$, and $\Psi_n(x) = \chi_{n-1}(x)\psi_n(x)$, for $n \ge 0$; see Figure 3.5 in [4]. Then we associate with each line a propagator

$$\mathcal{G}_\ell := \begin{cases} \dfrac{\Psi_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)^2}, & n_\ell \ge 0, \\ 1, & n_\ell = -1. \end{cases} \tag{3.4}$$

Given any subgraph $S$ of any tree $\theta$ we define the *value* of $S$ as

$$\mathscr{V}(S) = \left( \prod_{v \in N(S)} \mathcal{F}_v \right) \left( \prod_{\ell \in L(S)} \mathcal{G}_\ell \right). \tag{3.5}$$

Set $\Theta_{k,\boldsymbol{\nu},\boldsymbol{\alpha}} := \Theta_{k,\boldsymbol{\nu},\alpha_1} \times \dots \times \Theta_{k,\boldsymbol{\nu},\alpha_d}$ and for any $\boldsymbol{\theta} = (\theta_1, \dots, \theta_d) \in \Theta_{k,\boldsymbol{\nu},\boldsymbol{\alpha}}$ define $\boldsymbol{\mathscr{V}}(\boldsymbol{\theta}) := (\mathscr{V}(\theta_1), \dots, \mathscr{V}(\theta_d))$, so that one has

$$\boldsymbol{a}^{(k)}_{\boldsymbol{\nu}} = \sum_{\boldsymbol{\theta} \in \Theta_{k,\boldsymbol{\nu},\boldsymbol{\alpha}}} \boldsymbol{\mathscr{V}}(\boldsymbol{\theta}), \qquad b^{(k)}_{\boldsymbol{\nu}} = \sum_{\theta \in \Theta_{k,\boldsymbol{\nu},\beta}} \mathscr{V}(\theta), \qquad \boldsymbol{\nu} \neq \boldsymbol{0}, \tag{3.6a}$$

$$[-\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha}, \beta)]^{(k)}_{\boldsymbol{0}} = \sum_{\boldsymbol{\theta} \in \Theta_{k+1,\boldsymbol{0}\,\boldsymbol{\alpha}}} \boldsymbol{\mathscr{V}}(\boldsymbol{\theta}), \qquad [\partial_\beta f(\boldsymbol{\alpha}, \beta)]^{(k)}_{\boldsymbol{0}} = \sum_{\theta \in \Theta_{k+1,\boldsymbol{0},\beta}} \mathscr{V}(\theta), \tag{3.6b}$$

as is easy to check. In particular the quantities in (3.6) are well defined for any (fixed) $k \ge 1$ (see Appendix H in [3]).

**Remark 3.2.** Given a subgraph $S$ of any tree $\theta$ such that $\mathscr{V}(S) \neq 0$, for any line $\ell \in L(S)$ (except possibly the root line of $\theta$) one has $\Psi_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) \neq 0$, so that

$$\frac{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})}{16} \leq |\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell| \leq \frac{\alpha_{m_{n_\ell - 1}}(\boldsymbol{\omega})}{8} < \frac{\alpha_{m_{n_\ell - 1} + p_{n_\ell - 1}}(\boldsymbol{\omega})}{4} = \frac{\alpha_{m_{n_\ell} - 1}(\boldsymbol{\omega})}{4},$$

where $\alpha_{m_{-1}}(\boldsymbol{\omega})$ has to be interpreted as $+\infty$, and hence, by definition of $\alpha_m(\boldsymbol{\omega})$, one has $|\boldsymbol{\nu}_\ell| > 2^{m_{n_\ell} - 1}$. Moreover, by definition of $\{\alpha_{m_n}(\boldsymbol{\omega})\}_{n \geq 0}$, the number of scales which can be associated with a line $\ell$ in such way that the propagator does not vanish is at most 2.

### 3.2 Dimensional bounds

For any subgraph $S$ of any tree $\theta$ call $\mathfrak{N}_n(S)$ the number of lines on scale $\geq n$ in $S$, and set

$$K(S) := \sum_{v \in N(S)} |\boldsymbol{\nu}_v|. \tag{3.7}$$

We shall say that a line $\ell$ is *resonant* if it exits a self-energy cluster, otherwise $\ell$ is *non-resonant*. For any line $\ell \in \theta$ define the *minimum scale* of $\ell$ as

$$\zeta_\ell := \min\{n \in \mathbb{Z}_+ : \Psi_n(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell) \neq 0\}.$$

Given any subgraph $S$ of any tree $\theta$, we denote by $\mathfrak{N}^\bullet_n(S)$ the number of non-resonant lines $\ell \in L(S)$ such that $\zeta_\ell \geq n$. By definition, if $\mathscr{V}(S) \neq 0$, for each line $\ell \in L(S)$ either $n_\ell = \zeta_\ell$ or $n_\ell = \zeta_\ell + 1$. We have the following results.

**Lemma 3.3.** *For all $h \in \{\alpha_1, \ldots, \alpha_d, \beta\}$, $\boldsymbol{\nu} \in \mathbb{Z}^d$, $k \geq 1$ and for any $\theta \in \Theta_{k,\boldsymbol{\nu},h}$ with $\mathscr{V}(\theta) \neq 0$, one has $\mathfrak{N}^\bullet_n(\theta) \leq 2^{-(m_n - 2)} K(\theta)$ for all $n \geq 0$.*

**Lemma 3.4.** *For all $e, u \in \{\alpha_1, \ldots, \alpha_d, \beta\}$, $n \geq 0$, $k \geq 1$ and for any $T \in \mathfrak{S}^k_{n,u,e}$ with $\mathscr{V}(T) \neq 0$, one has $K(T) > 2^{m_n - 1}$ and $\mathfrak{N}^\bullet_p(T) \leq 2^{-(m_p - 2)} K(T)$ for all $0 \leq p \leq n$.*

The proofs of the two results above can be easily adapted from the proofs of Lemmas 6.4 and 6.5 in [4], respectively (and the same notations have been used), notwithstanding the different definition of resonant lines and the fact that here the lines different from the root line can have only scale $\geq 0$.

**Lemma 3.5.** *For any tree $\theta \in \Theta_{k,\boldsymbol{\nu},h}$ and any self-energy cluster $T \in \mathfrak{S}^k_{n,u,e}$ denote by $L_{NR}(\theta)$ and $L_{NR}(T)$ the sets of non-resonant lines in $\theta$ and $T$, respectively, and set*

$$\mathscr{V}_{NR}(\theta) := \Bigg( \prod_{v \in N(\theta)} \mathcal{F}_v \Bigg) \Bigg( \prod_{\ell \in L_{NR}(\theta)} \mathcal{G}_\ell \Bigg), \qquad \mathscr{V}_{NR}(T) := \Bigg( \prod_{v \in N(T)} \mathcal{F}_v \Bigg) \Bigg( \prod_{\ell \in L_{NR}(T)} \mathcal{G}_\ell \Bigg).$$

*Then*

$$|\mathscr{V}_{NR}(\theta)| \leq C_1^k \mathrm{e}^{-\xi|\boldsymbol{\nu}|/2}, \qquad |\mathscr{V}_{NR}(T)| \leq C_2^k \mathrm{e}^{-\xi K(T)/2}, \tag{3.8}$$

*for some positive constants $C_1$ and $C_2$.*

*Proof.* We prove only the first bound in (3.8) since the proof of the second one proceeds in the same way, with $T$ playing the role of $\theta$. For any $n_0 \geq 0$ one has

$$\prod_{\ell\in L_{NR}(\theta)} |\mathcal{G}_\ell| \leq \left(\frac{16}{\alpha_{m_{n_0}}(\boldsymbol{\omega})}\right)^{2k} \prod_{n\geq n_0+1} \left(\frac{16}{\alpha_{m_n}(\boldsymbol{\omega})}\right)^{2\mathfrak{N}_n^\bullet(\theta)} \leq D(n_0)^{2k} \exp(\xi(n_0)K(\theta)),$$

with

$$D(n_0) = \frac{16}{\alpha_{m_{n_0}}(\boldsymbol{\omega})}, \qquad \xi(n_0) = 8 \sum_{n\geq n_0+1} \frac{1}{2^{m_n}} \log \frac{16}{\alpha_{m_n}(\boldsymbol{\omega})}.$$

Then, since $\mathcal{B}(\boldsymbol{\omega}) < \infty$, one can choose $n_0$ such that $\xi(n_0) \leq \xi/2$, so that, since

$$\prod_{v\in N(\theta)} |\mathcal{F}_v| \leq C_0^k \mathrm{e}^{-\xi K(\theta)},$$

for some positive constant $C_0$, the bound follows. ∎

If $T$ is a self-energy cluster, we can (and shall) write $\mathscr{V}(T) = \mathscr{V}_T(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T})$ and $\mathscr{V}_{NR}(T) = \mathscr{V}_{T,NR}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T})$ to stress the dependence on $\boldsymbol{\nu}_{\ell'_T}$ — see Remark 3.1.

**Remark 3.6.** Since the proofs of Lemmas 3.3 and 3.4 work under the weaker condition

$$\frac{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})}{32} < |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell| < \frac{\alpha_{m_{n_\ell-1}}(\boldsymbol{\omega})}{4}$$

one can show that also $\partial_x^j \mathscr{V}_{T,NR}(\tau x)$ admits the same bound as $\mathscr{V}_{T,NR}(x)$ in (3.8) for $j = 0, 1, 2$ and $\tau \in [0, 1]$, possibly with a different constant $C_2$.

What emerges from Lemma 3.5 is that, if we could ignore the resonant lines, the convergence of the series (2.4) would immediately follow (for $\varepsilon$ small enough). On the contrary, the presence of resonant lines may be a real obstruction for the convergence. Suppose indeed that a resonant line $\ell$ exits a self-energy cluster $T$ on scale $n \ll n_\ell$. If $n \geq 0$, then $T$ must contain at least one line $\ell'$ on scale $n$ such that $2|\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_{\ell'}| \geq |\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'}| \geq |\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_{\ell'}|/2$ since $n \ll n_\ell$ (recall the definition of $\boldsymbol{\nu}^0_{\ell'}$ in Remark 3.1) and hence $|\boldsymbol{\nu}^0_{\ell'}| \geq 2^{m_n-1}$ (reason as in Remark 3.2 to bound $|\boldsymbol{\nu}^0_{\ell'}|$ in terms of the scale $n_\ell$). Therefore we can extract a factor $\mathrm{e}^{-\xi 2^{m_n}/8}$ from the product of the node factors of the nodes in $T$: however this is not enough to control the propagator $\mathcal{G}_\ell$ for which we only have the bound $2^8/\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^2$. Also the case $n = -1$ gives the same problem. Moreover in principle a tree can contain a "chain" of self-energy clusters and hence of resonant lines, which implies accumulation of small divisors. Therefore one would need a "gain factor" proportional to $(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)^2$ for each resonant line $\ell$ for the power series (2.4) to converge.

### 3.3 Symmetries

For all $k \geq 1$ define the *self-energies*

$$\begin{aligned} M^{(k)}_{u,e}(x,n) &:= \sum_{T\in\mathfrak{S}^k_{n,u,e}} \mathscr{V}_T(x), \qquad \mathcal{M}^{(k)}_{u,e}(x,n) := \sum_{p=-1}^{n} M^{(k)}_{u,e}(x,p), \\ \mathcal{M}^{(k)}_{u,e}(x) &:= \lim_{n\to\infty} \mathcal{M}^{(k)}_{u,e}(x,n). \end{aligned} \tag{3.9}$$

Here we shall exhibit the existence of suitable symmetries for the self-energy clusters, i.e some remarkable identities between the quantities $\mathcal{M}^{(k)}_{u,e}(x,n)$ and $\mathcal{M}^{(k)}_{u,e}(x)$ introduced in (3.9). In turn such symmetries will allow us to obtain a gain factor proportional to $(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)^2$ for "some" resonant line $\ell$: under suitable assumptions (which we shall exploit later on) this will imply the convergence of the power series (2.4).

**Lemma 3.7.** *For all $k\geq 1$ one has*

$$\mathcal{M}^{(k)}_{\alpha_i,\alpha_j}(0)=\partial_{\alpha_{0,j}}[-\partial_{\alpha_i}f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}},\qquad \mathcal{M}^{(k)}_{\alpha_i,\beta}(0)=\partial_{\beta_0}[-\partial_{\alpha_i}f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}},$$
$$\mathcal{M}^{(k)}_{\beta,\alpha_j}(0)=\partial_{\alpha_{0,j}}[\partial_{\beta}f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}},\qquad \mathcal{M}^{(k)}_{\beta,\beta}(0)=\partial_{\beta_0}[\partial_{\beta}f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}}.$$

*Proof.* First of all let us write, for $e_0=\alpha_{0,1},\ldots,\alpha_{0,d},\beta_0$ and $u=\alpha_1,\ldots,\alpha_d,\beta$,

$$\partial_{e_0}\left(\sum_{\theta\in\Theta_{k,\mathbf{0},u}}\mathscr{V}(\theta)\right)=\sum_{\theta\in\Theta_{k,\mathbf{0},u}}\sum_{v\in N(\theta)}\partial_{e_0}\mathcal{F}_v\left(\prod_{v'\in N(\theta)\setminus\{v\}}\mathcal{F}_{v'}\right)\left(\prod_{\ell\in L(\theta)}\mathcal{G}_\ell\right), \tag{3.10}$$

where we have used the fact that $\mathscr{V}(\theta)$ depends on $\boldsymbol{\alpha}_0,\beta_0$ only through the node factors. Each summand in the r.h.s. of (3.10) differs from $\mathscr{V}(\theta)$ because a further derivative (with respect to $\alpha_{0,j}$ or $\beta_0$) acts on the node factor of a node $v\in N(\theta)$. This can be graphically represented as the same tree $\theta$, but with a further line $\ell'$ entering the node $v$; such a line carries $\mathbf{0}$-momentum and has component $e=\alpha_1,\ldots,\alpha_d,\beta$ for $e_0=\alpha_{0,1},\ldots,\alpha_{0,d},\beta_0$, respectively, and hence it is a contribution to $\mathcal{M}^{(k)}_{u,e}(0)$. On the other hand it is easy to realise that each contribution to $\mathcal{M}^{(k)}_{u,e}(0)$ is of the form above. Therefore the assertion follows. ∎

**Lemma 3.8.** *For all $k\geq 1$ one has*

$$\mathcal{M}^{(k)}_{\alpha_i,\alpha_j}(x,n)=\mathcal{M}^{(k)}_{\alpha_j,\alpha_i}(-x,n)=\left(\mathcal{M}^{(k)}_{\alpha_j,\alpha_i}(x,n)\right)^*,\qquad i,j=1,\ldots,d, \tag{3.11a}$$

$$\mathcal{M}^{(k)}_{\beta,\beta}(x,n)=\mathcal{M}^{(k)}_{\beta,\beta}(-x,n)=\left(\mathcal{M}^{(k)}_{\beta,\beta}(x,n)\right)^*, \tag{3.11b}$$

$$\mathcal{M}^{(k)}_{\alpha_i,\beta}(x,n)=-\mathcal{M}^{(k)}_{\beta,\alpha_i}(-x,n)=-\left(\mathcal{M}^{(k)}_{\beta,\alpha_i}(x,n)\right)^*,\qquad i=1,\ldots,d, \tag{3.11c}$$

*where $*$ denotes complex conjugation.*

*Proof.* Let us start from (3.11a) — in fact (3.11b) can be obtained reasoning in the same way. Given any $T\in\mathfrak{S}^k_{n,\alpha_i,\alpha_j}$ let $T'\in\mathfrak{S}^k_{n,\alpha_j,\alpha_i}$ be obtained from $T$ by considering $\ell_T,\ell'_T$ as entering and exiting lines, respectively, and reversing the orientation of the lines in $\mathcal{P}_T$. Denote by $N(\mathcal{P}_T)$ the set of nodes in $N(T)$ connected by the lines in $\mathcal{P}_T$. The node factors of the nodes in $N(T)\setminus N(\mathcal{P}_T)$ and the propagators of the lines outside $\mathcal{P}_T$ do not change. Given $v\in N(\mathcal{P}_T)$ let $\ell_v,\ell'_v\in\mathcal{P}_T\cup\{\ell_T,\ell'_T\}$ be the lines exiting and entering $v$, respectively. If $h_{\ell_v}=h_{\ell'_v}=\beta$ or $h_{\ell_v},h_{\ell'_v}\in\{\alpha_1,\ldots,\alpha_d\}$ then $\mathcal{F}_v$ does not change when considering $v$ as a node in $T'$. If $h_{\ell_v}=\beta$ while $h_{\ell'_v}\in\{\alpha_1,\ldots,\alpha_d\}$ or vice versa, the node factor $\mathcal{F}_v$ changes its sign when considering $v$ as a node in $T'$. Since both $h_{\ell_T},h_{\ell'_T}\in\{\alpha_1,\ldots,\alpha_d\}$, then the number of nodes in $N(\mathcal{P}_T)$ whose node factor changes sign must be even, so that the overall product of such node factors does not change. Finally if $\ell \in \mathcal{P}_T$ one has $\boldsymbol{\nu}_\ell = \boldsymbol{\nu}^0_\ell + \boldsymbol{\nu}_{\ell'_T}$ when considering it as a line in $L(T)$, while $\boldsymbol{\nu}_\ell = -\boldsymbol{\nu}^0_\ell + \boldsymbol{\nu}_{\ell'_{T'}}$ when considering it as a line in $\mathcal{P}_{T'}$, so that, computing at $\boldsymbol{\nu}_{\ell'_{T'}} = -\boldsymbol{\nu}_{\ell'_T}$, the propagators are equal since they are even in their arguments. This proves the first equality in (3.11a). Now let $T'' \in \mathfrak{S}^k_{n,\alpha_j,\alpha_i}$ be obtained from $T'$ by replacing the mode labels $\boldsymbol{\nu}_v$ of the nodes in $N(T)$ with $-\boldsymbol{\nu}_v$. The node factors are changed into their complex conjugated, while (reasoning as before), when computing at $\boldsymbol{\nu}_{\ell'_{T''}} = -\boldsymbol{\nu}_{\ell'_T}$, the propagators (which are real) do not change.

To prove (3.11c) we reason as above, the only difference being that, for $T \in \mathfrak{S}^k_{n,\alpha_j,\beta}$, the numer of nodes in $N(\mathcal{P}_T)$ which change sing when considering them as nodes in $T'$ is odd, and hence the overall product of the node factors change its sign. ∎

**Remark 3.9.** From Lemma 3.8 it follows that for all $k \geq 1$ and all $n \geq 0$ one has

$$\partial_x \mathcal{M}^{(k)}_{\beta,\beta}(0,n) = 0,$$
$$\partial_x \mathcal{M}^{(k)}_{\alpha_i,\beta}(0,n) = -\left(\partial_x \mathcal{M}^{(k)}_{\beta,\alpha_i}(0,n)\right)^*, \qquad i = 1,\ldots,d.$$

**Lemma 3.10.** *For all $k \geq 1$ one has $\partial_x \mathcal{M}^{(k)}_{\alpha_i,\alpha_j}(0,n) = 0$ for $i,j = 1,\ldots,d$.*

*Proof.* Given a cluster $T \in \mathfrak{S}^k_{n,\alpha_i,\alpha_j}$, with $i,j = 1,\ldots,d$, contributing to $M^{(k)}_{\alpha_i,\alpha_j}(0,n)$ through (3.9), set

$$\partial_x \mathscr{V}_T(0) := \sum_{\ell \in \mathcal{P}_T} \left( \prod_{v \in N(T)} \mathcal{F}_v \right) \left( \partial_x \mathcal{G}_\ell \prod_{\ell' \in L(T) \setminus \{\ell\}} \mathcal{G}_{\ell'} \right), \tag{3.13}$$

where the propagators have to be computed at $\boldsymbol{\omega} \cdot \boldsymbol{\nu}_{\ell'_T} = 0$ and

$$\partial_x \mathcal{G}_\ell := \frac{\Psi'_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}^0_\ell)}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu}^0_\ell)^2} - \frac{2\Psi_{n_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}^0_\ell)}{(\boldsymbol{\omega} \cdot \boldsymbol{\nu}^0_\ell)^3},$$

where $\Psi'_n$ denotes the derivative of $\Psi_n$ with respect to its argument. Clearly $\partial_x \mathscr{V}_T(0)$ is a contribution to $\partial_x \mathcal{M}^{(k)}_{\alpha_i,\alpha_j}(0,n)$.

Now, the line $\ell$ divides $L(T)$ in two disjoint sets of nodes $N_1$ and $N_2$ such that $\ell_T$ exits a node of $N_1$ and $\ell'_T$ enters a node in $N_2$. In other words if $\ell$ exits a node $v$ one has $N_2 = \{w \in N(T) : w \preceq v\}$ and $N_1 = N(T) \setminus N_2$. Set

$$\boldsymbol{\nu}_1 = \sum_{v \in N_1} \boldsymbol{\nu}_v; \qquad \boldsymbol{\nu}_2 = \sum_{v \in N_2} \boldsymbol{\nu}_v.$$

Since $T$ is a self-energy cluster one has $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 = 0$. Then consider the family $\mathcal{F}_1(T)$ of self-energy clusters obtained from $T$ by detaching the exiting line $\ell_T$ and reattaching it to all nodes $w \in N_1$, and by detaching the entering line $\ell'_T$, then reattaching it to all nodes $w \in N_2$. Consider also a second family $\mathcal{F}_2(T)$ of self-energy clusters obtained from $T$ by detaching the exiting line $\ell_T$ then reattaching it to all nodes $w \in N_2$ and by detaching the entering line $\ell'_T$ then reattaching it to all nodes $w \in N_1$.

It can happen that, detaching $\ell_T$ from a node $w_1 \in N_1$ and reattaching it to a node $w_2 \in N(T)$, some node factors change their sign because some lines change their direction (see the proof of Lemma 3.8). But, since $h_{\ell_T} = \alpha_i$ and $h_{\ell'_T} = \alpha_j$, the number of changes of sign is even, so that the overall product of the node factors does not change its sign. The shift of the lines $\ell_T$ and $\ell'_T$ also changes the combinatorial factors of some node factors. However, if we group together all the self-energy clusters in $\mathcal{F}_1(T)$ with the two lines $\ell_T$ and $\ell'_T$ attached to the same nodes $v \in N_1$ and $w \in N_2$, respectively, we see that the corresponding values differ from each other because of a factor $-\boldsymbol{\nu}_v\boldsymbol{\nu}_w$. Reasoning in the same way we find that there are no changes of sign in the product of the node factors also in the construction of the family $\mathcal{F}_2(T)$. Moreover, for those lines that change their direction after such shift operation, the momentum $\boldsymbol{\nu}_\ell$ is replaced by $-\boldsymbol{\nu}_\ell$ but no changes are produced in the propagators since they are even, except for the differentiated propagator which can change sign: the sign changes for the self-energy clusters in $\mathcal{F}_1(T)$, while it remains the same for those in $\mathcal{F}_2(T)$. Then by summing over all possible clusters in $\mathcal{F}_1(T)$ we obtain $-\boldsymbol{\nu}_1\boldsymbol{\nu}_2$ times a common factor, while summing over all possible clusters in $\mathcal{F}_2(T)$ we obtain $\boldsymbol{\nu}_1\boldsymbol{\nu}_2$ times the same common factor, so that the overall sum gives zero. ∎

**Lemma 3.11.** *For all $k \geq 1$ one has*

$$[-\partial_{\boldsymbol{\alpha}} f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}} = \mathbf{0} \tag{3.14a}$$

$$\mathcal{M}^{(k)}_{\alpha_i,h}(0) = 0, \qquad i = 1,\dots,d, \quad h = \alpha_1,\dots,\alpha_d,\beta \tag{3.14b}$$

*Proof.* We first prove (3.14a). Given $\theta \in \Theta_{k,\mathbf{0},\alpha_j}$, denote by $\mathfrak{F}(\theta)$ the set of all possible $\theta' \in \Theta_{k,\mathbf{0},\alpha_j}$ which can be obtained from $\theta$ by detaching the root line $\ell_\theta$ and reattaching it to each node $v \in N(\theta)$. The values of such trees differ from each other because of a factor $\mathrm{i}\boldsymbol{\nu}_v$, where $v$ is the node which the root line is attached to (again, as in the proof of Lemmas 3.8 and 3.10, there is an even number of nodes whose node factor changes sign, and hence the overall product does not changes). But then, since $\sum_{v\in N(\theta)} \boldsymbol{\nu}_v = \mathbf{0}$, the sum over all such contributions is zero. Moreover this holds identically in $\boldsymbol{\alpha}_0, \beta_0$, therefore by Lemma 3.7 also (3.14b) follows. ∎

**Remark 3.12.** Identity (3.14a) is formally equal to (2.2c): therefore we proved that (2.2c) formally holds. So, besides the convergence of the series, we are left with (2.2d) to be solved.

We can summarise the results above as follows. Let us write

$$\mathcal{M}^{(k)}_{u,e}(x,n) = \mathscr{L}^{(k)}_{u,e} + x\mathcal{D}^{(k)}_{u,e} + x^2\mathscr{D}^{(k)}_{u,e}(x) + \mathscr{R}^{(k)}_{u,e}(x,n), \tag{3.15}$$

with

$$\begin{gathered}\mathscr{L}^{(k)}_{u,e} := \mathcal{M}^{(k)}_{u,e}(0), \qquad \mathcal{D}^{(k)}_{u,e} := \partial_x\mathcal{M}^{(k)}_{u,e}(0), \qquad \mathscr{D}^{(k)}_{u,e}(x) := \int_0^1 \mathrm{d}\tau\,(1-\tau)\partial_x^2\mathcal{M}^{(k)}_{u,e}(\tau x), \\ \mathscr{R}^{(k)}_{u,e}(x,n) := \mathcal{M}^{(k)}_{u,e}(x,n) - \mathcal{M}^{(k)}_{u,e}(x).\end{gathered} \tag{3.16}$$

Then we have

$$\begin{aligned}
&\mathcal{M}^{(k)}_{\alpha_i,\alpha_j}(x,n) = x^2 \mathscr{D}^{(k)}_{\alpha_i,\alpha_j}(x) + \mathscr{R}^{(k)}_{\alpha_i,\alpha_j}(x,n), \qquad i,j=1,\dots,d, \\
&\mathcal{M}^{(k)}_{\alpha_i,\beta}(x,n) = x\mathcal{D}^{(k)}_{\alpha_i,\beta} + x^2 \mathscr{D}^{(k)}_{\alpha_i,\beta}(x) + \mathscr{R}^{(k)}_{\alpha_i,\beta}(x,n), \qquad i=1,\dots,d, \\
&\mathcal{M}^{(k)}_{\beta,\alpha_i}(x,n) = -x\big(\mathcal{D}^{(k)}_{\alpha_i,\beta}\big)^* + x^2 \mathscr{D}^{(k)}_{\beta,\alpha_i}(x) + \mathscr{R}^{(k)}_{\beta,\alpha_i}(x,n), \qquad i=1,\dots,d, \\
&\mathcal{M}^{(k)}_{\beta,\beta}(x,n) = \mathscr{L}^{(k)}_{\beta,\beta} + x^2 \mathscr{D}^{(k)}_{\beta,\beta}(x) + \mathscr{R}^{(k)}_{\beta,\beta}(x,n).
\end{aligned} \tag{3.17}$$

In other words, if we could ignore the "rest" $\mathscr{R}^{(k)}_{u,e}(x,n)$, we would obtain a gain factor proportional to $x^2$ for the self-energies with $u,e \in \{\alpha_1,\dots,\alpha_d\}$, a gain proportional to $x$ for $u=\alpha_1,\dots,\alpha_d$ and $e=\beta$ (or vice versa) and no gain for $u=e=\beta$ (but in the latter case no factor proportional to $x$ would appear). This suggests us that if $\mathscr{L}^{(k)}_{\beta,\beta} \equiv 0$ and $\mathcal{D}^{(k)}_{\alpha_i,\beta} \equiv 0$ for all $i=1,\dots,d$ and all $k \geq 1$, we would obtain a gain proportional to $x^2$ for any self-energy (provided the "rest" is small) and this should imply the convergence of the power series.

**Condition 1.** *For all $k \geq 1$ one has $\mathscr{L}^{(k)}_{\beta,\beta} \equiv 0$ and $\mathcal{D}^{(k)}_{\alpha_i,\beta} \equiv 0$ for all $i=1,\dots,d$.*

**Lemma 3.13.** *Assume Condition 1. Then for all $h,h' \in \{\alpha_1,\dots,\alpha_d,\beta\}$ and for any $(\boldsymbol{\alpha}_0,\beta_0) \in \mathbb{T}^{d+1}$ one has $|\mathcal{M}^{(k)}_{h,h'}(x,n)| \leq C^k x^2$, for some positive constant $C$.*

The proof of the result above essentially follows the lines of the proof of Lemma 6.6 in [3].

**Remark 3.14.** One can prove also that, setting $\mathcal{D}^{(k)}_{\boldsymbol{\alpha},\beta} = (\mathcal{D}^{(k)}_{\alpha_1,\beta},\dots,\mathcal{D}^{(k)}_{\alpha_d,\beta})$, one has

$$\boldsymbol{\omega} \cdot \mathcal{D}^{(k)}_{\boldsymbol{\alpha},\beta} = 2\mathrm{i}(k-1)[\partial_\beta f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}},$$

for all $k \geq 1$. We shall not give the proof of the identity above since it will not be used here.

If Condition 1 is satisfied, Lemma 3.13 implies the convergence of the series (2.4) for $\varepsilon$ small enough: the argument is the same as after Lemma 6.6 in [3]. Moreover, by Lemma 3.7, the assumption $\mathscr{L}^{(k)}_{\beta,\beta} \equiv 0$ reads $[\partial_\beta f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}} \equiv$ const. Due to the variational nature of the Hamilton equation, $[\partial_\beta f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}}$ is the $\beta_0$-derivative of the $k$-th order of the time average of the Lagrangian $\gamma^{(k)}$ (which is analytic and periodic) computed along a solution of the range equation (one can reason as in [4]). This implies $[\partial_\beta f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}} \equiv 0$, so that also (2.2d) holds for any $\beta_0 \in \mathbb{T}^d$. Therefore, at least in the particular case that Condition 1 holds, we provided a quasi-periodic solution to the equation (1.2) as a convergent power series in $\varepsilon$. Note that in such a case the initial phase $\beta_0$ remains arbitrary, so that the full $(d+1)$-resonant unperturbed torus persists.

# 4 Resummation of the formal expansion

In Section 3 we have seen how to deal with the "completely degenerate case" of Condition 1, which yields infinitely many identities. If these identities do not hold we are not able to prove the convergence of the series (2.4). Now we shall see how to deal with such a case.

### 4.1 Renormalised trees

As seen in Section 3.2 all the obstruction to the convergence are due to the presence of self-energy clusters. Now we shall perform a different tree expansion with respect to the one performed in Section 3.1 in order to deal with this problem.

More precisely, we modify the tree expansion envisaged in Section 3.1 as follows. Given a tree $\theta$ we associate with each node $v \in N(\theta)$ a mode label and a component label as in Section 3.1; with each line $\ell \in L(\theta)$ we associate a momentum label as in Section 3.1 and a *pair* of component labels $(e_\ell, u_\ell) \in \{\alpha_1, \dots, \alpha_d\}$ with the constraint that $u_{\ell_v} = h_v$. We shall call $e_\ell$ and $u_\ell$ the *e-component* and the *u-component* of $\ell$, respectively. We denote by $p_v$ and $q_v$ the number of lines with $e$-component $\alpha_j$ for some $j = 1, \dots, d$ and $\beta$ entering $v$, respectively, and set $s_v = p_v + q_v$. We still impose the conservation law (3.1). We do not change the definition of cluster, while from now on a self-energy cluster is a cluster $T$ with only one entering line $\ell'_T$ and one exiting line $\ell_T$ such that $\boldsymbol{\nu}_{\ell_T} = \boldsymbol{\nu}_{\ell'_T}$, i.e. we drop the constraint (ii) from the definition of self-energy cluster given in Section 3.1.

A *renormalised tree* is a tree in which no self-energy cluster appears. Analogously a *renormalised subgraph* of a tree is a subgraph $S$ of a tree $\theta$ such that $S$ does not contains any self-energy cluster.

Given a renormalised tree we call *total momentum* and *total component* the momentum and the $e$-component associated with the root line. We denote by $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ the set of all renormalised trees with order $k$ total momentum $\boldsymbol{\nu}$ and total component $h$, and by $\mathfrak{R}_{n,u,e}$ the set of renormalised self-energy clusters on scale $n$ such that $u_{\ell_T} = u$ and $e_{\ell'_T} = e$.

Given $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ we associate with each $v \in N(\theta)$ a node factor $\mathcal{F}_v$ defined as in (3.2) and with each $\ell \in L(\theta)$ a propagator $\mathcal{G}_\ell$ defined as follows. First of all, given a $(d+1)\times(d+1)$ matrix $A$ with entries $A_{h,h'}$, for $h, h' \in \{\alpha_1, \dots, \alpha_d, \beta\}$, we denote by $A_{\boldsymbol{\alpha},\boldsymbol{\alpha}}$ the $d \times d$ matrix with entries $(A_{\boldsymbol{\alpha},\boldsymbol{\alpha}})_{i,j} := A_{\alpha_i,\alpha_j}$, for $i, j = 1, \dots, d$, by $A_{\boldsymbol{\alpha},\beta}$ the vector with components $(A_{\boldsymbol{\alpha},\beta})_i := A_{\alpha_i,\beta}$, for $i = 1, \dots, d$, and by $A_{\beta,\boldsymbol{\alpha}}$ the vector with components $(A_{\beta,\boldsymbol{\alpha}})_j := A_{\beta,\alpha_j}(x)$, for $j = 1, \dots, d$; with a slight abuse of notation we denote in the same way both column and row vectors. Then we define recursively the propagator of the line $\ell$ as $\mathcal{G}_\ell := \mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega} \cdot \boldsymbol{\nu}_\ell)$, with

$$\mathcal{G}^{[n]}(x) = \begin{pmatrix} \mathcal{G}^{[n]}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(x) & \mathcal{G}^{[n]}_{\boldsymbol{\alpha},\beta}(x) \\ \mathcal{G}^{[n]}_{\beta,\boldsymbol{\alpha}}(x) & \mathcal{G}^{[n]}_{\beta,\beta}(x) \end{pmatrix} := \Psi_n(x) \left( x^2 \mathbb{1} - \mathcal{M}^{[n-1]}(x) \right)^{-1}, \tag{4.1}$$

where $\Psi_n$ is defined as in Section 3.1, $\mathbb{1}$ is the $(d+1)\times(d+1)$ identity matrix and

$$\mathcal{M}^{[n-1]}(x) := \sum_{q=-1}^{n-1} \chi_q(x) M^{[q]}(x), \tag{4.2}$$

with $\chi_q$ defined as in Section 3.1 and

$$M^{[n]}(x) = \begin{pmatrix} M^{[n]}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(x) & M^{[n]}_{\boldsymbol{\alpha},\beta}(x) \\ M^{[n]}_{\beta,\boldsymbol{\alpha}}(x) & M^{[n]}_{\beta,\beta}(x) \end{pmatrix}, \quad \text{with} \quad M^{[n]}_{u,e}(x) := \sum_{T \in \mathfrak{R}_{n,u,e}} \varepsilon^{k(T)} \mathscr{V}_T(x), \quad n \geq -1, \tag{4.3}$$

and

$$\mathscr{V}_T(x) := \left( \prod_{v \in N(T)} \mathcal{F}_v \right) \left( \prod_{\ell \in L(T)} \mathcal{G}_\ell \right) \tag{4.4}$$

is the *renormalised value* of $T$.

Set $\mathcal{M} := \{\mathcal{M}^{[n]}(x)\}_{n\geq -1}$. We call *self-energies* the matrices $\mathcal{M}^{[n]}(x)$.

**Remark 4.1.** By construction $\mathcal{G}^{[n]}(x)$ depends also on $\varepsilon$ and $\beta_0$, even though we are not making explicit such a dependence; it does not depend on $\boldsymbol{\alpha}_0$ because $f_{\boldsymbol{\nu}}(\boldsymbol{\alpha}_0, \beta_0) = \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\alpha}_0} \hat{f}_{\boldsymbol{\nu}}(\beta_0)$ and $\sum_{v \in N(T)} \boldsymbol{\nu}_v = 0$ for any self-energy cluster $T$. The last comment applies also to the quantities $\boldsymbol{F}^{[k]}(\varepsilon, \beta_0)$ and $G^{[k]}(\varepsilon, \beta_0)$ introduced in (4.5) below.

Setting also $\mathcal{G}^{[-1]} = \mathbb{1}$, for any renormalised subgraph $S$ of any $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ we define the *renormalised value* of $S$ as in (3.5), but with the new definition for the propagators.

Set $\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\boldsymbol{\alpha}} := \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\alpha_1} \times \ldots \times \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\alpha_d}$ and for any $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d) \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\boldsymbol{\alpha}}$ denote $\boldsymbol{\mathscr{V}}(\boldsymbol{\theta}) := (\mathscr{V}(\theta_1), \ldots, \mathscr{V}(\theta_d))$. Then define (formally)

$$\begin{aligned}
&\boldsymbol{a}^{[k]}_{\boldsymbol{\nu}}(\varepsilon, \boldsymbol{\alpha}_0, \beta_0) := \sum_{\boldsymbol{\theta} \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\boldsymbol{\alpha}}} \boldsymbol{\mathscr{V}}(\boldsymbol{\theta}), \qquad b^{[k]}_{\boldsymbol{\nu}}(\varepsilon, \boldsymbol{\alpha}_0, \beta_0) := \sum_{\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\beta}} \mathscr{V}(\theta), \qquad \boldsymbol{\nu} \neq \boldsymbol{0}, \\
&\boldsymbol{F}^{[k]}(\varepsilon, \beta_0) := \sum_{\boldsymbol{\theta} \in \Theta^{\mathcal{R}}_{k+1,\boldsymbol{0},\boldsymbol{\alpha}}} \boldsymbol{\mathscr{V}}(\boldsymbol{\theta}), \qquad G^{[k]}(\varepsilon, \beta_0) := \sum_{\theta \in \Theta^{\mathcal{R}}_{k+1,\boldsymbol{0},\beta}} \mathscr{V}(\theta).
\end{aligned} \tag{4.5}$$

Finally set (again formally)

$$\boldsymbol{a}^{\mathcal{R}}(t; \varepsilon, \boldsymbol{\alpha}_0, \beta_0) := \sum_{k\geq 1} \varepsilon^k \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega} t} \boldsymbol{a}^{[k]}_{\boldsymbol{\nu}}(\varepsilon, \boldsymbol{\alpha}_0, \beta_0), \tag{4.6a}$$

$$b^{\mathcal{R}}(t; \varepsilon, \boldsymbol{\alpha}_0, \beta_0) := \sum_{k\geq 1} \varepsilon^k \sum_{\boldsymbol{\nu} \in \mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega} t} b^{[k]}_{\boldsymbol{\nu}}(\varepsilon, \boldsymbol{\alpha}_0, \beta_0), \tag{4.6b}$$

$$\boldsymbol{F}^{\mathcal{R}}(\varepsilon, \beta_0) := \sum_{k\geq 0} \varepsilon^k \boldsymbol{F}^{[k]}(\varepsilon, \beta_0), \tag{4.6c}$$

$$G^{\mathcal{R}}(\varepsilon, \beta_0) := \sum_{k\geq 0} \varepsilon^k G^{[k]}(\varepsilon, \beta_0), \tag{4.6d}$$

and define

$$\begin{aligned}
\boldsymbol{\alpha}^{\mathcal{R}}(t; \varepsilon, \boldsymbol{\alpha}_0, \beta_0) &= \boldsymbol{\alpha}_0 + \boldsymbol{\omega} t + \boldsymbol{a}^{\mathcal{R}}(t; \varepsilon, \boldsymbol{\alpha}_0, \beta_0), \\
\beta^{\mathcal{R}}(t; \varepsilon, \boldsymbol{\alpha}_0, \beta_0) &= \beta_0 + b^{\mathcal{R}}(t; \varepsilon, \boldsymbol{\alpha}_0, \beta_0).
\end{aligned} \tag{4.7}$$

The series (4.6) will be called *resummed series*, the term "resummed" coming from the fact that if we formally expand (4.6) in powers of $\varepsilon$ then we get (2.4), as is easy to check.

For any renormalised subgraph $S$ of any tree $\theta$ we denote by $\mathfrak{N}_n(S)$ the number of lines on scale $\geq n$ in $S$ and define $K(S)$ as in (3.7). Then we have the following results which are the counterparts of Lemmas 3.3 and 3.4, respectively, for renormalised trees.

**Lemma 4.2.** *For any $h \in \{\alpha_1, \ldots, \alpha_d, \beta\}$, $\boldsymbol{\nu} \in \mathbb{Z}^d$, $k \geq 1$ and for any $\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h}$ such that $\mathscr{V}(\theta) \neq 0$, one has $\mathfrak{N}_n(\theta) \leq 2^{-(m_n-2)} K(\theta)$ for all $n \geq 0$.*

**Lemma 4.3.** *For any $e, u \in \{\alpha_1, \ldots, \alpha_d, \beta\}$, $n \geq 0$ and for any $T \in \mathfrak{R}_{n,u,e}$ such that $\mathscr{V}_T(x) \neq 0$, one has $K(T) > 2^{m_n-1}$ and $\mathfrak{N}_p(T) \leq 2^{-(m_p-2)} K(T)$ for $0 \leq p \leq n$.*

The two results above can be proved as Lemmas 4.1 and 4.2 in [3], respectively.

## 4.2 A suitable assumption: bounds

Here we shall see that, under the assumption that the propagators $\mathcal{G}^{[n]}_{e,u}(\boldsymbol{\omega} \cdot \boldsymbol{\nu})$ are bounded proportionally to $1/|\boldsymbol{\omega} \cdot \boldsymbol{\nu}|^c$ for some constant $c$, the series (4.6) converge and solve the range equations (2.2a) and (2.2b): the key point is that now self-energy clusters (and hence resonant lines) are not allowed and hence a result of that kind is expected. Then, in what follows, we shall see that the assumption is justified at least along a curve $\beta_0(\varepsilon)$ where also the bifurcation equations (2.2c) and (2.2d) are satisfied.

Define $\|\cdot\|$ as an algebraic matrix norm (i.e. a norm which verifies $\|AB\| \leq \|A\|\|B\|$ for all matrices $A$ and $B$); for instance $\|\cdot\|$ can be the uniform norm.

**Definition 4.4.** *We shall say that $\mathcal{M}$ satisfies Property 1 if there are positive constants $c_1$ and $c_2$ such that*

$$\left\| \mathcal{G}^{[n]}(x) \right\| \leq \frac{c_1}{|x|^{c_2}}$$

*for all $n \geq 0$. Call $\mathcal{S} := \{(\varepsilon, \beta_0) \in \mathbb{R} \times \mathbb{T} :$ Property 1 holds$\}$.*

**Definition 4.5.** *We shall say that $\mathcal{M}$ satisfies Property 1-p if there are positive constants $c_1$ and $c_2$ such that*

$$\left\| \mathcal{G}^{[n]}(x) \right\| \leq \frac{c_1}{|x|^{c_2}},$$

*for $0 \leq n \leq p$. Call $\mathcal{S}_p := \{(\varepsilon, \beta_0) \in \mathbb{R} \times \mathbb{T} :$ Property 1-p holds$\}$.*

**Lemma 4.6.** *Assume $(\varepsilon, \beta_0) \in \mathcal{S}_p$. Then, for $0 \leq n \leq p$ and $\varepsilon$ small enough, the self-energies are well defined and one has*

$$\left| \partial^j_x M^{[n]}_{u,e}(x) \right| \leq \varepsilon^2 \, K_j \mathrm{e}^{-\overline{K}_j 2^{m_n}}, \qquad j = 0, 1, 2,$$

*for some positive constants $K_0, \overline{K}_0, K_1, \overline{K}_1, K_2$ and $\overline{K}_2$.*

The proof is essentially the same as the proof of Lemma 4.8 in [3] and Lemma 4.3 in [4]. In particular we need a property analogous to Remark 3.6 when bounding the derivatives.

**Remark 4.7.** If $\mathcal{M}$ satisfies Property 1-$p$ the matrices $\mathcal{M}^{[n]}(x)$ and $\mathcal{G}^{[n]}(x)$ are well defined for all $-1 \leq n \leq p$. In particular there exists $\gamma_0 > 0$ such that $|\mathcal{G}^{[n]}_{e,u}(x)| \leq \gamma_0 \, \alpha_{m_n}(\boldsymbol{\omega})^{-c_2}$ for all $0 \leq n \leq p$. If $\mathcal{M}$ satisfies Property 1, the same considerations apply for all $n \geq 0$.

**Lemma 4.8.** *Assume* $(\varepsilon, \beta_0) \in \mathcal{S}_p$*. Then, for* $0 \le n \le p$ *and* $\varepsilon$ *small enough, one has*

$$\left| M^{[n]}_{u,e}(x) - M^{[n]}_{u,e}(0) - x\,\partial_x M^{[n]}_{u,e}(0) \right| \le \varepsilon^2\, K_3 \mathrm{e}^{-\overline{K}_3 2^{m_n}}\, x^2$$

*for some positive constants* $K_3$ *and* $\overline{K}_3$*.*

The proof is essentially the same as the proof of Lemma 4.6 in [4].

**Lemma 4.9.** *Assume* $(\varepsilon, \beta_0) \in \mathcal{S}$*. Then the series (4.6), with the coefficients given by (4.5), converge for* $\varepsilon$ *small enough.*

The proof is essentially the same as the proof of Lemma 4.5 in [3] and Lemma 4.9 in [4].

**Lemma 4.10.** *Assume* $(\varepsilon, \beta_0) \in \mathcal{S}$*. Then for* $\varepsilon$ *small enough the function (4.6a) and (4.6b) solve the range equations (2.2a) and (2.2b), respectively.*

The proof is essentially the same as the proof of Lemma 4.6 in [3] and Lemma 4.10 in [4].

### 4.3 A suitable assumption: symmetries

Here we shall prove that, under the assumptions that $\mathcal{M}$ satisfies Property 1-$p$, there are suitable symmetries for the self-energy clusters: such symmetries are the counterpart of those founded in Section 3.3 for the formal expansion. Property 1-$p$ is assumed only because, under such assumption, all the quantities are well defined.

**Lemma 4.11.** *Let* $\mathfrak{B}_n$ *the set of* $B : \mathbb{R} \to \mathrm{GL}(n, \mathbb{C})$ *such that*

$$\begin{aligned} &B_{i,j}(-x) = B_{j,i}(x), \qquad i,j = 1,\dots,n-1, \qquad B_{n,n}(-x) = B_{n,n}(x) \\ &B_{n,i}(-x) = -B_{i,n}(x), \qquad i = 1,\dots,n-1. \end{aligned}$$

*Then if* $B \in \mathfrak{B}_n$ *also* $B^{-1} \in \mathfrak{B}_n$*.*

*Proof.* If $B \in \mathfrak{B}_n$ define the matrix $A$ by setting

$$\begin{aligned} &B_{i,j}(x) = A_{i,j}(x), \qquad i,j = 1,\dots,n-1, \qquad B_{n,n}(x) = A_{n,n}(x), \\ &B_{n,i}(x) = x\,A_{n,i}(x) \text{ and } B_{i,n}(x) = x\,A_{i,n}(x), \qquad i = 1,\dots,n-1, \end{aligned}$$

so that $A^T(-x) = A(x)$. Denote also by $C_{i,j}(x)$ the cofactor of the entry $A_{i,j}(x)$ for $i,j = 1,\dots,n$. By construction $C_{i,j}(-x) = C_{j,i}(x)$ for $i,j = 1,\dots,n$: then

$$\begin{aligned} \det B(x) &= (-1)^{n-1} x^2 \left[ A_{n,1}(x)\, C_{n,1}(x) - \ldots + A_{n,n-1}(x)\, C_{n,n-1}(x) \right] + A_{n,n}(x)\, C_{n,n}(x) \\ &= x^2 \det A(x) + \left(1 - x^2\right) A_{n,n}(x)\, C_{n,n}(x), \end{aligned}$$

so that $\det B(-x) = \det B(x)$. By noting that

$$\left(B^{-1}(x)\right)_{j,i} = \frac{1}{\det B(x)} \begin{cases} x^2 C_{i,j}(x) + (1-x^2) D_{i,j}(x), & i,j = 1,\dots,n-1, \\ x\,C_{i,j}(x), & i = n \text{ and } j = 1,\dots,n-1, \\ x\,C_{i,j}(x), & i = 1,\dots,n-1 \text{ and } j = n, \\ C_{i,j}(x), & i,j = n, \end{cases}$$

where $D_{i,j}(x)$ is the cofactor of $A_{i,j}(x)$ seen as entry of the $(n-1)\times(n-1)$ matrix obtained from $A(x)$ by deleting its $n$-th row and $n$-th column, the assertion follows. ■

**Lemma 4.12.** *Assume $(\varepsilon,\beta_0)\in\mathcal{S}_p$. Then for all $-1\le n\le p$ one has*

$$\left(\mathcal{M}^{[n]}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(x)\right)^T=\mathcal{M}^{[n]}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(-x)=\left(\mathcal{M}^{[n]}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(x)\right)^*, \tag{4.10a}$$

$$\mathcal{M}^{[n]}_{\beta,\beta}(x)=\mathcal{M}^{[n]}_{\beta,\beta}(-x)=\left(\mathcal{M}^{[n]}_{\beta,\beta}(x)\right)^*, \tag{4.10b}$$

$$\mathcal{M}^{[n]}_{\boldsymbol{\alpha},\beta}(x)=-\mathcal{M}^{[n]}_{\beta,\boldsymbol{\alpha}}(-x)=-\left(\mathcal{M}^{[n]}_{\beta,\boldsymbol{\alpha}}(x)\right)^*, \tag{4.10c}$$

*where $*$ denotes complex conjugation.*

*Proof.* We shall proceed by induction on $n$. First of all note that for $n=-1$ (4.10) trivially holds, since

$$\mathcal{M}^{[-1]}(x)=\begin{pmatrix}0_d & \mathbf{0}\\ \mathbf{0} & \varepsilon\partial^2_\beta f_{\mathbf{0}}\end{pmatrix}, \tag{4.11}$$

where $0_d$ is the $d\times d$ null matrix. Assume than that (4.10) hold for all $-1\le n'<n$ and let us start from the first equality in (4.10a). Given any $T\in\mathfrak{R}_{n,\alpha_i,\alpha_j}$ let $T'$ be obtained from $T$ by reversing the orientation of the lines along $\mathcal{P}_T\cup\{\ell_T,\ell'_T\}$. Denote by $N(\mathcal{P}_T)$ the set of nodes in $N(T)$ connected by the lines in $\mathcal{P}_T$. The node factors of the nodes in $N(T)\setminus N(\mathcal{P}_T)$ and the propagators on the lines outside $\mathcal{P}_T$ do not change when considering them as nodes and lines in $T'$. Given $v\in N(\mathcal{P}_T)$ let $\ell_v,\ell'_v\in\mathcal{P}_T\cup\{\ell_T,\ell'_T\}$ be the lines exiting and entering $v$, respectively. If $u_{\ell_v}=e_{\ell'_v}=\beta$ or $u_{\ell_v},e_{\ell'_v}\in\{\alpha_1,\ldots,\alpha_d\}$ then $\mathcal{F}_v$ does not change when considering $v$ as a node in $T'$. If $u_{\ell_v}=\beta$ while $e_{\ell'_v}\in\{\alpha_1,\ldots,\alpha_d\}$ or vice versa, the node factor $\mathcal{F}_v$ changes its sign when considering $v$ as a node in $T'$. Now, given $\ell\in\mathcal{P}_T$ we compute the propagator associated with $\ell$ at $x_\ell:=\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell=\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell+x$ and we obtain $\mathcal{G}_\ell=\Psi_{n_\ell}(x_\ell)(x_\ell^2\mathbb{1}-\mathcal{M}^{[n_\ell-1]}(x_\ell))^{-1}_{e_\ell,u_\ell}$; when considering $\ell$ as a line in $T'$, if we set $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_{T'}}=-x$, then the momentum of $\ell$ changes sign and hence the propagator becomes $\Psi_{n_\ell}(-x_\ell)((-x_\ell)^2\mathbb{1}-\mathcal{M}^{[n_\ell-1]}(-x_\ell))^{-1}_{u_\ell,e_\ell}$: thanks to the inductive hypothesis and Lemma 4.11, if $e_\ell=u_\ell=\beta$ or $e_\ell=u_\ell\in\{\alpha_1,\ldots,\alpha_d\}$ the propagator does not change when considering $\ell$ as a line in $T'$, otherwise it changes its sign. Let $h_0,\ldots,h_{2|\mathcal{P}_T|+1}$ be such that $h_0=u_{\ell_T}$, $\{h_1,\ldots,h_{2|\mathcal{P}_T|}\}$ is the ordered set of the components of the lines in $\mathcal{P}_T$ and $h_{2|\mathcal{P}_T|+1}=e_{\ell'_T}$. Note that there is a change of sign (in the node factor or in the propagator) corresponding to each ordered pair $h_r,h_{r+1}$ such that either $h_r=\alpha_i$ for some $i=1,\ldots,d$ and $h_{r+1}=\beta$ or vice versa. Since $h_0=\alpha_i$ and $h_{2|\mathcal{P}_T|+1}=\alpha_j$ the number of changes of sign is even and therefore the overall product does not change. This proves the first equality in (4.10a). The first equality in (4.10b) can be proved in the same way.

Now let $T''$ be the self-energy cluster obtained from $T'$ by replacing the mode labels $\boldsymbol{\nu}_v$ of the nodes in $N(T')$ with $-\boldsymbol{\nu}_v$. The node factors are changed into their complex conjugated and, thanks to the inductive hypothesis, when computing at $\boldsymbol{\nu}_{\ell'_{T''}}=-\boldsymbol{\nu}_{\ell'_{T'}}$, also the propagators are changed into their complex conjugated. Hence also the second equality in (4.10a) is proved. Again analogous considerations lead to the second equality in (4.10b).

To prove (4.10c) one can reason in the same way, the only difference being that for $T \in \mathfrak{R}_{n,\alpha_i,\beta}$ the number of changes of sign of the propagators of the lines in $\mathcal{P}_T$ and of the node factors of the nodes in $N(\mathcal{P}_T)$ is odd. This implies the change of sing in the first equality in (4.10c). ■

**Lemma 4.13.** *Assume $(\varepsilon, \beta_0) \in \mathcal{S}_p$. Then one has for $-1 \le n \le p$*

$$\mathcal{M}^{[n]}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(x) = O(\varepsilon^2 x^2), \tag{4.12a}$$

$$\mathcal{M}^{[n]}_{\beta,\boldsymbol{\alpha}}(x) = O(\varepsilon^2 x), \tag{4.12b}$$

$$\mathcal{M}^{[n]}_{\boldsymbol{\alpha},\beta}(x) = O(\varepsilon^2 x), \tag{4.12c}$$

$$\mathcal{M}^{[n]}_{\beta,\beta}(x) = \mathcal{M}^{[n]}_{\beta,\beta}(0) + O(\varepsilon^2 x^2), \tag{4.12d}$$

*where $\mathcal{M}^{[n]}_{\beta,\beta}(0) = O(\varepsilon)$.*

*Proof.* Let us start from the proof of (4.12a). First of all we shall show that $\sum_{T\in\mathfrak{R}_{n,u,e}} \mathscr{V}_T(0) = 0$ where $(u,e) \in \{\alpha_1, \ldots, \alpha_d\}^2$. Given a self-energy cluster $T \in \mathfrak{R}_{n,\alpha_i,\alpha_j}$ for $i,j = 1,\ldots,d$ consider all the self-energy cluster which can be obtained from $T$ by detaching the entering line $\ell'_T$ and reattaching it to each node $v \in N(T)$. After such operation $\mathscr{V}_T(0)$ changes by a factor $(\mathrm{i}\boldsymbol{\nu}_v)$ if $v$ is the node which the entering line is attached to, while the other node factors and propagators do not change (the combinatorial factors can be discussed as along the proof of Lemma 3.10). The sum of all clusters values is zero because $\sum_{v\in N(T)} \boldsymbol{\nu}_v = 0$. This implies $\mathcal{M}^{[n]}_{\boldsymbol{\alpha},\boldsymbol{\alpha}}(0) = 0_d$ (see the beginning of the proof of Lemma 4.12 for notation).

Now let us write $\partial_x \mathscr{V}_T(0)$ as in (3.13), where again the propagators have to be computed at $\boldsymbol{\omega}\cdot\boldsymbol{\nu}_{\ell'_T} = 0$, but now

$$\partial_x \mathcal{G}_\ell := \frac{\mathrm{d}}{\mathrm{d}x} \mathcal{G}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}^0_\ell + x)\bigg|_{x=0}.$$

The line $\ell$ divides $L(T)$ in two disjoint set of nodes $N_1$ and $N_2$ such that $\ell_T$ exits a node in $N_1$ and $\ell'_T$ enters a node in $N_2$. In other words $N_2 = \{w \in N(T) : w \prec \ell\}$ and $N_1 = N(T) \setminus N_2$. Set

$$\boldsymbol{\nu}_1 = \sum_{v\in N_1} \boldsymbol{\nu}_v; \qquad \boldsymbol{\nu}_2 = \sum_{v\in N_2} \boldsymbol{\nu}_v. \tag{4.13}$$

Since $T$ is a self-energy cluster one has $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 = 0$. Now consider the family $\mathcal{F}_1(T)$ of self-energy cluster obtained from $T$ by detaching the exiting line $\ell_T$ and reattaching it to all nodes $w \in N_1$, and by detaching the entering line $\ell'_T$ and reattaching it to all nodes $w \in N_2$. Consider also the family $\mathcal{F}_2(T)$ obtained from $T$ by detaching the exiting line $\ell_T$ and reattaching it to all nodes $w \in N_2$, and by detaching the entering line $\ell'_T$ then reattaching it to all nodes $w \in N_1$. One can note that the product of the node factors of a cluster $T' \in \mathcal{F}_1(T)$ differs from that of $T$ only because of an extra factor $-\boldsymbol{\nu}_v\boldsymbol{\nu}_w$, where $v \in N_1$ is the node which $\ell_T$ is attached to and $w \in N_2$ is the node which $\ell'_T$ enters (again we are considering together all self-energy clusters with the entering and exiting lines attached to the same nodes, respectively). Indeed, detaching $\ell_T$ from a node $w_1 \in N_1$ and then reattaching it to $w_2 \in N_1$, some node factors of the nodes in $N(\mathcal{P}(w_1,w_2))$ (we are denoting by $\mathcal{P}(w_1,w_2)$ the path connecting $w_1, w_2$ and by $N(\mathcal{P}(w_1,w_2))$ the set of nodes connected by lines in $\mathcal{P}(w_1,w_2)$) can change their sign since some lines can change their direction (see Lemma 4.12). Of course if the components of a line $\ell \in \mathcal{P}(w_1,w_2)$ are inverted, the corresponding propagator $\mathcal{G}_\ell = \Psi_{n_\ell}(x_\ell)(x_\ell^2 \mathbb{1} - \mathcal{M}^{[n_\ell-1]}(x_\ell))^{-1}_{e_\ell,u_\ell}$ is replaced by $\Psi_{n_\ell}(-x_\ell)((-x_\ell)^2\mathbb{1} - \mathcal{M}^{[n_\ell-1]}(-x_\ell))^{-1}_{u_\ell,e_\ell}$; thanks to Lemma 4.12, if $e_\ell = u_\ell = \beta$ or $e_\ell = u_\ell \in \{\alpha_1,\dots,\alpha_d\}$ the propagator does not change when considering $\ell$ as a line in $T'$, otherwise it changes its sign. But since one has $u_{\ell_T}, e_{\ell'_T} \in \{\alpha_1,\dots,\alpha_d\}$, then the number of changes of sign (both in the node factors or in the propagators along $\mathcal{P}(w_1,w_2)$) is even, so that the overall product does not change sign.

Reasoning as above, we can conclude that the value of a a cluster $T'' \in \mathcal{F}_2(T)$ differs from that of $T$ only because of a factor $-\boldsymbol{\nu}_v\boldsymbol{\nu}_w$, where $v \in N_1$ is the node which $\ell'_T$ enters and $w \in N_2$ is the node which $\ell_T$ exits.

No other changes are produced, except for the differentiated propagator which can change sign: the sign changes for the clusters in $\mathcal{F}_1(T)$ while it remains the same for those in $\mathcal{F}_2(T)$. Then by summing over all possible clusters in $\mathcal{F}_1(T)$ we obtain $-\boldsymbol{\nu}_1\boldsymbol{\nu}_2$ times a common factor, while by summing over all possible clusters in $\mathcal{F}_2(T)$ we obtain $\boldsymbol{\nu}_1\boldsymbol{\nu}_2$ times the same common factor, so that the overall sum gives zero. Hence (4.12a) is proved.

Now pass to (4.12b). Given a cluster $T \in \mathfrak{R}_{n,u,e}$ with $e \in \{\alpha_1,\dots,\alpha_d\}$ and $u = \beta$ consider all the self-energy clusters which can be obtained from $T$ by detaching the entering line $\ell'_T$ (note that $e_{\ell_T} = e$) and reattaching it to all the nodes $v \in N(T)$. Note that again some momenta can change sign, but the corresponding propagators does not change (again reasoning as done for the proof of Lemma 4.12 above). Hence we obtain a common factor times $\mathrm{i}\boldsymbol{\nu}_v$, where $v$ is the node which the exiting line is attached to, so that $\sum_T \mathscr{V}_T(0) = 0$.

To prove (4.12c) one simply notes that it follows from (4.12b) and (4.10c).

Finally, given a cluster $T \in \mathfrak{R}_{n,\beta,\beta}$, consider a contribution to $\partial_x \mathscr{V}_T(0)$ in which a line $\ell$ is differentiated (see (3.13)). The line $\ell$ divides $N(T)$ into two disjoint sets of nodes $N_1$ and $N_2$ such that $\ell_T$ exits a node $v_1 \in N_1$ and $\ell'_T$ enters a node $v_2 \in N_2$ i.e. $N_2 = \{w \in N(T) : w \prec \ell\}$ and $N_1 = N(T) \setminus N_2$. Again, with the same notations as in (4.13), one has $\boldsymbol{\nu}_1 + \boldsymbol{\nu}_2 = 0$. Then consider the cluster obtained by detaching the exiting line $\ell_T$ from $v_1$ and reattaching it to the node $v_2$, and, at the same time, by detaching the entering line $\ell'_T$ from $v_2$ and reattaching to the node $v_1$: note that this new cluster again belongs to $\mathfrak{R}_{n,\beta,\beta}$. Due to this operation, the directions of the line along the path connecting $v_1$ to $v_2$ are reversed, so that for such lines the momentum $\boldsymbol{\nu}_\ell$ is replaced with $-\boldsymbol{\nu}_\ell$ but the product of the propagators times the node factors does not change. This means that no overall change is produced, except for the differentiated propagator which changes its sign. By summing over the two considered clusters we obtain zero because of the change of sign of the differentiated propagator. Hence the assertion follows. ■

**Remark 4.14.** Lemma 4.13 is the counterpart of (3.17) for the renormalised self-energies.

Set $\Theta^{\mathcal{R},n}_{k,\boldsymbol{\nu},h} = \{\theta \in \Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},h} : n_\ell \le n \text{ for all } \ell \in L(\theta)\}$ and define

$$\boldsymbol{F}^{\mathcal{R},n}(\varepsilon,\beta_0) := \sum_{k\ge 0} \varepsilon^k \sum_{\boldsymbol{\theta}\in\Theta^{\mathcal{R},n}_{k+1,\mathbf{0},\boldsymbol{\alpha}}} \boldsymbol{\mathscr{V}}(\boldsymbol{\theta}), \qquad G^{\mathcal{R},n}(\varepsilon,\beta_0) := \sum_{k\ge 0} \varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R},n}_{k+1,\mathbf{0},\beta}} \mathscr{V}(\theta). \tag{4.14}$$

**Lemma 4.15.** *Assume* $(\varepsilon, \beta_0) \in \mathcal{S}_p$. *Then one has* $\varepsilon \partial_{\beta_0} G^{\mathcal{R},n}(\varepsilon, \beta_0) = \mathcal{M}^{[n]}_{\beta,\beta}(0) + O(\varepsilon^2 \mathrm{e}^{-C2^{m_{n+1}}})$, *for some positive constant* $C$, *for all* $n \leq p$.

The proof of the result above essentially follows the lines of the proof of Lemma 4.12 in [3] and Lemma 4.8 in [4]. In particular it does not depend on the Hamiltonian structure of the equations of motion.

**Remark 4.16.** From Lemma 4.15 it follows that, if $(\varepsilon, \beta_0) \in \mathcal{S}$, one can define

$$\mathcal{M}^{[\infty]}(x) := \lim_{n\to\infty} \mathcal{M}^{[n]}(x), \qquad \boldsymbol{G}^{\mathcal{R}}(\varepsilon, \beta_0) := \lim_{n\to\infty} \boldsymbol{G}^{\mathcal{R},n}(\varepsilon, \beta_0),$$

with $\boldsymbol{G}^{\mathcal{R},n}(\varepsilon, \beta_0) := (\boldsymbol{F}^{\mathcal{R},n}(\varepsilon, \beta_0), G^{\mathcal{R},n}(\varepsilon, \beta_0))$ and one has

$$\mathcal{M}^{[\infty]}_{\beta,\beta}(0) = \varepsilon \partial_{\beta_0} G^{\mathcal{R}}(\varepsilon, \beta_0). \tag{4.15}$$

Note that (4.15) is pretty much the same equality provided by Lemma 4.8 in [3], adapted to the present case.

### 4.4 A suitable assumption: bifurcation equations

Here we shall see how to solve the bifurcation equations (2.2c) and (2.2d) under the assumption that Property 1 is satisfied; again Property 1 assures that all quantities are well defined. We shall see that (2.2c) is automatically satisfied, while (2.2d) requires for $\beta_0$ to be properly chosen as a function of $\varepsilon$.

**Lemma 4.17.** *For any* $(\varepsilon, \beta_0) \in \mathcal{S}$ *one has* $\boldsymbol{F}^{\mathcal{R}}(\varepsilon, \beta_0) = 0$.

*Proof.* Consider a tree $\theta \in \Theta^{\mathcal{R}}_{k,0,\alpha_i}$ (that is a contribution to $F^{[k-1]}_i(\varepsilon, \beta)$) with root line $\ell_\theta$ such that $u_{\ell_\theta} = \alpha_i$ (of course $e_{\ell_\theta} = \alpha_i$), so that the propagator of the root line is 1. Now consider all trees $\theta'$ obtained $\theta$ by detaching the root line $\ell_\theta$ and reattaching it to all nodes $v \in N(\theta)$. By detaching $\ell_\theta$ from $v \in N(\theta)$ and reattaching it to another node $w \in N(\theta)$, the lines $\ell \in \mathcal{P}(v,w)$ (we are using the same notation as in the proof of Lemma 4.13) change their direction. In this case, given a node $v_1 \in N(\mathcal{P}(v,w)) \backslash \{v,w\}$, call $\ell_{v_1}, \ell'_{v_1} \in \mathcal{P}(v,w)$ the lines exiting and entering $v_1$ respectively. The node factor $\mathcal{F}_{v_1}$ does not change its sign if $u_{\ell_{v_1}} = e_{\ell'_{v_1}} = \beta$ or $u_{\ell_{v_1}}, e_{\ell_{v_2}} \in \{\alpha_1, \dots, \alpha_d\}$ when considering $v_1$ as a node in $\theta'$, otherwise the sign of $\mathcal{F}_{v_1}$ changes. The node factor $\mathcal{F}_v$ does not change its sign only if $e_{\ell_v} \in \{\alpha_1, \dots, \alpha_d\}$, while the node factor $\mathcal{F}_w$ does not change its sign only if $u_{\ell_w} \in \{\alpha_1, \dots, \alpha_d\}$. Moreover, given a line $\ell \in \mathcal{P}(v,w)$, thanks to Lemma 4.12 the corresponding propagator does not change its sign when one considers $\ell$ as a line of $\theta'$ only if $e_\ell = u_\ell = \beta$ or $e_\ell, u_\ell \in \{\alpha_1, \dots, \alpha_d\}$. Since one has $u_{\ell_\theta} = \alpha_i$ then the number of changes of sign, of both the propagators and of the node factors, is even, so that the overall product does not change. But in this case, the value of $\theta'$ differs from the value of $\theta$ by a factor $\mathrm{i}\boldsymbol{\nu}_v$, if $v$ is the node which the root line is attached to. The sum of all such values is zero because $\sum_{v\in N(\theta)} \boldsymbol{\nu}_v = 0$.

Let us now consider a tree $\theta \in \Theta^{\mathcal{R}}_{k,0,\alpha_i}$ with $u_{\ell_\theta} = \alpha_j$ with $j \neq i$ or $u_{\ell_\theta} = \beta$. In this case the value of the tree is zero because the propagator of the root line is $(\mathbb{1})_{e_{\ell_\theta}, u_{\ell_\theta}} = 0$. Of course we can reason in the same way for any $i = 1, \dots, d$, therefore the assertion follows. ∎

Now consider the equation

$$G^{\mathcal{R}}(\varepsilon,\beta_0)=0. \tag{4.16}$$

One cannot reason as in Lemma 4.17 above, because in principle there can be nonzero terms since the first order: in such a case, we have to consider (4.16) as an implicit function problem and fix $\beta_0=\beta_0(\varepsilon)$ in a suitable way.

**Lemma 4.18.** *Assume that there exists $\bar\varepsilon>0$ such that $\mathcal{S}=[-\bar\varepsilon,\bar\varepsilon]\times\mathbb{T}$. Then there exist at least two values $\beta_0=\beta_0(\varepsilon)$ such that (4.16) is satisfied for $\varepsilon$ small enough.*

*Proof.* Thanks to the variational nature of the Hamilton equations, the function $G^{\mathcal{R}}$ is the $\beta_0$-derivative of the average of the Lagrangian $\gamma$ computed along the solution of the range equations (see the comments at the end of Section 3). Under the assumption that Property 1 holds for all $\beta_0\in\mathbb{T}$, $\gamma$ is $C^\infty$ for any $\beta_0\in\mathbb{T}$ and hence it has at least two critical points. ■

If Property 1 does not hold for all $\beta_0\in\mathbb{T}$ — or simply if this is not known —, we have to reason in a different way. First of all, let us formally expand $G^{\mathcal{R}}$ in power series in $\varepsilon$, by writing $G^{\mathcal{R}}(\beta_0)=\sum_{k\ge0}\varepsilon^k G^{\mathcal{R}(k)}(\beta_0)$. Note that $G^{\mathcal{R}(k)}(\beta_0)$ equals $[\partial_\beta f(\boldsymbol{\alpha},\beta)]^{(k)}_{\mathbf{0}}$ and hence can be written as a sum over non-renormalised trees as in (3.6b).

If one has $G^{\mathcal{R}(k)}(\boldsymbol{\alpha}_0,\beta_0)\equiv0$ for all $k\ge0$, then (4.16) is formally satisfied. Otherwise the following condition makes sense.

**Condition 2.** *Either $G^{\mathcal{R}(0)}(\beta_0)$ is not identically vanishing or there exists $k_0\in\mathbb{N}$ such that $G^{\mathcal{R}(k)}(\beta_0)\equiv0$ for $0\le k<k_0$, while $G^{\mathcal{R}(k_0)}(\beta_0)$ is not identically vanishing.*

**Remark 4.19.** We know that $G^{\mathcal{R}(k_0)}$ is the derivative with respect to $\beta_0$ of the time average of the $k_0$-th order Lagrangian $\gamma^{(k_0)}$ computed along the formal solution. Since $\gamma^{(k_0)}$ is analytic and periodic in $\beta_0$, and it is not identically constant, then it admits at least one maximum and one minimum. In particular, for $\sigma=\pm$, there exist $\beta^*_{0,\sigma}\in\mathbb{T}$ and $\mathfrak{n}_\sigma\in\mathbb{N}$, with $\mathfrak{n}_\sigma$ odd, such that $(\sigma1)^{k_0+1}\partial_{\beta_0}^{\mathfrak{n}_\sigma}G^{\mathcal{R}(k_0)}(\beta^*_{0,\sigma})<0$.

**Remark 4.20.** Under Condition 2 we can write

$$G^{\mathcal{R}}(\varepsilon,\beta_0)=\varepsilon^{k_0}\left(G^{\mathcal{R}(k_0)}(\beta_0)+G^{\mathcal{R}(>k_0)}(\varepsilon,\beta_0)\right),$$

where $k_0\ge0$ and $G^{\mathcal{R}(>k_0)}(\varepsilon,\beta_0)=O(\varepsilon)$; hence we can solve the equation of motion up to order $k_0$ without fixing the parameter $\beta_0$.

With the notations in (3.16), the condition that $G^{\mathcal{R}}(\varepsilon,\beta_0)$ identically vanishes to all orders is equivalent to the condition that $\mathscr{L}^{(k)}_{\beta,\beta}\equiv0$ for all $k\ge1$ (see comments at the end of Section 3). Therefore the only condition left when neither Condition 1 nor Condition 2 are satisfied is the following.

**Condition 3.** *One has $\mathscr{L}^{(k)}_{\beta,\beta}\equiv0$ for all $k\ge1$ and there exists $i=1,\dots,d$ and $k_1\in\mathbb{N}$ such that $\mathcal{D}^{(k)}_{\alpha_i,\beta}\equiv0$ for $k<k_1$ while $\mathcal{D}^{(k_1)}_{\alpha_i,\beta}$ does not vanishes identically.*

**Remark 4.21.** If we take the formal expansion of the functions $\boldsymbol{F}^{\mathcal{R}}(\varepsilon,\beta_0)$, $G^{\mathcal{R}}(\varepsilon,\beta_0)$ and $\mathcal{M}^{[\infty]}_{u,e}(0)$, $u,e\in\{\alpha_1,\dots,\alpha_d,\beta\}$, we obtain the tree expansions of Section 3, where the self-energy clusters are allowed. Then, as we have seen in Lemma 3.5, the identity (4.15) holds to any perturbation order. If we assume Condition 2 we obtain

$$\sum_{k=1}^{k_0-1}\varepsilon^k[\mathcal{M}^{[\infty]}_{\beta,\beta}(0)]^{(k)}\equiv 0 \qquad \Longrightarrow \qquad \left|\sum_{k=1}^{k_0-1}\varepsilon^k[\mathcal{M}^{[n]}_{\beta,\beta}(0)]^{(k)}\right|\le\varepsilon^2A_1\,\mathrm{e}^{-A_2 2^{m_n}}, \tag{4.17}$$

for some positive constants $A_1$ (depending on $k_0$) and $A_2$. If Condition 3 is satisfied, then (4.17) is satisfied for any (finite) $k_0$, so that $\mathcal{M}^{[\infty]}_{\beta,\beta}(0)\to 0$ faster than any power as $\varepsilon\to 0$; moreover in such a case $[\partial_x\mathcal{M}^{[\infty]}_{\boldsymbol{\alpha},\beta}(0)]^{(k)}\equiv 0$ for all $k=1,\dots,k_1-1$ and

$$\left|\sum_{k=1}^{k_1-1}\varepsilon^k[\partial_x\mathcal{M}^{[n]}_{\boldsymbol{\alpha},\beta}(0)]^{(k)}\right|\le\varepsilon^2B_1\,\mathrm{e}^{-B_2 2^{m_n}}, \tag{4.18}$$

for some positive constants $B_1$ (depending on $k_1$) and $B_2$.

Assume Condition 2 and fix $\sigma\in\{\pm1\}$. Suppose for the time being $\mathcal{S}$ to be an open set containing $(0,\beta^*_{0,\sigma})$. Then, by reasoning as for Lemma 4.15 of [3], one can show that (i) there exists a neighbourhood $U$ of $\varepsilon=0$ such that the implicit function equation (4.16) admits in $\mathcal{S}$ a solution $\beta_0=\beta_{0,\sigma}(\varepsilon)$, with $\varepsilon\in U$ and $\beta_{0,\sigma}(0)=\beta^*_{0,\sigma}$; (ii) for $\operatorname{sign}\varepsilon=\sigma 1$ one has $\varepsilon\partial_{\beta_0}G^{\mathcal{R}}(\varepsilon,\beta_{0,\sigma}(\varepsilon))\le 0$. Then for $\varepsilon\in U$, with $\operatorname{sign}\varepsilon=\sigma1$, and $\beta_0=\beta_{0,\sigma}(\varepsilon)$, the functions $\boldsymbol{\alpha}^{\mathcal{R}}$, $\beta^{\mathcal{R}}$ in (4.7) are well defined and one has

$$\begin{aligned}\boldsymbol{F}^{\mathcal{R}}(\varepsilon,\beta_0)&=[-\partial_{\boldsymbol{\alpha}}f(\boldsymbol{\alpha}^{\mathcal{R}}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_0),\beta^{\mathcal{R}}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_0))]_{\mathbf{0}},\\ G^{\mathcal{R}}(\varepsilon,\beta_0)&=[\partial_{\beta}f(\boldsymbol{\alpha}^{\mathcal{R}}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_0),\beta^{\mathcal{R}}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_0))]_{\mathbf{0}},\end{aligned}$$

and hence by Lemma 4.10 the functions $\boldsymbol{\alpha}^{\mathcal{R}}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_{0,\sigma}(\varepsilon))$ and $\beta^{\mathcal{R}}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_{0,\sigma}(\varepsilon))$ solve the equation of motion (1.2). However, the argument above is not sufficient to prove the existence of a quasi-periodic solution with frequency $\boldsymbol{\omega}$, because we have assumed — without proving — that Property 1 is satisfied on a non-empty open set. In Section 4.5 we shall show that, thanks to the symmetry property of Lemma 4.13 and the identity of Lemma 4.15, Property 1 is satisfied along a suitable curve $\beta_0=\overline{\beta}_0(\varepsilon)$ such that $G^{\mathcal{R}}(\varepsilon,\overline{\beta}_0(\varepsilon))=0$ and $\overline{\beta}_0(\varepsilon)$ is continuous for $\varepsilon\neq 0$. More precisely, we shall proceed by induction as follows. Under Condition 3, assuming that Property 1-$n$ holds for all $n<p$ will imply, thanks to the bounds and symmetry properties seen in the previous sections, that also Property 1-$p$ holds. The discussion of Condition 2 is more delicate: we shall need to introduce some auxiliary quantities for which an analogous result is obtained and then show that this yields the same result for the self-energies.

## 4.5 Convergence of the resummed series

First of all we recall that if we formally expand the resummed series, we obtain the same formal expansion as in Section 3. In particular, either Condition 1 is satisfied — and hence we can reason as in Section 3 — or at least one among $\mathscr{L}^{(k)}_{\beta,\beta}$ and $\mathcal{D}^{(k)}_{\alpha_i,\beta}$ for $i=1,\dots,d$ is not identically vanishing. Let us start from the case in which Condition 3 holds.

**Lemma 4.22.** *Assume Condition 3. Then $\mathcal{M}$ satisfies Property* 1 *for all $\beta_0 \in \mathbb{T}$ and $\varepsilon$ small enough.*

*Proof.* We shall prove that $\mathcal{M}$ satisfies Property 1-$p$ for all $p \geq 0$, by induction on $p$. Property 1-0 is trivially satisfied for $\varepsilon$ small enough. Indeed the matrix $\mathcal{M}^{[-1]}(x)$ defined in (4.11) is the null matrix, so that $\mathcal{G}^{[0]}(x) = \mathbb{1}\Psi_0(x)/x^2$, and hence $\|\mathcal{G}^{[0]}(x)\| \leq c_0/x^{2(d+1)}$, for some constant $c_0 > 0$. Assume that $\mathcal{M}$ satisfies Property 1-$p$. By Lemmas 4.8 and 4.13

$$\mathcal{M}^{[p]}(x) = \begin{pmatrix} 0_d & \mathbf{0} \\ \mathbf{0} & \mathcal{M}^{[p]}_{\beta,\beta}(0) \end{pmatrix} + x \begin{pmatrix} 0_d & \partial_x \mathcal{M}^{[p]}_{\boldsymbol{\alpha},\beta}(0) \\ \partial_x \mathcal{M}^{[p]}_{\beta,\boldsymbol{\alpha}}(0) & 0 \end{pmatrix} + O(\varepsilon^2 x^2).$$

We have to bound from below the determinant of the matrix $x^2\mathbb{1} - \mathcal{M}^{[p]}(x)$: we have

$$\det(x^2\mathbb{1} - \mathcal{M}^{[p]}(x)) = x^{2d}\Big(x^2 - \Big(\mathcal{M}^{[p]}_{\beta,\beta}(0) - \Big|\partial_x \mathcal{M}^{[p]}_{\boldsymbol{\alpha},\beta}(0)\Big|^2\Big) + O(\varepsilon^2 x^2)\Big), \tag{4.19}$$

so that we have to show that

$$\mathcal{M}^{[p]}_{\beta,\beta}(0) \leq \frac{x^2}{2} + \Big|\partial_x \mathcal{M}^{[p]}_{\boldsymbol{\alpha},\beta}(0)\Big|^2, \qquad \text{with} \quad x^2 \geq \frac{\alpha_{m_{p+1}}(\boldsymbol{\omega})^2}{2^8}.$$

Since

$$\Big|\partial_x \mathcal{M}^{[p]}_{\boldsymbol{\alpha},\beta}(0)\Big| \leq \left|\sum_{k=1}^{k_1} \varepsilon^k [\partial_x \mathcal{M}^{[p]}_{\boldsymbol{\alpha},\beta}(0)]^{(k)}\right| + O(\varepsilon^{k_1+1}) \leq \varepsilon^2 B_1 \mathrm{e}^{-B_2 2^{m_p}} + O(\varepsilon^{k_1+1}),$$

and (use Remark 4.21 with $k_0 = 2k_1 + 2$)

$$\Big|\mathcal{M}^{[p]}_{\beta,\beta}(0)\Big| \leq \left|\sum_{k=1}^{2k_1+2} \varepsilon^k [\mathcal{M}^{[p]}_{\beta,\beta}(0)]^{(k)}\right| + O(\varepsilon^{2k_1+3}) \leq \varepsilon^2 A_1\, \mathrm{e}^{-A_2 2^{m_p}} + O(\varepsilon^{2k_1+3}),$$

the assertion follows by the condition $\mathcal{B}(\boldsymbol{\omega}) < \infty$. ∎

By Lemma 4.22 we can apply Lemma 4.18 and deduce, in the case of Condition 3, the existence of at least two $d$-dimensional invariant tori. Therefore we are left with Condition 2.

First of all, for all $n \geq 0$, we define the $C^\infty$ non-decreasing functions $\xi_n$ such that

$$\xi_n(x) := \begin{cases} 1, & x \leq \alpha_{m_{n+1}}(\boldsymbol{\omega})^2/2^{12}, \\ 0, & x \geq \alpha_{m_{n+1}}(\boldsymbol{\omega})^2/2^{11}, \end{cases} \tag{4.20}$$

and set $\xi_{-1}(x) = 1$. Define recursively, for all $n \geq 0$, the *regularised propagators*

$$\overline{\mathcal{G}}^{[n]}(x) := \Psi_n(x) \left(x^2\mathbb{1} - \overline{\mathcal{M}}^{[n-1]}(x)\xi_{n-1}(\Delta_{n-1})\right)^{-1}$$

with $\overline{\mathcal{M}}^{[-1]}(x) = \mathcal{M}^{[-1]}(x)$ as given by (4.11) and, for all $n \geq 0$,

$$\overline{\mathcal{M}}^{[n]}(x) := \overline{\mathcal{M}}^{[n-1]}(x) + \chi_n(x)\overline{M}^{[n]}(x),$$

where we have set for all $u,e\in\{\alpha_1,\dots,\alpha_d,\beta\}$,

$$\overline{M}^{[n]}_{u,e}(x) := \sum_{T\in\mathfrak{R}_{n,u,e}} \varepsilon^{k(T)}\overline{\mathscr{V}}_T(x),$$

with

$$\overline{\mathscr{V}}_T(x) := \Bigg(\prod_{v\in N(T)} \mathcal{F}_v\Bigg)\Bigg(\prod_{\ell\in L(T)} \overline{\mathcal{G}}^{[n_\ell]}_{e_\ell,u_\ell}(\boldsymbol{\omega}\cdot\boldsymbol{\nu}_\ell)\Bigg)$$

and

$$\Delta_{n-1}=\Delta_{n-1}(\varepsilon,\beta_0) := \overline{\mathcal{M}}^{[n-1]}_{\beta,\beta}(0;\varepsilon,\beta_0) - \sum_{k=0}^{k_0-1}\varepsilon^k[\overline{\mathcal{M}}^{[n-1]}_{\beta,\beta}(0;\varepsilon,\beta_0)]^{(k)}.$$

Set also $\overline{\mathcal{M}} := \{\overline{\mathcal{M}}^{[n]}(x)\}_{n\geq -1}$ and $\overline{\mathcal{M}}^{\xi} := \{\overline{\mathcal{M}}^{[n]}(x)\xi_n(\Delta_{n-1})\}_{n\geq-1}$.

**Lemma 4.23.** *$\overline{\mathcal{M}}^{\xi}$ satisfies Property 1 for $\varepsilon$ small enough and any $\beta_0\in\mathbb{T}$.*

*Proof.* We shall prove that $\overline{\mathcal{M}}^{\xi}$ satisfies Property 1-$p$ for all $p\geq 0$, by induction on $p$. Property 1-0 is trivially satisfied for $\varepsilon$ small enough. Indeed the matrix $\overline{\mathcal{M}}^{[-1]}(x)$ is self-adjoint, so that also $\overline{\mathcal{G}}^{[0]}(x)$ is self-adjoint and we can estimate its eigenvalues and conclude $\|\overline{\mathcal{G}}^{[0]}(x)\|\leq \bar{c}_0/x^{2(d+1)}$, for some $\bar{c}_0>0$. Assume then that $\overline{\mathcal{M}}^{\xi}$ satisfies Property 1-$p$. Then we can repeat almost word by word the proof of Lemmas 4.8 and 4.13, as done in [3, 4] so as to obtain

$$\overline{\mathcal{M}}^{[p]}(x) = \begin{pmatrix} 0_d & \mathbf{0} \\ \mathbf{0} & \overline{\mathcal{M}}^{[p]}_{\beta,\beta}(0)\end{pmatrix} + x\begin{pmatrix} 0_d & \partial_x\overline{\mathcal{M}}^{[p]}_{\boldsymbol{\alpha},\beta}(0) \\ \partial_x\overline{\mathcal{M}}^{[p]}_{\beta,\boldsymbol{\alpha}}(0) & 0\end{pmatrix} + O(\varepsilon^2x^2). \tag{4.21}$$

We have to bound from below the determinant of the matrix $x^2\mathbb{1}-\overline{\mathcal{M}}^{[p]}(x)\xi_p(\Delta_p)$. From (4.21) it is easy to check that such determinant is

$$x^{2d}\Big(x^2 - \Big(\overline{\mathcal{M}}^{[p]}_{\beta,\beta}(0) - \Big|\partial_x\overline{\mathcal{M}}^{[p]}_{\boldsymbol{\alpha},\beta}(0)\Big|^2\Big)\xi_p(\Delta_p) + O(\varepsilon^2x^2)\Big). \tag{4.22}$$

Thanks to the definition of the functions $\xi_p$, since

$$\sum_{k=0}^{k_0-1}\varepsilon^k[\overline{\mathcal{M}}^{[p]}_{\beta,\beta}(0)]^{(k)} = \sum_{k=0}^{k_0-1}\varepsilon^k[\mathcal{M}^{[p]}_{\beta,\beta}(0)]^{(k)}$$

by Remark 4.21, one has

$$x^2 - \Big(\overline{\mathcal{M}}^{[p]}_{\beta,\beta}(0) - \Big|\partial_x\overline{\mathcal{M}}^{[p]}_{\boldsymbol{\alpha},\beta}(0)\Big|^2\Big)\xi_p(\Delta_p) \geq x^2 - \Big(\overline{\mathcal{M}}^{[p]}_{\beta,\beta}(0)\Big)\xi_p(\Delta_p) \geq \frac{x^2}{2}.$$

Then $\|\overline{\mathcal{G}}^{[p+1]}(x)\|\leq \bar{c}_1/x^{2(d+1)}$, for some positive constant $\bar{c}_1$, that is Property 1-$(p+1)$ with $c_2=2(d+1)$ in Definition 4.4. ■

Set

$$\overline{\boldsymbol{a}}_{\boldsymbol{\nu}}^{[k]}(\varepsilon;\boldsymbol{\alpha}_0,\beta_0) := \sum_{\boldsymbol{\theta}\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\boldsymbol{\alpha}}} \overline{\boldsymbol{\mathscr{V}}}(\boldsymbol{\theta}), \qquad \overline{b}_{\boldsymbol{\nu}}^{[k]}(\varepsilon;\boldsymbol{\alpha}_0,\beta_0) := \sum_{\theta\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\beta}} \overline{\mathscr{V}}(\theta), \qquad \boldsymbol{\nu}\neq\boldsymbol{0}, \tag{4.23}$$

where, for $\boldsymbol{\theta}=(\theta_1,\dots,\theta_d)\in\Theta^{\mathcal{R}}_{k,\boldsymbol{\nu},\boldsymbol{\alpha}}$ we denoted $\overline{\boldsymbol{\mathscr{V}}}(\boldsymbol{\theta}) := (\overline{\mathscr{V}}(\theta_1),\dots,\overline{\mathscr{V}}(\theta_d))$, and define

$$\begin{aligned}
&\overline{\boldsymbol{a}}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_0) = \sum_{k\ge1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}\overline{\boldsymbol{a}}_{\boldsymbol{\nu}}^{[k]}, \qquad \overline{b}(t;\varepsilon,\boldsymbol{\alpha}_0,\beta_0) = \sum_{k\ge1}\varepsilon^k \sum_{\boldsymbol{\nu}\in\mathbb{Z}^d_*} \mathrm{e}^{\mathrm{i}\boldsymbol{\nu}\cdot\boldsymbol{\omega}t}\overline{b}_{\boldsymbol{\nu}}^{[k]},\\
&\overline{G}(\varepsilon,\beta_0) := \sum_{k\ge0}\varepsilon^k\overline{G}^{(k)}(\varepsilon,\beta_0) := \sum_{k\ge0}\varepsilon^k \sum_{\theta\in\Theta^{\mathcal{R}}_{k+1,\boldsymbol{0},\beta}} \overline{\mathscr{V}}(\theta).
\end{aligned} \tag{4.24}$$

A result analogous to Lemma 4.9 holds and can be proved in the same way (see [3, 4]), so we conclude that the series (4.24) converge. However, because of the presence of the functions $\xi_n$, in principle no equivalent of Lemma 4.10 applies in this case. In other words, in general the functions (4.24) are no longer solutions of the equations of motions, unless $\xi_n(\Delta_n)\equiv 1$. Therefore we would like to show that, for any $\varepsilon$ small enough, it is possible to fix suitably $\beta_0=\overline{\beta}_0(\varepsilon)$ in such a way that $\xi_n(\Delta_n)$ be identically one.

**Lemma 4.24.** *One has* $[\overline{G}(\varepsilon,\beta_0)]^{(k)} = [G^{\mathcal{R}}(\varepsilon,\beta_0)]^{(k)}$ *for all* $k=0,\dots,k_0$.

*Proof.* Set $\Theta^{\mathcal{R}(n)}_{k,\boldsymbol{\nu},\beta} := \{\theta\in\Theta^{\mathcal{R},n}_{k,\boldsymbol{\nu},\beta} : \exists\ell\in L(\theta) \text{ such that } n_\ell=n\}$ and write

$$\overline{G}(\varepsilon,\beta_0) = \sum_{k\ge0}\varepsilon^k\sum_{n\ge0}\sum_{\theta\in\Theta^{\mathcal{R}(n)}_{k+1,\boldsymbol{0},\beta}} \overline{\mathscr{V}}(\theta).$$

Note that if $\theta\in\Theta^{\mathcal{R}(n)}_{k,\boldsymbol{0},\beta}$ one has $\prod_{v\in N(\theta)}|\mathcal{F}_v| \le E_1^k\mathrm{e}^{-E_2 2^{m_n}}$, for some constants $E_1,E_2$. Moreover one can write formally

$$\overline{\mathcal{G}}^{[n_\ell]}(x) = \Psi_{n_\ell}(x)\frac{1}{x^2}\Big(\mathbb{1} + \sum_{m\ge1}\Big(\frac{1}{x^2}\overline{\mathcal{M}}^{[n_\ell-1]}(x)\xi_{n_\ell-1}(\Delta_{n_\ell-1})\Big)^m\Big),$$

and $\xi_{n_\ell-1}(\Delta_{n_\ell-1}) = 1+\xi'_{n_\ell-1}(\Delta^*)\Delta_{n_\ell-1}$ for some $\Delta^*$, where $\Delta_{n_\ell-1}=O(\varepsilon^{k_0})$ and

$$|\xi'_{n_\ell-1}(\Delta^*)| \le \frac{E_3}{\alpha_{m_{n_\ell}}(\boldsymbol{\omega})^2} \le \frac{E_3}{\alpha_{m_n}(\boldsymbol{\omega})^2},$$

for some positive constant $E_3$ independent of $n$. Hence the assertion follows. ∎

Define

$$\overline{\mathcal{M}}^{[\infty]}(x) := \lim_{n\to\infty}\overline{\mathcal{M}}^{[n]}(x), \tag{4.25}$$

and note that, by Lemma 4.23, the limit in (4.25) is well defined, and it is $C^\infty$ in both $\varepsilon$ and $\beta_0$.

For $\sigma=\pm$ let us introduce the $C^\infty$ functions $R(\varepsilon,\beta_0)$ such that $\overline{\mathcal{M}}^{[\infty]}_{\beta,\beta}(0) = \varepsilon\partial_{\beta_0}R(\varepsilon,\beta_0)$. Note that $R(\varepsilon,\beta_0)=\varepsilon^{k_0}\Gamma(\varepsilon,\beta_0)$, with $\Gamma(\varepsilon,\beta_0) = G^{\mathcal{R}(k_0)}(\beta_0)+O(\varepsilon)$, so that $\Gamma(0,\beta^*_{0,\sigma})=0$ and $\partial^{\mathfrak{n}_\sigma}_{\beta_0}\Gamma_\sigma(0,\beta^*_{0,\sigma})\neq0$, with $\beta^*_{0,\sigma}$ and $\mathfrak{n}_\sigma$ defined in Remark 4.19. For any of such function consider the implicit function equation

$$R(\varepsilon,\beta_0)=0. \tag{4.26}$$

**Lemma 4.25.** *Assume Condition 2. There exist a neighbourhood $U$ of $\varepsilon = 0$ and, for $\varepsilon \in U$, a solution $\beta_0 = \overline{\beta}_0(\varepsilon)$ to the implicit function equation (4.26), such that*

$$\lim_{\varepsilon \to 0^\sigma} \overline{\beta}_0(\varepsilon) = \beta^*_{0,\sigma}, \quad \sigma = \pm, \qquad \varepsilon \partial_{\beta_0} R(\varepsilon, \overline{\beta}_0(\varepsilon)) \le 0.$$

*Moreover $\overline{\beta}_0(\varepsilon)$ is continuous in $U$ for $k_0$ odd and in $U \setminus \{0\}$ for $k_0$ even.*

*Proof.* By construction, all the functions $\Gamma(\varepsilon, \beta_0)$ are smooth for $\beta_0 \in \mathbb{T}$ and $\varepsilon$ small enough. Then there exist two half-neighbourhoods $V_{\sigma,-}$ and $V_{\sigma,+}$ of $\beta_0 = \beta^*_{0,\sigma}$ such that $\Gamma(0, \beta_0) > 0$ for $\beta_0 \in V_{\sigma,+}$ and $\Gamma(0, \beta_0) < 0$ for $\beta_0 \in V_{\sigma,-}$. By continuity, there exist a neighbourhood $U_\sigma = (-\bar{\varepsilon}_\sigma, \bar{\varepsilon}_\sigma)$ and a continuous curve $\beta_{0,\sigma}(\varepsilon)$ such that $\beta_{0,\sigma}(0) = \beta^*_{0,\sigma}$ and $\Gamma(\varepsilon, \beta_{0,\sigma}(\varepsilon)) \equiv 0$ for $\varepsilon \in U_\sigma$. Moreover if $\partial^{\mathrm{n}_\sigma}_{\beta_0} G^{\mathcal{R}(k_0)}(\beta^*_{0,\sigma}) > 0$, then $V_{\sigma,+}$ and $V_{\sigma,-}$ are of the form $(\beta^*_{0,\sigma}, v_{\sigma,+})$ and $(v_{\sigma,-}, \beta^*_{0,\sigma})$, respectively, and therefore $\partial_{\beta_0}\Gamma(\varepsilon, \beta_{0,\sigma}(\varepsilon)) \ge 0$ for all $\varepsilon \in U_\sigma$. If on the contrary $\partial^{\mathrm{n}_\sigma}_{\beta_0} G^{\mathcal{R}(k_0)}(\beta^*_{0,\sigma}) < 0$, one has $V_{\sigma,+} = (v_+, \beta^*_{0,\sigma})$ and $V_{\sigma,-} = (\beta^*_{0,\sigma}, v_-)$, and then $\partial_{\beta_0}\Gamma(\varepsilon, \beta_{0,\sigma}(\varepsilon)) \le 0$ for all $\varepsilon \in U_\sigma$.

If $k_0$ is odd, then $\beta^*_{0,+} = \beta^*_{0,-}$ and hence one can take $\overline{\beta}_0(\varepsilon) = \beta_{0,+}(\varepsilon) = \beta_{0,-}(\varepsilon)$ in such a way that it is a continuous function of $\varepsilon \in U_+ = U_-$. If $k_0$ is even, then one has $\overline{\beta}_0(\varepsilon) = \beta_{0,+}(\varepsilon)$ for $\varepsilon > 0$ and $\overline{\beta}_0(\varepsilon) = \beta_{0,-}(\varepsilon)$ for $\varepsilon < 0$, so that $\overline{\beta}_0(\varepsilon)$ has a discontinuity at $\varepsilon = 0$. ■

**Lemma 4.26.** *Assume Condition 2. Let $U$ and $\overline{\beta}_0(\varepsilon)$ be the neighbourhood and the solution referred to in Lemma 4.25, respectively. Then whenever $\varepsilon \in U$ and $\beta_0 = \overline{\beta}_0(\varepsilon)$, one has $\xi_n(\Delta_n) \equiv 1$ for all $n \ge 0$.*

*Proof.* By Lemma 4.25, for $\varepsilon \in U$ and $\beta_0 = \overline{\beta}_0(\varepsilon)$, one has $\overline{\mathcal{M}}^{[\infty]}_{\beta,\beta}(0) = \varepsilon \partial_{\beta_0} R(\varepsilon, \overline{\beta}_0(\varepsilon))$. Hence, since the matrices $\overline{M}^{[n]}(x)$ satisfy bounds analogous to those in Lemma 4.6, possibly renaming the constants, one has for $\beta = \overline{\beta}_0(\varepsilon)$

$$\begin{aligned}
\overline{\mathcal{M}}^{[n]}_{\beta,\beta}(0) - \sum_{k=1}^{k_0-1} \varepsilon^k [\overline{\mathcal{M}}^{[n]}_{\beta,\beta}(0)]^{(k)} &\le \overline{\mathcal{M}}^{[n]}_{\beta,\beta}(0) - \overline{\mathcal{M}}^{[\infty]}_{\beta,\beta}(0) + \varepsilon^2 A_1 \mathrm{e}^{-A_2 2^{m_{n+1}}} \\
&\le \sum_{p \ge n+1} |\overline{M}^{[p]}_{\beta,\beta}(0)| + \varepsilon^2 A_1 \mathrm{e}^{-A_2 2^{m_{n+1}}} \le 2K_0 \varepsilon^2 \mathrm{e}^{-\overline{K}_0 2^{m_{n+1}}} + \varepsilon^2 A_1 \mathrm{e}^{-A_2 2^{m_{n+1}}} \le \frac{\alpha^2_{m_{n+1}}(\boldsymbol{\omega})}{2^{13}},
\end{aligned}$$

so the assertion follows by the definition of $\xi_n$. ■

The following result concludes the proof of the existence of an invariant $d$-dimensional torus under Condition 2.

**Lemma 4.27.** *Assume Condition 2 and let $\overline{\beta}_0(\varepsilon)$ be as in Lemma 4.25. One can choose the function $R(\varepsilon, \beta_0)$ such that $R(\varepsilon, \overline{\beta}_0(\varepsilon)) = G^{\mathcal{R}}(\varepsilon, \overline{\beta}_0(\varepsilon)) \equiv 0$, where*

$$G^{\mathcal{R}}(\varepsilon, \overline{\beta}_0(\varepsilon)) := \lim_{n \to \infty} G^{\mathcal{R},n}(\varepsilon, \overline{\beta}_0(\varepsilon))$$

*and the functions $G^{\mathcal{R},n}$ are defined in (4.14). In particular $(\boldsymbol{\alpha}(t,\varepsilon), \beta(t,\varepsilon)) = (\boldsymbol{\alpha}_0 + \boldsymbol{\omega} t, \overline{\beta}_0(\varepsilon)) + (\boldsymbol{a}^{\mathcal{R}}(t; \varepsilon, \boldsymbol{\alpha}_0, \overline{\beta}_0(\varepsilon)), b^{\mathcal{R}}(t; \varepsilon, \boldsymbol{\alpha}_0, \overline{\beta}_0(\varepsilon)))$ defined in (4.7) solves the equation of motion (1.2)*

*Proof.* It follows from the results above. Indeed, for any primitive $R$ there is a curve $\overline{\beta}_0(\varepsilon)$ along which $\mathcal{M} = \overline{\mathcal{M}} = \overline{\mathcal{M}}^{\xi}$ (hence $\mathcal{M}$ satisfies Property 1) and $R(\varepsilon; \overline{\beta}_0(\varepsilon)) \equiv 0$. By Lemma 4.15 and the fact that $\mathcal{M}$ satisfies Property 1, also $G^{\mathcal{R}}$ is among the primitives of $\mathcal{M}^{[\infty]}_{\beta,\beta}$ and hence the assertion follows. ∎

**Remark 4.28.** If one considers a convex unperturbed Hamiltonian, e.g. with a plus sign instead of the minus sign in (1.1), one can try to proceed in the same way. Some parts of the construction simplify: for instance, the self-energies $\mathcal{M}^{(k)}(x,n)$ turn out to be self-adjoint and $(\mathcal{M}^{(k)}(x,n))^T = \mathcal{M}^{(k)}(-x,n)$. On the other hand, when dealing with Conditions 2 and 3, one has to bound from below determinants which have the form (4.22) or (4.19), respectively, with the major difference that a sign plus appears in front of the squared term; for instance (4.19) becomes

$$x^{2d}\Big(x^2 - \Big(\mathcal{M}^{[p]}_{\beta,\beta}(0) + \Big|\partial_x \mathcal{M}^{[p]}_{\boldsymbol{\alpha},\beta}(0)\Big|^2\Big) + O(\varepsilon^2 x^2)\Big).$$

Then information on the sign of $\mathcal{M}^{[p]}_{\beta,\beta}(0)$ is not enough to control the corrections to $x^2$ and hence no lower bound follows for the determinant. Therefore in order to recover Cheng's result further cancellations seem to be necessary. In turn this means that one should expect other symmetries to hold for the self-energies.

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